

NEWTON'S APPROXIMANTS AND CONTINUED FRACTION EXPANSION OF $\frac{1+\sqrt{d}}{2}$

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ABSTRACT. Let d be a positive integer such that $d \equiv 1 \pmod{4}$ and d is not a perfect square. It is well known that the continued fraction expansion of $\frac{1+\sqrt{d}}{2}$ is periodic and symmetric, and if it has the period length $\ell \leq 2$, then all Newton's approximants $R_n = \frac{p_n^2 + \frac{d-1}{4}q_n^2}{q_n(2p_n - q_n)}$ are convergents of $\frac{1+\sqrt{d}}{2}$ and then it holds $R_n = \frac{p_{2n+1}}{q_{2n+1}}$ for all $n \geq 0$. We say that R_n is good approximant if R_n is a convergent of $\frac{1+\sqrt{d}}{2}$. When $\ell > 2$ then there is a good approximant in the half and at the end of the period. In this paper we prove that being a good approximant is a palindromic and a periodic property. We show that when $\ell > 2$, there are R_n 's, which are not good approximants. Further, we define the numbers $j = j(d, n)$ by $R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$ if R_n is a good approximant; and $b = b(d) = |\{n : 0 \leq n \leq \ell - 1 \text{ and } R_n \text{ is a good approximant}\}|$. We construct sequences which show that $|j|$ and b are unbounded.

1. INTRODUCTION

Let $d \equiv 1 \pmod{4}$ be a positive integer which is not a perfect square. The simple continued fraction expansion of $\frac{1+\sqrt{d}}{2}$ has the form

$$\frac{1 + \sqrt{d}}{2} = [a_0, \overline{a_1, a_2, \dots, a_{\ell-1}, 2a_0 - 1}].$$

Here $\ell = \ell(\frac{1+\sqrt{d}}{2})$ denotes the length of the shortest period in the expansion of $\frac{1+\sqrt{d}}{2}$. It is well known (see e.g. [9, §30]) that the sequence $a_1, \dots, a_{\ell-1}$ is palindromic, i.e. $a_i = a_{\ell-i}$ for $i = 1, \dots, \ell - 1$. This expansion can be obtained using the following algorithm: $a_0 = \lfloor \frac{1+\sqrt{d}}{2} \rfloor$, $s_0 = t_0 = 1$,

$$s_{i+1} = 2a_i t_i - s_i, \quad t_{i+1} = \frac{d - s_{i+1}^2}{4t_i}, \quad a_{i+1} = \left\lfloor \frac{s_{i+1} + \sqrt{d}}{2t_{i+1}} \right\rfloor, \quad \text{for } i \geq 0. \quad (1)$$

The numbers s_i and t_i are also palindromic ([9, Satz 3.32]):

$$\begin{aligned} s_{i+1} &= s_{\ell-i}, \\ t_i &= t_{\ell-i}, \end{aligned} \quad (2)$$

for $i = 0, 1, \dots, \ell - 1$, and when we get:

- (i) $s_i = s_{i+1}$, then $\ell = 2i$,
- (ii) $t_i = t_{i+1}$, then $\ell = 2i + 1$.

2000 *Mathematics Subject Classification.* 11A55.

Key words and phrases. Continued fractions, Newton's formula.

See [9, Satz 3.33].

Let $\frac{p_n}{q_n}$ be the n th convergent of $\frac{1+\sqrt{d}}{2}$. Then

$$\frac{1}{(a_{n+1} + 2)q_n^2} < \left| \frac{1 + \sqrt{d}}{2} - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}.$$

In particular,

$$\left| \frac{1 + \sqrt{d}}{2} - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}. \quad (3)$$

Furthermore, if a rational number $\frac{p}{q}$ with $q \geq 1$ satisfies

$$\left| \frac{1 + \sqrt{d}}{2} - \frac{p}{q} \right| < \frac{1}{2q^2}, \quad (4)$$

then $\frac{p}{q}$ equals one of the convergents of $\frac{1+\sqrt{d}}{2}$ (for the proof see e.g. [9, §13]).

Newton's iterative method for solving nonlinear equations

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

is another approximation method. Applying this method to the equation $f(x) = x^2 - x - \frac{d-1}{4} = 0$, which has a root $\frac{1+\sqrt{d}}{2}$, we obtain

$$x_{k+1} = \frac{x_k^2 + \frac{d-1}{4}}{2x_k - 1}.$$

We are interested in connections between these two approximation methods. The main question is: if we assume that x_0 is a convergent of $\frac{1+\sqrt{d}}{2}$, is x_1 also a convergent of $\frac{1+\sqrt{d}}{2}$? If $x_0 = \frac{p_n}{q_n}$, we are asking whether

$$R_n \stackrel{\text{def}}{=} \frac{p_n^2 + \frac{d-1}{4}q_n^2}{q_n(2p_n - q_n)}$$

is a convergent of $\frac{1+\sqrt{d}}{2}$.

The same question was discussed by several authors for \sqrt{d} and $R'_n = \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{dq_n}{p_n}\right)$. It is well known (see e.g. [1, p. 468]) that

$$R'_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1. \quad (5)$$

It was proved by Mikusiński [7] (see also Elezović [4], Sharma [10]) that if $\ell = 2t$, then

$$R'_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1.$$

These results imply that if $\ell(d) \leq 2$, then all approximants R'_n are convergents of \sqrt{d} . In 2001, Dujella [2] proved the converse of this result. Namely, if all R'_n approximants are convergents of \sqrt{d} , then $\ell(d) \leq 2$. Thus, if $\ell(d) > 2$, we know that some of approximants R'_n are convergents and some of them are not. Using a result of Komatsu [6] from 1999, Dujella showed that being a good approximant is a periodic and a palindromic property, so he defined the number b as the number of good approximants in the period. Formula (5) suggests that R'_n should be convergent whose index is twice as large when it is a good approximant. However,

this is not always true, and Dujella defined the number j as a distance from two times larger index. Dujella also pointed out that j is unbounded. In 2005, Dujella and the author [3] proved that b is unbounded, too.

Moreover, Sharma [10] observed arbitrary quadratic surd $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$, d is not a square of a rational number, whose period begins with a_1 , and $f(x) = x^2 + Ax + B$, such that $f(\alpha) = 0$ ($A = -2c$, $B = c^2 - d$). He showed that for every such α and the corresponding Newton's approximant $N_n = \frac{p_n^2 - Bq_n^2}{q_n(2p_n + Aq_n)}$ it holds

$$N_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1, \quad (6)$$

and when $\ell = 2t$ and the period is symmetric (except for the last term) then it holds

$$N_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1. \quad (7)$$

In this paper, we show that analogous results hold in the case of the approximation of $\frac{1+\sqrt{d}}{2}$. We show that every approximant is good if and only if $\ell \leq 2$. We give much an easier way to prove that to be a good approximant is a palindromic and a periodic property; we construct a sequence that shows that j could be arbitrary large, and we prove that for every b there exists d such that $b(d) = b$ and $b(d) > \ell(d)/2$.

2. WHICH CONVERGENTS MAY APPEAR?

Sometimes good approximants can be found in places other than the half and the end of the period.

Example 1. Let $d = 324n^2 + 108n - 27$, $n \in \mathbb{N}$. Then we have $\ell = 6$ and $R_1 = \frac{p_3}{q_3}$ and $R_3 = \frac{p_7}{q_7}$. Using algorithm (1) it is straightforward to check that

$$\frac{1 + \sqrt{d}}{2} = [9n + 1, \overline{1, 2n - 1, 3, 2n - 1, 1, 18n + 1}].$$

Now the direct computation shows that

$$\begin{aligned} R_0 &= 9n + 2 - \frac{8}{18n + 1}, \\ R_1 &= 9n + 2 - \frac{3}{6n + 1} = \frac{p_3}{q_3}, \\ R_2 &= 9n + 2 - \frac{6n + 1}{12n^2 + 4n} = \frac{p_5}{q_5}, \\ R_3 &= 9n + 2 - \frac{108n^2 + 36n}{216n^3 + 108n^2 + 6n - 1} = \frac{p_7}{q_7}, \\ R_4 &= 9n + 2 - \frac{1296n^4 - 432n^3 - 216n^2 + 60n + 5}{2592n^5 - 432n^4 - 648n^3 + 84n^2 + 34n - 1}, \\ R_5 &= 9n + 2 - \frac{1296n^4 + 864n^3 + 108n^2 - 12n + 1}{2592n^5 + 2160n^4 + 432n^3 - 23n^2 - 8n} = \frac{p_{11}}{q_{11}}. \end{aligned}$$

Theorem 1. If $R_n = \frac{p_k}{q_k}$, then k is odd.

Proof.

$$\begin{aligned} 4\left(R_n - \frac{1 + \sqrt{d}}{2}\right) &= \left(\frac{2p_n - q_n}{q_n} - \sqrt{d}\right) + \left(\frac{dq_n}{2p_n - q_n} - \sqrt{d}\right) \\ &= \left(\frac{2p_n - q_n}{q_n} - \sqrt{d}\right) - \frac{q_n\sqrt{d}}{2p_n - q_n} \left(\frac{2p_n - q_n}{q_n} - \sqrt{d}\right) = \frac{q_n}{2p_n - q_n} \left(\frac{2p_n - q_n}{q_n} - \sqrt{d}\right)^2. \end{aligned}$$

Since $d \geq 5$, we have

$$\frac{p_n}{q_n} > \frac{1 + \sqrt{d}}{2} - \frac{1}{q_n^2} > 1 - \frac{1}{q_n^2} > \frac{1}{2}$$

except if $q_n = 1$. But if $q_n = 1$, then we also have $2p_n \geq 2 > q_n$. Anyway $2p_n > q_n$, so $R_n > \frac{1 + \sqrt{d}}{2}$. Since $\frac{p_k}{q_k} > \frac{1 + \sqrt{d}}{2}$ if and only if k is odd, from $R_n = \frac{p_k}{q_k}$ we conclude that k is odd. \square

In the same way as in [2], when R_n is the convergent of $\frac{1 + \sqrt{d}}{2}$ we can write

$$R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$$

for an integer $j = j(d, n)$. From (6) and (7) we have $j = 0$ when $\ell\left(\frac{1 + \sqrt{d}}{2}\right) \leq 2$. In Example 1, j always equals 0. This suggests that Newton's method converges exactly twice faster, and it gives the convergent with a double index. However, this is not always true.

Example 2. Let $d = 4n^4 + 16n^3 + 28n^2 + 28n + 13$, $n \in \mathbb{N}$. We have $\ell(d) = 7$, $R_0 = \frac{p_3}{q_3}$ and $R_5 = \frac{p_9}{q_9}$. Using algorithm (1) it is straightforward to check that

$$\frac{1 + \sqrt{d}}{2} = [n^2 + 2n + 2, \overline{2n + 2, n, 1, 1, n, 2n + 2, 2n^2 + 4n + 3}].$$

Now the direct computation shows that:

$$\begin{aligned} R_0 &= n^2 + 2n + 2 + \frac{n + 1}{2n^2 + 4n + 3} = \frac{p_3}{q_3}, \\ R_1 &= n^2 + 2n + 2 + \frac{4n^3 + 12n^2 + 12n + 5}{8n^4 + 32n^3 + 52n^2 + 44n + 16}, \\ R_2 &= n^2 + 2n + 2 + \frac{4n^5 + 12n^4 + 16n^3 + 13n^2 + 5n + 1}{8n^6 + 32n^5 + 60n^4 + 68n^3 + 46n^2 + 18n + 3}, \\ R_3 &= n^2 + 2n + 2 + \frac{4n^5 + 20n^4 + 44n^3 + 53n^2 + 35n + 10}{(2n^2 + 4n + 3)(4n^4 + 16n^3 + 28n^2 + 26n + 11)}, \\ R_4 &= n^2 + 2n + 2 + \frac{16n^5 + 64n^4 + 116n^3 + 120n^2 + 68n + 17}{4(2n^2 + 3n + 2)(4n^4 + 14n^3 + 22n^2 + 19n + 7)}, \\ R_5 &= n^2 + 2n + 2 + \frac{16n^7 + 80n^6 + 192n^5 + 284n^4 + 272n^3 + 168n^2 + 61n + 10}{(4n^3 + 8n^2 + 8n + 3)(8n^5 + 32n^4 + 60n^3 + 66n^2 + 40n + 11)} = \frac{p_9}{q_9}, \\ R_6 &= n^2 + 2n + 2 + \\ &= \frac{64n^9 + 448n^8 + 1536n^7 + 3344n^6 + 5040n^5 + 5424n^4 + 4152n^3 + 2176n^2 + 708n + 109}{8(4n^4 + 12n^3 + 18n^2 + 14n + 5)(4n^6 + 20n^5 + 48n^4 + 70n^3 + 64n^2 + 35n + 9)} \\ &= \frac{p_{13}}{q_{13}}. \end{aligned}$$

In Example 2, we have shown that j could be ± 1 ($R_0 = \frac{p_3}{q_3}$ and $R_5 = \frac{p_9}{q_9}$). We shall prove (Theorem 3) that $|j|$ can be arbitrary large. Let us first show some other interesting details. Let us show that being a good approximant is a periodic and a palindromic property. I.e. $j(d, n) = -j(d, \ell - n - 2)$.

3. GOOD APPROXIMANTS ARE PERIODIC AND SYMMETRIC

Formula [10, (8)] says: For $n \in \mathbb{N}$ it holds

$$a_0 q_{n\ell-1} = p_{n\ell-1} - q_{n\ell-2}, \quad (8)$$

$$(a_0 - 1)p_{n\ell-1} = \frac{d-1}{4}q_{n\ell-1} - p_{n\ell-2}. \quad (9)$$

Lemma 1. For $k \in \mathbb{N}$ and $i \geq 0$ it holds

$$R_{k\ell+i-1} = \frac{R_{k\ell-1}R_{i-1} + \frac{d-1}{4}}{R_{k\ell-1} + R_{i-1} - 1}.$$

Proof. We have

$$\begin{aligned} \frac{p_{k\ell+i-1}}{q_{k\ell+i-1}} &= [a_0, a_1, \dots, a_{k\ell-1}, a_0 - 1 + a_0, a_1, \dots, a_{i-2}, a_{i-1}] \\ &= \left[a_0, a_1, \dots, a_{k\ell-1}, a_0 - 1 + \frac{p_{i-1}}{q_{i-1}} \right] = \frac{p_{k\ell-1}(a_0 - 1 + \frac{p_{i-1}}{q_{i-1}}) + p_{k\ell-2}}{q_{k\ell-1}(a_0 - 1 + \frac{p_{i-1}}{q_{i-1}}) + q_{k\ell-2}} \\ &\stackrel{(9)}{=} \frac{p_{k\ell-1}\frac{p_{i-1}}{q_{i-1}} + \frac{d-1}{4}q_{k\ell-1}}{(8) \frac{p_{k\ell-1}}{q_{k\ell-1}}(\frac{p_{i-1}}{q_{i-1}} - 1) + p_{k\ell-1}} = \frac{p_{k\ell-1}p_{i-1} + \frac{d-1}{4}q_{k\ell-1}q_{i-1}}{q_{k\ell-1}(p_{i-1} - q_{i-1}/2) + q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2)}. \end{aligned} \quad (10)$$

Now we have

$$\begin{aligned} &(R_{k\ell-1}R_{i-1} + \frac{d-1}{4}) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1}) = \\ &= (p_{k\ell-1}^2 + \frac{d-1}{4}q_{k\ell-1}^2)(p_{i-1}^2 + \frac{d-1}{4}q_{i-1}^2) + \frac{d-1}{4}q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1}) \\ &= (p_{k\ell-1}p_{i-1} + \frac{d-1}{4}q_{k\ell-1}q_{i-1})^2 + \frac{d-1}{4}(q_{k\ell-1}(p_{i-1} - q_{i-1}/2) + q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2))^2, \end{aligned} \quad (11)$$

$$\begin{aligned} &(R_{k\ell-1} + R_{i-1} - 1) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1}) = \\ &= (p_{k\ell-1}^2 + \frac{d-1}{4}q_{k\ell-1}^2)q_{i-1}(2p_{i-1} - q_{i-1}) + (p_{i-1}^2 + \frac{d-1}{4}q_{i-1}^2)q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1}) \\ &\quad - q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1}) \\ &= (q_{k\ell-1}(p_{i-1} - q_{i-1}/2) + q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2)) \cdot \\ &\quad \cdot (2p_{k\ell-1}p_{i-1} + \frac{d-1}{2}q_{k\ell-1}q_{i-1} - q_{k\ell-1}(p_{i-1} - q_{i-1}/2) - q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2)), \end{aligned} \quad (12)$$

and the quotient of (11) and (12) is, by (10), equal to $R_{k\ell+i-1}$. \square

Lemma 2. For $k \in \mathbb{N}$ and $i \geq 0$ it holds

$$R_{k\ell-i-1} = \frac{R_{k\ell-1}(1 - R_{i-1}) + \frac{d-1}{4}}{R_{k\ell-1} - R_{i-1}}.$$

Proof. Using $\frac{x \cdot p_n + p_{n-1}}{x \cdot q_n + q_{n-1}} = [a_0, a_1, \dots, a_{n-1}, a_n, x]$ for $x = 0$, $n = k\ell - i$, we have

$$\begin{aligned}
\frac{p_{k\ell-i-1}}{q_{k\ell-i-1}} &= \frac{0 \cdot p_{k\ell-i} + p_{k\ell-i-1}}{0 \cdot q_{k\ell-i} + q_{k\ell-i-1}} = [a_0, a_1, \dots, a_{k\ell-i-1}, a_{k\ell-i}, 0] \\
&= [a_0, a_1, \dots, a_{k\ell-i}, a_{k\ell-i-1}, 0, -a_{k\ell-i-1}] \\
&\quad \vdots \\
&= [a_0, a_1, \dots, a_{k\ell-i}, a_{k\ell-i-1}, \dots, a_{k\ell-1}, a_0, 0, -a_0, -a_{k\ell-1}, \dots, -a_{k\ell-i-1}] \\
&= [a_0, a_1, \dots, a_{k\ell-i}, a_{k\ell-i-1}, \dots, a_{k\ell-1}, a_0, 0, -a_0, -a_1, \dots, -a_{i-1}] \\
&= \left[a_0, a_1, \dots, a_{k\ell-i}, a_{k\ell-i-1}, \dots, a_{k\ell-1}, a_0 - \frac{p_{i-1}}{q_{i-1}} \right] \\
&= \frac{p_{k\ell-1} \left(a_0 - \frac{p_{i-1}}{q_{i-1}} \right) + p_{k\ell-2}}{q_{k\ell-1} \left(a_0 - \frac{p_{i-1}}{q_{i-1}} \right) + q_{k\ell-2}} \stackrel{(9)}{=} \frac{p_{k\ell-1} \left(1 - \frac{p_{i-1}}{q_{i-1}} \right) + \frac{d-1}{4} q_{k\ell-1}}{p_{k\ell-1} - q_{k\ell-1} \frac{p_{i-1}}{q_{i-1}}} \\
&= \frac{p_{k\ell-1} (q_{i-1} - p_{i-1}) + \frac{d-1}{4} q_{k\ell-1} q_{i-1}}{p_{k\ell-1} q_{i-1} - q_{k\ell-1} p_{i-1}}. \tag{13}
\end{aligned}$$

Now we have

$$\begin{aligned}
&(R_{k\ell-1}(1 - R_{i-1}) + \frac{d-1}{4}) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1}) = \\
&= (p_{k\ell-1}^2 + \frac{d-1}{4} q_{k\ell-1}^2)(q_{i-1}(2p_{i-1} - q_{i-1}) - p_{i-1}^2 - \frac{d-1}{4} q_{i-1}^2) + \\
&\quad + \frac{d-1}{4} q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1}) \\
&= - \left[(p_{k\ell-1}(q_{i-1} - p_{i-1}) + \frac{d-1}{4} q_{k\ell-1} q_{i-1})^2 + \frac{d-1}{4} (p_{k\ell-1} q_{i-1} - q_{k\ell-1} p_{i-1})^2 \right], \tag{14}
\end{aligned}$$

$$\begin{aligned}
&(R_{k\ell-1} - R_{i-1}) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1}) = \\
&= (p_{k\ell-1}^2 + \frac{d-1}{4} q_{k\ell-1}^2)q_{i-1}(2p_{i-1} - q_{i-1}) - (p_{i-1}^2 + \frac{d-1}{4} q_{i-1}^2)q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1}) \\
&= -(p_{k\ell-1} q_{i-1} - q_{k\ell-1} p_{i-1}) \cdot \\
&\quad \cdot (2p_{k\ell-1}(q_{i-1} - p_{i-1}) + \frac{d-1}{2} q_{k\ell-1} q_{i-1} - p_{k\ell-1} q_{i-1} + q_{k\ell-1} p_{i-1}), \tag{15}
\end{aligned}$$

and the quotient of (14) and (15) is, by (13), equal to $R_{k\ell-i-1}$. \square

Lemma 3. For arbitrary a_0, a_1, \dots, a_k and α we have

$$[a_k, a_{k-1}, \dots, a_1, a_0 + \alpha] = \frac{p_k + \alpha q_k}{p_{k-1} + \alpha q_{k-1}}.$$

Proof. Using $[a_k, a_{k-1}, \dots, a_1, a_0 + \alpha] = [a_k, a_{k-1}, \dots, a_1, a_0, \frac{1}{\alpha}]$ we get

$$\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}^\tau \begin{pmatrix} \frac{1}{\alpha} & 1 \\ 1 & 0 \end{pmatrix},$$

so we have

$$[a_k, a_{k-1}, \dots, a_1, a_0 + \alpha] = \frac{\frac{1}{\alpha} p_k + q_k}{\frac{1}{\alpha} p_{k-1} + q_{k-1}} = \frac{p_k + \alpha q_k}{p_{k-1} + \alpha q_{k-1}}.$$

\square

Theorem 2. For $i = 0, \dots, \lfloor \ell/2 \rfloor$ and

$$\alpha_i = -\frac{q_{2i-1}\left(\frac{d-1}{4}q_{i-1}^2 + p_{i-1}^2\right) - p_{2i-1}(2p_{i-1} - q_{i-1})q_{i-1}}{q_{2i}\left(\frac{d-1}{4}q_{i-1}^2 + p_{i-1}^2\right) - p_{2i}(2p_{i-1} - q_{i-1})q_{i-1}} = -\frac{p_{2i-1} - q_{2i-1}R_{i-1}}{p_{2i} - q_{2i}R_{i-1}} \quad (16)$$

it holds

$$R_{k\ell+i-1} = \frac{\alpha_i p_{2k\ell+2i} + p_{2k\ell+2i-1}}{\alpha_i q_{2k\ell+2i} + q_{2k\ell+2i-1}}, \text{ for all } k \geq 0, \text{ and}$$

$$R_{k\ell-i-1} = \frac{p_{2k\ell-2i-1} - \alpha_i p_{2k\ell-2i-2}}{q_{2k\ell-2i-1} - \alpha_i q_{2k\ell-2i-2}}, \text{ for all } k \geq 1.$$

Proof. Let us first consider the continued fraction expansion of α_i .

$$\alpha_i = -\left[0, \frac{p_{2i} - R_{i-1}q_{2i}}{p_{2i-1} - R_{i-1}q_{2i-1}}\right] \stackrel{\text{Lm. 3}}{=} -[0, a_{2i}, a_{2i-1}, \dots, a_1, a_0 - R_{i-1}]$$

$$= [0, -a_{2i}, -a_{2i-1}, \dots, -a_1, -a_0 + R_{i-1}].$$

If $k = 0$ we have

$$\frac{\alpha_i p_{2i} + p_{2i-1}}{\alpha_i q_{2i} + q_{2i-1}} = [a_0, a_1, \dots, a_{2i-1}, a_{2i}, \alpha_i]$$

$$= [a_0, a_1, \dots, a_{2i-1}, a_{2i}, 0, -a_{2i}, -a_{2i-1}, \dots, -a_1, -a_0 + R_{i-1}] = R_{i-1},$$

and if $k > 0$ we have

$$\frac{\alpha_i p_{2k\ell+2i} + p_{2k\ell+2i-1}}{\alpha_i q_{2k\ell+2i} + q_{2k\ell+2i-1}} = [a_0, a_1, \dots, a_{2k\ell-1}, a_0 - 1 + a_0, a_1, \dots, a_{2i-1}, a_{2i}, \alpha_i]$$

$$= [a_0, a_1, \dots, a_{2k\ell-1}, a_0 - 1 + R_{i-1}] = \frac{p_{2k\ell-1}(a_0 - 1 + R_{i-1}) + p_{2k\ell-2}}{q_{2k\ell-1}(a_0 - 1 + R_{i-1}) + q_{2k\ell-2}}$$

$$\stackrel{(9)}{=} \frac{p_{2k\ell-1}R_{i-1} + \frac{d-1}{4}q_{2k\ell-1}}{q_{2k\ell-1}(R_{i-1} - 1) + p_{2k\ell-1}} \stackrel{(6)}{=} \frac{R_{k\ell-1}R_{i-1} + \frac{d-1}{4}}{R_{i-1} - 1 + R_{k\ell-1}} \stackrel{\text{Lm. 1}}{=} R_{k\ell+i-1},$$

$$\stackrel{(8)}{=} \frac{p_{2k\ell-2i-1} - \alpha_i p_{2k\ell-2i-2}}{q_{2k\ell-2i-1} - \alpha_i q_{2k\ell-2i-2}} = \left[a_0, a_1, \dots, a_{2(k\ell-i)-1}, -\frac{1}{\alpha_i} \right]$$

$$= [a_0, a_1, \dots, a_{2(k\ell-i)-1}, 0, 0, a_{2i}, a_{2i-1}, \dots, a_1, a_0 - R_{i-1}]$$

$$= [a_0, a_1, \dots, a_{2(k\ell-i)-1}, a_{2(k\ell-i)}, a_{2(k\ell-i)+1}, \dots, a_{2k\ell-1}, a_0 - R_{i-1}]$$

$$\stackrel{(9)}{=} \frac{p_{2k\ell-1}(a_0 - R_{i-1}) + p_{2k\ell-2}}{q_{2k\ell-1}(a_0 - R_{i-1}) + q_{2k\ell-2}} \stackrel{(8)}{=} \frac{p_{2k\ell-1}(1 - R_{i-1}) + \frac{d-1}{4}q_{2k\ell-1}}{p_{2k\ell-1} - R_{i-1}q_{2k\ell-1}}$$

$$\stackrel{(6)}{=} \frac{R_{k\ell-1}(1 - R_{i-1}) + \frac{d-1}{4}}{R_{k\ell-1} - R_{i-1}} \stackrel{\text{Lm. 2}}{=} R_{k\ell-i-1}.$$

□

Remark. Theorem 2 could be proved using the same ideas as in [6], but the ideas in the proof of Theorem 2 can also be used to prove [6, Tm. 1] in an easier way.

The following Corollary reduces our problem to the half-periods.

Corollary 1. For $n = 0, \dots, \lfloor \ell/2 \rfloor$ and $k \geq 0$

$$R_{k\ell+n} = \frac{p_{2(k\ell+n)+1+2j}}{q_{2(k\ell+n)+1+2j}} \iff R_{(k+1)\ell-n-2} = \frac{p_{2((k+1)\ell-n-2)+1-2j}}{q_{2((k+1)\ell-n-2)+1-2j}},$$

or in other words: $j(d, k\ell + n) = j(d, n) = -j(d, \ell - n - 2)$.

Proof. Same as [2, Lemma 3]. \square

4. HOW LARGE CAN j BE?

Lemma 4.

$$R_{n+1} < R_n. \quad (17)$$

Proof. Using $R_n = \frac{1}{4} \left(\frac{2p_n - q_n}{q_n} + \frac{dq_n}{2p_n - q_n} + 2 \right)$, the statement of the lemma is equivalent to

$$(-1)^n (dq_n q_{n+1} - (2p_n - q_n)(2p_{n+1} - q_{n+1})) > 0. \quad (18)$$

If n is even, then $\frac{p_n}{q_n} < \frac{\sqrt{d+1}}{2}$ and $\frac{p_{n+1}}{q_{n+1}} > \frac{\sqrt{d+1}}{2}$. Furthermore, since $\frac{p_{n+1}}{q_{n+1}} - \frac{\sqrt{d+1}}{2} < \frac{\sqrt{d+1}}{2} - \frac{p_n}{q_n}$, we have $\frac{2p_n - q_n}{q_n} + \frac{2p_{n+1} - q_{n+1}}{q_{n+1}} < 2\sqrt{d}$. Therefore

$$\frac{2p_n - q_n}{q_n} \cdot \frac{2p_{n+1} - q_{n+1}}{q_{n+1}} < \left[\left(\frac{2p_n - q_n}{q_n} + \frac{2p_{n+1} - q_{n+1}}{q_{n+1}} \right) / 2 \right]^2 < d,$$

and (18) holds. If n is odd, the proof is completely analogous. \square

Proposition 1. *When $\ell \left(\frac{1+\sqrt{d}}{2} \right) > 2$, then for all $n \geq 0$ we have*

$$|j(d, n)| \leq \frac{\ell - 3}{2}.$$

Proof. Let $R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$. According to Corollary 1 it suffices to consider the case $j > 0$ and $n < \ell$.

Assume first that ℓ is even, say $\ell = 2m$. Then $R_{m-1} = \frac{p_{\ell-1}}{q_{\ell-1}}$ and $R_{\ell-1} = \frac{p_{2\ell-1}}{q_{2\ell-1}}$. If $n < m - 1$, using (17) we have $2n + 1 + 2j \leq \ell - 2$, and $2j \leq \ell - 3$. Since ℓ is even, we have $j \leq \frac{\ell-4}{2}$. For $n = m - 1$ and $n = \ell - 1$ we have $j = 0$, and for $m - 1 < n < \ell - 1$ we have $2n + 1 + 2j \leq 2\ell - 2$ and $2j \leq 2\ell - 3 - 2n \leq \ell - 3$. Thus we have $j \leq \frac{\ell-4}{2}$ again.

Assume now that ℓ is odd, say $\ell = 2m + 1$. If for some $n, 0 \leq n < m$ we got $j > \frac{\ell-3}{2}$, i.e. $j \geq m$, we would have $2n + 1 + 2j \geq \ell$. By Corollary 1, we have $R_{\ell-n-2} = \frac{p_{2(\ell-n-2)+1-2j}}{q_{2(\ell-n-2)+1-2j}}$, and $2(\ell - n - 2) + 1 - 2j \leq \ell - 2$. Now from $\frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} > \dots$ it follows that $R_n > R_{\ell-n-2}$. However Lemma 4 implies that this is not possible, since $\ell - n - 2 \geq m$. For $m - 1 < n < \ell - 1$, the proof is completely analogous to the even case. \square

Let us show now that the Proposition 1 estimate is sharp. If we want that $j = j(d, n)$ is large, the continued fraction expansion should have many small a_i 's following a_n . Let us first see for fixed a_i 's what property a_0 should satisfy, in order to get the continued fraction expansion of a number of the form $\frac{1+\sqrt{d}}{2}, d \in \mathbb{N}, d \equiv 1 \pmod{4}$.

Proposition 2. *Let $\ell \in \mathbb{N}$ and $a_1, a_2, \dots, a_{\ell-1}$ such that $a_1 = a_{\ell-1}, a_2 = a_{\ell-2}, \dots$. Then the number $[a_0, \overline{a_1, a_2, \dots, a_{\ell-1}, 2a_0 - 1}]$ is of the form $\frac{1+\sqrt{d}}{2}, d \in \mathbb{N}, d \equiv 1 \pmod{4}$ if and only if*

$$2a_0 \equiv 1 - (-1)^\ell p'_{\ell-2} q'_{\ell-2} \pmod{p'_{\ell-1}}, \quad (19)$$

where $\frac{p'_i}{q'_i}$ are convergents of the number $[a_1, a_2, \dots, a_i]$. Then it holds:

$$d = 1 + 4 \left(a_0^2 - a_0 + \frac{(2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}} \right). \quad (20)$$

Proof. Let $\alpha = \frac{1+\sqrt{d}}{2} = [a_0, \overline{a_1, a_2, \dots, a_2, a_1, 2a_0 - 1}]$. Since $a_0, a_1 \in \mathbb{N}$, we have $\alpha > 1$. Let us observe:

$$\beta = a_0 - 1 + \alpha = [2a_0 - 1, \overline{a_1, a_2, \dots, a_2, a_1}] = [\overline{b_0, b_1, b_2, \dots, b_{\ell-2}, b_{\ell-1}}].$$

Since β is purely periodic, β is reduced, so we have $(\frac{p_i}{q_i})$ are convergents of β :

$$\beta, \bar{\beta} = \frac{p_{\ell-1} - q_{\ell-2} \pm \sqrt{(p_{\ell-1} - q_{\ell-2})^2 + 4q_{\ell-1}p_{\ell-2}}}{2q_{\ell-1}}.$$

We then have

$$\beta = [\overline{b_0, b_1, b_2, \dots, b_{\ell-2}, b_{\ell-1}}] \quad \text{and} \quad -1/\bar{\beta} = [\overline{b_{\ell-1}, b_{\ell-2}, \dots, b_2, b_1, b_0}],$$

i.e. because the expansion is palindromic we have

$$\beta = [b_0, \overline{b_1, b_2, \dots, b_2, b_1, b_0}] = [b_0, \beta_1] \quad \text{and} \quad -\bar{\beta} = [0, \overline{b_1, b_2, \dots, b_2, b_1, b_0}] = [0, \beta_1].$$

We see that $2a_0 - 1 = b_0 = \beta + \bar{\beta}$. Now we have:

$$d = (2\alpha - 1)^2 = (2\beta - 2a_0 + 1)^2 = (\beta - \bar{\beta})^2 = \frac{(p_{\ell-1} - q_{\ell-2})^2 + 4q_{\ell-1}p_{\ell-2}}{q_{\ell-1}^2}.$$

From

$$p_i = (2a_0 - 1)p'_i + q'_i, \quad q_i = p'_i,$$

we get

$$d = \left(\frac{(2a_0 - 1)p'_{\ell-1} + q'_{\ell-1} - p'_{\ell-2}}{p'_{\ell-1}} \right)^2 + 4 \frac{(2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}.$$

Because the expansion is palindromic, $p'_{\ell-2} = q'_{\ell-1}$, so we have

$$d = (2a_0 - 1)^2 + 4 \frac{(2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}. \quad (=20)$$

It is clear that $(2a_0 - 1)^2 \equiv 1 \pmod{4}$, so d will be an integer congruent to 1 mod 4, if and only if $p'_{\ell-1} \mid (2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}$, i.e.

$$(2a_0 - 1)p'_{\ell-2} \equiv -q'_{\ell-2} \pmod{p'_{\ell-1}}.$$

From $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ follows $p'_{\ell-2}q'_{\ell-1} \equiv (-1)^\ell \pmod{p'_{\ell-1}}$, so we have

$$(2a_0 - 1) \equiv -(-1)^\ell q'_{\ell-2}q'_{\ell-1} \pmod{p'_{\ell-1}}.$$

and we get (19) because the expansion is palindromic, i.e. $p'_{\ell-2} = q'_{\ell-1}$. \square

Lemma 5. Let F_k denote the k -th Fibonacci number, and $F_{-2} = -1, F_{-1} = 1, F_0 = 0$. For $m \in \mathbb{N}$ or $2m \in \mathbb{N}$ when $k \mid 3$, and $d_k(m) = 4((m \cdot F_k + 1)^2 + m \cdot F_{k-3}) + 1$ it holds

$$\frac{1 + \sqrt{d_k(m)}}{2} = [m \cdot F_k + 1, \underbrace{1, 1, \dots, 1}_{k-1 \text{ times}}, 2m \cdot F_k + 1],$$

and $\ell\left(\frac{1 + \sqrt{d_k(m)}}{2}\right) = k$.

Proof. From (19), it follows:

$$2a_0 \equiv 1 - (-1)^k F_{k-1} F_{k-2} \equiv 1 - (-1)^k F_{k-1} (F_k - F_{k-1}) \equiv 1 + (-1)^k F_{k-1}^2 \pmod{F_k}.$$

Now from Cassini's identity $F_k F_{k-2} - F_{k-1}^2 = (-1)^{k-1}$ we have:

$$2a_0 \equiv 2 \pmod{F_k},$$

or

$$a_0 = \begin{cases} 1 + m \cdot F_k, & m \in \mathbb{N}, \quad \text{when } 3 \nmid k, \\ 1 + \frac{m}{2} \cdot F_k, & m \in \mathbb{N}, \quad \text{when } 3 \mid k. \end{cases}$$

From (20) it follows:

$$\begin{aligned} d &= 4 \left((m \cdot F_k + 1)^2 - m \cdot F_k - 1 + \frac{(2m \cdot F_k + 2 - 1)F_{k-1} + F_{k-2}}{F_k} \right) + 1 \\ &= 4((m \cdot F_k + 1)^2 + m \cdot F_k - 2m \cdot F_{k-2}) + 1 = 4((m \cdot F_k + 1)^2 + m \cdot F_{k-3}) + 1, \end{aligned}$$

and when $k \equiv 0 \pmod{3}$,

$$d = 4 \left(\left(\frac{m}{2} \cdot F_k + 1 \right)^2 + \frac{m}{2} \cdot F_{k-3} \right) + 1.$$

□

For arbitrary k , we find m , such that for $d_k(m)$ it holds $R_0 = \frac{p_{k-2}}{q_{k-2}}$. We have $\frac{p_{k-2}}{q_{k-2}} = a_0 + \frac{F_{k-2}}{F_{k-1}}$. On the other hand, using $a_0 = 1 + m_k F_k$ we have: $R_0 = \frac{a_0^2 + (1 + m_k F_k)^2 + m_k F_{k-3}}{2a_0 - 1} = a_0 + \frac{a_0 + m_k F_{k-3}}{2a_0 - 1}$. So we get $R_0 = \frac{p_{k-2}}{q_{k-2}}$ if and only if $\frac{F_{k-2}}{F_{k-1}} = \frac{a_0 + m_k F_{k-3}}{2a_0 - 1}$, or

$$m_k = \frac{F_{k-1} - F_{k-2}}{2F_k F_{k-2} - F_{k-1} F_k - F_{k-1} F_{k-3}} = \frac{F_{k-3}}{F_k F_{k-4} - F_{k-1} F_{k-3}}.$$

It remains to see when m_k is an integer, or half of an integer if $3 \mid k$. Using Binet's formula $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}$, we have

$$m_k = \frac{F_{k-3}}{2 \cdot (-1)^{k-1}}.$$

So if k is odd, m_k is greater than 0, and when $3 \nmid k$, then F_{k-3} has to be even, but this is not possible. So, k has to be of the form $6n + 3, n \in \mathbb{N}$. Thus, we just proved:

Theorem 3. *Let $k = 6n + 3, n \in \mathbb{N}$. For $d_k = (F_k F_{k-3} + 2)^2 + 2F_{k-3}^2 + 1$ it holds $R_0 = \frac{p_{k-2}}{q_{k-2}}$.* □

Remark 1. *From Corollary 1 and (6) we get $R_{k-2} = \frac{p_k}{q_k}$ and $R_{k-1} = \frac{p_{2k-1}}{q_{2k-1}}$, respectively. So these are the only three good approximants.*

Corollary 2.

$$\begin{aligned} \sup \{ |j(d, n)| \} &= +\infty, \\ \limsup \left\{ \frac{|j(d, n)|}{\ell \left(\frac{1+\sqrt{d}}{2} \right)} \right\} &= \frac{1}{2}. \end{aligned}$$

□

It remains the question how large can be j compared with d . Let

$$d(j) = \min\{d \mid \text{there exists } n \text{ such that } j(d, n) \geq j\}.$$

In Table 1 we list $d(j)$ values for $1 \leq j \leq 104$ such that $d(j) > d(j')$ for $j' < j$. We also give corresponding n and k values such that $R_n = \frac{p_k}{q_k} = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$.

$d(j)$	$\ell(\frac{1+\sqrt{d}}{2})$	n	k	$j(d, n)$	$\frac{\ln d(j)}{\ln j(d, n)}$	$\frac{\sqrt{d(j)}}{j(d, n)}$	$\frac{j(d, n)}{\ell(\frac{1+\sqrt{d}}{2})}$
57	6	0	3	1		7.549834	0.166667
193	15	2	9	2	7.592457	6.946222	0.133333
721	36	6	19	3	5.989956	8.950481	0.0833333
1121	28	6	21	4	5.0652853	8.370335	0.142857
2521	85	22	55	5	4.866551	10.0419122	0.0588235
2641	82	23	59	6	4.397305	8.56511	0.0731707
4201	105	32	79	7	4.287494	9.259303	0.0666667
5401	120	16	49	8	4.133004	9.186437	0.0666667
10369	161	63	109	9	4.208298	11.314254	0.0559006
12241	167	37	97	11	3.925337	10.0580957	0.0658683
24841	231	62	151	13	3.945595	12.123868	0.0562771
33121	340	124	277	14	3.943803	12.999411	0.0411765
38689	310	79	189	15	3.900707	13.113013	0.0483871
46729	406	163	293	17	3.795027	12.715819	0.0418719
52201	345	88	217	20	3.626111	11.423769	0.057971
66721	413	123	295	24	3.495307	10.76267	0.0581114
121369	513	109	271	26	3.593077	13.399252	0.0506823
139921	559	158	373	28	3.555854	13.359291	0.0500894
203449	879	280	631	35	3.437967	12.887235	0.039818
212881	907	309	691	36	3.423587	12.816397	0.0396913
311761	962	300	685	42	3.38446	13.294181	0.043659
430081	1389	436	961	44	3.427875	14.904673	0.0316775
503881	1438	500	907	47	3.410284	15.103101	0.0326843
606481	1266	407	915	50	3.403719	15.575378	0.0394945
706729	1815	539	1181	51	3.425483	16.48376	0.0280992
760369	1802	559	1231	56	3.364069	15.571275	0.0310766
795409	1180	346	807	57	3.360485	15.646615	0.0483051
990721	1840	569	1267	64	3.319687	15.552339	0.0347826
1132609	2256	681	1507	72	3.259556	14.781126	0.0319149
1157641	2441	727	1603	74	3.243886	14.539693	0.0303154
1318249	2607	808	1773	78	3.234509	14.719875	0.0299194
1700689	2856	892	1951	83	3.246676	15.712105	0.0290616
1912681	2921	838	1845	84	3.264413	16.464251	0.0287573
2058001	3190	983	2155	94	3.199714	15.26142	0.0294671
2357569	3224	1044	2297	104	3.159325	14.763824	0.0322581

TABLE 1. $d(j)$ for $1 \leq j \leq 104$.

5. NUMBER OF GOOD APPROXIMANTS

Theorem 4. R_n is a convergent of $\frac{1+\sqrt{d}}{2}$ for all $n \geq 0$ if and only if $\ell(\frac{1+\sqrt{d}}{2}) \leq 2$.

Proof. Formulas (6) and (7) implies that if $\ell \leq 2$, then all R_n are convergents of $\frac{\sqrt{d+1}}{2}$.

Assume now that R_{n-1} is a convergent of $\frac{\sqrt{d+1}}{2}$ for all $n \in \mathbb{N}$. Then by Theorems 1, 4 and (6), we must have $R_{n-1} = \frac{p_{2n-1}}{q_{2n-1}}$ for all $n \in \mathbb{N}$, so every α_n in Theorem 2 has to be 0. Thus, for $n = 1$, we should have $R_0 = \frac{p_1}{q_1}$, i.e. $\alpha_1 = 0$. So let $d = 1 + 4(a_0^2 - a_0 + t)$. Then $\frac{1+\sqrt{d}}{2} = [a_0, \overline{a_1, a_2, \dots, a_{\ell-1}, 2a_0 - 1}]$. Hence, from (16) it follows:

$$\begin{aligned} 0 &= q_1 \left(\frac{d-1}{4} q_0^2 + p_0^2 \right) - p_1(2p_0 - q_0)q_0 = \\ &= a_1(a_0^2 - a_0 + t + a_0^2) - (1 + a_0 a_1)(2a_0 - 1) \\ &= a_1 t - 2a_0 + 1, \end{aligned}$$

that is

$$t = \frac{2a_0 - 1}{a_1}.$$

It is well known [9, p. 107] that $\ell(\frac{1+\sqrt{d}}{2}) \leq 2$, when $d = 1 + 4(a_0^2 - a_0 + t)$ and $t \mid 2a_0 - 1$. \square

Let

$$b(d) = \left| \left\{ n \mid 0 \leq n \leq \ell - 1, R_n \text{ is a convergent of } \frac{1+\sqrt{d}}{2} \right\} \right|.$$

Theorem 4 shows that $\frac{\ell(d)}{b(d)} > 1$ when $\ell > 2$ and $\frac{\ell(d)}{b(d)} = 1$, for $\ell \leq 2$. In Example 1 we showed that for $d = 324n^2 + 108n - 27$ we have $b(d) = 4$ and $\ell(d) = 6$, and in Example 2 we showed that for $d = 4n^4 + 16n^3 + 28n^2 + 28n + 13$ we have $b(d) = 3$ and $\ell(d) = 7$.

Let

$$\ell_b = \min \left\{ \ell \mid \text{there exists } d \text{ such that } \ell(\frac{1+\sqrt{d}}{2}) = \ell \text{ and } b = b(d) \right\}.$$

According to Theorem 4 we have $\ell_1 = 1$, $\ell_2 = 2$ and $\ell_b > b$ for $b > 2$. From Example 2 follows that $\ell_3 \leq 7$. From Corollary 1, (6) and (7) it follows that ℓ_b and b have the same parity, when $\ell_b < +\infty$. From Example 1, it follows that $\ell_4 = 6$. Let us show that $\ell_3 = 5$.

Example 3. Let $d = 16n^4 + 16n^3 + 12n^2 - 4n + 1$, $n \in \mathbb{N}$. Then $\ell(d) = 5$ and $b(d) = 3$. Using algorithm (1) it is straightforward to check that

$$\frac{1+\sqrt{d}}{2} = [2n^2 + n, \overline{1, 2n, 2n, 1, 4n^2 + 2n - 1}].$$

Now the direct computation shows:

$$\begin{aligned} R_0 &= 2n^2 + n + 1 - \frac{2n-1}{4n^2 + 2n - 1}, \\ R_1 &= 2n^2 + n + 1 - \frac{2n}{4n^2 + 2n + 1} = \frac{p_3}{q_3}, \\ R_2 &= 2n^2 + n + 1 - \frac{8n^3 + 8n^2 + 2n - 1}{16n^4 + 24n^3 + 16n^2 + 2n - 1} = \frac{p_5}{q_5}, \\ R_3 &= 2n^2 + n + 1 - \frac{2n(16n^4 + 16n^3 + 12n^2 + 2n + 1)}{(4n^2 + 2n + 1)(16n^4 + 16n^3 + 12n^2 + 1)}, \\ R_4 &= 2n^2 + n + 1 - \frac{32n^5 + 64n^4 + 64n^3 + 28n^2 + 4n - 1}{8n(8n^5 + 20n^4 + 26n^3 + 18n^2 + 7n + 1)} = \frac{p_9}{q_9}. \end{aligned}$$

In Table 2, we list the upper bounds for ℓ_b , $3 \leq b \leq 100$, obtained by experiments. It is not hard to check that sequences of numbers such that $b = 5$ and $\ell = 9$ or $b = 6$ and $\ell = 10$ exists, but the number 945 is the only one found which shows that $\ell_{10} \leq 14$ (and we tested all numbers $\leq 2^{21.5}$). Also, contrary to \sqrt{d} , where it holds $\ell_6 = 8$ [2, Exam. 1], for $\frac{1+\sqrt{d}}{2}$ we were not able to find such d .

In this section, we find some sequences which will significantly improve some of the entries in Table 2.

6. SEQUENCES WITH MANY GOOD APPROXIMANTS

Let us first prove some lemmas.

Proposition 3. *Let d, s_n, t_n, p_n, q_n be as in Algorithm (1). Then for $n \geq -1$ it holds*

$$\begin{aligned} (2p_n - q_n)^2 - dq_n^2 &= (-1)^{n+1}4t_{n+1}, \\ (2p_n - q_n)(2p_{n-1} - q_{n-1}) - dq_nq_{n-1} &= (-1)^n2s_{n+1}. \end{aligned}$$

Proof. Similarly as in [9, §20, III], since \sqrt{d} is irrational, from

$$\begin{aligned} \frac{1 + \sqrt{d}}{2} &= \left[a_0, a_1, \dots, a_n, \frac{s_{n+1} + \sqrt{d}}{2t_{n+1}} \right] = \frac{\frac{s_{n+1} + \sqrt{d}}{2t_{n+1}}p_n + p_{n-1}}{\frac{s_{n+1} + \sqrt{d}}{2t_{n+1}}q_n + q_{n-1}} \\ &= \frac{p_n(\sqrt{d} + s_{n+1}) + 2t_{n+1}p_{n-1}}{q_n(\sqrt{d} + s_{n+1}) + 2t_{n+1}q_{n-1}}, \end{aligned}$$

we get

$$2p_n - q_n = q_n s_{n+1} + 2t_{n+1}q_{n-1} \tag{21}$$

$$dq_n = 2(p_n s_{n+1} + 2t_{n+1}p_{n-1}) - (q_n s_{n+1} + 2t_{n+1}q_{n-1}). \tag{22}$$

Multiplying (21) with $2p_n - q_n$ and (22) with q_n , by subtraction we get:

$$(2p_n - q_n)^2 - dq_n^2 = 4t_{n+1}(q_{n-1}p_n - p_{n-1}q_n) = 4t_{n+1}(-1)^{n+1},$$

and multiplying (21) with $2p_{n-1} - q_{n-1}$ and (22) with q_{n-1} , by subtraction we get:

$$(2p_n - q_n)(2p_{n-1} - q_{n-1}) - dq_nq_{n-1} = 2s_{n+1}(q_n p_{n-1} - p_n q_{n-1}) = 2s_{n+1}(-1)^n.$$

□

b	$\ell_b \leq$	d	$\ell_b/b \leq$	b	$\ell_b \leq$	d	$\ell_b/b \leq$
3	5	41	1.66667	52	180	2 414 425	3.46154
4	6	57	1.5	53	429	2 328 625	8.09434
5	9	353	1.8	54	176	554 625	3.25926
6	10	129	1.66667	55	397	1 004 809	7.21819
7	13	4481	1.85714	56	180	1 839 825	3.21429
8	14	873	1.75	57	471	1 977 625	8.26316
9	17	67 073	1.88889	58	232	365 625	4.26316
10	14	945	1.4	59	499	2 601 625	8.45763
11	21	1 054 721	1.9091	60	210	1 388 625	3.5
12	20	2625	1.66667	61	607	2 739 601	9.9509
13	33	204 425	2.53847	62	246	2 660 065	3.96775
14	22	215 985	1.57143	63	527	2 229 625	8.36508
15	45	127 465	3.57143	64	226	2 544 993	3.5313
16	28	28 665	1.75	65	387	1 665 625	5.95385
17	31	244 205	1.82353	66	260	2 165 625	3.9394
18	34	87 057	1.88889	67	625	2 944 201	9.32836
19	53	2 483 125	2.78948	68	266	2 237 625	3.91177
20	38	1 588 457	1.9	69	679	2 586 625	9.8406
21	69	1 007 165	3.28572	70	340	1 517 697	4.85715
22	44	1 343 433	2.28572	71	763	2 193 241	10.74648
23	91	2 720 801	3.95653	72	298	2 721 705	4.13889
24	50	770 133	2.083334	73	961	2 792 425	13.16439
25	87	2 193 425	3.48	74	310	408 969	4.18919
26	64	190 125	2.46154	75	985	1 783 825	13.13334
27	95	2 632 825	3.51852	76	390	1 083 537	5.13158
28	60	182 457	2.14286	77	993	2 751 625	12.89611
29	113	1 286 305	3.89656	78	400	2 768 985	5.12821
30	76	2 837 097	2.53334	79	1083	1 859 425	13.70887
31	99	1 503 125	3.19355	80	356	639 009	4.45
32	86	235 305	2.6875	81	1075	2 188 825	13.27161
33	129	186 745	3.9091	82	356	1 105 425	4.34147
34	94	133 353	2.76471	83	1131	2 394 625	13.62651
35	153	1 512 745	4.37143	84	398	610 929	4.7381
36	94	174 097	2.61112	85	1187	2 602 825	13.96471
37	147	2 263 105	3.973	86	462	2 967 289	5.3721
38	112	57 321	2.94737	87	1105	2 889 625	12.70115
39	173	614 125	4.4359	88	462	1 112 697	5.25
40	96	2 033 361	2.4	89	1259	2 558 425	14.14607
41	227	2 526 625	5.53659	90	386	1 157 625	4.28889
42	122	677 457	2.90477	91	1409	2 766 625	15.48352
43	309	680 425	7.18605	92	672	2 100 249	7.30435
44	142	2 512 705	3.22728	93	1395	2 402 425	15.30435
45	243	1 743 625	5.4	94	592	1 796 977	6.29788
46	128	2 754 297	2.78261	95	1717	2 056 609	18.073685
47	273	2 815 625	5.80852	96	518	2 739 625	5.39584
48	166	1 962 873	3.45834	97	2013	2 903 209	20.75258
49	353	2 796 625	7.20409	98	530	2 268 945	5.40817
50	142	2 411 937	2.84	99	3495	2 869 441	35.3031
51	245	1 540 625	4.80393	100	746	2 718 441	7.46

TABLE 2. Upper bounds for ℓ_b , for $3 \leq b \leq 100$.

Let $g_n \stackrel{\text{def}}{=} \gcd\left(p_n^2 + \frac{d-1}{4}q_n^2, q_n(2p_n - q_n)\right)$.

Lemma 6. g_n divides $\gcd(d, t_{n+1}, s_{n+1}, s_{n+2})$.

Proof. Assume first that q_n is odd. Then g_n is odd as well, and we have:

$$\begin{aligned} g_n &= \gcd\left(p_n^2 + \frac{d-1}{4}q_n^2, q_n(2p_n - q_n)\right) \\ &= \gcd\left(4\left(p_n^2 + \frac{d-1}{4}q_n^2\right) - 2q_n(2p_n - q_n), q_n(2p_n - q_n)\right) \\ &= \gcd\left((2p_n - q_n)^2 + dq_n^2, q_n(2p_n - q_n)\right). \end{aligned}$$

Since q_n is odd and $\gcd(p_n, q_n) = 1$, it follows that $\gcd(2p_n - q_n, q_n) = 1$. Thus g_n divides $2p_n - q_n$ and d .

If q_n is even, then p_n is odd, and since $d \equiv 1 \pmod{4}$, g_n is also odd. So we have:

$$\begin{aligned} g_n &= \gcd\left(p_n^2 + \frac{d-1}{4}q_n^2, 2q_n\left(p_n - \frac{q_n}{2}\right)\right) = \gcd\left(p_n^2 + \frac{d-1}{4}q_n^2 - q_n\left(p_n - \frac{q_n}{2}\right), q_n\left(p_n - \frac{q_n}{2}\right)\right) \\ &= \gcd\left(\left(p_n - \frac{q_n}{2}\right)^2 + \frac{dq_n^2}{4}, q_n\left(p_n - \frac{q_n}{2}\right)\right). \end{aligned}$$

Since q_n is odd and $\gcd(p_n, q_n) = 1$ it follows that $\gcd(2p_n - q_n, q_n) = 2$. So g_n divides $2p_n - q_n$ and d .

Proposition 3 implies that $g_n \mid \gcd(t_{n+1}, s_{n+1}, s_{n+2})$. \square

Proposition 4. **i):** If $a_{n+1} > \frac{\sqrt{2}}{g_n} \sqrt{\sqrt{d} + 2}$, then R_n is a convergent of $\frac{1+\sqrt{d}}{2}$,

ii): if $a_{n+1} < \frac{1}{g_n} \sqrt{\sqrt{d} - 2} - 2$, then R_n is not a convergent of $\frac{1+\sqrt{d}}{2}$.

Proof. Let $R_n = \frac{u}{v}$, $\gcd(u, v) = 1$. Then $v = \frac{q_n(2p_n - q_n)}{g_n}$.

i) Let $a_{n+1} > \frac{\sqrt{2}}{g_n} \sqrt{\sqrt{d} + 2}$. We have

$$\begin{aligned} R_n - \frac{1 + \sqrt{d}}{2} &= \frac{q_n}{2p_n - q_n} \left(\frac{p_n}{q_n} - \frac{1 + \sqrt{d}}{2} \right)^2 \\ &< \frac{q_n}{2p_n - q_n} \frac{1}{a_{n+1}^2 g_n^4} = \frac{1}{2v^2} \frac{2}{g_n^2 a_{n+1}^2} \left(2 \frac{p_n}{q_n} - 1 \right) \\ &< \frac{1}{2v^2} \frac{2}{g_n^2 a_{n+1}^2} \left(2 \left(\frac{1 + \sqrt{d}}{2} + 1 \right) - 1 \right) = \frac{1}{2v^2} \frac{2}{g_n^2 a_{n+1}^2} (\sqrt{d} + 2) < \frac{1}{2v^2}. \end{aligned}$$

From (4), we see that R_n is a convergent.

ii) Let $a_{n+1} < \frac{1}{g_n} \sqrt{\sqrt{d} - 2} - 2$. Then

$$\begin{aligned} R_n - \frac{1 + \sqrt{d}}{2} &= \frac{q_n}{2p_n - q_n} \left(\frac{p_n}{q_n} - \frac{1 + \sqrt{d}}{2} \right)^2 > \frac{q_n}{2p_n - q_n} \frac{1}{(a_{n+1} + 2)^2 g_n^4} \\ &= \frac{1}{v^2} \frac{1}{g_n^2 (a_{n+1} + 2)^2} \left(2 \frac{p_n}{q_n} - 1 \right) > \frac{1}{v^2} \frac{1}{g_n^2 (a_{n+1} + 2)^2} \left(2 \left(\frac{1 + \sqrt{d}}{2} - 1 \right) - 1 \right) \\ &= \frac{1}{v^2} \frac{1}{g_n^2 (a_{n+1} + 2)^2} (\sqrt{d} - 2) > \frac{1}{v^2}. \end{aligned}$$

Now (3) proves (ii) of the proposition. \square

There are many quadruples (e, f, g, h) such that experimental results show that $d_n = (e \cdot f^n + g)^2 + h \cdot f^n$ should have many good approximants (numbers of this form sometimes have a very interesting continued fraction expansion; see e.g. [5],

[8], [11]). But it is not easy to show that either (i) or (ii) from Proposition 4 holds for every n . However, we found some for which that holds.

Proposition 5. *If*

$$d_n = (24 \cdot 9^n + 1)^2 + 12 \cdot 9^n, \quad (23)$$

then for $n \in \mathbb{N}$ we have $\ell\left(\frac{1+\sqrt{d_n}}{2}\right) = 4n + 6$ and

$$\frac{1 + \sqrt{d_n}}{2} = \left[\frac{12 \cdot 9^n + 1, \overline{8, 24 \cdot 9^{n-1}, 8 \cdot 9^1, 24 \cdot 9^{n-2}, 8 \cdot 9^2} \dots}{\dots 24 \cdot 9, 8 \cdot 9^{n-1}, 24, 8 \cdot 9^n, 2, 1, 2, 8 \cdot 9^n, 24, 8 \cdot 9^{n-1}, 24 \cdot 9 \dots} \right. \\ \left. \dots \overline{8 \cdot 9^2, 24 \cdot 9^{n-2}, 8 \cdot 9^1, 24 \cdot 9^{n-1}, 8, 24 \cdot 9^n + 1} \right].$$

Proof. Let $s_0 = t_0 = 1$, and we have $a_0 = 12 \cdot 9^n + 1$.

$$\begin{aligned} s_1 &= 12 \cdot 9^n + 1, & t_1 &= 3 \cdot 9^n, & a_1 &= 8, \\ s_2 &= 12 \cdot 9^n - 1, & t_2 &= 9, & a_2 &= 24 \cdot 9^{n-1}. \end{aligned}$$

For $1 \leq k \leq n$, from

$$s_{2k} = 12 \cdot 9^n - 1, \quad t_{2k} = 9^k, \quad a_{2k} = 24 \cdot 9^{n-k},$$

using (1) we have:

$$\begin{aligned} s_{2k+1} &= 12 \cdot 9^n + 1, & t_{2k+1} &= 3 \cdot 9^{n-k}, & a_{2k+1} &= \left\lfloor \frac{24 \cdot 9^n + 1}{3 \cdot 9^{n-k}} \right\rfloor = 8 \cdot 9^k, \\ s_{2k+2} &= 12 \cdot 9^n - 1, & t_{2k+2} &= 9^{k+1}, & a_{2k+2} &= \left\lfloor \frac{24 \cdot 9^n}{9^{k+1}} \right\rfloor. \end{aligned}$$

For $k < n$ we have:

$$a_{2k+2} = 24 \cdot 9^{n-(k+1)},$$

and for $k = n$:

$$\begin{aligned} a_{2n+2} &= 2, \\ s_{2n+3} &= 12 \cdot 9^n + 1, & t_{2n+3} &= 12 \cdot 9^n + 1, & a_{2n+3} &= \left\lfloor \frac{36 \cdot 9^n + 2}{24 \cdot 9^n + 2} \right\rfloor = 1, \\ s_{2n+4} &= 12 \cdot 9^n + 1, \end{aligned}$$

and since $s_{2n+3} = s_{2n+4}$ we have $\ell = 2(2n + 3) = 4n + 6$. \square

Lemma 7. *For the sequence (23) and $g_k = \gcd\left(p_k^2 + \frac{d_n-1}{4}q_k^2, q_k(2p_k - q_k)\right)$, for $k = 0, 1, \dots, 2n + 1, 2n + 3, \dots, 4n + 4$ we have $g_k = 1$.*

Proof. From (2) it follows $s_{4n+6} = s_1, s_{4n+5} = s_2, \dots, t_{4n+5} = t_1, \dots$, and using Lemma 6, for $k = 0, 1, \dots, 2n+1, 2n+3, \dots, 2n+4$ we have $g_k \mid \gcd(s_{k+1}, s_{k+2}, t_{k+1}) = 1$. \square

Theorem 5. *For the sequence $d_n = (24 \cdot 9^n + 1)^2 + 12 \cdot 9^n$ we have $b(d_n) = 2n + 4$.*

Proof. Using Proposition 5, we have $\ell = 4n + 6$. Using (7) and (6), R_{2n+2} and R_{4n+5} are good approximants. By Corollary 1, it suffices to check approximants R_k , $k = 0, 1, \dots, 2n + 1$. By Lemma 7, we have $g_k = 1$. By Proposition 4 (i) R_k is a good approximant if

$$a_{k+1}^2 \geq 48 \cdot 9^n + 8 = 2(24 \cdot 9^n + 4) > 2(\sqrt{d} + 2), \quad (24)$$

and by Proposition 4 (ii) R_k is not good approximant if

$$a_{k+1} < \sqrt{24 \cdot 9^n + 3} - 2 < \sqrt{\sqrt{d} - 2} - 2. \quad (25)$$

For $i = 2k$, $k = 0, 1, \dots, n$ using Proposition 5 we have $a_{2k+1} = 8 \cdot 9^k$, so let us see when (24) holds.

$$(8 \cdot 9^k)^2 = 64 \cdot 9^{2k} > 48 \cdot 9^n + 8$$

holds when $2k \geq n$, i.e. $k \geq \lfloor \frac{n+1}{2} \rfloor$. Thus $R_{2\lfloor \frac{n+1}{2} \rfloor}, R_{2\lfloor \frac{n+1}{2} \rfloor+2}, \dots, R_{2n}$ are good approximants. For $2k \leq n-1$, i.e. $k \leq \lfloor \frac{n-1}{2} \rfloor$, (25) holds, thus $R_0, R_2, \dots, R_{2\lfloor \frac{n-1}{2} \rfloor}$ are not good approximants.

For $i = 2k - 1$, $k = 1, 2, \dots, n$ we have $a_{2k} = 24 \cdot 9^{n-k}$ and $a_{2n+2} = 2$, so (24) holds when $2k \leq n + 1$, i.e. $k \leq \lfloor \frac{n+1}{2} \rfloor$. Thus $R_1, R_3, \dots, R_{2\lfloor \frac{n+1}{2} \rfloor-1}$ are good approximants. For $2k \geq n+2$, i.e. $k \geq \lfloor \frac{n+3}{2} \rfloor$, (25) holds, so $R_{2\lfloor \frac{n+1}{2} \rfloor+1}, R_{2\lfloor \frac{n+1}{2} \rfloor+3}, \dots, R_{2n-1}$ are not good approximants.

Therefore there are exactly $2 + 2(n + 1 - \lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor) = 2 + 2(n + 1) = 2n + 4$ good approximants. \square

Corollary 3. *For the sequence (23) we have $\ell(d_n) = 4n + 6$ and $b(d_n) = 2n + 4$ for every $n \in \mathbb{N}$. So for every even positive integer b there exists $d \in \mathbb{N}$, $d \equiv 1 \pmod{4}$, $d \neq \square$, such that $b(d) = b$ and $b(d) > \frac{\ell(d)}{2}$.*

Proof. For $b = 2$ we have $\ell = 2$, and for $b = 4$, in Table 2 we have the number 57, which has $\ell = 6$, and for other even b 's we use sequence (23). \square

Proposition 6. *Let*

$$d_n = (3 \cdot 16^n + 1)^2 + 4 \cdot 16^n. \quad (26)$$

Then for $n \in \mathbb{N}$ it holds $\ell(\frac{1+\sqrt{d_n}}{2}) = 4n + 1$ and

$$\frac{1 + \sqrt{d_n}}{2} = \left[\frac{3}{2} \cdot 16^n + 1, \overline{3, 3 \cdot 4^{2n-1}, 3 \cdot 4^1, 3 \cdot 4^{2n-2}, 3 \cdot 4^2, \dots}, \dots, \overline{3 \cdot 4^{n+1}, 3 \cdot 4^{n-1}, 3 \cdot 4^n, 3 \cdot 4^n, 3 \cdot 4^{n-1}, 3 \cdot 4^{n+1}, \dots}, \dots, \overline{3 \cdot 4^2, 3 \cdot 4^{2n-2}, 3 \cdot 4^1, 3 \cdot 4^{2n-1}, 3, 3 \cdot 16^n + 1} \right].$$

Proof. Let $s_0 = t_0 = 1$, and we have $a_0 = \frac{3}{2} \cdot 16^n + 1$.

$$s_1 = 3 \cdot 16^n + 1, \quad t_1 = 4^{2n}, \quad a_1 = 3 \cdot 4^0.$$

For $0 \leq k < 2n - 1$ we get

$$s_{2k+1} = 3 \cdot 16^n + 1, \quad t_{2k+1} = 4^{2n-k}, \quad a_{2k+1} = 3 \cdot 4^k,$$

and we have:

$$\begin{aligned} s_{2k+2} &= 3 \cdot 16^n - 1, & t_{2k+2} &= 4^{k+1}, & a_{2k+2} &= \left\lfloor \frac{3 \cdot 4^{2n}}{4^{k+1}} \right\rfloor = 3 \cdot 4^{2n-(k+1)}, \\ s_{2k+3} &= 3 \cdot 16^n + 1, & t_{2k+3} &= 4^{2n-(k+1)}, & a_{2k+3} &= \left\lfloor \frac{3 \cdot 4^{2n} + 1}{4^{2n-(k+1)}} \right\rfloor = 3 \cdot 4^{k+1}, \end{aligned}$$

so when $k = n - 1$, we have $t_{2k+2} = t_{2k+3}$, thus $\ell = 2(2n - 2 + 2) + 1 = 4n + 1$. \square

Lemma 8. *For the sequence (26) and $g_k = \gcd(p_k^2 + \frac{d_n-1}{4}q_k^2, q_k(2p_k - q_k))$, for $k \geq 0$ we have $g_k = 1$.*

Proof. From (2) it follows that $s_{4n-1} = s_{4n+1} = s_{4n+3} = 3 \cdot 16^n + 1$ and $s_{4n-2} = s_{4n} = s_{4n+2} = 3 \cdot 16^n - 1$. Using Lemma 6, we have $g_k \mid \gcd(s_{k+1}, s_{k+2}) = 1$, for every $k \geq 0$. \square

Theorem 6. *For the sequence $d_n = (3 \cdot 16^n + 1)^2 + 4 \cdot 16^n$ we have $b(d_n) = 2n + 1$.*

Proof. By Proposition 6, we have $\ell = 4n + 1$. Thus, by (6), R_{4n} is good approximant. By Lemma 8 we have $g_k = 1$. Using Proposition 4 (i) (divided by 3) R_k is good approximant if

$$\frac{a_{k+1}}{3} \geq 4^n > \sqrt{\frac{2}{3} \cdot 16^n + \frac{8}{9}} = \frac{\sqrt{2}}{3} \cdot \sqrt{3 \cdot 16^n + 4} > \frac{\sqrt{2}}{3} \cdot \sqrt{\sqrt{d} + 2}, \quad (27)$$

and by Proposition 4 (ii) R_k is not a good approximant if

$$\frac{a_{k+1}}{3} \leq 4^{n-1} < \sqrt{\frac{1}{3} \cdot 16^n - \frac{1}{9}} - \frac{2}{3} = \frac{1}{3} \cdot \sqrt{3 \cdot 16^n - 1} - 2 < \frac{1}{3} \sqrt{\sqrt{d} - 2} - 2. \quad (28)$$

For $k = 2i + 1$, $i = 0, 1, 2, \dots, 2n - 1$, by Proposition 6, we have $a_{2i+1} = 3 \cdot 4^i$, thus for $i = 0, 1, 2, \dots, n - 1$ (28) holds, so $R_0, R_2, \dots, R_{2n-2}$ are not good approximants, and for $i = n, n+1, \dots, 2n-1$ (27) holds, thus $R_{2n}, R_{2n+2}, \dots, R_{4n-2}$ are good approximants. For k of the form $2i$, using Corollary 1, it follows that $R_1, R_3, \dots, R_{2n-1}$ are good approximants, and the others are not.

Therefore there are exactly $2n + 1$ good approximants. \square

Corollary 4. *For the sequence (26) we have $\ell(d_n) = 4n + 1$ and $b(d_n) = 2n + 1$ for every $n \in \mathbb{N}$. Thus for every odd positive integer b there exists $d \in \mathbb{N}$, $d \equiv 1 \pmod{4}$, $d \neq \square$, such that $b(d) = b$ and $b(d) > \frac{\ell(d)}{2}$.* \square

From Corollaries 3 and 4, we immediately obtain the following result.

Corollary 5.

$$\sup \left\{ \frac{\ell_b}{b} : b \geq 1 \right\} \leq 2.$$

\square

ACKNOWLEDGMENTS

The author would like to thank Professor Dujella for many valuable suggestions.

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