Householder's approximants and continued fraction expansion of quadratic irrationals

Vinko Petričević
vpetrice@math.hr

Department of Mathematics, University of Zagreb, Croatia

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Let α be arbitrary quadratic irrationality ($\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$ and d is not a square of a rational number). It is well known that regular continued fraction expansion of α is periodic, i.e. has the form $\alpha = [\,a_0, a_1, \ldots, a_k, \overline{a_{k+1}, a_{k+2}, \ldots, a_{k+\ell}}\,]$. Here $\ell = \ell(\alpha)$ denotes the length of the shortest period in the expansion of α.

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Connections between these two approximation methods were discussed by several authors. Let $\frac{p_n}{q_n}$ be the nth convergent of α. The principal question is:
Let $f(x) = (x - \alpha)(x - \alpha')$, where $\alpha' = c - \sqrt{d}$ and $x_0 = \frac{p_n}{q_n}$, is x_1 also a convergent of α?

It is well known that for $\alpha = \sqrt{d}$, $d \in \mathbb{N}$, $d \neq \square$, and the corresponding Newton's approximant $R_n = \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{dq_n}{p_n}\right)$ it follows that

$$R_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1. \tag{1.1}$$

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It was proved by Mikusiński [Mik1954] that if $\ell = 2t$, then

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These results imply that if $\ell(\sqrt{d}) \leq 2$, then all approximants R_n are convergents of $\sqrt{d}$. Dujella [Duje2001] proved the converse of this result. Namely, if $\ell(\sqrt{d}) > 2$, we know that some of approximants R_n are not convergents.

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Example 1.1

If $d = 16x^4 - 16x^3 - 12x^2 + 16x - 4$, where $x \geq 2$, then $\ell(\sqrt{d}) = 8$ and $\sqrt{d} = [\,(2x+1)(2x-2), \overline{x, 1, 1, 2x2 - x - 2, 1, 1, x, 2(2x+1)(2x-2)}\,]$.

$$R_0 = \frac{p_3}{q_3}, \quad R_1 = \frac{p_5}{q_5}, \quad R_3 = \frac{p_7}{q_7},$$
$$R_5 = \frac{p_9}{q_9}, \quad R_6 = \frac{p_{11}}{q_{11}}, \quad R_7 = \frac{p_{15}}{q_{15}}.$$

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Theorem 1.2

If d is a square-free positive integer such that $\ell(\sqrt{d}) > 2$, then $|j(d,n)| \leq \frac{\ell-3}{2}$ for all $n \geq 0$.

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He also and pointed out that $j(\sqrt{d})$ is unbounded.

Theorem 1.3

Let $t \geq 1$ and $m \geq 5$ be integers such that $m \equiv \pm 1 \pmod 6$ and let $d = F_{m-2}^2[(2F_{m-2}t - F_{m-4})^2 + 4]/4$. Then $\sqrt{d} =$

$$\left[\tfrac{1}{2}F_{m-2}(2F_{m-2}t - F_{m-4}), \overline{2t-1, \underbrace{1, 1, \ldots, 1, 1}_{m-3 \text{ times}}, 2t-1, F_{m-2}(2F_{m-2}t - F_{m-4})}\right].$$

Therefore, $\ell(\sqrt{d}) = m$. Furthermore, $R_0 = \frac{p_{m-2}}{q_{m-2}}$ and hence $j(d,0) = \frac{m-3}{2}$, $j(d,km) = \frac{m-3}{2}$ and $j(d,km-2) = -\frac{m-3}{2}$ for $k \geq 1$.

Let $b(d)$ denote the number of good approximants among the numbers $R_n, n = 0, 1, \ldots, \ell - 1$. In [DujPet2005], Dujella and P. showed that the quantity $b(d)$ can be arbitrary large. Moreover, we construct families of examples which show that for every positive integer b there exist a positive integer d such that $b(d) = b$ and $b(d) > \ell(\sqrt{d})/2$.

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Proposition 1.4

Let $g := gcd(p_n^2 + dq_n^2, 2p_nq_n$. If $a_{n+1} > \frac{2}{g}\sqrt{\sqrt{d}+1}$, then R_n is a convergent of $\sqrt{d}$. If $a_{n+1} < \frac{1}{g}\sqrt{2(\sqrt{d}-1)} - 2$, then R_n is not a convergent of $\sqrt{d}$.

Theorem 1.5

For $n \geq 1$:
If $d_n = (12 \cdot 9^n + 1)^2 + 6 \cdot 9^n$, then $\ell(\sqrt{d_n}) = 4n + 6$ and $b(d_n) = 2n + 4$.
If $d_n = (2 \cdot 9^n + 1)^2 + 9^n$, then $\ell(\sqrt{d_n}) = 2n + 1$ and $b(d_n) = 2n + 1$.

In 2012, P. [Pet1.2012] proved the analogous results for $\alpha = \frac{1+\sqrt{d}}{2}$, $d \in \mathbb{N}$, $d \neq \square$ and $d \equiv 1 \pmod 4$.

Sharma [Sha1959] observed arbitrary quadratic surd $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$, d is not a square of a rational number, whose period begins with a_1. He showed that for every such α and the corresponding Newton's approximant $N_n = \frac{p_n^2 - \alpha\alpha' q_n^2}{2q_n(p_n - cq_n)}$ it holds $N_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad$ for $k \geq 1$, and when $\ell = 2t$ and the period is palindromic then it holds $N_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad$ for $k \geq 1$.

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Frank and Sharma [F-S1965] discussed generalization of Newton's formula. They showed that for every α, whose period begins with a_1, for $k, n \in \mathbb{N}$ it holds

$$\frac{p_{nk\ell-1}}{q_{nk\ell-1}} = \frac{\alpha(p_{k\ell-1} - \alpha' q_{k\ell-1})^n - \alpha'(p_{k\ell-1} - \alpha q_{k\ell-1})^n}{(p_{k\ell-1} - \alpha' q_{k\ell-1})^n - (p_{k\ell-1} - \alpha q_{k\ell-1})^n}, \tag{2.1}$$

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and when $\ell = 2t$ and the period is palindromic then for $k, n \in \mathbb{N}$ it holds

$$\frac{p_{nkt-1}}{q_{nkt-1}} = \frac{\alpha(p_{kt-1} - \alpha' q_{kt-1})^n - \alpha'(p_{kt-1} - \alpha q_{kt-1})^n}{(p_{kt-1} - \alpha' q_{kt-1})^n - (p_{kt-1} - \alpha q_{kt-1})^n}. \tag{2.2}$$

Householder's iterative method (see e.g. [Hous1970, §4.4]) of order p for rootsolving: $x_{n+1} = H^{(p)}(x_n) = x_n + p \cdot \frac{(1/f)^{(p-1)}(x_n)}{(1/f)^{(p)}(x_n)}$, where $(1/f)^{(p)}$ denotes p-th derivation of $1/f$. Householder's method of order 1 is just Newton's method. For Householder's method of order 2 one gets Halley's method, and Householder's method of order p has rate of convergence $p+1$.

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It is not hard to show that for $f(x) = (x-\alpha)(x-\alpha')$ it holds:

$$H^{(m+1)}(x) = \frac{xH^{(m)}(x) - \alpha\alpha'}{H^{(m)}(x) + x - \alpha - \alpha'}, \quad \text{for } m \in \mathbb{N}. \tag{2.3}$$

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Let us define $R_n^{(1)} \stackrel{\text{def}}{=} \frac{p_n}{q_n}$ and for $m > 1$ $R_n^{(m)} \stackrel{\text{def}}{=} H^{(m-1)}\left(\frac{p_n}{q_n}\right)$. We will say that $R_n^{(m)}$ is *good approximation*, if it is a convergent of α.

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Formula (2.1) shows that for arbitrary quadratic surd, whose period begins with a_1 and $k, m \in \mathbb{N}$, it holds

$$R_{k\ell-1}^{(m)} = \frac{p_{mk\ell-1}}{q_{mk\ell-1}}, \tag{2.4}$$

and when $\ell = 2t$ and period is periodic, from (2.2) it follows

$$R_{kt-1}^{(m)} = \frac{p_{mkt-1}}{q_{mkt-1}}. \tag{2.5}$$

Formula [Sha1959, (8)] says: For $k \in \mathbb{N}$ it holds

$$(a_\ell - a_0)p_{k\ell-1} + p_{k\ell-2} = q_{k\ell-1}(d - c^2), \tag{2.6}$$

$$(a_\ell - a_0)q_{k\ell-1} + q_{k\ell-2} = p_{k\ell-1} - 2cq_{k\ell-1}, \tag{2.7}$$

and formula (2.3) says

$$R_n^{(m+1)} = \frac{R_n^{(1)} R_n^{(m)} - \alpha\alpha'}{R_n^{(1)} + R_n^{(m)} - 2c}, \quad \text{for } m \in \mathbb{N},\ n = 0, 1, \dots. \tag{2.8}$$

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Lemma 2.1

For $m, k \in \mathbb{N}$ and $i = 1, 2, \dots, \ell$, when the period begins with a_1, it holds $R_{k\ell+i-1}^{(m)} = \frac{R_{k\ell-1}^{(m)} R_{i-1}^{(m)} - \alpha\alpha'}{R_{k\ell-1}^{(m)} + R_{i-1}^{(m)} - 2c}$.

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Proof.

For $m = 1$, statement of the lemma is proven in [Frank1962, Thm. 2.1]. Using mathematical induction and (2.8) it is not hard to show that the statement of the lemma holds too. $\square$

When period is palindromic and begins with a_1, formulas (2.6) and (2.7) become

$$a_0 p_{k\ell-1} + p_{k\ell-2} = 2c p_{k\ell-1} + q_{k\ell-1}(d - c^2), \tag{2.9}$$

$$a_0 q_{k\ell-1} + q_{k\ell-2} = p_{k\ell-1}. \tag{2.10}$$

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Lemma 2.2

For $m, k \in \mathbb{N}$ and $i = 1, 2, \ldots, \ell - 1$, when period is palindromic and begins with a_1, it holds $R^{(m)}_{k\ell-i-1} = \frac{R^{(m)}_{k\ell-1}(R^{(m)}_{i-1}-2c)+\alpha\alpha'}{R^{(m)}_{i-1}-R^{(m)}_{k\ell-1}}$.

Proof.

For $m = 1$ we have:

$$R^{(1)}_{k\ell-i-1} = \frac{p_{k\ell-i-1}}{q_{k\ell-i-1}} = \frac{0 \cdot p_{k\ell-i} + p_{k\ell-i-1}}{0 \cdot q_{k\ell-i} + q_{k\ell-i-1}} = [\, a_0, \ldots, a_{k\ell-i}, 0 \,]$$

$$= [\, a_0, \ldots, a_{k\ell-i}, a_{k\ell-i-1}, \ldots, a_{k\ell-1}, a_0, 0, -a_0, -a_1, \ldots, -a_{i-1} \,]$$

$$= \left[\, a_0, \ldots, a_{k\ell-i}, a_{k\ell-i-1}, \ldots, a_{k\ell-1}, a_0 - \frac{p_{i-1}}{q_{i-1}} \,\right]$$

$$= \frac{p_{k\ell-1}\left(a_0 - R^{(1)}_{i-1}\right) + p_{k\ell-2}}{q_{k\ell-1}\left(a_0 - R^{(1)}_{i-1}\right) + q_{k\ell-2}} \underset{(2.10)}{\overset{(2.9)}{=}} \frac{R^{(1)}_{k\ell-1}\left(R^{(1)}_{i-1} - 2c\right) + \alpha\alpha'}{R^{(1)}_{i-1} - R^{(1)}_{k\ell-1}}.$$

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For $m = 1$ we have:

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&= [\,a_0, \ldots, a_{k\ell-i}, a_{k\ell-i-1}, \ldots, a_{k\ell-1}, a_0, 0, -a_0, -a_1, \ldots, -a_{i-1}\,] \\
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\end{aligned}
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Using mathematical induction and (2.8) it is not hard to show that the statement of the lemma holds too. □

Proposition 2.3

Let $m \in \mathbb{N}$. When period begins with a_1, for $i = 1, 2, \ldots, \ell - 1$ and $\beta_i^{(m)} = -\frac{p_{mi-1} - R_{i-1}^{(m)} q_{mi-1}}{p_{mi} - R_{i-1}^{(m)} q_{mi}}$, it holds

$$R_{k\ell+i-1}^{(m)} = \frac{\beta_i^{(m)} p_{m(k\ell+i)} + p_{m(k\ell+i)-1}}{\beta_i^{(m)} q_{m(k\ell+i)} + q_{m(k\ell+i)-1}}, \text{ for all } k \geq 0,$$

and when period is palindromic, then

$$R_{k\ell-i-1}^{(m)} = \frac{p_{m(k\ell-i)-1} - \beta_i^{(m)} p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1} - \beta_i^{(m)} q_{m(k\ell-i)-2}}, \text{ for all } k \geq 1.$$

Proof.

We have $\beta_i^{(m)} = \left[0, -a_{mi}, -a_{mi-1}, \ldots, -a_1, -a_0 + R_{i-1}^{(m)}\right]$.

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$$\frac{\beta_i^{(m)} p_{mi} + p_{mi-1}}{\beta_i^{(m)} q_{mi} + q_{mi-1}} = \left[\,a_0, \ldots, a_{mi}, \beta_i^{(m)}\,\right]$$
$$= \left[\,a_0, \ldots, a_{mi}, 0, -a_{mi}, -a_{mi-1}, \ldots, -a_1, -a_0 + R_{i-1}^{(m)}\,\right] = R_{i-1}^{(m)},$$

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and similarly if $k > 0$ we have

$$\frac{\beta_i^{(m)} p_{m(k\ell+i)} + p_{m(k\ell+i)-1}}{\beta_i^{(m)} q_{m(k\ell+i)} + q_{m(k\ell+i)-1}} = \left[a_0, \ldots, a_{mk\ell-1}, a_{mk\ell} - a_0 + R_{i-1}^{(m)}\right]$$
$$= \frac{p_{mk\ell-1}\left(a_{mk\ell} - a_0 + R_{i-1}^{(m)}\right) + p_{mk\ell-2}}{q_{mk\ell-1}\left(a_{mk\ell} - a_0 + R_{i-1}^{(m)}\right) + q_{mk\ell-2}}$$
$$\underset{(2.7)}{\overset{(2.6),(2.4)}{=}} \frac{R_{k\ell-1}^{(m)} R_{i-1}^{(m)} + d - c^2}{R_{k\ell-1}^{(m)} + R_{i-1}^{(m)} - 2c} \overset{\text{Lm. } 2.1}{=} R_{k\ell+i-1}^{(m)}.$$

Proof.

When period is palindromic we have:

$$\frac{p_{m(k\ell-i)-1} - \beta_i^{(m)} p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1} - \beta_i^{(m)} q_{m(k\ell-i)-2}} = \left[a_0, \ldots, a_{m(k\ell-i)-1}, -\frac{1}{\beta_i^{(m)}} \right]$$

$$= \left[a_0, \ldots, a_{m(k\ell-i)-1}, a_{m(k\ell-i)}, a_{m(k\ell-i)+1}, \ldots, a_{mk\ell-1}, a_0 - R_{i-1}^{(m)} \right]$$

$$= \frac{p_{mk\ell-1}\left(a_0 - R_{i-1}^{(m)}\right) + p_{mk\ell-2}}{q_{mk\ell-1}\left(a_0 - R_{i-1}^{(m)}\right) + q_{mk\ell-2}} \overset{(2.9),(2.4)}{\underset{(2.10)}{=}} \frac{R_{k\ell-1}^{(m)}(R_{i-1}^{(m)} - 2c) + c^2 - d}{R_{i-1}^{(m)} - R_{k\ell-1}^{(m)}},$$

which is using Lemma 2.2 equal to the $R_{k\ell-i-1}^{(m)}$. □

Analogously as in [Duje2001, Lm. 3], from Proposition 2.3 it follows:

Theorem 2.4

To be a good approximant is a periodic property, i.e. for all $r \in \mathbb{N}$ it holds

$$R_n^{(m)} = \frac{p_k}{q_k} \qquad \Longleftrightarrow \qquad R_{r\ell+n}^{(m)} = \frac{p_{rm\ell+k}}{q_{rm\ell+k}},$$

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and when period is palindromic, it is also a palindromic property, i.e. it holds:

$$R_n^{(m)} = \frac{p_k}{q_k} \qquad \Longleftrightarrow \qquad R_{\ell-n-2}^{(m)} = \frac{p_{m\ell-k-2}}{q_{m\ell-k-2}}.$$

Let us show how Theorem 2.4 can be applied. The first example shows palindromic situation, the second is not palindromic (but we accidentally get good approximation in the half of the period), and the third shows that good approximants do depend on m.

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Example 2.5

Let us observe $\sqrt{44} = [6, \overline{1, 1, 1, 2, 1, 1, 1, 12}]$. The period is palindromic and we have $\ell = 8$. Let us consider e.g. the case $m = 5$. We have:
$R_n^{(5)} = \frac{p_n^5 + 440 p_n^3 q_n^2 + 9680 p_n q_n^4}{5 p_n^4 q_n + 440 p_n^2 q_n^3 + 1936 q_n^5}$.
From (2.4) and (2.5) we have $R_3^{(5)} = \frac{p_{19}}{q_{19}} = \frac{3\,160\,100}{476\,403}$,
$R_7^{(5)} = \frac{p_{39}}{q_{39}} = \frac{4\,993\,116\,004\,999}{752\,740\,560\,150}$, $R_{11}^{(5)} = \frac{p_{59}}{q_{59}}$, $R_{15}^{(5)} = \frac{p_{79}}{q_{79}}$, $\ldots$, $R_{4k-1}^{(5)} = \frac{p_{20k-1}}{q_{20k-1}}$.
$R_0^{(5)} = \frac{p_8}{q_8} = \frac{2514}{379}$. From Theorem 2.4 we have $R_6^{(5)} = \frac{p_{30}}{q_{30}} = \frac{7944493914}{1197677521}$, and also
$R_{8k}^{(5)} = \frac{p_{40k+8}}{q_{40k+8}}$ and $R_{8k-2}^{(5)} = \frac{p_{40k-10}}{q_{40k-10}}$.
$R_1^{(5)} = \frac{235\,487}{35\,501}$ is not a convergent of $\sqrt{44}$, so neither $R_{8k+1}^{(5)}$ nor $R_{8k-3}^{(5)}$ will be.
$R_2^{(5)} = \frac{6\,251\,453}{942\,442}$ is not a convergent of $\sqrt{44}$, so neither $R_{8k+2}^{(5)}$ nor $R_{8k-4}^{(5)}$ will be.

Example 2.6

Let us observe $\alpha = \frac{5+\sqrt{21}}{3} = [9, \overline{5, 6, 1, 2}]$ and $m = 3$, we have:
$R_m^{(3)} = \frac{37p_n^3 - 4572p_n q_n^2 + 23368q_n^3}{81p_n^2 q_n - 1242p_n q_n^2 + 4824q_n^3}$.
We have $R_3^{(3)} = \frac{p_{11}}{q_{11}} = \frac{4\,4004\,659}{435\,564}$, and so on $R_{4k-1}^{(3)} = \frac{p_{12k-1}}{q_{12k-1}}$. The period is not palindromic, and accidentally we have $R_1^{(3)} = \frac{p_7}{q_7} = \frac{36\,409}{3960}$ (in palindromic case would be $\frac{p_5}{q_5}$), and so on $R_{4k+1}^{(3)} = \frac{p_{12k+7}}{q_{12k+7}}$.

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Example 2.7

Let us observe $\alpha = \frac{7+\sqrt{11}}{5} = [2, \overline{15, 1, 3, 1, 3, 1}]$. For $m = 3$ we have:
$R_{6k-1}^{(3)} = \frac{p_{18k-1}}{q_{18k-1}}$; $R_1^{(3)} = \frac{p_7}{q_7}$ and $R_{6k+1}^{(3)} = \frac{p_{18k+7}}{q_{18k+7}}$.
For $m = 4$ we have: $R_{6k-1}^{(4)} = \frac{p_{24k-1}}{q_{24k-1}}$; $R_0^{(4)} = \frac{p_5}{q_5}$ and $R_{6k}^{(4)} = \frac{p_{24k+5}}{q_{24k+5}}$; $R_1^{(4)} = \frac{p_{11}}{q_{11}}$ and $R_{6k+1}^{(4)} = \frac{p_{24k+11}}{q_{24k+11}}$; $R_3^{(4)} = \frac{p_{17}}{q_{17}}$ and $R_{6k+3}^{(4)} = \frac{p_{24k+17}}{q_{24k+17}}$.

Let us define coprime positive numbers $P_n^{(m)}$, $Q_n^{(m)}$ by

$$\frac{P_n^{(m)}}{Q_n^{(m)}} \stackrel{\text{def}}{=} R_n^{(m)}.$$

From (2.8) it is not hard to show that it holds

$$P_n^{(m)} - \alpha Q_n^{(m)} = \left(P_n^{(1)} - \alpha Q_n^{(1)}\right)^m = (p_n - \alpha q_n)^m.$$

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Lemma 2.8

$R_n^{(m)} < \alpha$ if and only if n is even and m is odd. Therefore, $R_n^{(m)}$ can be an even convergent only if n is even and m is odd.

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Similarly as in [Duje2001], if $R_n^{(m)} = \frac{p_k}{q_k}$, we can define $j^{(m)} = j^{(m)}(\alpha, n)$ as the distance from convergent with m times larger index:

$$j^{(m)} = \frac{k + 1 - m(n+1)}{2}. \tag{2.11}$$

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This is an integer, by Lemma 2.8. Using Theorem 2.4 we have
$j^{(m)}(\alpha, n) = j^{(m)}(\alpha, k\ell + n)$, and in palindromic case:
$j^{(m)}(\alpha, n) = -j^{(m)}(\alpha, \ell - n - 2)$.

From now on, let us observe only quadratic irrationals of the form $\alpha = \sqrt{d}$, $d \in \mathbb{N}$, $d \neq \square$. It is well known that period of such α is palindromic and begins with a_1.

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Theorem 2.9 (for proof see [Pet2.2013])

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Proposition 2.10 (for proof see [Pet2.2013])

When $d \neq \square$, for $n \geq 0$ we have $\left|j^{(m)}(\sqrt{d}, n)\right| < \frac{m(\ell/2-1)}{2}$.

Theorem 2.11 (Euler, see §26 in [Perron1954])

Let $\ell \in \mathbb{N}$ and $a_1, \ldots, a_{\ell-1} \in \mathbb{N}$ such that $a_1 = a_{\ell-1}$, $a_2 = a_{\ell-2}$, ... The number $[\, a_0, \overline{a_1, a_2, \ldots, a_{\ell-1}, 2a_0}\,]$ is of the form $\sqrt{d}$, $d \in \mathbb{N}$ if and only if

$$2a_0 \equiv (-1)^{\ell-1} p'_{\ell-2} q'_{\ell-2} \pmod{p'_{\ell-1}}, \tag{2.12}$$

where $\frac{p'_n}{q'_n}$ are convergents of the number $[\, a_1, a_2, \ldots, a_{n-1}, a_n\,]$. Then it holds:

$$d = a_0^2 + \frac{2a_0 p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}. \tag{2.13}$$

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Lemma 2.12

Let F_k denote the k-th Fibonacci number. Let $n \in \mathbb{N}$ and $k > 1$, $k \equiv 1, 2 \pmod 3$. For $d_k(n) = \left(\frac{(2n-1)F_k+1}{2}\right)^2 + (2n-1)F_{k-1} + 1$ it holds $\sqrt{d_k(n)} = \left[\,\frac{(2n-1)F_k+1}{2}, \overline{\underbrace{1, 1, \dots, 1, 1}_{k-1 \text{ times}}, (2n-1)F_k + 1}\,\right]$, and $\ell(\sqrt{d_k(n)}) = k$.

Proof.

From (2.12), it follows:

$$2a_0 \equiv (-1)^{k-1}F_{k-1}F_{k-2} \equiv (-1)^{k-1}F_{k-1}(F_k - F_{k-1}) \equiv (-1)^{k-1}(-F_{k-1}^2) \pmod{F_k}.$$

Now from Cassini's identity $F_kF_{k-2} - F_{k-1}^2 = (-1)^{k-1}$ we have $2a_0 \equiv 1 \pmod{F_k}$. When $3 \mid k$, this congruence is not solvable, and if $3 \nmid k$, the solution is $a_0 \equiv \frac{F_k+1}{2} \pmod{F_k}$, i.e.

$$a_0 = \frac{F_k+1}{2} + (n-1)F_k = \frac{(2n-1)F_k+1}{2}, \qquad n \in \mathbb{N}.$$

From (2.13) it follows:

$$\begin{aligned} d &= \left(\frac{(2n-1)F_k+1}{2}\right)^2 + \frac{\big((2n-1)F_k+1\big)F_{k-1} + F_{k-2}}{F_k} \\ &= \left(\frac{(2n-1)F_k+1}{2}\right)^2 + (2n-1)F_{k-1} + 1. \end{aligned}$$

□

Theorem 2.13

Let F_ℓ denote the ℓ-th Fibonacci number. Let $\ell > 3, \ell \equiv \pm 1 \pmod 6$. Then for $d_\ell = \left(\frac{F_{\ell-3}F_\ell+1}{2}\right)^2 + F_{\ell-3}F_{\ell-1} + 1$ and $M \in \mathbb{N}$ it holds $\ell\left(\sqrt{d_\ell}\right) = \ell$ and

$$j^{(3M-1)}(\sqrt{d_\ell}, 0) = j^{(3M)}(\sqrt{d_\ell}, 0) = j^{(3M+1)}(\sqrt{d_\ell}, 0) = \frac{\ell-3}{2} \cdot M.$$

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Proof.

By (2.11), we have to prove

$$R_0^{(3M-1)} = \frac{p_{M\ell-2}}{q_{M\ell-2}}, \qquad R_0^{(3M)} = \frac{p_{M\ell-1}}{q_{M\ell-1}}, \qquad R_0^{(3M+1)} = \frac{p_{M\ell}}{q_{M\ell}}.$$

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We have $a_0 = \frac{F_{\ell-3}F_\ell+1}{2}$, and by Lemma 2.12 it holds $\sqrt{d_\ell} = \left[\, a_0, \overline{\underbrace{1, 1, \ldots, 1, 1}_{\ell-1 \text{ times}}, 2a_0} \,\right]$. From Cassini's identity, it follows

$$R_0^{(1)} = \frac{p_0}{q_0} = a_0, \qquad R_0^{(2)} = a_0 + \frac{F_{\ell-2}}{F_{\ell-1}} = \frac{p_{\ell-2}}{q_{\ell-2}},$$

Proof.

$$R_0^{(3)} = a_0 + \frac{F_{\ell-1}F_{\ell-2}^3}{F_{\ell-1}^2 F_{\ell-2}^2 + F_{\ell-2}^2} = a_0 + \frac{F_{\ell-1}}{F_\ell} = \frac{p_{\ell-1}}{q_{\ell-1}}. \tag{2.14}$$

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Let us prove the theorem using induction on M. For proving the inductive step, first observe that from (2.8) for $m \geq 3$ we have:

$$R_k^{(m)} = \frac{R_k^{(2)} R_k^{(m-2)} + d}{R_k^{(2)} + R_k^{(m-2)}}, \qquad R_k^{(m)} = \frac{R_k^{(3)} R_k^{(m-3)} + d}{R_k^{(3)} + R_k^{(m-3)}}. \tag{2.15}$$

Proof.

$$R_0^{(3)} = a_0 + \frac{F_{\ell-1}F_{\ell-2}^3}{F_{\ell-1}^2 F_{\ell-2}^2 + F_{\ell-2}^2} = a_0 + \frac{F_{\ell-1}}{F_\ell} = \frac{p_{\ell-1}}{q_{\ell-1}}. \tag{2.14}$$

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Suppose that for some $i \in \{0, \ell-2, \ell-1\}$ it holds $\frac{p_{(M-1)\ell+i}}{q_{(M-1)\ell+i}} = R_0^{(m-3)}$. We have:

$$\frac{p_{M\ell+i}}{q_{M\ell+i}} = \Big[a_0, \underbrace{1, 1, \ldots, 1, 1}_{\ell-1 \text{ times}}, a_0 + R_0^{(m-3)} \Big] =$$

$$\underset{(2.10)}{\overset{(2.9)}{=}} \frac{p_{\ell-1} R_0^{(m-3)} + d q_{\ell-1}}{q_{\ell-1} R_0^{(m-3)} + p_{\ell-1}} \overset{(2.14)}{=} \frac{R_0^{(3)} R_0^{(m-3)} + d}{R_0^{(3)} + R_0^{(m-3)}} \overset{(2.15)}{=} R_0^{(m)}. \qquad \square$$

Corollary 2.14

For each $m \geq 2$ it holds

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$$\limsup\left\{\frac{|j^{(m)}(\sqrt{d}, n)|}{\ell\left(\sqrt{d}\right)}\right\} \geq \frac{m}{6}.$$

Analogously as before, let us define

$$b^{(m)}(\alpha) = \left|\{n : 0 \leq n \leq \ell - 1, R_n^{(m)} \text{ is a convergent of } \alpha\}\right|.$$

For arbitrary m experimental results suggest that similar properties could hold as for $m = 2$. But $b^{(m)}(\alpha)$ is not a monotonic function in m. And there are some differences, as the following example shows.

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Example 2.15

We have $\ell\left(\sqrt{45}\right) = 6$ and

$$b^{(m)}(\sqrt{45}) = \begin{cases} 4, & \text{if } m \equiv 2 \pmod 4, \\ 6, & \text{if } m \not\equiv 2 \pmod 4. \end{cases}$$

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Thank you