DIOPHANTINE QUADRUPLES WITH THE PROPERTIES $D(n_1)$ AND $D(n_2)$

ANDREJ DUJELLA AND VINKO PETRIČEVIĆ

Abstract. For a nonzero integer n, a set of m distinct nonzero integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$, is called a $D(n)$-m-tuple. In this paper, we show that there infinitely many essentially different quadruples which are simultaneously $D(n_1)$-quadruples and $D(n_2)$-quadruples with $n_1 \neq n_2$.

1. Introduction

For a nonzero integer n, a set of distinct nonzero integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$, is called a Diophantine m-tuple with the property $D(n)$ or $D(n)$-m-tuple. The $D(1)$-m-tuples are called simply Diophantine m-tuples, and have been studied since the ancient times. Diophantus of Alexandria found a set of four rationals $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ with the property that the product of any two of its distinct elements is a square of a rational number. By multiplying elements of this set by 16 we obtain the $D(256)$-quadruple $\{1, 33, 68, 105\}$. Fermat found the first $D(1)$-quadruple, it was the set $\{1, 3, 8, 120\}$. In 1969, Baker and Davenport [5], using linear forms in logarithms of algebraic numbers and the reduction method introduced in that paper, showed that the set $\{1, 3, 8\}$ can be extended to a Diophantine quintuple only by adding 120 to the set. In 2004, Dujella [13] showed that there are no Diophantine sextuples and that there are at most finitely many Diophantine quintuples. Recently, He, Togbé and Ziegler proved that there are no Diophantine quintuples [25]. (See also [6] for an analogous result concerning the conjecture of nonexistence of $D(4)$-quintuples.) On the other hand, it was known already to Euler that there are infinitely many rational Diophantine quintuples. In particular, the Fermat's set $\{1, 3, 8, 120\}$ can be extended to a rational Diophantine quintuple by adding 777480/8288641 to the set. Recently, Stoll [30] proved that the extension of Fermat's set to a rational Diophantine quintuple is unique. The first example of a rational Diophantine sextuple, the set $\{11/192, 35/192, 155/27, 512/27, 1235/48, 180873/16\}$, was found by Gibbs [22], while Dujella, Kazalicki, Mikić and Szikszai [18] recently proved that there are infinitely many rational Diophantine sextuples (see also [17]). It is not known whether there exists any rational Diophantine septuple. For an overview of results on $D(1)$-m-tuples and its generalizations see [15].

Let us mention some results concerning $D(n)$-sets with $n \neq 1$. It is easy to show that there are no $D(n)$-quadruples if $n \equiv 2 \pmod 4$ (see e.g. [7]). On the other hand, it is known that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple [9]. It is believed that the size of sets with the property $D(n)$ is bounded by an absolute constant (independent on n). It is known that the size of sets with the property $D(n)$ is ≤ 31 for $|n| \le 400$; $< 15.476 \log |n|$ for $|n| > 400$, and $< 3 \cdot 2^{168}$ for n prime (see [11, 12, 19] and also [4]).

2010 *Mathematics Subject Classification.* Primary 11D09; Secondary 11G05.
Key words and phrases. Diophantine quadruples, elliptic curves.

In [27], A. Kihel and O. Kihel asked if there are Diophantine triples $\{a, b, c\}$ which are $D(n)$-triples for several distinct n's. They conjectured that there are no Diophantine triples which are also $D(n)$-triples for some $n \neq 1$. However, the conjecture is not true, since, for example, $\{8, 21, 55\}$ is a $D(1)$ and $D(4321)$-triple (as noted in the MathSciNet review of [27]), while $\{1, 8, 120\}$ is a $D(1)$ and $D(721)$-triple, as observed by Zhang and Grossman [31]. In [1], several infinite families of Diophantine triples were presented which are also $D(n)$-sets for two additional n's. Furthermore, there are examples of Diophantine triples which are $D(n)$-sets for three additional n's. For example, the set $\{4, 12, 420\}$ is a $D(n)$-quadruple for $n = 1, 436, 3796, 40756$ (see also [2]).

If we omit the condition that one of the n's is equal to 1, then the size of a set N for which there exists a triple $\{a, b, c\}$ of nonzero integers which is a $D(n)$-set for all $n \in N$ can be arbitrarily large. Indeed, take any triple $\{a, b, c\}$ such that the elliptic curve

$$E: \quad y^2 = (x + ab)(x + ac)(x + bc)$$

has positive rank over $\mathbb{Q}$. Then there are infinitely many rational points on $E(\mathbb{Q})$. For an arbitrary large positive integer M we may choose M distinct rational points $R_1, \dots, R_M \in 2E(\mathbb{Q})$, so that we have

$$x(R_i) + bc = \square, \quad x(R_i) + ca = \square, \quad x(R_i) + ab = \square,$$

where $\square$ stands for a square of a rational number (see e.g. [26, 4.1, p. 37]). We choose $z \in \mathbb{Z} \setminus \{0\}$ such that $z^2 x(R_i) \in \mathbb{Z}$ for all $i = 1, 2, \dots, m$. Then the triple $\{az, bz, cz\}$ is a $D(n)$-triples for $n = x(R_i)z^2$ for all $i = 1, 2, \dots, m$ (see [1, Section 4] for the details).

On the other hand, assuming Lang's conjecture on varieties of general type, for a given quadruple $\{a, b, c, d\}$ of distinct integers, the size of the set N of integers n for which $\{a, b, c, d\}$ is a $D(n)$-quadruple is bounded by an absolute constant. Indeed, let $ab + n = x^2$. By multiplying remaining five conditions, we get the hyperelliptic curve

$$y^2 = (x^2 + ac - ab)(x^2 + bc - ab)(x^2 + ad - ab)(x^2 + bd - ab)(x^2 + cd - ab),$$

which has genus 4 unless it has two equal roots. Assume e.g. that $ad = bc$. Then we get the hyperelliptic curve

$$y^2 = (x^2 - ab + ac)(x^2 + bc - ab)(ax^2 - a^2 b + b^2 c)(ax^2 - a^2 b + bc^2)$$

with distinct roots (unless $b = -a$ or $c = -a$) and, hence, with genus equal to 3. Finally, if e.g. $c = -a$ and $d = -b$, we get the hyperelliptic curve

$$y^2 = (x^2 - ab - a^2)(x^2 - 2ab)(x^2 - ab - b^2)$$

with distinct roots and with genus equal to 2. Assuming the above mentioned Lang' conjecture, Caporaso, Harris and Mazur [8] proved that for $g \geq 2$ the number $B(g, \mathbb{K}) = \max_C |C(\mathbb{K})|$ is finite, where C runs over all curves of genus g over a number field $\mathbb{K}$. Therefore, we get that, under Lang's conjecture, $|N| \leq \max(B(2, \mathbb{Q}), B(3, \mathbb{Q}), B(4, \mathbb{Q}))$.

Thus, it seems natural to ask is there any set of four distinct nonzero integers which is a $D(n_i)$-quadruple for two distinct (nonzero) integers n_1 and n_2. However, it seems that this question has not been studies yet and that there are no examples of such quadruples in the literature. In Section 2 we will present results of our computer search for such quadruples. Motivated by certain regularities in found examples, we will show in Section 3 that there are infinitely many such examples. If $\{a, b, c, d\}$ is $D(n_1)$ and $D(n_2)$-quadruple and u is a nonzero rational such that $au, bu, cu, du, n_1 u^2$ and $n_2 u^2$ are integers, then $\{au, bu, cu, du\}$ is a $D(n_1 u^2)$ and

$D(n_2u^2)$-quadruples. We will say that these two quadruples are equivalent, and list only one representative of each found class of quadruples.

Our main result is

Theorem 1. *There are infinitely many nonequivalent sets of four distinct nonzero integers $\{a, b, c, d\}$ with the property that there exist two distinct nonzero integers such that $\{a, b, c, d\}$ a $D(n_1)$-quadruple and a $D(n_2)$-quadruple.*

2. Numerical examples

We started with computational search for $D(n)$-quadruples, where $-500\,000 \leq n \leq 500\,000$. For a fixed nonzero integer n, by observing divisors of integers of the form $m^2 - n$, it is not hard to get some $D(n)$-quadruples (we were searching in the range $m \leq 333\,333$).

We have implemented the algorithm in `C++`. For a fixed n, we construct a graph, connecting the numbers k and l with an edge provided they satisfy $k \cdot l = m^2 - n$. The graph can be represented using standard containers (for example `map<long, set<long> > g`; so for $k < l$, k and l are connected if set `g[l]` contains k). We also connect k and l with $k + l + 2m$, since $k(k + l + 2m) + n = (k + m)^2$ and $l(k + l + 2m) + n = (l + m)^2$ (a $D(n)$-triple of the form $\{k, l, k + l \pm 2\sqrt{kl + n}\}$ is called *regular*).

But we actually used container `unordered_map<long, vector<long> >`, which is somewhat faster and takes less memory. For $m = 1, \ldots, 333\,333$, it usually takes about 10–12 seconds (on one core of 3.6GHz) to build such a graph, and it usually takes about 500MB of memory (but graph density depends on n). Then we search for a 4-clique in graph (e.g. $D(n)$-quadruple). We do this by sorting each `vector`, and using binary search. So for finding all 4-cliques it takes about a second, and for the most of n's we get several hundreds of quadruples.

Then we searched for n_2 using M. Stoll's program `ratpoints` (see [29]). For a quadruple $\{a, b, c, d\}$, the search for an integer point on the hyperelliptic curve $y^2 = (ab+x)(ac+x)(ad+x)(bc+x)(bd+x)(cd+x)$ with $x = n_2 \leq 10^8$ takes about 0.02 seconds. Here is summarize results of our search:

$\{a, b, c, d\}$	$\{n_1, n_2\}$	$\{a, b, c, d\}$	$\{n_1, n_2\}$
-1701, -901, 224, 243	413424, 463968	-1, 7, 22532, 23407 *	30632, 214376
-189, -133, 27, 32 *	6192, 8352	15, 380, 5735, 634880	361536, 7123200
-176, -169, 169, 176	31265, 36305	15, 720, 9135, 40656	17424, 13708816
-52, 135, 351, 575	37296, 67536	27, 115, 160, 1755	-2016, 37296
-27, 28, 189, 493 *	13752, 61272	28, 6348, 18750, 88872	330625, 38101225
-27, 189, 4189, 6364 *	194328, 1325304	45, 276, 8820, 18228	112896, 2966656
-15, 1140, 2057, 15609	234256, 989296	51, 192, 315, 2331	-6656, 1080144
-11, 28, 385, 540	11124, 34164	69, 300, 949, 2925	63400, 417544
-4, 209, 5129, 49049	252840, 6062280	70, 430, 2178, 18634	-20691, 1678149
-3, 21, 1152, 1517 *	5392, 37312	125, 2709, 2816, 5621	-273600, 1443600
-3, 21, 2597, 3132 *	11512, 80152	169, 448, 8640, 11305	97344, 28482624
-1, 7, 64, 119 *	128, 848	175, 231, 300, 396	-16400, -40400
-1, 7, 4484, 4879 *	6248, 43688	234, 322, 406, 1222	-10323, -69723

We indicate by * quadruples which contain two elements a and b such that $a/b = -1/7$. These quadruples will play crucial role in the proof of Theorem 1 in the next section.

3. Quadruples containing the pair $\{-1, 7\}$

Motivated by the examples indicated by * in the previous section, we will show that there infinitely many quadruples of the form $\{a, b, c, d\}$, where $a/b = -1/7$ that are $D(n)$-quadruples for two distinct (nonzero) n's. Then we will show that in fact we may take $a = -1$ and $b = 7$ and get the the same conclusion.

We use regular triples mentioned in the previous section. Namely, if $AB+n = R^2$, then $\{A, B, A+B+2R\}$ and $\{A, B, A+B-2R\}$ are $D(n)$-triples. Let $cd+n_1 = r^2$ and $cd + n_2 = s^2$. If $c + d - 2r = 7$ and $c + d - 2s = -1$, then $\{7, c, d\}$ is a $D(n_1)$-triple and $\{-1, c, d\}$ is a $D(n_2)$-triple. We have to satisfy the remaining six conditions from the definition of $D(n_i)$-quadruples.

We search for a solution in the from $n_2 = kn_1 - l$ with (rational) constants k and l. We have $n_1 = -cd + r^2$ and $c = -d + 2r + 7$. From $c + d - 2s = -1$, we get $r = -4 + s$. By inserting this in $cd + n_2 = s^2$ and solving this equation for d, we get that $(28k^2 - 24k - 4)s - 63k^2 + 62k + 4kl - 4l + 1$ is a perfect square, say t^2. Thus we obtain

$$s = \frac{4l + 63k^2 - 62k - 4kl - 1 + t^2}{4(7k^2 - 6k - 1)},$$

$$d = \frac{-4kl + 4l + 1 - 50k + 49k^2 - 2t - 14tk + t^2}{4(7k^2 - 6k - 1)}.$$

Consider now the condition that $bd + n_2$ is a perfect square. We obtain a quadratic function in t with the discriminant $-128(k - 1)^2(112k^2 - 112k - 15kl + 7l)$. From $112k^2 - 112k - 15kl + 7l = 0$ we get

$$l = \frac{112k(k - 1)}{15k - 7}.$$

There are four remaining conditions: $ab+n_2$, $ab+n_1$, $ac+n_1$ and $ad+n_1$ are perfect squares. All four conditions lead to quadratic functions in t. The corresponding discriminants are

$$56k(15k - 7)(7k + 1)(13k - 7)(k - 1)^2,$$
$$-56(15k - 7)(7k + 1)(k - 7)(k - 1)^2,$$
$$2048(7k + 1)(k - 7)(k - 1)^3(15k - 7),$$
$$2048(7k + 1)(k - 7)(k - 1)^3(15k - 7)$$

(last two discriminants are identical). Hence, by taking $k = 7$, we can satisfy last three conditions simultaneously. Only one condition remains, $ab + n_2$ is a perfect square, and it is equivalent to $-6 + \frac{7}{900}t^2$ being perfect square. From

$$-6 + \frac{7}{900}t^2 = (1 + \frac{u(t - 30)}{30})^2,$$

we obtain

$$c = \frac{(2u^2 - 3u + 7)(2u^2 - u - 7)}{(u^2 - 7)^2},$$

$$d = -\frac{(u^2 - 3u + 14)(u^2 + u - 14)}{(u^2 - 7)^2},$$

$$n_1 = \frac{4(2u^4 - u^3 - 20u^2 - 7u + 98)}{(u^2 - 7)^2},$$

$$n_2 = \frac{4(2u^2 - 7u + 14)(u^2 + 7)}{(u^2 - 7)^2}.$$

For $u \notin \{0, 1, 2, -7/5, -5, 7/2, 7, 4, 7/3, -7, -2, 3, -7/2, 7/4\}$ the elements of the set $\{-1, 7, c, d\}$ are distinct rationals. By taking $u = v/w$ and getting rid of denominators, we obtain the following result.

Proposition 2. *Let v and w be coprime integers and*

$$v/w \notin \{0, 1, 2, -2, 3, 4, -5, 7, -7, 7/2, -7/2, 7/3, 7/4, -7/5\}.$$

Then the set

$$\begin{aligned} \{-(-v^2+7w^2)^2, 7(-v^2+7w^2)^2, -(-2v^2+vw+7w^2)(2v^2-3vw+7w^2), \\ (v^2-3vw+14w^2)(-v^2-vw+14w^2)\} \end{aligned} \tag{1}$$

is a $D(n_1)$-quadruple and a $D(n_2)$-quadruple for

$$\begin{aligned} n_1 &= 4(-v^2+7w^2)^2(2v^4-v^3w-20v^2w^2-7vw^3+98w^4), \\ n_2 &= 4(-v^2+7w^2)^2(2v^2-7vw+14w^2)(v^2+7w^2). \end{aligned}$$

We have obtained infinitely many quadruples with the required property satisfying $a/b = -1/7$ (in other word, infinitely many rational quadruples with $a = -1$, $b = 7$). Now we will show that there are infinitely many integer quadruples with $a = -1$, $b = 7$. Indeed, let v and w be a solution of the Pellian equation

$$v^2 - 7w^2 = 2. \tag{2}$$

The equation (2) has infinitely many integer solutions given by

$$v_0 = 3, \quad v_1 = 3, \quad v_{i+2} = 16v_{i+1} - v_i,$$

$$w_0 = -1, \quad w_1 = 1, \quad w_{i+1} = 16w_{i+1} - w_i.$$

By inserting $v = v_i$, $w = \pm w_i$ in (1), and dividing elements of the quadruple by the common factor 4, we obtain quadruples of the form $\{-1, 7, c, d\}$ which are $D(n)$-quadruples for two distinct n's. Here are few smallest examples:

$\{a, b, c, d\}$	$\{n_1, n_2\}$
-1, 7, 119, 64	128, 848
-1, 7, 1191959, 1185664	1585088, 11095568
-1, 7, 5840864, 5826919	7778528, 54449648
-1, 7, 76695715424, 76694116519	102259887968, 715819215728
-1, 7, 376369378007, 376365836032	501823476032, 3512764332176

4. The case $n_1 = 0$

In the definition of $D(n)$-m-tuples, the case of $n = 0$ is usually excluded, although certainly the definition make sense in this case also. The reason for excluding $n = 0$ is in very different behavior of $D(0)$-tuples compared with $D(n)$-tuples for $n \neq 0$. While for a fixed $n \neq 0$ the size of sets with the property $D(n)$ is bounded, sets with the property $D(0)$ can be arbitrarily large, just take any subset of the set of squares $\{1, 4, 9, 16, \ldots\}$. However, in the context of finding quadruples which are $D(n_1)$ and $D(n_2)$-quadruples for $n_1 \neq n_2$, it seems to be natural to consider also the case $n_1 = 0$. We might expect that in this case it could be easier to find such quadruples, but it seems that there is not straightforward way to see why there should be infinitely many of them.

A simple search for $D(n)$-quadruples which elements are perfect squares gives many such examples. Here we list some of them:

$\{a,b,c,d\}$	$\{n_1,n_2\}$	$\{a,b,c,d\}$	$\{n_1,n_2\}$
1, 4, 169, 1024	0, 6720	196, 625, 1024, 3969	0, 705600
1, 36, 529, 1024	0, 60480	324, 841, 1369, 4096	0, -262080
25, 64, 961, 2025	0, 188496	1, 324, 2209, 4096	0, 887040
100, 625, 1024, 2025	0, 360000	36, 729, 2500, 4096	0, 518400
64, 169, 441, 2401	0, 1164240	256, 729, 2401, 5625	0, 1587600
961, 1849, 2704, 2916	0, -1774080	121, 169, 2704, 5625	0, 1436400
32, 98, 1152, 3528	0, 257985	1681, 4096, 5625, 5929	0, -6879600

Our starting point in constructing infinitely many quadruples $\{a,b,c,d\}$ which are $D(0)$-quadruples and also $D(n_2)$-quadruples for $n_2 \neq 0$, is the following simple fact (see [10, Theorem 1] and [9, Section 5]). The set

$$\{a, ak^2-2k-2, a(k+1)^2-2k, a(2k+1)^2-8k-4\}$$

is a $D(2a(2k+1)+1)$-quadruple provided all its elements are distinct nonzero integers. Thus, we take $b = ak^2-2k-2$, $c = a(k+1)^2-2k$, $d = a(2k+1)^2-8k-4$, $n = 2a(2k+1)+1$, and we want to find integers a and k such that $\{a,b,c,d\}$ is also a $D(0)$-quadruple, i.e. such that ab, ac and ad are perfect squares. By putting $ab = (ak+r)^2$ we get

$$k = -\frac{2a+r^2}{2(a(1+r))}.$$

Then we put $ac = (2ra+s)^2$ and we get

$$a = -\frac{-4r^2-4r^3-r^4+s^2}{4(r^3-2-2r+rs)}.$$

The final condition that ad is a perfect square, now becomes

$$\begin{aligned}(3)\quad &(r^2-2r+1)s^4+(8r^4+8r^3-16r)s^3\\ &+(22r^6+68r^5+54r^4-40r^3-24r^2+32r+32)s^2\\ &+(-192r^3+24r^8-448r^4+88r^7-336r^5)s\\ &+9r^{10}+30r^9-39r^8-248r^7-200r^6+352r^5+752r^4+512r^3+128r^2=\square.\end{aligned}$$

Since r^2-2r+1 is a square, this quartic curve in s has rational point at infinity, so it can be in standard way transformed into an elliptic curve over $\mathbb{Q}(r)$:

$$\begin{aligned}(4)\quad y^2 = x^3 &+ (4r^6+56r^5+84r^4+80r^3+48r^2-64r-64)x^2\\ &+(-1024r^9-2048r^8+1024r^7+5120r^6+3072r^5-3072r^4\\ &-5120r^3-1024r^2+2048r+1024)x.\end{aligned}$$

The curve (4) has a point $[0,0]$ of order 2 and two independent points of infinite order:

$$\begin{aligned}P_1 = [&-6r^6-4r^5+74r^4+168r^3+88r^2-32r-32,\\ &-256r^7-1792r^6-4352r^5-4352r^4-1024r^3+1024r^2+512r],\\ P_2 = [&-4r^5+10r^4+8r^3+24r^2-6r^6-32r,\\ &-320r^3+128r+448r^6-512r^4-64r^2+128r^7+192r^5].\end{aligned}$$

If fact, by using the algorithm of Gusić and Tadić from [24] (see also [23, 30] for other variants of the algorithm), we can check that the rank of (4) over $\mathbb{Q}(r)$ is equal to 2 and that P_1 and P_2 are its free generators. Indeed, the specialization $r = 13$ satisfies the assumptions of [24, Theorem 1.3].

Hence, there are infinitely many rational points on curves (4) and (3), and thus infinitely many quadruples with the required property. We present an explicit formula. By taking the point $P_2 - P_1$ on (4) and we get

$$s = -\frac{r(3r^3 + 9r^2 + 7r + 2)}{r^2 + r - 1},$$

and (after multiplying with the common denominator) the quadruple

$$\{4r^4(r+2)^2, (r^3 - 4r + 1)^2, (r^3 + 4r^2 - 1)^2, 4(2r-1)^2\} \tag{5}$$

which is a $D(0)$-quadruple and a $D(16r^{10} + 96r^9 + 112r^8 - 192r^7 - 256r^6 + 192r^5 + 112r^4 - 96r^3 + 16r^2)$-quadruple. By taking r to be an integer in (5) we obtain the following result

Proposition 3. *There are infinitely many nonequivalent sets of four distinct nonzero integers $\{a, b, c, d\}$ with the property that a, b, c, d are perfect squares (so that $\{a, b, c, d\}$ is a $D(0)$-quadruple) and there exist $n_2 \neq 0$ such that $\{a, b, c, d\}$ a $D(n_2)$-quadruple.*

Let us mention that in [16, 28] sets which all elements are squares appeared in similar context (construction of (strong) Eulerian m-tuples, which are shifted $D(-1)$-m-tuples). Other connections of (rational) Diophantine m-tuples and elliptic curves can be found in [3, 14, 18, 20, 21].

Acknowledgements. The authors were supported by the Croatian Science Foundation under the project no. IP-2018-01-1313. The authors acknowledge support from the QuantiXLie Center of Excellence, a project co-financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004). The authors acknowledge the usage of the supercomputing resources of Division of Theoretical Physics at Ruđer Bošković Institute.

References

[1] N. Adžaga, A. Dujella, D. Kreso and P. Tadić, *Triples which are $D(n)$-sets for several n's*, J. Number Theory **184** (2018), 330–341.

[2] N. Adžaga, A. Dujella, D. Kreso and P. Tadić, *On Diophantine m-tuples and $D(n)$-sets*, RIMS Kokyuroku **2092** (2018), to appear.

[3] J. Aguirre, A. Dujella and J. C. Peral, *On the rank of elliptic curves coming from rational Diophantine triples*, Rocky Mountain J. Math. **42** (2012), 1759–1776.

[4] R. Becker, M. Ram Murty, *Diophantine m-tuples with the property $D(n)$*, Glas. Mat. Ser. III, to appear.

[5] A. Baker and H. Davenport, *The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[6] M. Bliznac Trebješanin and A. Filipin, *Nonexistence of $D(4)$-quintuples*, J. Number Theory **194** (2019), 170–217.

[7] E. Brown, *Sets in which $xy + k$ is always a square*, Math. Comp. **45** (1985), 613–620.

[8] L. Caporaso, J. Harris and B. Mazur, *Uniformity of rational points*, J. Amer. Math. Soc. **10** (1997), 1–35.

[9] A. Dujella, *Generalization of a problem of Diophantus*, Acta Arith. **65** (1993), 15–27.

[10] A. Dujella, *Some polynomial formulas for Diophantine quadruples*, Grazer Math. Ber. **328** (1996), 25–30.

[11] A. Dujella, *On the size of Diophantine m-tuples*, Math. Proc. Cambridge Philos. Soc. **132** (2002), 23–33.

[12] A. Dujella, *Bounds for the size of sets with the property $D(n)$*, Glas. Mat. Ser. III **39** (2004), 199–205.

[13] A. Dujella, *There are only finitely many Diophantine quintuples*, J. Reine Angew. Math. **566** (2004), 183–214.

[14] A. Dujella, *On Mordell-Weil groups of elliptic curves induced by Diophantine triples*, Glas. Mat. Ser. III **42** (2007), 3–18.

[15] A. Dujella, *What is ... a Diophantine m-tuple?*, Notices Amer. Math. Soc. **63** (2016), 772–774.

[16] A Dujella, I. Gusić, V. Petričević and P. Tadić, *Strong Eulerian triples*, Glas. Mat. Ser. III **53** (2018), 33–42.

[17] A. Dujella and M. Kazalicki, *More on Diophantine sextuples*, in: Number Theory - Diophantine problems, uniform distribution and applications, Festschrift in honour of Robert F. Tichy's 60th birthday (C. Elsholtz, P. Grabner, Eds.), Springer-Verlag, Berlin, 2017, pp. 227–235.

[18] A. Dujella, M. Kazalicki, M. Mikić and M. Szikszai, *There are infinitely many rational Diophantine sextuples*, Int. Math. Res. Not. IMRN **2017 (2)** (2017), 490–508.

[19] A. Dujella and F. Luca, *Diophantine m-tuples for primes*, Int. Math. Res. Not. **47** (2005), 2913–2940.

[20] A. Dujella and J. C. Peral, *High rank elliptic curves with torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ induced by Diophantine triples*, LMS J. Comput. Math. **17** (2014), 282–288.

[21] A. Dujella and J. C. Peral, *Elliptic curves induced by Diophantine triples*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM, to appear.

[22] P. Gibbs, *Some rational Diophantine sextuples*, Glas. Mat. Ser. III **41** (2006), 195–203.

[23] I. Gusić and P. Tadić, *A remark on the injectivity of the specialization homomorphism*, Glas. Mat. Ser. III **47** (2012), 265–275.

[24] I. Gusić and P. Tadić, *Injectivity of the specialization homomorphism of elliptic curves*, J. Number Theory **148** (2015), 137–152.

[25] B. He, A. Togbé and V. Ziegler, *There is no Diophantine quintuple*, Trans. Amer. Math. Soc., to appear.

[26] D. Husemöller, Elliptic Curves, Springer–Verlag, 1987.

[27] A. Kihel and O. Kihel, *On the intersection and the extendibility of P_t-sets*, Far East J. Math. Sci. **3** (2001), 637–643.

[28] A. J. MacLeod, *Square Eulerian quadruples*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **20** (2016), 1–7.

[29] M. Stoll, *Documentation for the ratpoints program*, preprint (2008), `arXiv:0803.3165`

[30] M. Stoll, *Diagonal genus 5 curves, elliptic curves over $\mathbb{Q}(t)$, and rational diophantine quintuples*, Acta Arith., to appear.

[31] Y. Zhang and G. Grossman, *On Diophantine triples and quadruples*, Notes Number Theory Discrete Math. **21** (2015), 6–16.

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
Email address, A. Dujella: `duje@math.hr`

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
Email address, V. Petričević: `vpetrice@math.hr`