$D(n)$-QUINTUPLES WITH SQUARE ELEMENTS

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For an integer n, a set of m distinct nonzero integers with the property that the product of any two of its distinct elements plus n is a square, is called a Diophantine m-tuple with the property $D(n)$ or $D(n)$-m-tuple. The $D(1)$-m-tuples (with rational elements) are called simply (rational) Diophantine m-tuples, and have been studied since the ancient time.
The first example of a rational Diophantine quadruple was the set

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

found by Diophantus. Fermat found the first Diophantine quadruple in integers $\{1, 3, 8, 120\}$. Euler proved that the exist infinitely many rational Diophantine quintuples (see [18]), in particular he was able to extend the integer Diophantine quadruple found by Fermat, to the rational quintuple

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}.$$

Stoll [20] recently showed that this extension is unique.
In 1969, using linear forms in logarithms of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport [3] proved that if d is a positive integer such that $\{1, 3, 8, d\}$ forms a Diophantine quadruple, then d has to be 120. This result motivated the conjecture that there does not exist a Diophantine quintuple in integers. The conjecture has been proved recently by He, Togbé and Ziegler [17] (see also [4, 7]).
On the other hand, it is not known how large can a rational Diophantine tuple be. In 1999, Gibbs found the first example of rational Diophantine sextuple [16]

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

In 2017, Dujella, Kazalicki, Mikić and Szikszai proved that there are infinitely many rational Diophantine sextuples. Recently, Dujella, Kazalicki and Petričević in [13] proved that there are infinitely many rational Diophantine sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect squares, and in [12] they proved that there are infinitely many Diophantine sextuples containing two regular quadruples and one regular quintuple. No example of a rational Diophantine septuple is known. The Lang conjecture on varieties of general type implies that the number of elements of a rational Diophantine tuple is bounded by an absolute constant (see the introduction of [10]). Diophantine m-tuples have been studied over the rings other that $\mathbb{Z}$ and $\mathbb{Q}$, for example Dujella and Kazalicki [9] computed the number of Diophantine quadruples over finite fields. For more information on Diophantine m-tuples see the survey article [8].
Sets with $D(n)$ properties have also been extensively studied. It is easy to show that there are no integer $D(n)$-quadruples if $n \equiv 2 \pmod 4$, and it is know that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there is at least one $D(n)$-quadruple [6]. Recently, Bonciocat, Cipu and Mignotte [2] proved that there are no $D(-1)$-quadruples (as well as $D(-4)$-quadruples) thus leaving the existence of $D(n)$-quadruples in the remaining six sporadic cases open.
Dražić and Kazalicki [5] described rational $D(n)$-quadruples with fixed product of elements in terms of points on certain elliptic curves. It is not known if there is a rational Diophantine $D(n)$-quintuple for every n, and no example of rational $D(n)$-sextuple is known if n is not a perfect square.
One can also study m-tuples that have $D(n)$-property for more than one n. Adžaga, Dujella, Kreso and Tadić [1] presented several families of Diophantine triples which have $D(n)$-property for two distinct n's with $n \neq 1$ as well as some Diophantine triples which are $D(n)$-sets for three distinct n's with $n \neq 1$. Dujella and Petričević in [14] proved that there are infinitely many (essentially different) integer quadruples which are simultaneously $D(n_1)$-quadruples and $D(n_2)$-quadruples with $n_1 \neq n_2$, and in [15] showed that the same thing is true for three distinct n's (since the elements of their quadruples are squares one of n's is equal to zero). Our main result extends the previous results to quintuples.

Theorem 1 *There are infinitely many nonequivalent quintuples that have $D(n_1)$ property for some $n_1 \in \mathbb{N}$ such that all the elements in the quintuple are perfect squares. In particular, there are infinitely many nonequivalent integer quintuples that are simultaneously $D(n_1)$-quintuples and $D(n_2)$-quintuples with $n_1 \neq n_2$ since then we can take $n_2 = 0$.*

Note that if $\{a, b, c, d, e\}$ is a $D(n_1)$-quintuple, and u a nonzero rational, then $\{ua, ub, uc, ud, ue\}$ is a $D(n_1u^2)$-quintuple and we say that these two quintuples are equivalent. Since every rational Diophantine quintuple is equivalent to some $D(u^2)$-quintuple whenever u is an integer divisible by the common denominator of the elements in the quintuple, Theorem 1 will follow if we prove that there are infinitely many rational Diophantine quintuples with the property that the product of any two of its elements is a perfect square.

1 Search methodology

In searching for D-sets with m elements, it is natural to first find some sets with $m-1$ element. So we first looked what a D-pair could be.
In this quest, we have actually been searching for $D(1)$ and $D(0)$ rational sets in which all elements have the same denominator, and all numerators are squares or $D \times \square$, where D is squarefree.
So for some $a_1, a_2 \in \mathbb{N}$ to be a pair, for some $b \in \mathbb{N}$, then it has to hold $\frac{Da_1^2}{b} \cdot \frac{Da_2^2}{b} + 1 = c^2$, for some $c \in \mathbb{Q}$. Or in the other words, it has to hold $(Da_1 \cdot a_2)^2 + b^2 = c^2$, for $c \in \mathbb{N}$. So for a fixed b, we calculated all Pythagorean triangles with one leg b. And then $D \cdot a_1 \cdot a_2$ is the other leg. Well known formulas for Pythagorean triples are

$$b = 2dkl \text{ and } Da_1a_2 = d(k+l)(k-l),$$

for some $k, l, d \in \mathbb{N}$, and opposite.
So we just had to find all divisors of the other leg.
We written program in `C++`. To remember pairs, we constructed a graph, so we just had to find the bigger clique in it. Because such a graph is very sparse, it's not hard to do it. On 6-core computer the first quintuple was shown in about 10 seconds:

$$M = \left\{\frac{225^2}{480480}, \frac{2548^2}{480480}, \frac{286^2}{480480}, \frac{1408^2}{480480}, \frac{819^2}{480480}\right\}$$

which by clearing denominators gives Diophantine $D(480480^2)$-quintuple with square elements.
We used the simplest algorithm for finding divisors. Using sieve of Eratosthenes we generate all primes $\leq P$. And then check only those primes (experimental results from other tests suggested that big D-sets usually have only *small* prime factors; for example, for our first quintuple, $P = 11$ is good enough).
To check all prime divisors it would be hard because if for example $b = 2k \cdot l$, divisors of $k + l$ could be big (and it is very small possibility that this number be in other D-pairs).
We check two versions of pairs in algorithm. In one, each numerator is a square, and in the other $D > 1$. The first one is much faster, and the second finds more results.
We first checked only for $P \leq 10^6$. But for example, let us see for $b \leq N = 480480$ and the first algorithm. On a 6-core computer. For $P = 10^3 \ldots 10^8$ there are no bigger changes in times, while the last one used about 1GB of memory. So let us see differences between $P = 11$, and $P = 10^3$ (B_i represent number of found set including maybe some the same, and the last two columns (G_4 and G_5) are number of distinct sets):

$<$	B_2	B_3	B_4	B_5	G_4	G_5	time
11	1957115	36897	1948	2	14741	1	13sec
10^6	3629788	51068	2256	5	1618	1	28sec,

while for $P = 99$, there is only one second better and only number of $B_2 = 3629040$, and all is the same as fo $P = 10^8$.
In the second table we show for $P = 11$ how numbers are changing when we double N, or double it once more:

N	B_2	B_3	B_4	B_5	G_4	G_5	time
$2\times$	3848686	65022	3279	3	2521	1	28sec
$4\times$	7370326	112135	5418	5	4240	2	60sec.

This last number $G_2 = 2$ meens that we have found equivalent quintuple.

We noticed that first few found nonequivalent quintuples have special structure.
A Diophantine quadruple $\{a, b, c, d\}$ is called regular if

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1).$$

Definition 1 *We say that rational Diophantine quintuple $\{a, b, c, d, e\}$ is exotic if $abcd = 1$, quadruples $\{a, b, d, e\}$ and $\{a, c, d, e\}$ are regular, and if the product of any two of its elements is a perfect square.*

So we could create many quintuples using parametrizations on some surfaces, and we proved that there are infinitely of them [11].

2 Regular quintuples

After about a week of brute-force searching (on 24-core computer), the fourth found quintuple had not this structure:

$$\left\{\frac{12384^2}{1337776440}, \frac{18130^2}{1337776440}, \frac{30745^2}{1337776440}, \frac{110880^2}{1337776440}, \frac{259259^2}{1337776440}\right\}.$$

A Diophantine quintuple $\{a, b, c, d, e\}$ is called regular if

$$(abcde + 2abc + a + b + c - d - e)^2 = 4(ab+1)(ac+1)(bc+1)(de+1).$$

The last quintuple is regular quintuple. Later, using parametrizations on some surfaces we were able to find many such quintuples, but we don't know is there infinitely many of them.

3 Concluding remarks

While we have found infinitely many rational Diophantine quintuples with $D(0)$ property, it remains open if there is a rational Diophantine quintuple with square elements.
On the other hand, there are infinitely many rational Diophantine quadruples with square elements, for example the following two parametric family has this property

$$\begin{aligned}
a &= \frac{3^2(s-1)^2(s+1)^2v^2}{2^2(2s^3-2s+v^2)^2},\\
b &= \frac{v^2(-4s^3+4s+v^2)^2}{2^2(s+1)^2(s-1)^2(-s^3+s+v^2)^2},\\
c &= \frac{(2s^3-2s+v^2)^2}{3^2v^2s^2},\\
d &= \frac{4^2(-s^3+s+v^2)^2s^2}{v^2(-4s^3+4s+v^2)^2}.
\end{aligned}$$

In all examples we had using brute-force search for Diophantine sets with square elements, quadruples have an extra property that the product $abcd = 1$. This would suggest that there is no quintuple with square elements.
But when we write similar program for such search, after few hours on 6-core computer, we find some for which the product $abcd \neq 1$ (and thousands for which product is 1), for example:

$$\left\{\left(\frac{18}{77}\right)^2, \left(\frac{55}{96}\right)^2, \left(\frac{56}{15}\right)^2, \left(\frac{340}{77}\right)^2\right\}.$$

Acknowledgments. The authors were supported by the Croatian Science Foundation under the project no. IP-2018-01-1313. The authors acknowledge support from the QuantiXLie Center of Excellence, a project co-financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004).
In some calculations, we have a big help of computer cluster BURA.

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