Householder's approximants and continued fraction expansion of quadratic irrationals

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Let α be arbitrary quadratic irrationality ($\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$ and d is not a square of a rational number). It is well known that regular continued fraction expansion of α is periodic, i.e. has the form $\alpha = [\,a_0, a_1, \ldots, a_k, \overline{a_{k+1}, a_{k+2}, \ldots, a_{k+\ell}}\,]$. Here $\ell = \ell(\alpha)$ denotes the length of the shortest period in the expansion of α.

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for solving nonlinear equations $f(x) = 0$ is another approximation method. Connections between these two approximation methods were discussed by several authors. Let $\frac{p_n}{q_n}$ be the nth convergent of α. The principal question is: Let $f(x) = (x - \alpha)(x - \alpha')$, where $\alpha' = c - \sqrt{d}$ and $x_0 = \frac{p_n}{q_n}$, is x_1 also a convergent of α?

It is well known that for $\alpha = \sqrt{d}$, $d \in \mathbb{N}$, $d \neq \square$, and the corresponding Newton's approximant $R_n = \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{dq_n}{p_n}\right)$ it follows that

$$R_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1. \tag{1.1}$$

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It was proved by Mikusiński [Mik1954] that if $\ell = 2t$, then

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These results imply that if $\ell(\sqrt{d}) \leq 2$, then all approximants R_n are convergents of $\sqrt{d}$. Dujella [Duje2001] proved the converse of this result. Namely, if $\ell(\sqrt{d}) > 2$, we know that some of approximants R_n are not convergents. He showed that being again a convergent is a periodic and a palindromic property.

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In 2011, P. [Pet1.2011] proved the analogous results for $\alpha = \frac{1+\sqrt{d}}{2}$, $d \in \mathbb{N}$, $d \neq \square$ and $d \equiv 1 \pmod 4$.

Sharma [Sha1959] observed arbitrary quadratic surd $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$, d is not a square of a rational number, whose period begins with a_1. He showed that for every such α and the corresponding Newton's approximant $N_n = \frac{p_n^2 - \alpha\alpha' q_n^2}{2q_n(p_n - cq_n)}$ it holds $N_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad$ for $k \geq 1$, and when $\ell = 2t$ and the period is palindromic then it holds $N_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad$ for $k \geq 1$.

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$$\frac{p_{nk\ell-1}}{q_{nk\ell-1}} = \frac{\alpha(p_{k\ell-1} - \alpha' q_{k\ell-1})^n - \alpha'(p_{k\ell-1} - \alpha q_{k\ell-1})^n}{(p_{k\ell-1} - \alpha' q_{k\ell-1})^n - (p_{k\ell-1} - \alpha q_{k\ell-1})^n}, \tag{1.3}$$

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and when $\ell = 2t$ and the period is palindromic then for $k, n \in \mathbb{N}$ it holds

$$\frac{p_{nkt-1}}{q_{nkt-1}} = \frac{\alpha(p_{kt-1} - \alpha' q_{kt-1})^n - \alpha'(p_{kt-1} - \alpha q_{kt-1})^n}{(p_{kt-1} - \alpha' q_{kt-1})^n - (p_{kt-1} - \alpha q_{kt-1})^n}. \tag{1.4}$$

Householder's iterative method (see e.g. [Hous1970, §4.4]) of order p for rootsolving: $x_{n+1} = H^{(p)}(x_n) = x_n + p \cdot \frac{(1/f)^{(p-1)}(x_n)}{(1/f)^{(p)}(x_n)}$, where $(1/f)^{(p)}$ denotes p-th derivation of $1/f$. Householder's method of order 1 is just Newton's method. For Householder's method of order 2 one gets Halley's method, and Householder's method of order p has rate of convergence $p + 1$.

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It is not hard to show that for $f(x) = (x-\alpha)(x-\alpha')$ it holds:

$$H^{(m+1)}(x) = \frac{xH^{(m)}(x) - \alpha\alpha'}{H^{(m)}(x) + x - \alpha - \alpha'}, \quad \text{for } m \in \mathbb{N}. \tag{2.1}$$

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Let us define $R_n^{(1)} \overset{\text{def}}{=} \frac{p_n}{q_n}$ and for $m > 1$ $R_n^{(m)} \overset{\text{def}}{=} H^{(m-1)}\left(\frac{p_n}{q_n}\right)$. We will say that $R_n^{(m)}$ is *good approximation*, if it is a convergent of α.

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Formula (1.3) shows that for arbitrary quadratic surd, whose period begins with a_1 and $k, m \in \mathbb{N}$, it holds

$$R_{k\ell-1}^{(m)} = \frac{p_{mk\ell-1}}{q_{mk\ell-1}}, \tag{2.2}$$

and when $\ell = 2t$ and period is periodic, from (1.4) it follows

$$R_{kt-1}^{(m)} = \frac{p_{mkt-1}}{q_{mkt-1}}.$$

Formula [Sha1959, (8)] says: For $k \in \mathbb{N}$ it holds

$$(a_\ell - a_0)p_{k\ell-1} + p_{k\ell-2} = q_{k\ell-1}(d - c^2), \tag{2.3}$$

$$(a_\ell - a_0)q_{k\ell-1} + q_{k\ell-2} = p_{k\ell-1} - 2cq_{k\ell-1}, \tag{2.4}$$

and formula (2.1) says

$$R_n^{(m+1)} = \frac{R_n^{(1)}R_n^{(m)} - \alpha\alpha'}{R_n^{(1)} + R_n^{(m)} - 2c}, \quad \text{for } m \in \mathbb{N},\ n = 0, 1, \ldots. \tag{2.5}$$

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Lemma 2.1

For $m, k \in \mathbb{N}$ and $i = 1, 2, \ldots, \ell$, when the period begins with a_1, it holds $R^{(m)}_{k\ell+i-1} = \frac{R^{(m)}_{k\ell-1} R^{(m)}_{i-1} - \alpha\alpha'}{R^{(m)}_{k\ell-1} + R^{(m)}_{i-1} - 2c}$.

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Proof.

For $m = 1$, statement of the lemma is proven in [Frank1962, Thm. 2.1]. Using mathematical induction and (2.5) it is not hard to show that the statement of the lemma holds too. □

When period is palindromic and begins with a_1, formulas (2.3) and (2.4) become

$$a_0 p_{k\ell-1} + p_{k\ell-2} = 2cp_{k\ell-1} + q_{k\ell-1}(d - c^2), \tag{2.6}$$

$$a_0 q_{k\ell-1} + q_{k\ell-2} = p_{k\ell-1}. \tag{2.7}$$

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Lemma 2.2

For $m, k \in \mathbb{N}$ and $i = 1, 2, \ldots, \ell - 1$, when period is palindromic and begins with a_1, it holds $R^{(m)}_{k\ell-i-1} = \frac{R^{(m)}_{k\ell-1}(R^{(m)}_{i-1} - 2c) + \alpha\alpha'}{R^{(m)}_{i-1} - R^{(m)}_{k\ell-1}}$.

Proof.

For $m = 1$ we have:

$$R^{(1)}_{k\ell-i-1} = \frac{p_{k\ell-i-1}}{q_{k\ell-i-1}} = \frac{0 \cdot p_{k\ell-i} + p_{k\ell-i-1}}{0 \cdot q_{k\ell-i} + q_{k\ell-i-1}} = [\,a_0, \dots, a_{k\ell-i}, 0\,]$$
$$= [\,a_0, \dots, a_{k\ell-i}, a_{k\ell-i-1}, \dots, a_{k\ell-1}, a_0, 0, -a_0, -a_1, \dots, -a_{i-1}\,]$$
$$= \left[\,a_0, \dots, a_{k\ell-i}, a_{k\ell-i-1}, \dots, a_{k\ell-1}, a_0 - \frac{p_{i-1}}{q_{i-1}}\,\right]$$
$$= \frac{p_{k\ell-1}\left(a_0 - R^{(1)}_{i-1}\right) + p_{k\ell-2}}{q_{k\ell-1}\left(a_0 - R^{(1)}_{i-1}\right) + q_{k\ell-2}} \overset{(2.6)}{\underset{(2.7)}{=}} \frac{R^{(1)}_{k\ell-1}\left(R^{(1)}_{i-1} - 2c\right) + \alpha\alpha'}{R^{(1)}_{i-1} - R^{(1)}_{k\ell-1}}.$$

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$$= [\,a_0,\ldots,a_{k\ell-i},a_{k\ell-i-1},\ldots,a_{k\ell-1},a_0,0,-a_0,-a_1,\ldots,-a_{i-1}\,]$$

$$= \left[\,a_0,\ldots,a_{k\ell-i},a_{k\ell-i-1},\ldots,a_{k\ell-1},a_0-\frac{p_{i-1}}{q_{i-1}}\,\right]$$

$$= \frac{p_{k\ell-1}\left(a_0-R^{(1)}_{i-1}\right)+p_{k\ell-2}}{q_{k\ell-1}\left(a_0-R^{(1)}_{i-1}\right)+q_{k\ell-2}} \overset{(2.6)}{\underset{(2.7)}{=}} \frac{R^{(1)}_{k\ell-1}\left(R^{(1)}_{i-1}-2c\right)+\alpha\alpha'}{R^{(1)}_{i-1}-R^{(1)}_{k\ell-1}}.$$

Using mathematical induction and (2.5) it is not hard to show that the statement of the lemma holds too. □

Proposition 2.3

Let $m \in \mathbb{N}$. When period begins with a_1, for $i = 1, 2, \ldots, \ell - 1$ and $\beta_i^{(m)} = -\frac{p_{mi-1} - R_{i-1}^{(m)} q_{mi-1}}{p_{mi} - R_{i-1}^{(m)} q_{mi}}$, it holds

$$R_{k\ell+i-1}^{(m)} = \frac{\beta_i^{(m)} p_{m(k\ell+i)} + p_{m(k\ell+i)-1}}{\beta_i^{(m)} q_{m(k\ell+i)} + q_{m(k\ell+i)-1}}, \text{ for all } k \geq 0,$$

and when period is palindromic, then

$$R_{k\ell-i-1}^{(m)} = \frac{p_{m(k\ell-i)-1} - \beta_i^{(m)} p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1} - \beta_i^{(m)} q_{m(k\ell-i)-2}}, \text{ for all } k \geq 1.$$

Proof.

We have $\beta_i^{(m)} = \left[\,0, -a_{mi}, -a_{mi-1}, \ldots, -a_1, -a_0 + R_{i-1}^{(m)}\,\right]$.

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$$\frac{\beta_i^{(m)} p_{mi} + p_{mi-1}}{\beta_i^{(m)} q_{mi} + q_{mi-1}} = \left[\,a_0, \ldots, a_{mi}, \beta_i^{(m)}\,\right]$$

$$= \left[\,a_0, \ldots, a_{mi}, 0, -a_{mi}, -a_{mi-1}, \ldots, -a_1, -a_0 + R_{i-1}^{(m)}\,\right] = R_{i-1}^{(m)},$$

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and similarly if $k > 0$ we have

$$\frac{\beta_i^{(m)} p_{m(k\ell+i)} + p_{m(k\ell+i)-1}}{\beta_i^{(m)} q_{m(k\ell+i)} + q_{m(k\ell+i)-1}} = \left[a_0, \ldots, a_{mk\ell-1}, a_{mk\ell} - a_0 + R_{i-1}^{(m)}\right]$$
$$= \frac{p_{mk\ell-1}\left(a_{mk\ell} - a_0 + R_{i-1}^{(m)}\right) + p_{mk\ell-2}}{q_{mk\ell-1}\left(a_{mk\ell} - a_0 + R_{i-1}^{(m)}\right) + q_{mk\ell-2}}$$
$$\overset{(2.3),(2.2)}{\underset{(2.4)}{=}} \frac{R_{k\ell-1}^{(m)} R_{i-1}^{(m)} + d - c^2}{R_{k\ell-1}^{(m)} + R_{i-1}^{(m)} - 2c} \overset{\text{Lm. 2.1}}{=} R_{k\ell+i-1}^{(m)}.$$

Proof.

When period is palindromic we have:

$$\frac{p_{m(k\ell-i)-1}-\beta_i^{(m)}p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1}-\beta_i^{(m)}q_{m(k\ell-i)-2}}=\left[a_0,\ldots,a_{m(k\ell-i)-1},-\frac{1}{\beta_i^{(m)}}\right]$$

$$=\left[a_0,\ldots,a_{m(k\ell-i)-1},a_{m(k\ell-i)},a_{m(k\ell-i)+1},\ldots,a_{mk\ell-1},a_0-R_{i-1}^{(m)}\right]$$

$$=\frac{p_{mk\ell-1}\left(a_0-R_{i-1}^{(m)}\right)+p_{mk\ell-2}}{q_{mk\ell-1}\left(a_0-R_{i-1}^{(m)}\right)+q_{mk\ell-2}}\underset{(2.7)}{\overset{(2.6),(2.2)}{=}}\frac{R_{k\ell-1}^{(m)}(R_{i-1}^{(m)}-2c)+c^2-d}{R_{i-1}^{(m)}-R_{k\ell-1}^{(m)}},$$

which is using Lemma 2.2 equal to the $R_{k\ell-i-1}^{(m)}$. □

Analogously as in [Duje2001, Lm. 3], from Proposition 2.3 it follows:

Theorem 2.4

To be a good approximant is a periodic property, i.e. for all $r \in \mathbb{N}$ it holds

$$R_n^{(m)} = \frac{p_k}{q_k} \qquad \Longleftrightarrow \qquad R_{r\ell+n}^{(m)} = \frac{p_{rm\ell+k}}{q_{rm\ell+k}},$$

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and when period is palindromic, it is also a palindromic property, i.e. it holds:

$$R_n^{(m)} = \frac{p_k}{q_k} \qquad \Longleftrightarrow \qquad R_{\ell-n-2}^{(m)} = \frac{p_{m\ell-k-2}}{q_{m\ell-k-2}}.$$

Let us define coprime positive numbers $P_n^{(m)}$, $Q_n^{(m)}$ by

$$\frac{P_n^{(m)}}{Q_n^{(m)}} \stackrel{\text{def}}{=} R_n^{(m)}.$$

From (2.5) it is not hard to show that it holds

$$P_n^{(m)} - \alpha Q_n^{(m)} = \left(P_n^{(1)} - \alpha Q_n^{(1)}\right)^m = (p_n - \alpha q_n)^m.$$

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Lemma 2.5

$R_n^{(m)} < \alpha$ if and only if n is even and m is odd. Therefore, $R_n^{(m)}$ can be an even convergent only if n is even and m is odd.

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Similarly as in [Duje2001], if $R_n^{(m)} = \frac{p_k}{q_k}$, we can define $j^{(m)} = j^{(m)}(\alpha, n)$ as the distance from convergent with m times larger index:

$$j^{(m)} = \frac{k + 1 - m(n+1)}{2}. \tag{2.8}$$

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$$j^{(m)} = \frac{k + 1 - m(n+1)}{2}. \tag{2.8}$$

This is an integer, by Lemma 2.5. Using Theorem 2.4 we have
$j^{(m)}(\alpha, n) = j^{(m)}(\alpha, k\ell + n)$, and in palindromic case:
$j^{(m)}(\alpha, n) = -j^{(m)}(\alpha, \ell - n - 2)$.

From now on, let us observe only quadratic irrationals of the form $\alpha = \sqrt{d}$, $d \in \mathbb{N}$, $d \neq \square$. It is well known that period of such α is palindromic and begins with a_1.

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Proposition 2.7 (for proof see [Pet2.2012])

When $d \neq \square$, for $n \geq 0$ we have $\left|j^{(m)}(\sqrt{d}, n)\right| < \frac{m(\ell/2-1)}{2}$.

Theorem 2.8 (Euler, see §26 in [Perron1954])

Let $\ell \in \mathbb{N}$ and $a_1, \dots, a_{\ell-1} \in \mathbb{N}$ such that $a_1 = a_{\ell-1}$, $a_2 = a_{\ell-2}$, ... The number $[\, a_0, \overline{a_1, a_2, \dots, a_{\ell-1}, 2a_0}\,]$ is of the form $\sqrt{d}$, $d \in \mathbb{N}$ if and only if

$$2a_0 \equiv (-1)^{\ell-1} p'_{\ell-2} q'_{\ell-2} \pmod{p'_{\ell-1}}, \tag{2.9}$$

where $\frac{p'_n}{q'_n}$ are convergents of the number $[\, a_1, a_2, \dots, a_{n-1}, a_n\,]$. Then it holds:

$$d = a_0^2 + \frac{2a_0 p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}. \tag{2.10}$$

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Lemma 2.9

Let F_k *denote the* k*-th Fibonacci number. Let* $n \in \mathbb{N}$ *and* $k > 1, k \equiv 1, 2 \pmod 3$. *For* $d_k(n) = \left(\frac{(2n+1)F_k+1}{2}\right)^2 + (2n+1)F_{k-1} + 1$ *it holds*

$$\sqrt{d_k(n)} = \left[\, \frac{(2n-1)F_k+1}{2}, \overline{\underbrace{1, 1, \ldots, 1, 1}_{k-1 \text{ times}}, (2n-1)F_k + 1}\,\right], \text{ and } \ell(\sqrt{d_k(n)}) = k.$$

Proof.

From (2.9), it follows:

$$2a_0 \equiv (-1)^{k-1}F_{k-1}F_{k-2} \equiv (-1)^{k-1}F_{k-1}(F_k - F_{k-1}) \equiv (-1)^{k-1}(-F_{k-1}^2) \pmod{F_k}.$$

Now from Cassini's identity $F_kF_{k-2} - F_{k-1}^2 = (-1)^{k-1}$ we have $2a_0 \equiv 1 \pmod{F_k}$. When $3 \mid k$, this congruence is not solvable, and if $3 \nmid k$, the solution is $a_0 \equiv \frac{F_k+1}{2} \pmod{F_k}$, i.e.

$$a_0 = \frac{F_k+1}{2} + (n-1)F_k = \frac{(2n-1)F_k+1}{2}, \qquad n \in \mathbb{N}.$$

From (2.10) it follows:

$$d = \left(\frac{(2n-1)F_k+1}{2}\right)^2 + \frac{\big((2n-1)F_k+1\big)F_{k-1} + F_{k-2}}{F_k} = \left(\frac{(2n-1)F_k+1}{2}\right)^2 + (2n-1)F_{k-1} + 1.$$

□

Theorem 2.10

Let F_ℓ denote the ℓ-th Fibonacci number. Let $\ell > 3, \ell \equiv \pm 1 \pmod 6$. Then for $d_\ell = \left(\frac{F_{\ell-3}F_\ell+1}{2}\right)^2 + F_{\ell-3}F_{\ell-1} + 1$ and $M \in \mathbb{N}$ it holds $\ell\left(\sqrt{d_\ell}\right) = \ell$ and

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Proof.

By (2.8), we have to prove

$$R_0^{(3M-1)} = \frac{p_{M\ell-2}}{q_{M\ell-2}}, \qquad R_0^{(3M)} = \frac{p_{M\ell-1}}{q_{M\ell-1}}, \qquad R_0^{(3M+1)} = \frac{p_{M\ell}}{q_{M\ell}}.$$

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We have $a_0 = \frac{F_{\ell-3}F_\ell+1}{2}$, and by Lemma 2.9 it holds $\sqrt{d_\ell} = \left[\, a_0, \overline{\underbrace{1, 1, \ldots, 1, 1}_{\ell-1 \text{ times}}, 2a_0}\,\right]$. From Cassini's identity, it follows

$$R_0^{(1)} = \frac{p_0}{q_0} = a_0, \qquad R_0^{(2)} = a_0 + \frac{F_{\ell-2}}{F_{\ell-1}} = \frac{p_{\ell-2}}{q_{\ell-2}},$$

Proof.

$$R_0^{(3)} = a_0 + \frac{F_{\ell-1}F_{\ell-2}^3}{F_{\ell-1}^2 F_{\ell-2}^2 + F_{\ell-2}^2} = a_0 + \frac{F_{\ell-1}}{F_\ell} = \frac{p_{\ell-1}}{q_{\ell-1}}. \qquad (2.11)$$

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Let us prove the theorem using induction on M. For proving the inductive step, first observe that from (2.5) for $m \geq 3$ we have:

$$R_k^{(m)} = \frac{R_k^{(2)} R_k^{(m-2)} + d}{R_k^{(2)} + R_k^{(m-2)}}, \qquad R_k^{(m)} = \frac{R_k^{(3)} R_k^{(m-3)} + d}{R_k^{(3)} + R_k^{(m-3)}}. \qquad (2.12)$$

Proof.

$$R_0^{(3)} = a_0 + \frac{F_{\ell-1}F_{\ell-2}^3}{F_{\ell-1}^2 F_{\ell-2}^2 + F_{\ell-2}^2} = a_0 + \frac{F_{\ell-1}}{F_\ell} = \frac{p_{\ell-1}}{q_{\ell-1}}. \tag{2.11}$$

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Suppose that for some $i \in \{0, \ell-2, \ell-1\}$ it holds $\frac{p_{(M-1)\ell+i}}{q_{(M-1)\ell+i}} = R_0^{(m-3)}$. We have:

$$\frac{p_{M\ell+i}}{q_{M\ell+i}} = \Big[a_0, \underbrace{1, 1, \ldots, 1, 1}_{\ell-1 \text{ times}}, a_0 + R_0^{(m-3)}\Big] =$$

$$\underset{(2.7)}{\overset{(2.6)}{=}} \frac{p_{\ell-1} R_0^{(m-3)} + d q_{\ell-1}}{q_{\ell-1} R_0^{(m-3)} + p_{\ell-1}} \overset{(2.11)}{=} \frac{R_0^{(3)} R_0^{(m-3)} + d}{R_0^{(3)} + R_0^{(m-3)}} \overset{(2.12)}{=} R_0^{(m)}.$$

□

Corollary 2.11

For each $m \geq 2$ it holds

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$$\sup\left\{|j^{(m)}(\sqrt{d}, n)|\right\} = +\infty,$$

$$\limsup\left\{\frac{|j^{(m)}(\sqrt{d}, n)|}{\ell\left(\sqrt{d}\right)}\right\} \geq \frac{m}{6}.$$

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Thanks