Householder's approximants and continued fraction expansion of quadratic irrationals

Vinko Petričević

Department of Mathematics,
University of Zagreb, Croatia

e-mail: vpetrice@math.hr

URL: http://web.math.hr/~vpetrice/

Newton's iterative method

Continued fractions give good rational approximations of arbitrary $\alpha \in \mathbb{R}$. Newton's iterative method $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ for solving nonlinear equations $f(x) = 0$ is another approximation method.

Let $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$ and d is not a square of a rational number. It is well known that regular continued fraction expansion of α is periodic, i.e. has the form $\alpha = [a_0, a_1, \dots, a_k, \overline{a_{k+1}, a_{k+2}, \dots, a_{k+\ell}}]$. Here $\ell = \ell(\alpha)$ denotes the length of the shortest period in the expansion of α. Connections between these two approximation methods were discussed by several authors. Let $\frac{p_n}{q_n}$ be the nth convergent of α. The principal question is: Let $f(x) = (x-\alpha)(x-\alpha')$, where $\alpha' = c - \sqrt{d}$ and $x_0 = \frac{p_n}{q_n}$, is x_1 also a convergent of α?

It is well known that for $\alpha = \sqrt{d}$, $d \in \mathbb{N}$, $d \neq \square$, and the corresponding Newton's approximant $R_n = \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{dq_n}{p_n}\right)$ it follows that

$$R_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1. \tag{1}$$

It was proved by Mikusiński [5] that if $\ell = 2t$, then

$$R_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1. \tag{2}$$

These results imply that if $\ell(\sqrt{d}) \leq 2$, then all approximants R_n are convergents of $\sqrt{d}$. Dujella [1] proved the converse of this result. Namely, if $\ell(\sqrt{d}) > 2$, we know that some of approximants R_n are not convergents. He showed that being again a convergent is a periodic and a palindromic property. Formulas (1) and (2) suggest that R_n should be convergent whose index is twice as large when it is a good approximant. However, this is not always true. Dujella defined the number $j(\sqrt{d})$ as a distance from two times larger index, and pointed out that $j(\sqrt{d})$ is unbounded. In 2011, the author [6] proved the analogous results for $\alpha = \frac{1+\sqrt{d}}{2}$, $d \in \mathbb{N}$, $d \neq \square$ and $d \equiv 1 \pmod 4$.

Sharma [8] observed arbitrary quadratic surd $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$, d is not a square of a rational number, whose period begins with a_1. He showed that for every such α and the corresponding Newton's approximant $N_n = \frac{p_n^2 - \alpha\alpha' q_n^2}{2q_n(p_n - cq_n)}$ it holds $N_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}$, for $k \geq 1$, and when $\ell = 2t$ and the period is palindromic then it holds $N_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}$, for $k \geq 1$. Frank and Sharma [3] discussed generalization of Newton's formula. They showed that for every α, whose period begins with a_1, for $k, n \in \mathbb{N}$ it holds

$$\frac{p_{nk\ell-1}}{q_{nk\ell-1}} = \frac{\alpha(p_{k\ell-1} - \alpha' q_{k\ell-1})^n - \alpha'(p_{k\ell-1} - \alpha q_{k\ell-1})^n}{(p_{k\ell-1} - \alpha' q_{k\ell-1})^n - (p_{k\ell-1} - \alpha q_{k\ell-1})^n}, \tag{3}$$

and when $\ell = 2t$ and the period is palindromic then for $k, n \in \mathbb{N}$ it holds

$$\frac{p_{nkt-1}}{q_{nkt-1}} = \frac{\alpha(p_{kt-1} - \alpha' q_{kt-1})^n - \alpha'(p_{kt-1} - \alpha q_{kt-1})^n}{(p_{kt-1} - \alpha' q_{kt-1})^n - (p_{kt-1} - \alpha q_{kt-1})^n}. \tag{4}$$

For detailed proofs and explanation of the rest of the poster see [7].

Householder's iterative methods

Householder's iterative method (see e.g. [4, §4.4]) of order p for root-solving: $x_{n+1} = H^{(p)}(x_n) = x_n + p \cdot \frac{(1/f)^{(p-1)}(x_n)}{(1/f)^{(p)}(x_n)}$, where $(1/f)^{(p)}$ denotes p-th derivation of $1/f$. Householder's method of order 1 is just Newton's method. For Householder's method of order 2 one gets Halley's method, and Householder's method of order p has rate of convergence $p+1$.

It is not hard to show that for $f(x) = (x-\alpha)(x-\alpha')$ it holds:

$$H^{(m+1)}(x) = \frac{xH^{(m)}(x) - \alpha\alpha'}{H^{(m)}(x) + x - \alpha - \alpha'}, \quad \text{for } m \in \mathbb{N}. \tag{5}$$

Let us define

$$R_n^{(1)} \stackrel{\text{def}}{=} \frac{p_n}{q_n}, \quad \text{and for } m > 1 \quad R_n^{(m)} \stackrel{\text{def}}{=} H^{(m-1)}\left(\frac{p_n}{q_n}\right).$$

We will say that $R_n^{(m)}$ is *good approximation*, if it is a convergent of α. Formula (3) shows that for arbitrary quadratic surd, whose period begins with a_1 and $k, m \in \mathbb{N}$, it holds

$$R_{k\ell-1}^{(m)} = \frac{p_{mk\ell-1}}{q_{mk\ell-1}}, \tag{6}$$

and when $\ell = 2t$ and period is periodic, from (4) it follows

$$R_{kt-1}^{(m)} = \frac{p_{mkt-1}}{q_{mkt-1}}.$$

Good approximants are periodic and palindromic

Formula [8, (8)] says: For $k \in \mathbb{N}$ it holds

$$(a_\ell - a_0)p_{k\ell-1} + p_{k\ell-2} = q_{k\ell-1}(d - c^2), \tag{7}$$
$$(a_\ell - a_0)q_{k\ell-1} + q_{k\ell-2} = p_{k\ell-1} - 2cq_{k\ell-1}, \tag{8}$$

and formula (5) says

$$R_n^{(m+1)} = \frac{R_n^{(1)}R_n^{(m)} - \alpha\alpha'}{R_n^{(1)} + R_n^{(m)} - 2c}, \quad \text{for } m \in \mathbb{N},\ n = 0, 1, \dots. \tag{9}$$

Lemma 1 *For $m, k \in \mathbb{N}$ and $i = 1, 2, \dots, \ell$, when the period begins with a_1, it holds $R_{k\ell+i-1}^{(m)} = \frac{R_{k\ell-1}^{(m)}R_{i-1}^{(m)} - \alpha\alpha'}{R_{k\ell-1}^{(m)} + R_{i-1}^{(m)} - 2c}$.*

Proof. For $m = 1$, statement of the lemma is proven in [2, Thm. 2.1]. Using mathematical induction and (9) it is not hard to show that the statement of the lemma holds too. ❑

When period is palindromic and begins with a_1, formulas (7) and (8) become

$$a_0 p_{k\ell-1} + p_{k\ell-2} = 2cp_{k\ell-1} + q_{k\ell-1}(d - c^2), \tag{10}$$
$$a_0 q_{k\ell-1} + q_{k\ell-2} = p_{k\ell-1}. \tag{11}$$

Lemma 2 *For $m, k \in \mathbb{N}$ and $i = 1, 2, \dots, \ell-1$, when period is palindromic and begins with a_1, it holds $R_{k\ell-i-1}^{(m)} = \frac{R_{k\ell-1}^{(m)}(R_{i-1}^{(m)} - 2c) + \alpha\alpha'}{R_{i-1}^{(m)} - R_{k\ell-1}^{(m)}}$.*

Proof. For $m = 1$ we have:

$$R_{k\ell-i-1}^{(1)} = \frac{0 \cdot p_{k\ell-i} + p_{k\ell-i-1}}{0 \cdot q_{k\ell-i} + q_{k\ell-i-1}} = [a_0, \dots, a_{k\ell-i}, 0]$$
$$= [a_0, \dots, a_{k\ell-i}, a_{k\ell-i-1}, \dots, a_{k\ell-1}, a_0, 0, -a_0, -a_1, \dots, -a_{i-1}]$$
$$= \left[a_0, \dots, a_{k\ell-i}, a_{k\ell-i-1}, \dots, a_{k\ell-1}, a_0 - \frac{p_{i-1}}{q_{i-1}}\right]$$
$$= \frac{p_{k\ell-1}(a_0 - R_{i-1}^{(1)}) + p_{k\ell-2}}{q_{k\ell-1}(a_0 - R_{i-1}^{(1)}) + q_{k\ell-2}} \overset{(10)}{\underset{(11)}{=}} \frac{R_{k\ell-1}^{(1)}(R_{i-1}^{(1)} - 2c) + \alpha\alpha'}{R_{i-1}^{(1)} - R_{k\ell-1}^{(1)}}.$$

Using mathematical induction and (9) it is not hard to show that the statement of the lemma holds too. ❑

Proposition 1 *Let $m \in \mathbb{N}$. When period begins with a_1, for $i = 1, 2, \dots, \ell-1$ and $\beta_i^{(m)} = -\frac{p_{mi-1} - R_{i-1}^{(m)}q_{mi-1}}{p_{mi} - R_{i-1}^{(m)}q_{mi}}$, it holds*

$$R_{k\ell+i-1}^{(m)} = \frac{\beta_i^{(m)}p_{m(k\ell+i)} + p_{m(k\ell+i)-1}}{\beta_i^{(m)}q_{m(k\ell+i)} + q_{m(k\ell+i)-1}}, \quad \text{for all } k \geq 0,$$

and when period is palindromic, then

$$R_{k\ell-i-1}^{(m)} = \frac{p_{m(k\ell-i)-1} - \beta_i^{(m)}p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1} - \beta_i^{(m)}q_{m(k\ell-i)-2}}, \quad \text{for all } k \geq 1.$$

Proof. We have $\beta_i^{(m)} = [0, -a_{mi}, -a_{mi-1}, \dots, -a_1, -a_0 + R_{i-1}^{(m)}]$.
If $k = 0$ we have

$$\frac{\beta_i^{(m)}p_{mi} + p_{mi-1}}{\beta_i^{(m)}q_{mi} + q_{mi-1}} = [a_0, \dots, a_{mi}, \beta_i^{(m)}]$$
$$= [a_0, \dots, a_{mi}, 0, -a_{mi}, -a_{mi-1}, \dots, -a_1, -a_0 + R_{i-1}^{(m)}] = R_{i-1}^{(m)},$$

and similarly if $k > 0$ we have

$$\frac{\beta_i^{(m)}p_{m(k\ell+i)} + p_{m(k\ell+i)-1}}{\beta_i^{(m)}q_{m(k\ell+i)} + q_{m(k\ell+i)-1}} = [a_0, \dots, a_{mk\ell-1}, a_{mk\ell} - a_0 + R_{i-1}^{(m)}]$$
$$= \frac{p_{mk\ell-1}(a_{mk\ell} - a_0 + R_{i-1}^{(m)}) + p_{mk\ell-2}}{q_{mk\ell-1}(a_{mk\ell} - a_0 + R_{i-1}^{(m)}) + q_{mk\ell-2}}$$
$$\overset{(7),(6)}{\underset{(8)}{=}} \frac{R_{k\ell-1}^{(m)}R_{i-1}^{(m)} + d - c^2}{R_{k\ell-1}^{(m)} + R_{i-1}^{(m)} - 2c} \overset{\text{Lm. 1}}{=} R_{k\ell+i-1}^{(m)}.$$

When period is palindromic we have:

$$\frac{p_{m(k\ell-i)-1} - \beta_i^{(m)}p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1} - \beta_i^{(m)}q_{m(k\ell-i)-2}} = \left[a_0, \dots, a_{m(k\ell-i)-1}, -\frac{1}{\beta_i^{(m)}}\right]$$
$$= [a_0, \dots, a_{m(k\ell-i)-1}, a_{m(k\ell-i)}, a_{m(k\ell-i)+1}, \dots, a_{mk\ell-1}, a_0 - R_{i-1}^{(m)}]$$
$$= \frac{p_{mk\ell-1}(a_0 - R_{i-1}^{(m)}) + p_{mk\ell-2}}{q_{mk\ell-1}(a_0 - R_{i-1}^{(m)}) + q_{mk\ell-2}} \overset{(10),(6)}{\underset{(11)}{=}} \frac{R_{k\ell-1}^{(m)}(R_{i-1}^{(m)} - 2c) + c^2 - d}{R_{i-1}^{(m)} - R_{k\ell-1}^{(m)}},$$

which is using Lemma 2 equal to the $R_{k\ell-i-1}^{(m)}$. ❑

Analogously as in [1, Lm. 3], from Proposition 1 it follows:

Theorem 1 *To be a good approximant is a periodic property, i.e. for all $r \in \mathbb{N}$ it holds*

$$R_n^{(m)} = \frac{p_k}{q_k} \iff R_{r\ell+n}^{(m)} = \frac{p_{rm\ell+k}}{q_{rm\ell+k}},$$

and when period is palindromic, it is also a palindromic property, i.e. it holds:

$$R_n^{(m)} = \frac{p_k}{q_k} \iff R_{\ell-n-2}^{(m)} = \frac{p_{m\ell-k-2}}{q_{m\ell-k-2}}.$$

Which convergents may appear?

Let us define coprime positive numbers $P_n^{(m)}$, $Q_n^{(m)}$ by

$$\frac{P_n^{(m)}}{Q_n^{(m)}} \stackrel{\text{def}}{=} R_n^{(m)}.$$

From (9) it is not hard to show that it holds

$$P_n^{(m)} - \alpha Q_n^{(m)} = \left(P_n^{(1)} - \alpha Q_n^{(1)}\right)^m = (p_n - \alpha q_n)^m.$$

Lemma 3 *$R_n^{(m)} < \alpha$ if and only if n is even and m is odd. Therefore, $R_n^{(m)}$ can be an even convergent only if n is even and m is odd.*

Similarly as in [1], if $R_n^{(m)} = \frac{p_k}{q_k}$, we can define $j^{(m)} = j^{(m)}(\alpha, n)$ as the distance from convergent with m times larger index:

$$j^{(m)} = \frac{k + 1 - m(n+1)}{2}. \tag{12}$$

This is an integer, by Lemma 3. Using Theorem 1 we have $j^{(m)}(\alpha, n) = j^{(m)}(\alpha, k\ell + n)$, and in palindromic case: $j^{(m)}(\alpha, n) = -j^{(m)}(\alpha, \ell - n - 2)$.

From now on, let us observe only quadratic irrationals of the form $\alpha = \sqrt{d}$, $d \in \mathbb{N}$, $d \neq \square$. It is well known that period of such α is palindromic and begins with a_1.

Theorem 2 $\left|R_{n+1}^{(m)} - \sqrt{d}\right| < \left|R_n^{(m)} - \sqrt{d}\right|$.

Proposition 2 *When $d \neq \square$, for $n \geq 0$ we have $\left|j^{(m)}(\sqrt{d}, n)\right| < \frac{m(\ell/2-1)}{2}$.*

Lemma 4 *Let F_k denote the k-th Fibonacci number. Let $n \in \mathbb{N}$ and $k > 1$, $k \equiv 1, 2 \pmod 3$. For $d_k(n) = \left(\frac{(2n+1)F_k+1}{2}\right)^2 + (2n+1)F_{k-1} + 1$ it holds $\sqrt{d_k(n)} = \left[\frac{(2n-1)F_k+1}{2}, \overline{\underbrace{1, 1, \dots, 1, 1}_{k-1 \text{ times}}, (2n-1)F_k + 1}\right]$, and $\ell(\sqrt{d_k(n)}) = k$.*

Theorem 3 *Let F_ℓ denote the ℓ-th Fibonacci number. Let $\ell > 3$, $\ell \equiv \pm 1 \pmod 6$. Then for $d_\ell = \left(\frac{F_{\ell-3}F_\ell+1}{2}\right)^2 + F_{\ell-3}F_{\ell-1} + 1$ and $M \in \mathbb{N}$ it holds $\ell(\sqrt{d_\ell}) = \ell$ and*

$$j^{(3M-1)}(\sqrt{d_\ell}, 0) = j^{(3M)}(\sqrt{d_\ell}, 0) = j^{(3M+1)}(\sqrt{d_\ell}, 0) = \frac{\ell-3}{2} \cdot M.$$

Proof. By (12), we have to prove

$$R_0^{(3M-1)} = \frac{p_{M\ell-2}}{q_{M\ell-2}}, \quad R_0^{(3M)} = \frac{p_{M\ell-1}}{q_{M\ell-1}}, \quad R_0^{(3M+1)} = \frac{p_{M\ell}}{q_{M\ell}}.$$

We have $a_0 = \frac{F_{\ell-3}F_\ell+1}{2}$, and by Lemma 4 it holds $\sqrt{d_\ell} = \left[a_0, \overline{\underbrace{1, 1, \dots, 1, 1}_{\ell-1 \text{ times}}, 2a_0}\right]$. From Cassini's identity, it follows

$$R_0^{(1)} = \frac{p_0}{q_0} = a_0, \qquad R_0^{(2)} = a_0 + \frac{F_{\ell-2}}{F_{\ell-1}} = \frac{p_{\ell-2}}{q_{\ell-2}},$$
$$R_0^{(3)} = a_0 + \frac{F_{\ell-1}F_{\ell-2}^3}{F_{\ell-1}^2F_{\ell-2}^2 + F_{\ell-2}^2} = a_0 + \frac{F_{\ell-1}}{F_\ell} = \frac{p_{\ell-1}}{q_{\ell-1}}. \tag{13}$$

Let us prove the theorem using induction on M. For proving the inductive step, first observe that from (9) for $m \geq 3$ we have:

$$R_k^{(m)} = \frac{R_k^{(2)}R_k^{(m-2)} + d}{R_k^{(2)} + R_k^{(m-2)}}, \qquad R_k^{(m)} = \frac{R_k^{(3)}R_k^{(m-3)} + d}{R_k^{(3)} + R_k^{(m-3)}}. \tag{14}$$

Suppose that for some $i \in \{0, \ell-2, \ell-1\}$ it holds $\frac{p_{(M-1)\ell+i}}{q_{(M-1)\ell+i}} = R_0^{(m-3)}$.
We have:

$$\frac{p_{M\ell+i}}{q_{M\ell+i}} = \left[a_0, \underbrace{1, 1, \dots, 1, 1}_{\ell-1 \text{ times}}, a_0 + R_0^{(m-3)}\right] =$$
$$\overset{(10)}{\underset{(11)}{=}} \frac{p_{\ell-1}R_0^{(m-3)} + dq_{\ell-1}}{q_{\ell-1}R_0^{(m-3)} + p_{\ell-1}} \overset{(13)}{=} \frac{R_0^{(3)}R_0^{(m-3)} + d}{R_0^{(3)} + R_0^{(m-3)}} \overset{(14)}{=} R_0^{(m)}. \quad ❑$$

Corollary 1 *For each $m \geq 2$ it holds*

$$\sup\left\{\left|j^{(m)}(\sqrt{d}, n)\right|\right\} = +\infty,$$
$$\limsup\left\{\frac{\left|j^{(m)}(\sqrt{d}, n)\right|}{\ell(\sqrt{d})}\right\} \geq \frac{m}{6}.$$

References

[1] A. Dujella, *Newton's formula and continued fraction expansion of $\sqrt{d}$*, Experiment. Math. **10** (2001), 125–131.

[2] E. Frank, *On continued fraction expansions for binomial quadratic surds*, Numer. Math. **4** (1962) 85–95.

[3] E. Frank, A. Sharma, *Continued fraction expansions and iterations of Newton's formula*, J. Reine Angew. Math. **219** (1965) 62–66.

[4] A. S. Householder, The Numerical Treatment of a Single Nonlinear Equation, McGraw-Hill, New York, 1970.

[5] J. Mikusiński, *Sur la méthode d'approximation de Newton*, Ann. Polon. Math. **1** (1954), 184–194.

[6] V. Petričević, *Newton's approximants and continued fraction expansion of $\frac{1+\sqrt{d}}{2}$*, Math. Commun., to appear

[7] V. Petričević, *Householder's approximants and continued fraction expansion of quadratic irrationals*, preprint, 2011. http://web.math.hr/~vpetrice/radovi/hous.pdf

[8] A. Sharma, *On Newton's method of approximation*, Ann. Polon. Math. **6** (1959) 295–300.