Strong Eulerian Triples

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Let q be a non-zero rational. A set $\{a_1, a_2, \dots, a_m\}$ of m non-zero rationals is called a rational $D(q)$-m-tuple if $a_i \cdot a_j + q$ is a perfect square for all $1 \le i < j \le m$. Diophantus found the first rational $D(1)$-quadruple $\{1/16, 33/16, 17/4, 105/16\}$, while Euler found a rational $D(1)$-quintuple by extending the integer $D(1)$-quadruple $\{1, 3, 8, 120\}$, found by Fermat, with the fifth rational number $777480/8288641$. The first example of a rational $D(1)$-sextuple, the set $\{11/192, 35/192, 155/27, 512/27, 1235/48, 180873/16\}$, was found by Gibbs, while Dujella, Kazalicki, Mikić and Szikszai recently proved that there are infinitely many rational $D(1)$-sextuples. It is not known whether there exist any rational $D(1)$-septuples. For an overview of results on $D(1)$-m-tuples and its generalizations see [2].
It is known that for every rational q there exist infinitely many rational $D(q)$-quadruples. In 2012, Dujella and Fuchs proved that for infinitely many square-free integers q there are infinitely many rational $D(q)$-quintuples.

Apart of the case $q = 1$, the most studied case in the literature is $q = -1$, which is closely related to another old problem investigated by Diophantus and Euler, concerning the sets of numbers with the property that the product of any two of its distinct elements plus their sum is a perfect square. We call a set $\{x_1, x_2, \dots, x_m\}$ an Eulerian m-tuple if $x_i x_j + x_i + x_j$ is a perfect square for all $1 \le i < j \le m$. The equality $x_i x_j + x_i + x_j = (x_i + 1)(x_j + 1) - 1$ gives an explicit connection between Eulerian m-tuples and $D(-1)$-m-tuples. It is known that there does not exist a $D(-1)$-quintuple in integers and that there are only finitely many such quadruples, and all of them have to contain the element 1 . In particular, there does not exist an Eulerian quadruple in positive integers. On the other hand, there exist infinitely many rational $D(-1)$-quintuples, and hence infinitely many Eulerian quintuples in rationals.
Note that in the definition of rational $D(q)$-m-tuples we excluded the requirement that the product of an element with itself plus q is a square. There is no obvious reason why the sets (called strong $D(1)$-m-tuples) which satisfy these stronger conditions would not exist. Dujella and Petričević proved in 2008 that there exist infinitely many strong $D(1)$-triples, while no example of a strong $D(1)$-quadruple is known.

Strong Eulerian triples

Definition 1 *A set of three rationals $\{x_1, x_2, x_3\}$ such that $x_1x_2 + x_1 + x_2$, $x_1x_3 + x_1 + x_3$, $x_2x_3 + x_2 + x_3$, $x_1^2 + 2x_1$, $x_2^2 + 2x_2$ and $x_3^2 + 2x_3$ are all perfect squares is called a* strong Eulerian triple*. Equivalently, by taking $a_i = x_i + 1$, we may consider strong rational $D(-1)$-triples, i.e. sets of three rationals $\{a_1, a_2, a_3\}$ such that $a_1a_2 - 1$, $a_1a_3 - 1$, $a_2a_3 - 1$, $a_1^2 - 1$, $a_2^2 - 1$ and $a_3^2 - 1$ are all perfect squares.*

It is clear that all elements of a strong rational $D(-1)$-triple has to have the same sign, and that $\{a_1, a_2, a_3\}$ is a strong rational $D(-1)$-triple if and only if $\{-a_1, -a_2, -a_3\}$ has the same property. Thus, there is no loss of generality in assuming that all elements of a strong rational $D(-1)$-triple are positive.

Example 1 *We start by searching experimentally for strong rational $D(-1)$-triples with elements with relatively small numerators and denominators (smaller than $2.5 \cdot 10^7$). We found seven examples with all elements different from 1:*

$$\{493/468, 1313/1088, 33137/32912\},$$
$$\{1517/1508, 42601/11849, 909745/757393\},$$
$$\{125/117, 689/400, 14353373/13130325\},$$
$$\{354005/22707, 193397/183315, 2084693/2074035\},$$
$$\{2833349/218660, 3484973/2619045, 3056365/3047653\},$$
$$\{2257/1105, 2873/2745, 3859145/862784\},$$
$$\{2257/1105, 115825/8177, 14307761/10303760\},$$

and 23 examples containing the number 1:

$$\{1, 5/4, 14645/484\}, \quad \{1, 689/400, 1025/64\},$$
$$\{1, 689/400, 969425/861184\}, \quad \{1, 689/400, 9047825/4857616\},$$
$$\{1, 2501/100, 59189/12100\}, \quad \{1, 2501/100, 3219749/2102500\},$$
$$\{1, 6625/1296, 3254641/435600\}, \quad \{1, 19825/17424, 46561/32400\},$$
$$\{1, 19825/17424, 50689/3600\}, \quad \{1, 17009/6400, 8530481/4494400\},$$
$$\{1, 26245/324, 26361205/18301284\}, \quad \{1, 28625/2704, 27060449/25603600\},$$
$$\{1, 60229/44100, 65125/39204\}, \quad \{1, 65125/39204, 2829205/30276\},$$
$$\{1, 168305/94864, 262145/1024\}, \quad \{1, 926021/96100, 13236725/7365796\},$$
$$\{1, 1692821/902500, 1932725/662596\}, \quad \{1, 2993525/2896804, 6519845/6461764\},$$
$$\{1, 3603685/2965284, 5791045/777924\}, \quad \{1, 4324625/1478656, 4919681/883600\},$$
$$\{1, 12376325/12096484, 12844709/11628100\},$$
$$\{1, 19193525/18887716, 22980245/15100996\},$$
$$\{1, 12231605/2353156, 13689845/894916\}.$$

Strong $D(-1)$-triples containing 1

Example 2 *Let us take a closer look at strong rational $D(-1)$-triples of the form $\{1, 689/400, c\}$. We get the following values for c with numerators and denominators less than 10^{21}:*

1025/64, 969425/861184, 9047825/4857616, 352915361/30030400,
109066004561/106119577600, 284429759489/271837104400,
1322025501425/1301315125504, 2253725966225/876912382096,
9055090973825/809791213456, 30776081662625/29873264334736,
41085820444721/37500436537600, 38029186636625/23706420917776,
710390547822449/245964644227600, 206973503563719329/2738904077011600,
180130335826717601/7772841524238400,
1383595259111988401/1191448219611040000,
349568886374130209/40505499045648400,
842490595967154166625/184668498086700979264.

Example 2 clearly indicates that we may expect that there exist infinitely many strong rational $D(-1)$-triples of the form $\{1, 689/400, c\}$. It is not so clear what to expect for triples of the form $\{1, 5/4, c\}$ or $\{1, 65/16, c\}$.
By connecting the problem with certain families of elliptic curves, we will show that there exist infinitely many strong rational $D(-1)$-triples. We find only eight strong rational $D(-1)$-triples that do not contain the number 1 (see Example 1 and Remark 3). Accordingly, our construction gives several infinite families of strong rational $D(-1)$-triples which all contain the number 1. This means that the corresponding strong Eulerian triples contain the number 0, and all other elements are squares. MacLeod found examples of rational Eulerian triples and quadruples which all elements are squares. However, in our situation there is an additional requirement that each element increased by 2 is also a square.

Associated elliptic curves

So, let $\{a, b, c\}$ be a strong rational $D(-1)$-triples with $a = 1$. Thus $b - 1$, $c - 1$, $b^2 - 1$, $c^2 - 1$ and $bc - 1$ are perfect squares. From the first and third condition we get $b - 1 = \alpha^2$, $b + 1 = \beta^2$ for rationals α, β. By taking $\beta^2 - 2 = \alpha^2 = (\beta - 2u)^2$, we get $\beta = \frac{2u^2+1}{2u}$ and

$$b = \frac{4u^4+1}{4u^2} \tag{1}$$

for a non-zero rational u. (If $a \neq 1$, instead of the genus 0 curve $\alpha^2 + 2 = \beta^2$, we would have a genus 1 curve $\alpha^4 + 2\alpha^2 + 1 - a^2 = \gamma^2$.)
Analogously we get

$$c = \frac{4v^4+1}{4v^2} \tag{2}$$

for a non-zero rational v.
The only remaining condition is that $bc - 1$ should be a perfect square. By inserting (1) and (2) in $bc - 1 = \square$, we get

$$(16u^4 + 4)v^4 - 16u^2v^2 + 4u^4 + 1 = z^2. \tag{3}$$

This curve is a quartic in v with a rational point $[u, 4u^4 - 1]$. Thus it can be in the standard way transformed into an elliptic curve:

$$Y^2 = X(X + 32u^4 + 8)(X + 16u^4 - 16u^2 + 4). \tag{4}$$

There is a point

$$P = [-4(4u^4 + 1), 16u(4u^4 + 1)]$$

on (4), which comes from the point $[u, -4u^4 + 1]$ on (3). For all non-zero rationals u, the point P is of infinite order on the specialized curve (4) over $\mathbb{Q}$. Now we consider multiples mP, $m \ge 2$, of P on (4), transfer them back to the quartic (3), and compute the components b, c of the corresponding strong rational $D(-1)$-triple. Since the point P is of infinite order, for fixed u, i.e. fixed b, in that way we get infinitely many strong rational $D(-1)$-triples of the form $\{1, b, c\}$.
The point P gives $[u, -4u^4 + 1]$, and thus does not provide a triple, since in this case we get $v = u$ and $b = c$. The point $2P$ gives $\left[\frac{-u(4u^4-3)}{(12u^4-1)}, \frac{(64u^{12}+272u^8-68u^4-1)}{(12u^4-1)^2}\right]$ and the strong rational $D(-1)$-triple

$$\left\{1, (4u^4 + 1)/(4u^2), \frac{(4u^4+1)(256u^{16}+4352u^{12}-1952u^8+272u^4+1)}{(4u^2(4u^4-3)^2(12u^4-1)^2)}\right\},$$

while $3P$ gives $\left[\frac{u(64u^{12}-656u^8+108u^4+5)}{320u^{12}+432u^8-164u^4+1}, \dots\right]$ and the strong rational $D(-1)$-triple

$$\begin{gathered}\{1, (4u^4 + 1)/(4u^2),\\ (4u^4 + 1)(256u^{16} + 4352u^{12} - 195u^8 + 272u^4 + 1) \times\\ (65536u^{32} + 6422528u^{28} - 13516800u^{24} + 49995776u^{20} - 23443968u^{16}\\ + 3124736u^{12} - 52800u^8 + 1568u^4 + 1)\\ /(4u^2(64u^{12} - 656u^8 + 108u^4 + 5)^2(320u^{12} + 432u^8 - 164u^4 + 1)^2)\}.\end{gathered}$$

By inserting $u = 1$, we get the triples

$$\{1, 5/4, 14645/484\} \quad \text{and} \quad \{1, 5/4, 330926870165/318391604644\}$$

(the first triple already appeared in Example 1, while the second triple is outside of the range covered by Example 1).
It is clear that further multiples $4P, 5P, \dots$ would provide more complicated formulas for triples. To get new relatively simple formulas for triples, we may try to find subfamilies of the elliptic curve (4) with rank ≥ 2.
We search for an additional point on the 2-isogenous curve

$$y^2 = x(x^2 - 24x + 32xu^2 - 96xu^4 + 16 + 128u^2 + 384u^4 + 512u^6 + 256u^8)$$

by considering divisors of $16 + 128u^2 + 384u^4 + 512u^6 + 256u^8 = 16(2u^2 + 1)^4$. Imposing $x = 8(2u^2 + 1)$ to be the x-coordinate of a point, leads to the condition that $4u^2 - 14$ is a square, which gives $u = (14 + w^2)/(4w)$ for a rational w. Thus

$$b = (w^8 + 56w^6 + 1240w^4 + 10976w^2 + 38416)/(16w^2(14 + w^2)^2).$$

By transferring the additional point of infinite order of the original quartic (3), we get

$$v = (w^6 + 18w^4 - 100w^2 - 392)/(4w(3w^4 + 28w^2 + 140))$$

and

$$\begin{gathered}c = (w^8 + 40w^6 + 4888w^4 + 7840w^2 + 38416) \times\\ (w^8 - 4w^7 + 24w^6 - 40w^5 + 152w^4 + 16w^3 + 608w^2 + 672w + 784) \times\\ (w^8 + 4w^7 + 24w^6 + 40w^5 + 152w^4 - 16w^3 + 608w^2 - 672w + 784)\\ /(16w^2(w^6 + 18w^4 - 100w^2 - 392)^2(3w^4 + 28w^2 + 140)^2).\end{gathered}$$

By inserting $w = 1$, we get the triple

$$\{1, 50689/3600, 104776974625/104672955024\}$$

(this triple is outside of the range covered by Example 1).

In this Section we used families of elliptic curves with rank ≥ 1 over $\mathbb{Q}(u)$, resp. rank ≥ 2 over $\mathbb{Q}(w)$, and known independent points of infinite order to construct families of strong rational $D(-1)$-triples. It is natural to ask what is the exact generic rank of these two families and whether the known points are in fact generators of the corresponding Mordell-Weil groups. We proved [1] that the elliptic curve given in (4) has rank one over $\mathbb{Q}(u)$ and the free generator is the point $P = [-4(4u^4 + 1), 16u(4u^4 + 1)]$. For the elliptic curve obtained from (4) by the substitution $u = (14 + w^2)/(4w)$, after removing the denominators, we get the elliptic curve C over $\mathbb{Q}(w)$ given by the equation

$$\begin{aligned}Y^2 = X^3 &+ (3w^8 + 152w^6 + 3272w^4 + 29792w^2 + 115248)X^2\\ &+ 2(w^8 + 56w^6 + 1240w^4 + 10976w^2 + 38416)(w^4 + 20w^2 + 196)^2X.\end{aligned}$$

We proved that C has rank equal to 2 over $\mathbb{Q}(w)$ and that the points with first coordinates

$$\begin{aligned}x(P) &= -(w^8 + 56w^6 + 1240w^4 + 10976w^2 + 38416),\\ x(Q) &= (w^2 - 14)^2(w^4 + 20w^2 + 196)^2/(64w^2)\end{aligned}$$

are its free generators.

Concluding remarks

Remark 1 *We may ask how large can the rank be over $\mathbb{Q}$ of a specialization for $u \in \mathbb{Q}$ of the elliptic curve (4). Since for $u = (14 + w^2)/(4w)$ the rank over $\mathbb{Q}(w)$ is equal to 2, there are infinitely many rationals u for which the rank of (4) is ≥ 2. We was able to find curves with rank equal to 3 (e.g. for $u = 2/5$, $u = 4$), 4 (e.g. for $u = 50/11$, $u = 12/65$), 5 (e.g. for $u = 12/65$, $u = 16/83$) and 6 (for $u = 86/743$, $u = 3570/1051$, $u = 1642/3539$). Note that $u = 2/5$ gives $b = 689/400$. The fact that for this specialization the specialized curve has rank 3, with generators with relatively small height, explains the observation from Example 2 that there are unusually many strong rational $D(-1)$-triples of the form $\{1, 689/400, c\}$ for c's with small numerators and denominators.*

Remark 2 *The results of this paper motivate following open questions:*

1) Are there infinitely many strong rational $D(-1)$-triples that do not contain the number 1?

2) Is there any strong rational $D(-1)$-quadruple?

Note that the triple $(a, b, c) = (125/117, 689/400, 14353373/13130325)$ from Example 1 has an additional property that $b - 1$ is also a square. Furthermore, $26(a - 1)$ and $26(c - 1)$ are also squares. Hence, although we do not know any strong $D(-1)$-quadruple over $\mathbb{Q}$, we get the set

$$\{1, 125/117, 689/400, 14353373/13130325\}$$

which is a strong $D(-1)$-quadruple over the quadratic field $\mathbb{Q}(\sqrt{26})$.

Remark 3 *Let $a \neq \pm 1$ be a rational such that $a^2 - 1$ is a square, i.e. $a = (t^2 + 1)/(2t)$ for a rational $t \neq 0, \pm 1$. It can be extended to infinitely many strong rational $D(-1)$-pairs. Indeed, by following the construction in the case $a = 1$, we now get the condition $\alpha^4 + 2\alpha^2 + 1 - a^2 = \gamma^2$. This quartic is birationally equivalent to the elliptic curve*

$$Y^2 = (X + 2t^2)(X^2 + t^6 - 2t^4 + t^2), \tag{5}$$

for which we showed that it has the rank over $\mathbb{Q}(t)$ equal to 1, with the free generator $R = [-t^2 + 1, t^4 - 1]$ (which of infinite order for all rationals $t \neq 0, \pm 1$). One explicit extension $\{a, b\}$ is by $b = (t^4 + 18t^2 + 1)/(8t(t^2 + 1))$. We have noted that the elements of the known examples of strong rational $D(-1)$-triples that do not contain the number 1 induce the elliptic curves with relatively large rank. In particular, for $a = 42601/11849$ and $a = 14353373/13130325$ the rank is equal to 5. We have performed an additional search for strong rational $D(-1)$-triples that do not contain the number 1 by considering elliptic curves in the family (5) with rank ≥ 3, and checking small linear combinations of their generators. In that way, we found one new example of a strong rational $D(-1)$-triple (corresponding to $t = 17/481$):

$$\{115825/8177, 408988121/327645760, 752442457/720825305\},$$

which is outside of the range covered by Example 1.

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