ON THE LARGEST ELEMENT IN $D(n)$-QUADRUPLES

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Abstract. Let n be a nonzero integer. A set of nonzero integers $\{a_1, \dots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$ is called a $D(n)$-m-tuple. In this paper, we consider the question, for given integer n which is not a perfect square, how large and how small can be the largest element in a $D(n)$-quadruple. We construct families of $D(n)$-quadruples in which the largest element is of order of magnitude $|n|^3$, resp. $|n|^{2/5}$.

1. Introduction

For a nonzero integer n, a set of distinct nonzero integers $\{a_1, a_2, \dots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$ is called a $D(n)$-m-tuple (or a Diophantine m-tuple with the property $D(n)$).

The most studied case is $n = 1$ and $D(1)$-m-tuples are called Diophantine m-tuples. Fermat found the first Diophantine quadruple, it was the set $\{1, 3, 8, 120\}$. In 1969, Baker and Davenport [1] proved that the set $\{1, 3, 8\}$ can be extended to a Diophantine quintuple only by adding 120 to the set. In 2004, Dujella [11] proved that there are no Diophantine sextuples and that there are at most finitely many Diophantine quintuples. Recently, He, Togbé and Ziegler proved that there are no Diophantine quintuples [23] (see also [3]). On the other hand, there are examples of $D(n)$-quintuples and sextuples for $n \ne 1$, e.g. $\{8, 32, 77, 203, 528\}$ is a $D(-255)$-quintuple [7], while $\{99, 315, 9920, 32768, 44460, 19534284\}$ is a $D(2985984)$-sextuple [21] (see also [18]). For an overview of results on Diophantine m-tuples and its generalizations see [14].

Several authors considered the problem of the existence of Diophantine quadruples with the property $D(n)$. It is easy to show that there are no $D(n)$-quadruples if $n \equiv 2 \pmod 4$ ([4, 22, 25]). Indeed, assume that $\{a_1, a_2, a_3, a_4\}$ is a $D(n)$-quadruple. Since the square of an integer is $\equiv 0$ or $1 \pmod 4$, we have that $a_i a_j \equiv 2$ or $3 \pmod 4$. This implies that none of the a_i's is divisible by 4. Therefore, we may assume that $a_1 \equiv a_2 \pmod 4$. But now we have that $a_1 a_2 \equiv 0$ or $1 \pmod 4$, a contradiction. On the other hand, it is shown in [6] that if $n \not\equiv 2 \pmod 4$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple. For $n \in S$, the question of the existence of $D(n)$-quadruples is still open.

The Lang conjecture on varieties of general type implies that the size of sets with the property $D(n)$ is bounded by an absolute constant (independent on n). It is known that the size of sets with the property $D(n)$ is ≤ 31 for $|n| \le 400$; $< 15.476 \log |n|$ for $|n| > 400$, and $< 3 \cdot 2^{168}$ for n prime (see [9, 10, 19] and also [2]).

It is easy to see that there exist infinitely many $D(1)$-quadruples. Indeed, the set $\{k-1, k+1, 4k, 16k^3 - 4k\}$ for $k \ge 2$ is a $D(1)$-quadruple (see e.g. [8]). More precisely, it was proved in [24] that the number of $D(1)$-quadruples with elements $\le N$ is $\sim C \sqrt[3]{N} \log N$, where $C \approx 0.338285$ (the main contribution comes from the quadruples of the form $\{a, b, a+b+2r, 4r(a+r)(b+r)\}$ where $ab + 1 = r^2$,

2010 *Mathematics Subject Classification.* Primary 11D09; Secondary 11G05.
Key words and phrases. $D(n)$-quadruples.

see [12]). If n is a perfect square, say $n = \ell^2$, then by multiplying elements of a $D(1)$-quadruple by ℓ we obtain a $D(\ell^2)$-quadruple, and thus we conclude that there exist infinitely many $D(\ell^2)$-quadruples. Moreover, it was proved in [6] that any $D(\ell^2)$-pair $\{a, b\}$, such that ab is not a perfect square, can be extended to a $D(\ell^2)$-quadruple.

The following conjecture was proposed in [13].

Conjecture 1. *If a nonzero integer n is not a perfect square, then there exist only finitely many $D(n)$-quadruples.*

As we already mentioned, it is easy to verify the conjecture in case $n \equiv 2 \pmod 4$ since there does not exist a $D(n)$-quadruple in that case. Only other cases where the conjecture is known to be true are the cases $n = -1$ and $n = -4$, see [15, 16] (these two cases are equivalent since, by [6], all elements of a $D(-4)$-quadruple are even).

Motivated by Conjecture 1, in this paper we consider the question, for given integer n which is not a perfect square, what can be said about the largest element in a $D(n)$-quadruple. In particular, the question how large it can be (compared with $|n|$) is closely related with Conjecture 1. On the other hand, the question how small it can be (again compared with $|n|$) makes sense also in the case when n is a perfect square. Since $\{a, b, c, d\}$ is a $D(n)$-quadruple if and only if $\{-a, -b, -c, -d\}$ has the same property, without loss of generality we may assume that $\max\{|a|, |b|, |c|, |d|\} = d$. Our main results are collected in the following theorem.

Theorem 1. *Let δ, ε be real numbers such that $2/5 \le \delta \le 3$ and $\varepsilon > 0$. Then there exist an integer n which is not a perfect square and a $D(n)$-quadruple $\{a, b, c, d\}$ such that*

$$\left| \frac{\log(\max\{|a|, |b|, |c|, |d|\})}{\log |n|} - \delta \right| < \varepsilon.$$

In Section 2 we will consider $D(n)$-quadruples, where n is not a perfect square, with large elements and construct family of quadruples with d of order of magnitude $|n|^3$, while in Section 3 we will consider $D(n)$-quadruples with small elements and construct family of quadruples with d of order of magnitude $|n|^{2/5}$. Since elements of both families of quadruples are polynomials in one variable, a standard construction with $D(n)$-quadruples will finish the proof of Theorem 1.

It should be noted that we do not know what are best possible results in both direction, i.e. is there any family of $D(n)$-quadruples with d of order of magnitude $|n|^\delta$ with $\delta > 3$ or $\delta < 2/5$. We will show in Section 3 that we cannot have $\delta < 1/4$.

2. $D(n)$-quadruples with large elements

The proof of the fact that for $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$ there exists at least one $D(n)$-quadruple [6], is based on explicit formulas for $D(n)$-quadruples, where $n = 4k + 3$, $n = 8k + 1$, $n = 8k + 5$, $n = 8k$, $n = 16k + 4$ and $n = 16k + 12$, while elements of $D(n)$-quadruples are polynomials in k. For example,

$$\{1, 9k^2 + 8k + 1, 9k^2 + 14k + 6, 36k^2 + 44k + 13\}$$

is a $D(4k+3)$-quadruple. This example shows that it is possible to have $\max\{a, b, c, d\} \sim \frac{9}{4}|n|^2$. The same conclusion also follows from the fact that

$$\begin{aligned}(1)\qquad &\{1, 144k^4 + 216k^3 + 113k^2 + 20k + 1, 144k^4 + 360k^3 + 329k^2 + 134k + 22,\\ &\qquad 576k^4 + 1152k^3 + 848k^2 + 272k + 33\}\end{aligned}$$

is a $D((4k + 1)(4k + 3))$-quadruple (see [17]).

In order to find families of quadruples with larger elements (compared with $|n|$), we performed an extensive search for $D(n)$-quadruples (similar as explained in [20]) and then sieve them according to the requirement that $\max\{a,b,c,d\}/|n|^2$ is relatively large (in particular, larger that $9/4$). Then we search for properties which are common to several found quadruples. For example, quadruples containing one small element (e.g. $a=1$) and quadruples containing a regular triple (a $D(n)$-triple $\{b,c,d\}$ is called regular if $(b+c-d)^2=4(bc+n)$). We have extracted the following interesting examples:

$\{a,b,c,d\}$	n	d/n^2
$\{1, 2912, 131977, 174097\}$	-208	4.024062
$\{1, 16896, 1980161, 2362881\}$	-512	9.013676
$\{1, 56640, 12525465, 14266673\}$	-944	16.009535

These examples suggest that for every positive integer k there might exist a quadruple such that $d/n^2 \approx k^2$. Furthermore, in these three examples (and some other found examples) we have that $n|b$. In fact, these examples suggests that we may take that $b/n=-k(4k-1)$. Starting with these assumptions, it is now easy to reconstruct the corresponding family $\{a,b,c,d\}$ of $D(n)$-quadruples:

$$\begin{aligned}
a &= 1,\\
b &= 256k^4-128k^3-48k^2+16k,\\
c &= 4096k^6-4096k^5-512k^4+1152k^3+16k^2-88k-7,\\
d &= 4096k^6-2048k^5-1792k^4+640k^3+288k^2-48k-15,\\
n &= -64k^2+16k+16.
\end{aligned}$$

Here k is an arbitrary nonzero integer. By taking $|k|\to\infty$, we obtain $d/n^2\to\infty$ and $d/|n|^3\to 1/64$.

After the substitution $k=z+1/8$, the family of quadruples becomes

$$\{1, 256z^4-72z^2+\frac{17}{16}, 4096z^6-1024z^5-2112z^4+416z^3+335z^2-\frac{153}{4}z-\frac{1007}{64},$$
$$4096z^6+1024z^5-2112z^4-416z^3+335z^2+\frac{153}{4}z-\frac{1007}{64}\},$$

with $n=-64z^2+17$. Thus, n, b and $d+c$ are even polynomials in z, while $d-c$ is odd (analogously as in (1)). From these properties, it is also possible to reconstruct the polynomials by the method of undetermined coefficients.

In our example we have $n\sim -64k^2$ and $d\sim 4096k^6$. Hence, $\log d/\log|n|\to 3$ as $k\to\infty$. If we take $k=y^{\ell_1}$, and multiply all elements of the quadruple by y^{ℓ_2}, we get quadruple in which $n\sim -64y^{2\ell_1+2\ell_2}$ and $d\sim 4096y^{6\ell_1+\ell_2}$. Hence, now we have $\log d/\log|n| \sim (6\ell_1+\ell_2)/(2\ell_1+2\ell_2)$. By varying nonnegative integers ℓ_1 and ℓ_2, we get that any point from the interval $[1/2,3]$ is an accumulation point of the set $\{\log d/\log|n| : \{a,b,c,d\}$ is a $D(n)$-quadruple for some non-square $n\}$.

3. $D(n)$-quadruples with small elements

In this section, we consider the question how small can be elements of a $D(n)$-quadruple, in particular how small can be its largest element. As we have already seen at the end of the previous section, it is easy to get quadruples in which $\max\{a,b,c,d\}$ is of order of magnitude $|n|^{1/2}$. Indeed, we can take any nonzero integer k and any fixed $D(k)$-quadruple $\{a_1,b_1,c_1,d_1\}$, and multiply its elements by large positive integer ℓ to get $D(k\ell^2)$-quadruple $\{a_1\ell, b_1\ell, c_1\ell, d_1\ell\}$, which yields $\log(\max\{|a|,|b|,|c|,|d|\})/\log|n|\to 1/2$ as $\ell\to\infty$.

Thus, we are interested if there are (families of) examples with d significantly smaller that $|n|^{1/2}$. If $n < 0$, then we can assume that $0 < a < b < c < d$, and from $cd + n > 0$ it follows that $d > |n|^{1/2}$. Hence, we may assume that $n > 0$. We claim that d cannot be smaller that $n^{1/4}$. Indeed, let $|a| \le |b| \le |c| \le d < n^{1/4}$. Since $d \ge 2$, we have $n > 16$. We may assume that $c > b$ (if $c < b$ the proof is analogous). From $cd+n = r^2$, $bd+n = s^2$ we get $n^{1/2} > d(c-b) = r^2-s^2 \ge (s+1)^2-s^2 = 2s+1$. We obtain $s < \frac{1}{2}n^{1/2}$, which implies $bd < -\frac{3}{4}n$, and this contradicts $|bd| < n^{1/2}$.

As the examples of $D(16k^4 - 72k^2 + 48k + 9)$-triple $\{4k - 4, 8k - 4, 12k\}$ and $D(144k^4 + 264k^3 + 181k^2 + 52k + 5)$-triple $\{-6k - 1, 2k + 1, 6k + 4\}$ show, in a $D(n)$-triple we may have all elements of the order $n^{1/4}$. However, we were not able to find $D(n)$-quadruples with the same property.

3.1. **$D(n)$-quadruples of the form $\{a, -a, b, -b\}$.** In considering certain problems with $D(n)$-quadruples, it might be convenient to study sets of the form $\{a, -a, b, -b\}$. In order to satisfy the definition of a $D(n)$-quadruple, such sets has to satisfy only four conditions: $-a^2+n$, $-b^2+n$, $ab+n$ and $-ab+n$ are perfect squares, compared with six conditions which has to be satisfied by general set of four elements.

By considering $D(n)$-quadruples of the form $\{a, -a, b, -b\}$, we found examples with $\max\{|a|, |b|\} \le n^{1/2}$:

$$\{-4u, 4u, -3 - u^2, 3 + u^2\}$$

is a $D((u^2 + 9)(1 + u^2))$-quadruple,

$$\{-4u(u - 1)(u - 2), 4u(u - 1)(u - 2), -(u^2 - 2u + 2)^2, (u^2 - 2u + 2)^2\}$$

is a $D((u^2 - 2u + 2)^4)$-quadruple, while

$$\{-4u^2 - 2u - 1, 4u^2 + 2u + 1, -4u(u + 1), 4u(u + 1)\}$$

is a $D((10u^2 + 2u + 1)(2u^2 + 2u + 1))$-quadruple.

We can improve slightly these results to get families of $D(n)$-quadruples of the form $\{a, -a, b, -b\}$ such that $\max\{|a|, |b|\}$ is of order of magnitude $n^{9/20}$.

Let $-a^2 + n = r^2$ and $-ab + n = (r - 1)^2$. We get

$$n = a^2 + r^2, \quad r = (ab - a^2 + 1)/2.$$

We write the third condition in the form $-b^2 + n = (r - t)^2$, which gives that $a^2t^2 + 4t + 4a^2 - 4a^2t - 4t^2$ is a perfect square, say

$$(t^2 + 4 - 4t)a^2 - 4t^2 + 4t = ((t - 2)a + u)^2.$$

We get

$$a = -(4t^2 - 4t + u^2)/(2u(t - 2)), \quad b = -(4t - 8t^2 + 4t^3 + u^2)/(2u(t - 2)).$$

It remains the satisfy the last condition that $ab+n$ is a perfect square. The condition leads to

$$16t^8 - 64t^7 + 8u^2t^6 + 96t^6 - 64t^5 + 16t^4 - 40u^2t^4 + u^4t^4 + 48u^2t^3 + 4u^4t^3 - 16u^2t^2 - 8u^4t + 4u^4 + 2u^6 = \square.$$

If we take $u = t(t - 1)/v$, the condition becomes

$$(v^2 + 2)t^4 + (4v^2 - 4)t^3 + (8v^4 + 2)t^2 + (16v^4 - 8v^2)t + 16v^6 - 16v^4 + 4v^2 = s^2.$$

This quartic over $\mathbb{Q}(v)$ has a $\mathbb{Q}(v)$-rational point $P_1 = [0, -2v(2v^2 - 1)]$, and therefore it can be in standard way (see e.g. [5]) transformed into an elliptic curve. There is another point on the quartic:

$$P_2 = [-4(v - 1)v/(2v + 1), -2v(16v^4 - 16v^3 + 14v^2 - 8v + 3)/((2v + 1)^2)].$$

If we take the point $2P_2$, we get

$$t = \frac{-8v(16v^4 - 16v^3 - 2v^2 + 8v - 3)}{32v^3 - 56v^2 + 20v + 1},$$

which gives the $D(n)$-quadruple $\{a, -a, b, -b\}$, where

$$\begin{aligned} a &= 16384v^9 - 32768v^8 + 20480v^7 - 5120v^6 + 4608v^5 - 4096v^4 + 1216v^3 - 304v^2 \\ &\quad + 164v - 24, \\ b &= 12288v^8 - 24576v^7 + 21504v^6 - 11520v^5 + 2880v^4 + 576v^3 - 528v^2 + 156v - 24, \\ n &= 268435456v^{20} - 1073741824v^{19} + 1879048192v^{18} - 1946157056v^{17} \\ &\quad + 1392508928v^{16} - 788529152v^{15} + 465567744v^{14} - 412090368v^{13} \\ &\quad + 412483584v^{12} - 328990720v^{11} + 205324288v^{10} - 110215168v^9 \\ &\quad + 53587968v^8 - 22474752v^7 + 7394304v^6 - 1852160v^5 + 468752v^4 \\ &\quad - 164480v^3 + 47872v^2 - 7552v + 580, \end{aligned}$$

which satisfies $\log(\max\{|a|, |b|\})/\log n \to 9/20$ as $v \to \infty$.

3.2. $D(n)$**-quadruples with** $d^5 \sim 8n^2$**.** Now we describe the construction of a family of $D(n)$-quadruples in which d is of order of magnitude $n^{2/5}$. The construction is motivated by the following experimentally found example:

$$\{468, 335, -85, -448\}$$

is a $D(1312164)$-quadruple.

Let $n = x^2 + y$ and put $ab + n = (x + s)^2$, $bd + n = (x - s)^2$. We get

$$y = -ab + 2xs + s^2, \quad d = \frac{ab - 4xs}{b}.$$

Now put $ac + n = (x - r)^2$, $cd + n = (x + r)^2$, and we get

$$a = \frac{s(s + 2x - r)}{b}, \quad c = \frac{-br}{s}.$$

By taking $bc + n = (x - t)^2$, we obtain

$$x = \frac{b^2r - s^2r + st^2}{2st}.$$

It remains to satisfy the condition that $ad+n$ is a perfect square. We put $ad+n = (x - r - s - t + 1)^2$. This leads to the equation

$$\begin{aligned} &-2b^2s^2t^2 - b^2t^3s + b^2st^2 - 2s^3r^2b^2 + sb^4r^2 - s^5t^2 + s^3b^2t^2 + b^2s^2t^3 - ts^2b^2r \\ &\quad + s^3t^4 + tb^4r + s^5r^2 + 4b^2rs^2t^2 - b^4rt^2 - tb^4rs + ts^3b^2r - s^3r^2t^2 - b^4r^2t \\ &\quad + b^2s^2r^2t + b^2st^3r + b^2r^2st^2 - 2b^2rst^2 = 0, \end{aligned} \tag{2}$$

In order to find some of its solution, we introduce the condition $x - y = t/2$. This gives $t = \frac{-s^2+b^2}{2s^2}$, and by inserting it in (2) (the discriminant of the equation in r becomes a square), we get

$$r = \frac{-(b^4 + 2b^2s^3 - 4s^2b^2 - 2s^5 - s^4)}{2s^2(-2s^3 - s^2 + b^2)}.$$

Thus, we obtain the rational $D(n)$-quadruple $\{a,b,c,d\}$, where

$$\begin{aligned}
a &= \frac{-b(s-1)(2s^3-3s^2+b^2)}{s(-2s^3-s^2+b^2)},\\
c &= \frac{b(b^4+2b^2s^3-4s^2b^2-2s^5-s^4)}{2s^3(-2s^3-s^2+b^2)},\\
d &= \frac{(b^2+2s^3+s^2)(b^2-s-2s^2)}{b(-2s^3-s^2+b^2)},\\
n &= (b+s)(b-s)(2b^3s-b^3+3b^2s-2s^2b^2-3s^2b-4bs^3+s^3+4bs^4-4s^5)\\
&\quad \times (2b^3s-b^3-3b^2s+2s^2b^2-3s^2b-4bs^3-s^3+4bs^4+4s^5)\\
&\quad \times (16s^4(-2s^3-s^2+b^2)^2)^{-1}.
\end{aligned}$$

In order to get quadruples with integers elements, we put $b = ks$ and search for the solution in the form $s = s_3k^3 + s_2k^2 + s_1k + s_0$. The condition that rational functions appearing in a, c, d, n become polynomials, leads to $s_0 = -1/2$, $s_2 = 1/2$, $s_3 = -s_1/3$. We take $s_1 = -3/2$ and $k = 2v - 1$, and we obtain the $D(64v^{10} - 128v^9 - 64v^8 + 240v^7 - 32v^6 - 136v^5 + 41v^4 + 22v^3 - 7v^2])$-quadruple

$$\{8v^4-4v^3-8v^2, 8v^4-12v^3+4v-1, -4v^3+2v^2+2v-1, -8v^4+4v^3+12v^2-4v-4\},$$

which satisfies $\log(\max\{|a|,|b|,|c|,|d|\})/\log n \to 2/5$ as $v \to \infty$. For $v = 3$ we get our starting motivating example: $D(1312164)$-quadruple $\{468, 335, -85, -448\}$.

We now apply the same argument as at the end of Section 2. We have $n \sim 64v^{10}$ and $\max\{|a|,|b|,|c|,|d|\} \sim 8v^4$. If we take $v = y^{\ell_1}$, and multiply all elements of the quadruple by y^{ℓ_2}, we get quadruple in which $n \sim 64y^{10\ell_1+2\ell_2}$ and $\max\{|a|,|b|,|c|,|d|\} \sim 8y^{4\ell_1+\ell_2}$. Hence, we have $\log(\max\{|a|,|b|,|c|,|d|\})/\log|n| \sim (4\ell_1+\ell_2)/(10\ell_1+2\ell_2)$. By varying nonnegative integers ℓ_1 and ℓ_2, we get that any point from the interval $[2/5, 1/2]$ is an accumulation point of the set

$$\left\{\frac{\log(\max\{|a|,|b|,|c|,|d|\})}{\log|n|} : \{a,b,c,d\} \text{ is a } D(n)\text{-quadruple for some non-square } n\right\},$$

which together with the mentioned result from the end of Section 2 finishes the proof of Theorem 1.

Remark 1. As we mentioned in the introduction, the question how small the largest element of a $D(n)$-quadruple can be make sense also in the case when n is a perfect square. By taking the $D(42849)$-quadruple $\{188, 140, -160, -198\}$ as a motivating example, we found that

$$\{60u^2-24u-4, 100u-60, -4(5u-2)(3u-1), -90u^2+96u-30\}$$

is a $D((15u-7)^2(5u-1)^2)$-quadruple, and this yields $\log(\max\{|a|,|b|,|c|,|d|\})/\log n \to 1/2$ as $u \to \infty$. Thus, a version of Theorem 1 with n a nonzero perfect square holds for $1/2 \le \delta < \infty$. Again, we do not know whether the lower bound for δ is best possible.

Acknowledgements. The authors were supported by the Croatian Science Foundation under the project no. IP-2018-01-1313. The authors acknowledge support from the QuantiXLie Center of Excellence, a project co-financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004). The authors acknowledge the usage of the supercomputing resources of Division of Theoretical Physics at Ruđer Bošković Institute.

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