

POTENTIAL THEORY OF DIRICHLET FORMS DEGENERATE AT THE BOUNDARY: THE CASE OF NO KILLING POTENTIAL

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ABSTRACT. In this paper we consider the Dirichlet form on the half-space \mathbb{R}_+^d defined by the jump kernel $J(x, y) = |x - y|^{-d-\alpha}\mathcal{B}(x, y)$, where $\mathcal{B}(x, y)$ can be degenerate at the boundary. Unlike our previous works [6, 7] where we imposed critical killing, here we assume that the killing potential is identically zero. In case $\alpha \in (1, 2)$ we first show that the corresponding Hunt process has finite lifetime and dies at the boundary. Then, as our main contribution, we prove the boundary Harnack principle and establish sharp two-sided Green function estimates. Our results cover the case of the censored α -stable process, $\alpha \in (1, 2)$, in the half-space studied in [2].

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1. INTRODUCTION

Let $\mathbb{R}_+^d = \{x = (\tilde{x}, x_d) : x_d > 0\}$ be the upper half-space in the d -dimensional Euclidean space \mathbb{R}^d . In this paper we study the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}_+^d, dx)$ defined by

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} (u(x) - u(y))(v(x) - v(y))J(x, y) dy dx, \quad (1.1)$$

where \mathcal{F} is the closure of $C_c^\infty(\mathbb{R}_+^d)$ under $\mathcal{E}_1 := \mathcal{E} + (\cdot, \cdot)_{L^2(\mathbb{R}_+^d, dx)}$. Our main assumption is on the jump kernel $J(x, y)$: We assume that $J(x, y) = |x - y|^{-d-\alpha}\mathcal{B}(x, y)$, $\alpha \in (0, 2)$, where $(x, y) \mapsto \mathcal{B}(x, y)$ is a symmetric function satisfying certain Hölder-type and scaling conditions, and most importantly, is comparable to the function

$$\begin{aligned} \tilde{B}(x, y) &:= \left(\frac{x_d \wedge y_d}{|x - y|} \wedge 1 \right)^{\beta_1} \left(\frac{x_d \vee y_d}{|x - y|} \wedge 1 \right)^{\beta_2} \left[\log \left(1 + \frac{(x_d \vee y_d) \wedge |x - y|}{x_d \wedge y_d \wedge |x - y|} \right) \right]^{\beta_3} \\ &\quad \times \left[\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right]^{\beta_4}. \end{aligned} \quad (1.2)$$

Here $\beta_1, \beta_2, \beta_3, \beta_4$ are non-negative parameters such that $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$. Here and below, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. The precise assumptions on $\mathcal{B}(x, y)$ are given in Section 2. Although we allow that $\mathcal{B}(x, y) \equiv 1$, our focus is on the case when $\beta_1 \vee \beta_2 > 0$. In such a case, the function $\mathcal{B}(x, y)$ vanishes at the boundary of \mathbb{R}_+^d , and we call the corresponding Dirichlet form degenerate at the boundary. We refer to $\mathcal{B}(x, y)$ as the boundary part of the jump kernel $J(x, y)$. This setting was introduced in [6, Section 5].

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The Hunt process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ will be denoted by $Y = (Y_t, \mathbb{P}_x)$ and its lifetime by ζ .

Our motivation to study the form (1.1) and the corresponding process Y comes from two sources.

Firstly, note that in case $J(x, y) = |x - y|^{-d-\alpha}$ (i.e. $\mathcal{B}(x, y) \equiv 1$), the process Y is the censored α -stable process in the half-space \mathbb{R}_+^d which was introduced and studied in [2] (also for a more general state space than \mathbb{R}_+^d). Two main results of [2] can be roughly described as follows: (1) There is a dichotomy between cases $\alpha \in (1, 2)$ and $\alpha \in (0, 1]$. In the former case the process Y has finite lifetime ζ and approaches the boundary of the state space at ζ , while in the latter, Y is conservative and will never approach the boundary; (2) In case when the state space D is a $C^{1,1}$ open set and $\alpha \in (1, 2)$, the boundary Harnack principle holds with the exact decay rate $\delta_D(x)^{\alpha-1}$ (here $\delta_D(x)$ denotes the distance of the point x to the boundary of D). Shortly after, in case of a bounded $C^{1,1}$ open set and $\alpha \in (1, 2)$, sharp two-sided Green function estimates were established in [3].

Secondly, a Dirichlet form related to (1.1) was introduced in [6] and further studied in [7]. We considered the form $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$ where

$$\mathcal{E}^\kappa(u, v) = \mathcal{E}(u, v) + \int_{\mathbb{R}_+^d} u(x)v(x)\kappa(x) dx, \quad (1.3)$$

and $\mathcal{F}^\kappa = \mathcal{F} \cap L^2(\mathbb{R}_+^d, \kappa(x)dx)$. The killing function is given by $\kappa(x) = C(\alpha, p, \mathcal{B})x_d^{-\alpha}$, where $C(\alpha, p, \mathcal{B})$ is a semi-explicit strictly positive and finite constant depending on α , \mathcal{B} and a parameter $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. The investigation of the form (1.3) was initiated in [6] and completed in [7] with two main results: Sharp two-sided Green function estimates for all admissible values of the parameters involved in $\tilde{B}(x, y)$, cf. [7, Theorem 1.1], and full identification of the parameters for which the boundary Harnack principle holds true, cf. [7, Theorem 1.2 and Theorem 1.3]. In proving those results, the strict positivity of the killing function was used in an essential way in several places. This includes the proof of finite lifetime, Carleson estimate, and the decay of the Green function at the boundary.

The goal of this paper is to extend the main results of [7] to the Dirichlet form (1.1) (which has no killing) in case $\alpha \in (1, 2)$. Due to the fact that $\lim_{p \downarrow (\alpha-1)_+} C(\alpha, p, \mathcal{B}) = 0$, this can be considered as a limiting case of the setting in [7]. Theorem 1.1 below can be viewed as a generalization of the corresponding result in [2] in case of the state space \mathbb{R}_+^d and $\alpha \in (1, 2)$, to jump kernels degenerate at the boundary, while Theorem 1.2 is related to the main result of [3].

The following two theorems are the main contribution of this paper. For $a, b > 0$ let $D_{\tilde{w}}(a, b) := \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : |\tilde{x} - \tilde{w}| < a, 0 < x_d < b\}$. Assumptions **(A1)**-**(A4)** are given in Section 2.

Theorem 1.1. *Suppose that $\alpha \in (1, 2)$ and that \mathcal{B} satisfies **(A1)**-**(A4)**. There exists $C \geq 1$ such that for all $r > 0$, $\tilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in \mathbb{R}_+^d which is harmonic in $D_{\tilde{w}}(2r, 2r)$ with respect to Y and vanishes continuously on $B((\tilde{w}, 0), 2r) \cap \partial\mathbb{R}_+^d$, we have*

$$\frac{f(x)}{x_d^{\alpha-1}} \leq C \frac{f(y)}{y_d^{\alpha-1}}, \quad x, y \in D_{\tilde{w}}(r/2, r/2). \quad (1.4)$$

Let $G(x, y)$, $x, y \in \mathbb{R}_+^d$, denote the Green function of the process Y (see Section 6 for the existence of the Green function).

Theorem 1.2. *Suppose that $\alpha \in (1, 2)$ and $d > (\alpha + \beta_1 + \beta_2) \wedge 2$. Assume that \mathcal{B} satisfies (A1)-(A4). Then there exists $C > 1$ such that for all $x, y \in \mathbb{R}_+^d$,*

$$\begin{aligned} & C^{-1} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \left(\frac{y_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \frac{1}{|x-y|^{d-\alpha}} \leq G(x, y) \\ & \leq C \left(\frac{x_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \left(\frac{y_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \frac{1}{|x-y|^{d-\alpha}}. \end{aligned} \quad (1.5)$$

Note that in both results we have assumed that $\alpha \in (1, 2)$. The case $\alpha \in (0, 1]$ is qualitatively different and new methods are needed to analyze it. We leave this case for future research.

Now we explain the content of the paper, our strategy of proving the results and differences to the methods used in [2] and [6, 7].

In Section 2 we precisely introduce the setup and assumptions on the boundary function $\mathcal{B}(x, y)$, and recall some of the relevant results from [6].

The goal of Section 3 is to prove that in case $\alpha \in (1, 2)$, the process Y has finite lifetime and is therefore transient. The proof is new and relies on a Hardy-type inequality, see Proposition 3.2. This inequality implies that $\mathcal{F} \neq \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ is the closure of $C_c^\infty(\mathbb{R}_+^d)$ under $\mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(\mathbb{R}_+^d, dx)}$. This implies that Y is a (proper) subprocess of \overline{Y} – the Hunt process associated with $(\mathcal{E}, \overline{\mathcal{F}})$, hence the lifetime of Y is finite. A consequence of finite lifetime is Corollary 3.4 which has two parts: The first one shows that the process Y approaches the boundary at the lifetime, while the second part replaces [6, Lemma 4.1] in the standard proof of the Carleson inequality, see Theorem 5.6.

Section 4 is devoted to proving Dynkin's formula for some non-compactly supported and non-smooth functions. Let

$$L_\alpha^\mathcal{B} f(x) := \text{p.v.} \int_{\mathbb{R}_+^d} (f(y) - f(x)) J(x, y) dy \quad (1.6)$$

be the operator corresponding to the form $(\mathcal{E}, \mathcal{F})$, defined for all $f : \mathbb{R}_+^d \rightarrow \mathbb{R}$ for which the principal value integral makes sense. It is straightforward to see that $L_\alpha^\mathcal{B} x_d^{\alpha-1} = 0$, which can be understood as $x \mapsto x_d^{\alpha-1}$ being harmonic in the analytic sense. See Lemma 3.1 below. In order to use probabilistic methods, it is crucial to show that this function is harmonic in the probabilistic sense. The proofs of [2, Lemmas 3.3 and 5.1] rely on using the isotropic stable process and its part process in \mathbb{R}_+^d . Since these two processes are of no help to us in the present setting, we use instead Dynkin's formula for barriers, cf. Proposition 4.6. The proof of this formula is a slight modification of the arguments in [6, Section 9].

Section 5 is devoted to the proof of Theorem 1.1. We first argue that the proofs of some results from [6] in the case $p > \alpha - 1$ are easily modified to the case $p = \alpha - 1$. Then we show that for any function f as in the statement of Theorem 1.1, it holds that

$$\frac{f(x)}{f(y)} \asymp \frac{\mathbb{P}_x(Y_{\tau_{D_{\tilde{w}}}(r/2, r/2)}} \in D(1, 1))}{\mathbb{P}_y(Y_{\tau_{D_{\tilde{w}}}(r/2, r/2)}} \in D(1, 1))}, \quad x, y \in D_{\tilde{w}}(r/2, r/2).$$

Since $x \mapsto x_d^{\alpha-1}$ satisfies the conditions in Theorem 1.1, the assertion of Theorem 1.1 is valid.

In the first part of Section 6 we present the proof of Theorem 1.2. The proof uses some results from [7], scaling and the boundary Harnack principle. In the second part we give sharp estimates of the Green potential of x_d^γ for $\gamma > -\alpha$. Again, we argue that, using the boundary Harnack principle, proofs of some lower bounds of the killed Green function obtained in [7] for $p > \alpha - 1$ are valid without any change for the case $p = \alpha - 1$. Having these Green function estimates one can apply results from [1] to get the estimates of the Green potentials.

We end the introduction with an explanation of the connection between the process Y and the process Y^κ associated with the Dirichlet form $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$. This connection is analogous to the one between the censored stable process and the killed stable process, cf. [2, Theorem 2.1]. Namely, the process Y can be obtained from Y^κ through either the Ikeda-Nagasawa-Watanabe piecing together procedure, or through the Feynman-Kac transform via $\exp \int_0^t \kappa(Y_s^\kappa) dt$. The case $\mathcal{B} \equiv 1$ and $\kappa(x) = C(\alpha/2, \alpha, \mathcal{B})x_d^{-\alpha}$ corresponds exactly to the isotropic α -stable process killed upon exiting \mathbb{R}_+^d .

Throughout this paper, the positive constants $\beta_1, \beta_2, \beta_3, \beta_4, \theta, r_0, n_0$ will remain the same. We will use the following convention: Lower case letters $c, c_i, i = 1, 2, \dots$ are used to denote constants in the proofs and the labeling of these constants starts anew in each proof. The notation $c_i = c_i(a, b, c, \dots), i = 0, 1, 2, \dots$ indicates constants depending on a, b, c, \dots . We will not specify the dependency on d . We will use “:=” to denote a definition, which is read as “is defined to be”. For any $x \in \mathbb{R}^d$ and $r > 0$, we use $B(x, r)$ to denote the open ball of radius r centered at x .

2. SETUP AND PRELIMINARY

In this section we precisely describe the setup and recall some preliminary results from earlier works.

Let $d \geq 1, \alpha \in (0, 2), j(|x - y|) = |x - y|^{-\alpha-d}$ and $J(x, y) = j(|x - y|)\mathcal{B}(x, y)$. We first give the assumptions on the boundary function $\mathcal{B}(x, y)$.

(A1) $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all $x, y \in \mathbb{R}_+^d$.

(A2) If $\alpha > 1$, there exist $\theta > \alpha - 1$ and $C > 0$ such that

$$|\mathcal{B}(x, x) - \mathcal{B}(x, y)| \leq C \left(\frac{|x - y|}{x_d \wedge y_d} \right)^\theta.$$

(A3) There exist $C \geq 1$ and parameters $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, with $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$, such that

$$C^{-1}\tilde{B}(x, y) \leq \mathcal{B}(x, y) \leq C\tilde{B}(x, y), \quad x, y \in \mathbb{R}_+^d, \quad (2.1)$$

where $\tilde{B}(x, y)$ is defined in (1.2).

(A4) For all $x, y \in \mathbb{R}_+^d$ and $a > 0, \mathcal{B}(ax, ay) = \mathcal{B}(x, y)$. In case $d \geq 2$, for all $x, y \in \mathbb{R}_+^d$ and $\tilde{z} \in \mathbb{R}^{d-1}, \mathcal{B}(x + (\tilde{z}, 0), y + (\tilde{z}, 0)) = \mathcal{B}(x, y)$.

For examples of functions \mathcal{B} satisfying **(A1)**-**(A4)**, see [6, 7]. Assumption **(A3)** implies that $\mathcal{B}(x, y)$ is bounded. Note that **(A4)** implies that $x \mapsto \mathcal{B}(x, x)$ is a constant on \mathbb{R}_+^d . Without loss of generality, we will assume that $\mathcal{B}(x, x) = 1$.

We observe that if $\beta_4 > 0$, then, for any $\varepsilon \in (0, \beta_2)$, there exists $c_\varepsilon > 0$ such that

$$(\log 2)^{-\beta_4} \tilde{B}_{\beta_1, \beta_2, \beta_3, 0}(x, y) \leq \tilde{B}_{\beta_1, \beta_2, \beta_3, \beta_4}(x, y) \leq c_\varepsilon \tilde{B}_{\beta_1, \beta_2 - \varepsilon, \beta_3, 0}(x, y). \quad (2.2)$$

Throughout the paper we always assume that

$J(x, y) = j(|x - y|)\mathcal{B}(x, y)$ on $\mathbb{R}_+^d \times \mathbb{R}_+^d$ with \mathcal{B} satisfying **(A1)** – **(A4)** and $\mathcal{B}(x, x) = 1$.

Let $\bar{\mathbb{R}}_+^d = \{x = (\tilde{x}, x_d) : x_d \geq 0\}$. Define

$$\begin{aligned} \mathcal{E}(u, v) &:= \frac{1}{2} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} (u(x) - u(y))(v(x) - v(y))J(x, y) dy dx \\ &= \frac{1}{2} \int_{\bar{\mathbb{R}}_+^d} \int_{\bar{\mathbb{R}}_+^d} (u(x) - u(y))(v(x) - v(y))J(x, y) dy dx. \end{aligned}$$

By Fatou's lemma, $(\mathcal{E}, C_c^\infty(\mathbb{R}_+^d))$ and $(\mathcal{E}, C_c^\infty(\overline{\mathbb{R}}_+^d))$ are closable in $L^2(\mathbb{R}_+^d, dx)(= L^2(\overline{\mathbb{R}}_+^d, dx))$. Let \mathcal{F} be the closure of $C_c^\infty(\mathbb{R}_+^d)$ under $\mathcal{E}_1 := \mathcal{E} + (\cdot, \cdot)_{L^2(\mathbb{R}_+^d, dx)}$ and let $\overline{\mathcal{F}}$ be the closure of $C_c^\infty(\overline{\mathbb{R}}_+^d)$ under $\mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(\mathbb{R}_+^d, dx)}$. Then $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \overline{\mathcal{F}})$ are regular Dirichlet forms.

Let $((Y_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}_+^d \setminus \mathcal{N}})$ be the Hunt process associated with $(\mathcal{E}, \mathcal{F})$ whose lifetime is ζ . By [6, Proposition 3.2], the exceptional set \mathcal{N} can be taken to be the empty set. We add a cemetery point ∂ to the state space \mathbb{R}_+^d and define $Y_t = \partial$ for $t \geq \zeta$. Let $((\overline{Y}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \overline{\mathbb{R}}_+^d \setminus \mathcal{N}_0})$ be the Hunt process associated with $(\mathcal{E}, \overline{\mathcal{F}})$ where \mathcal{N}_0 is an exceptional set. Let $(\mathcal{E}, \overline{\mathcal{F}}_{\mathbb{R}_+^d})$ be the part form of $(\mathcal{E}, \overline{\mathcal{F}})$ on \mathbb{R}_+^d . i.e., the form corresponding to the process \overline{Y} killed at the exit time $\tau_{\mathbb{R}_+^d} := \inf\{t > 0 : \overline{Y}_t \notin \mathbb{R}_+^d\}$. It follows from [4, Theorem 4.4.3(i)] that $(\mathcal{E}, \overline{\mathcal{F}}_{\mathbb{R}_+^d})$ is a regular Dirichlet form on $L^2(\mathbb{R}_+^d, dx)$ and that $C_c^\infty(\mathbb{R}_+^d)$ is its core. Hence $\overline{\mathcal{F}}_{\mathbb{R}_+^d} = \mathcal{F}$ implying that \overline{Y} killed upon exiting \mathbb{R}_+^d is equal to Y . Thus we conclude that Y is a subprocess of \overline{Y} , that the exceptional set \mathcal{N}_0 can be taken to be a subset of $\partial\mathbb{R}_+^d$, and that the lifetime of Y can be identified with $\tau_{\mathbb{R}_+^d}$.

Suppose that for all $x \in \mathbb{R}_+^d$ it holds that $\mathbb{P}_x(\tau_{\mathbb{R}_+^d} = \infty) = 1$. Then $(Y_t, \mathbb{P}_x, x \in \mathbb{R}_+^d) \stackrel{d}{=} (\overline{Y}_t, \mathbb{P}_x, x \in \mathbb{R}_+^d)$ implying that $\mathcal{F} = \overline{\mathcal{F}}_{\mathbb{R}_+^d} = \overline{\mathcal{F}}$.

For any $r > 0$, define a process $Y^{(r)}$ by $Y_t^{(r)} := rY_{r^{-\alpha}t}$. By the proof of [6, Lemma 5.1], Y has the following scaling property.

Lemma 2.1. $(Y^{(r)}, \mathbb{P}_{x/r})$ has the same law as (Y, \mathbb{P}_x) .

For any open subset V of \mathbb{R}_+^d and for $r > 0$, we define $rV := \{rx : x \in V\}$ and $\tau_V = \inf\{t > 0 : Y_t \notin V\}$. A consequence of Lemma 2.1 is that

$$\mathbb{E}_{rx}\tau_{rV} = r^\alpha \mathbb{E}_x\tau_V, \quad x \in V. \quad (2.3)$$

Definition 2.2. A non-negative Borel function defined on \mathbb{R}_+^d is said to be harmonic in an open set $V \subset \mathbb{R}_+^d$ with respect to Y if for every bounded open set $U \subset \overline{U} \subset V$,

$$f(x) = \mathbb{E}_x[f(Y_{\tau_U})] \quad \text{for all } x \in U. \quad (2.4)$$

A non-negative Borel function f defined on \mathbb{R}_+^d is said to be *regular harmonic* in an open set $V \subset \mathbb{R}_+^d$ if

$$f(x) = \mathbb{E}_x[f(Y_{\tau_V})] \quad \text{for all } x \in V.$$

The following result is taken from [6, 7].

Theorem 2.3 (Harnack inequality, [6, Theorem 1.1] & [7, Theorem 1.4]). (a) *There exists a constant $C_1 > 0$ such that for any $r > 0$, any $B(x_0, r) \subset \mathbb{R}_+^d$ and any non-negative function f in \mathbb{R}_+^d which is harmonic in $B(x_0, r)$ with respect to Y , we have*

$$f(x) \leq C_1 f(y), \quad \text{for all } x, y \in B(x_0, r/2).$$

(b) *There exists a constant $C_2 > 0$ such that for any $L > 0$, any $r > 0$, any $x_1, x_2 \in \mathbb{R}_+^d$ with $|x_1 - x_2| < Lr$ and $B(x_1, r) \cup B(x_2, r) \subset \mathbb{R}_+^d$ and any non-negative function f in \mathbb{R}_+^d which is harmonic in $B(x_1, r) \cup B(x_2, r)$ with respect to Y , we have*

$$f(x_2) \leq C_2(L + 1)^{\beta_1 + \beta_2 + d + \alpha} f(x_1).$$

3. HARDY INEQUALITY AND THE FINITE LIFETIME

For a given \mathcal{B} , we define $C(\alpha, p, \mathcal{B})$ for $\alpha \in (0, 2)$ and $p \in (-1, \alpha + \beta_1)$ by

$$C(\alpha, p, \mathcal{B}) = \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\alpha)/2}} \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}((1-s)\tilde{u}, 1, s\mathbf{e}_d) ds d\tilde{u}, \quad (3.1)$$

where $\mathbf{e}_d = (\tilde{0}, 1)$. In case $d = 1$, $C(\alpha, p, \mathcal{B})$ is defined by

$$C(\alpha, p, \mathcal{B}) = \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}(1, s) ds,$$

but we will only give the statement of the result for $d \geq 2$. The statement in the $d = 1$ case is similar and simpler.

We first note that $p \mapsto C(\alpha, p, \mathcal{B})$ is strictly increasing for $p \in ((\alpha - 1)_+, \alpha + \beta_1)$ (see [6, Lemma 5.4 and Remark 5.5]) and $\lim_{p \uparrow \alpha + \beta_1} C(\alpha, p, \mathcal{B}) = \infty$. Moreover,

$$C(\alpha, p, \mathcal{B}) \begin{cases} \in (0, \infty) & \text{for } p \in ((\alpha - 1)_+, \alpha + \beta_1); \\ = 0 & \text{for } p = 0, \alpha - 1; \\ \in (-\infty, 0) & \text{for } p \in (\alpha - 1, 0) \cup (0, \alpha - 1). \end{cases} \quad (3.2)$$

Let

$$C_c^2(\mathbb{R}_+^d; \mathbb{R}^d) = \{f : \mathbb{R}_+^d \rightarrow \mathbb{R} : \text{there exists } u \in C_c^2(\mathbb{R}^d) \text{ such that } u = f \text{ on } \mathbb{R}_+^d\}$$

be the space of functions on \mathbb{R}_+^d that are restrictions of $C_c^2(\mathbb{R}^d)$ functions. Clearly, if $f \in C_c^2(\mathbb{R}_+^d; \mathbb{R}^d)$ then $f \in C_b^2(\mathbb{R}_+^d) \cap L^2(\mathbb{R}_+^d)$.

For $\varepsilon > 0$, let

$$L_{\alpha, \varepsilon}^{\mathcal{B}} f(x) := \int_{\mathbb{R}_+^d, |y-x| > \varepsilon} (f(y) - f(x)) J(x, y) dy, \quad x \in \mathbb{R}_+^d, \quad (3.3)$$

so that

$$L_{\alpha}^{\mathcal{B}} f(x) = \text{p.v.} \int_{\mathbb{R}_+^d} (f(y) - f(x)) J(x, y) dy = \lim_{\varepsilon \rightarrow 0} L_{\alpha, \varepsilon}^{\mathcal{B}} f(x), \quad (3.4)$$

which is defined for all functions $f : \mathbb{R}_+^d \rightarrow \mathbb{R}$ for which the principal value integral makes sense. We have shown in [6, Proposition 3.4] that this is the case when $f \in C_c^2(\mathbb{R}_+^d; \mathbb{R}^d)$.

For $p \in \mathbb{R}$, let $g_p(x) = x_d^p$, $x \in \mathbb{R}_+^d$.

Lemma 3.1. *For $p \in (-1, \alpha + \beta_1)$, it holds that*

$$L_{\alpha}^{\mathcal{B}} g_p(x) = C(\alpha, p, \mathcal{B}) x_d^{p-\alpha}, \quad x \in \mathbb{R}_+^d. \quad (3.5)$$

In particular,

$$L_{\alpha}^{\mathcal{B}} g_{\alpha-1}(x) = 0, \quad x \in \mathbb{R}_+^d. \quad (3.6)$$

Moreover, there exists $\widehat{C} = \widehat{C}(\alpha, p, \mathcal{B}) > 0$ such that

$$|L_{\alpha, \varepsilon}^{\mathcal{B}} g_p(x)| \leq \widehat{C} x_d^{p-\alpha} \quad \text{for all } x \in \mathbb{R}_+^d \text{ and } \varepsilon \in (0, x_d/2]. \quad (3.7)$$

Proof. The equality (3.5) is proved in [6, Lemma 5.4] for $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. It is easy to see that the proof in fact works for $p \in (-1, \alpha + \beta_1)$. We now follow the proof in [6, Lemma 5.4] to show (3.7).

Fix $x = (\tilde{0}, x_d) \in \mathbb{R}_+^d$ and $\varepsilon \in (0, x_d/2]$. By the change of variables $y = x_d z$, and by using (A4), we have

$$\begin{aligned} L_{\alpha, \varepsilon}^{\mathcal{B}} g_p(x) &= x_d^{p-\alpha} \int_{\mathbb{R}_+^d, |\tilde{z}|^2 + |z_d - 1|^2 > (\varepsilon/x_d)^2} \frac{z_d^p - 1}{|(\tilde{z}, z_d) - \mathbf{e}_d|^{d+\alpha}} \mathcal{B}(\mathbf{e}_d, (\tilde{z}, z_d)) dz_d d\tilde{z} \\ &=: x_d^{p-\alpha} I_1(\varepsilon). \end{aligned}$$

Using the change of variables $\tilde{z} = |z_d - 1| \tilde{u}$, we get

$$I_1(\varepsilon) = \int_{\mathbb{R}_+^d, |z_d - 1|^2 |\tilde{u}|^2 + |z_d - 1|^2 > (\varepsilon/x_d)^2} (|\tilde{u}|^2 + 1)^{-(d+\alpha)/2} \frac{z_d^p - 1}{|z_d - 1|^{1+\alpha}} \mathcal{B}(\mathbf{e}_d, (|z_d - 1| \tilde{u}, z_d)) dz_d d\tilde{u}$$

$$= \int_{\mathbb{R}^{d-1}} (|\tilde{u}|^2 + 1)^{-(d+\alpha)/2} I_2(\varepsilon, \tilde{u}) d\tilde{u},$$

where

$$I_2(\varepsilon, \tilde{u}) = \left(\int_0^{1-(\varepsilon/x_d)(|\tilde{u}|^2+1)^{-1/2}} + \int_{1+(\varepsilon/x_d)(|\tilde{u}|^2+1)^{-1/2}}^\infty \right) \frac{z_d^p - 1}{|z_d - 1|^{1+\alpha}} \mathcal{B}(\mathbf{e}_d, (|z_d - 1|\tilde{u}, z_d)) dz_d.$$

Fix \tilde{u} and let $\varepsilon_0 = (\varepsilon/x_d)(|\tilde{u}|^2 + 1)^{-1/2} \leq 1/2$. By the same argument as that in the proof of [6, Lemma 5.4], we have that $I_2(\varepsilon, \tilde{u}) = I_{21}(\varepsilon, \tilde{u}) + I_{22}(\varepsilon, \tilde{u})$ where

$$I_{21}(\varepsilon, \tilde{u}) := \int_0^{1-\varepsilon_0} \frac{(s^p - 1) + (s^{\alpha-1-p} - s^{\alpha-1})}{(1-s)^{1+\alpha}} \mathcal{B}(((1-s)\tilde{u}, 1), \mathbf{se}_d) ds,$$

$$I_{22}(\varepsilon, \tilde{u}) := \int_{1-\varepsilon_0}^{\frac{1}{1+\varepsilon_0}} \frac{s^{\alpha-1-p} - s^{\alpha-1}}{(1-s)^{1+\alpha}} \mathcal{B}(((1-s)\tilde{u}, 1), \mathbf{se}_d) ds,$$

and there exists a constant $c_2 > 0$ independent of $\tilde{u} \in \mathbb{R}^{d-1}$ such that

$$|I_{21}(\varepsilon, \tilde{u})| \leq \int_0^1 \frac{|(s^p - 1)(1 - s^{\alpha-p-1})|}{(1-s)^{1+\alpha}} \mathcal{B}(((1-s)\tilde{u}, 1), \mathbf{se}_d) ds < c_1 < \infty. \quad (3.8)$$

Moreover, by [2, p.121] and the fact that $\varepsilon_0 \leq 1/2$,

$$\left| \int_{1-\varepsilon_0}^{\frac{1}{1+\varepsilon_0}} \frac{s^{\alpha-1-p} - s^{\alpha-1}}{(1-s)^{1+\alpha}} \mathcal{B}((1-s)\tilde{u}, 1), \mathbf{se}_d) ds \right| \leq c_2 \varepsilon_0^{2-\alpha} \leq c_2.$$

Therefore,

$$\sup_{\varepsilon \in (0, x_d/2]} |I_1(\varepsilon)| \leq c_3 \int_{\mathbb{R}^{d-1}} \frac{d\tilde{u}}{(|\tilde{u}|^2 + 1)^{(d+\alpha)/2}} = c_4 < \infty.$$

□

We now show that the following Hardy inequality holds when $\alpha \neq 1$.

Proposition 3.2. *Suppose $\alpha \neq 1$. Then there exists $C = C(\alpha) \in (0, \infty)$ such that for all $u \in \mathcal{F}$,*

$$\mathcal{E}(u, u) \geq C \int_{\mathbb{R}_+^d} \frac{u(x)^2}{x_d^\alpha} dx. \quad (3.9)$$

Proof. Since \mathcal{F} is the closure of $C_c^\infty(\mathbb{R}_+^d)$ under \mathcal{E}_1 , it suffices to prove (3.9) for $u \in C_c^\infty(\mathbb{R}_+^d)$.

Fix $u \in C_c^\infty(\mathbb{R}_+^d)$, choose a $p \in (\alpha - 1, 0) \cup (0, \alpha - 1)$ and let $v(x) = u(x)/g_p(x)$. Recall from (3.2) that $C(\alpha, p, \mathcal{B}) \in (-\infty, 0)$.

Using the elementary identity $(ab - cd)^2 = a^2b(b - d) + c^2d(d - b) + bd(a - c)^2$ and the symmetry of J , we have that, for all $\varepsilon > 0$,

$$\begin{aligned} & \int_{\mathbb{R}_+^d \times \mathbb{R}_+^d, |x-y|>\varepsilon} (v(y)g_p(y) - v(x)g_p(x))^2 J(x, y) dy dx \\ &= \int_{\mathbb{R}_+^d \times \mathbb{R}_+^d, |x-y|>\varepsilon} v(y)^2 g_p(y)(g_p(y) - g_p(x)) + v(x)^2 g_p(x)(g_p(x) - g_p(y)) J(x, y) dy dx \\ & \quad + \int_{\mathbb{R}_+^d \times \mathbb{R}_+^d, |x-y|>\varepsilon} g_p(x)g_p(y)(v(y) - v(x))^2 J(x, y) dy dx \\ & \geq -2 \int_{\mathbb{R}_+^d} v(x)^2 g_p(x) \left(\int_{\mathbb{R}_+^d, |x-y|>\varepsilon} (g_p(y) - g_p(x)) J(x, y) dy \right) dx \end{aligned}$$

$$= -2 \int_{\text{supp}(u)} \frac{u(x)^2}{g_p(x)} L_{\alpha, \varepsilon}^{\mathcal{B}} g_p(x) dx.$$

Let $a_0 := \text{dist}(\mathbb{R}_+^d, \text{supp}(u))/2 > 0$. By (3.7), the functions $\{\frac{u(x)^2}{g_p(x)} L_{\alpha, \varepsilon}^{\mathcal{B}} g_p(x) : \varepsilon \in (0, a_0)\}$ are uniformly bounded on $\text{supp}(u)$. Thus, by the bounded convergence theorem, (3.2) and (3.5),

$$\begin{aligned} \mathcal{E}(u, u) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_{\mathbb{R}_+^d \times \mathbb{R}_+^d, |x-y| > \varepsilon} (v(y)g_p(y) - v(x)g_p(x))^2 J(x, y) dy dx \\ &\geq -\lim_{\varepsilon \downarrow 0} \int_{\text{supp}(u)} \frac{u(x)^2}{g_p(x)} L_{\alpha, \varepsilon}^{\mathcal{B}} g_p(x) dx = c \int_{\mathbb{R}_+^d} \frac{u(x)^2}{g_p(x)} x_d^{p-\alpha} dx = c \int_{\mathbb{R}_+^d} \frac{u(x)^2}{x_d^\alpha} dx, \end{aligned}$$

where $c = -C(\alpha, p, \mathcal{B}) \in (0, \infty)$. \square

Recall that ζ is the lifetime of Y . Using the above Hardy inequality, we now show that ζ is finite when $\alpha > 1$.

Proposition 3.3. *Suppose $\alpha > 1$. Then $\mathcal{F} \neq \overline{\mathcal{F}}$ and $\mathbb{P}_x(\zeta < \infty) = 1$ for all $x \in \mathbb{R}_+^d$.*

Proof. Take a $u \in C_c^\infty(\overline{\mathbb{R}_+^d})$ such that $u \geq 1$ on $B(0, 1) \cap \mathbb{R}_+^d$, then $u \notin \mathcal{F}$. In fact, if $u \in \mathcal{F}$, then by Proposition 3.2,

$$\infty > \mathcal{E}(u, u) \geq c \int_{\mathbb{R}_+^d} \frac{u(x)^2}{x_d^\alpha} dx \geq c \int_{B(0,1) \cap \mathbb{R}_+^d} |x|^{-\alpha} dx = \infty,$$

which gives a contradiction.

The fact that $\mathcal{F} \neq \overline{\mathcal{F}}$ implies that there is a point $x_0 \in \mathbb{R}_+^d$ such that $\mathbb{P}_{x_0}(\zeta < \infty) > 0$. Then by the scaling property of Y in Lemma 2.1, we have that $\mathbb{P}_x(\zeta < \infty) = \mathbb{P}_{x_0}(\zeta < \infty) > 0$ for all $x \in \mathbb{R}_+^d$. Now, by the same argument as in the proof of [2, Proposition 4.2], we have that $\mathbb{P}_x(\zeta < \infty) = 1$ for all $x \in \mathbb{R}_+^d$. \square

The fact that the lifetime of Y is finite has two important consequences.

Corollary 3.4. (a) *For all $x \in \mathbb{R}_+^d$, $\mathbb{P}_x(Y_{\zeta^-} \in \partial \mathbb{R}_+^d) = 1$.*

(b) *There exists a constant $n_0 \geq 2$ such that for all $x \in \mathbb{R}_+^d$, $\mathbb{P}_x(\tau_{B(x, n_0 x_d)} = \zeta) > 1/2$.*

Proof. Using Lemma 2.1, we see that

$$\mathbb{P}_x(\tau_{B(x, n x_d)} = \zeta) = \mathbb{P}_{(\tilde{0}, 1)}(\tau_{B((\tilde{0}, 1), n)} = \zeta), \quad x \in \mathbb{R}_+^d.$$

The sequence of events $(\{\tau_{B((\tilde{0}, 1), n)} = \zeta\})_{n \geq 1}$ is increasing in n and

$$\bigcup_{n=1}^{\infty} \{\tau_{B((\tilde{0}, 1), n)} = \zeta\} = \{\zeta < \infty\}. \quad (3.10)$$

Thus, by Proposition 3.3 we have

$$\lim_{n \rightarrow \infty} \mathbb{P}_{(\tilde{0}, 1)}(\tau_{B((\tilde{0}, 1), n)} = \zeta) = \mathbb{P}_{(\tilde{0}, 1)}(\zeta < \infty) = 1. \quad (3.11)$$

Moreover, since there is no killing inside \mathbb{R}_+^d , it holds that $\{\tau_{B((\tilde{0}, 1), n)} = \zeta\} \subset \{Y_{\zeta^-} \in \partial \mathbb{R}_+^d\}$ for each $n \geq 1$. Thus it follows from (3.10) and (3.11) that $\mathbb{P}_{(\tilde{0}, 1)}(Y_{\zeta^-} \in \partial \mathbb{R}_+^d) = 1$. The claim (a) now follows by scaling.

To see (b), note that by (3.11) there exists a $n_0 \geq 2$ such that $\mathbb{P}_{(\tilde{0}, 1)}(\tau_{B((\tilde{0}, 1), n_0)} = \zeta) > 1/2$. Therefore,

$$\mathbb{P}_x(\tau_{B(x, n_0 x_d)} = \zeta) = \mathbb{P}_{(\tilde{0}, 1)}(\tau_{B((\tilde{0}, 1), n_0)} = \zeta) > 1/2, \quad x \in \mathbb{R}_+^d. \quad \square$$

4. DYNKIN'S FORMULA FOR BARRIERS

In this section we always assume that $\alpha > 1$.

Recall that for $a, b > 0$ and $\tilde{w} \in \mathbb{R}^{d-1}$,

$$D_{\tilde{w}}(a, b) := \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : |\tilde{x} - \tilde{w}| < a, 0 < x_d < b\}.$$

Without loss of generality, we will mostly deal with the case $\tilde{w} = \tilde{0}$. We will write $D(a, b)$ for $D_{\tilde{0}}(a, b)$ and $U(r) = D_{\tilde{0}}(\frac{r}{2}, \frac{r}{2})$. Further we use U for $U(1)$. For any $a > 0$, set $D(a) := \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > a\}$ and $U_a(r) := \{y \in U(r) : \delta_{U(r)} > a\}$. We write U_a for $U_a(1)$.

Let $v \in C_c^\infty(\mathbb{R}^d)$ be a non-negative smooth radial function such that $v(y) = 0$ for $|y| \geq 1$ and $\int_{\mathbb{R}^d} v(y) dy = 1$. For $b \geq 10$ and $k \geq 1$, set $v_k(y) := b^{kd}v(b^k y)$. Next we define $g_k := v_k * (g \mathbf{1}_{D(5^{-k})})$ for a bounded, compactly supported function g vanishing on $\mathbb{R}^d \setminus \mathbb{R}_+^d$. Since $b^{-k} < 5^{-k}$, we have $g_k \in C_c^\infty(\mathbb{R}_+^d)$ and hence $L_\alpha^\mathcal{B} g_k$ is defined everywhere. Also note that $v_k * g \in C_c^\infty(\mathbb{R}_+^d; \mathbb{R}^d)$ and thus $L_\alpha^\mathcal{B}(v_k * g)$ is well defined (cf. [6, Subsection 3.2]).

Let $(a_k)_{k \geq 1}$ be a decreasing sequence of positive numbers such that $\lim_{k \rightarrow \infty} a_k = 0$ and

$$a_k \geq 2^{-k(\beta_1/2+1)/(1+\alpha+3\beta_1/2)} \geq 2^{-k}.$$

Lemma 4.1. *Let $R, M \geq 1$ and $g : \mathbb{R}^d \rightarrow [0, M]$ be a bounded, compactly supported function vanishing on $\mathbb{R}^d \setminus \mathbb{R}_+^d$. For any $z \in U(R)$, it holds that*

$$\lim_{k \rightarrow \infty} L_\alpha^\mathcal{B}(v_k * g - g_k)(z) = 0. \quad (4.1)$$

Moreover, there exists $C > 0$ independent of $R, M \geq 1$ and g such that for all $k \geq 2$ and $z \in U_{a_k}(R)$,

$$0 \leq L_\alpha^\mathcal{B}(v_k * g - g_k)(z) \leq CM(2/3)^{k(\beta_1/2+1)} z_d^{\beta_1}. \quad (4.2)$$

Proof. Let $z \in U_{a_k}(R)$. We first estimate the difference

$$L_\alpha^\mathcal{B}(v_k * g - g_k)(z) = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}_+^d, |y-z| > \epsilon} \frac{((v_k * g)(y) - g_k(y)) - ((v_k * g)(z) - g_k(z))}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy.$$

Note that for $k \geq 2$, $u \in B(0, b^{-k})$ and $y \in \mathbb{R}_+^d$ with $y_d > 3^{-k}$, it holds that $y_d - u_d > 3^{-k} - 10^{-k} > 5^{-k}$. Therefore

$$1 - \mathbf{1}_{D(5^{-k})}(y - u) = 0. \quad (4.3)$$

Since v_k is supported in $B(0, b^{-k})$, for all $k \geq 2$ and $z \in \mathbb{R}_+^d$ with $z_d > a_k > 2^{-k}$,

$$\int_{\mathbb{R}^d} (1 - \mathbf{1}_{D(5^{-k})}(z - u)) g(z - u) v_k(u) du = 0.$$

Thus $(v_k * g - g_k)(z) = 0$. Due to the same reason we have that for $z \in U_{a_k}(R)$,

$$\begin{aligned} & \int_{\mathbb{R}_+^d, |y-z| > \epsilon} \frac{((v_k * g)(y) - g_k(y)) - ((v_k * g)(z) - g_k(z))}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \\ &= \int_{\mathbb{R}_+^d, |y-z| > \epsilon, y_d \leq 3^{-k}} \int_{\mathbb{R}^d} v_k(u) \frac{(1 - \mathbf{1}_{D(5^{-k})}(y - u)) g(y - u)}{|y-z|^{d+\alpha}} du \mathcal{B}(y, z) dy \\ &\leq M \int_{\mathbb{R}^d} v_k(u) du \int_{\mathbb{R}_+^d, y_d \leq 3^{-k}} \frac{\mathcal{B}(y, z)}{|y-z|^{d+\alpha}} dy \\ &\leq c_1 M \int_{y_d \leq 3^{-k}} \frac{1}{|y-z|^{d+\alpha}} \left(\frac{y_d}{|y-z|} \right)^{\beta_1/2} dy \\ &\leq c_2 M (3^{-k})^{\beta_1/2+1} \int_0^\infty \frac{t^{d-2}}{(t^2 + cz_d^2)^{(d+\alpha+\beta_1/2)/2}} dt \end{aligned}$$

$$\begin{aligned}
&= c_3 M (3^{-k})^{\beta_1/2+1} z_d^{-1-\alpha-\beta_1/2} \int_0^\infty \frac{s^{d-2}}{(s^2+1)^{(d+\alpha+\beta_1/2)/2}} ds \\
&\leq c_4 M (2/3)^{k(\beta_1/2+1)} z_d^{\beta_1}.
\end{aligned}$$

In the third line we used that $0 \leq g \leq M$, in the fourth the fact that (together with (2.2))

$$\left(\frac{y_d \wedge z_d}{|y-z|} \wedge 1 \right)^{\beta_1} \log \left(1 + \frac{(y_d \vee z_d) \wedge |y-z|}{y_d \wedge z_d \wedge |y-z|} \right)^{\beta_3} \leq c \left(\frac{y_d}{|y-z|} \wedge 1 \right)^{\beta_1/2},$$

in the fifth integration in polar coordinates in \mathbb{R}^{d-1} , in the sixth the change of variables $t = c^{1/2} z_d s$, and in the last line the fact that $2^{-k(\beta_1/2+1)} z_d^{-1-\alpha-3\beta_1/2} \leq 1$ which follows from $z_d \geq a_k$ and the choice of a_k . Note also that it is clear from the second line that the first line is non-negative. Thus by letting $\epsilon \rightarrow 0$ we get for $z \in U_{a_k}(R)$,

$$0 \leq L_\alpha^\mathcal{B}(v_k * g - g_k)(z) \leq c_4 M (2/3)^{k(\beta_1/2+1)} z_d^{\beta_1}.$$

Now take $z \in U(R)$. Then there exists $k_0 \geq 1$ such that $z \in U_{a_k}(R)$ for all $k \geq k_0$, and it follows from above that

$$\lim_{k \rightarrow \infty} L_\alpha^\mathcal{B}(v_k * g - g_k)(z) = 0.$$

□

Lemma 4.2. *Assume that $R, M \geq 1$ and $g : \mathbb{R}_+^d \rightarrow [0, M]$ is a function which is C^2 on $D(R, R)$. For any $k \geq 2$, $z \in U_{a_k}(R)$ and $|u| < b^{-k}$,*

$$\text{p.v.} \int_{\mathbb{R}_+^d} \frac{g(y-u) - g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \quad (4.4)$$

is well defined. Moreover, for $z \in U_{a_k}(R)$,

$$L_\alpha^\mathcal{B}(v_k * g)(z) = \int_{\mathbb{R}^d} v_k(u) \left(\text{p.v.} \int_{\mathbb{R}_+^d} \frac{g(y-u) - g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right) du, \quad (4.5)$$

and there exists $C(z) = C(z, g, M, R) > 0$ such that $|L_\alpha^\mathcal{B}(v_k * g)(z)| \leq C(z)$ for all $k \geq 2$.

Proof. Let $z \in U_{a_k}(R)$ and $|u| < b^{-k}$. Let $G(y, z, u) := (g(y-u) - g(z-u))|y-z|^{-d-\alpha}$. For $0 < \epsilon < \eta < z_d/10$, consider

$$\begin{aligned}
&\int_{\mathbb{R}_+^d, \epsilon < |y-z|} G(y, z, u) \mathcal{B}(y, z) dy - \int_{\mathbb{R}_+^d, \eta < |y-z|} G(y, z, u) \mathcal{B}(y, z) dy \\
&= \int_{\mathbb{R}_+^d, \epsilon < |y-z| < \eta} G(y, z, u) \mathcal{B}(y, z) dy \\
&= \int_{\epsilon < |y-z| < \eta} G(y, z, u) dy + \int_{\epsilon < |y-z| < \eta} G(y, z, u) (\mathcal{B}(y, z) - 1) dy \\
&=: I + II.
\end{aligned}$$

Since g is C^2 on $D(R, R)$ and $y-u, z-u \in D(R, R)$, we see that

$$\begin{aligned}
|I| &\leq \int_{\epsilon < |(y-u)-(z-u)| < \eta} \frac{|g(y-u) - g(z-u) - \nabla g(z-u) \mathbf{1}_{(|(y-u)-(z-u)| < 1)} \cdot (y-z)|}{|(y-u) - (z-u)|^{d+\alpha}} dy \\
&\leq c_1 \sup_{w \in B(z, z_d/5)} |\partial^2 g(w)| \int_{\epsilon < |y-z| < \eta} |y-z|^{-d-\alpha+2} dy = c_2(z) (\eta^{2-\alpha} - \epsilon^{2-\alpha}).
\end{aligned}$$

Further, by using the mean value theorem in the first line and **(A2)** in the second, we get

$$\begin{aligned}
 |II| &\leq \sup_{w \in B(z, z_d/5)} |\nabla g(w)| \int_{\epsilon < |y-z| < \eta} \frac{|\mathcal{B}(y, z) - 1|}{|y-z|^{d+\alpha-1}} dy \\
 &\leq c_3(z) \int_{\epsilon < |y-z| < \eta} |y-z|^{-d-\alpha} \left(\frac{|y-z|}{y_d \wedge z_d} \right)^\theta dy \\
 &\leq c_4 c_3(z) z_d^{-\theta} \int_{\epsilon < |y-z| < \eta} |y-z|^{-d-\alpha+\theta} dy = c_5(z) (\eta^{\theta-\alpha+1} - \epsilon^{\theta-\alpha+1}).
 \end{aligned}$$

The estimates for I and II imply that the principal value integral in (4.4) is well defined.

Let $z \in U_{a_k}(R)$. For $\epsilon < z_d/10$ and $|u| < b^{-k}$, we have

$$\begin{aligned}
 &\left| \int_{\mathbb{R}_+^d, |y-z| > \epsilon} G(y, z, u) \mathcal{B}(y, z) dy \right| \\
 &\leq \left| \int_{\mathbb{R}_+^d, |y-z| \geq z_d/10} G(y, z, u) \mathcal{B}(y, z) dy \right| + \left| \int_{\mathbb{R}_+^d, \epsilon < |y-z| < z_d/10} G(y, z, u) \mathcal{B}(y, z) dy \right| \\
 &=: III + IV.
 \end{aligned}$$

Estimating g by M , we get that

$$III \leq 2M \int_{|y-z| \geq z_d/10} |y-z|^{-d-\alpha} dy \leq c_7 z_d^{-\alpha} = c_8(z).$$

The integral in IV is estimated in I and II with $\eta = z_d/10$, so we have

$$IV \leq c_2(z) (z_d/10)^{2-\alpha} + c_5(z) (z_d/10)^{\theta-\alpha+1} = c_9(z).$$

Thus we have that

$$\left| \int_{\mathbb{R}_+^d, |y-z| > \epsilon} \frac{g(y-u) - g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right| \leq c_{10}(z). \quad (4.6)$$

Hence we can use the dominated convergence theorem to conclude that

$$\begin{aligned}
 L_\alpha^\mathcal{B}(v_k * g)(z) &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}_+^d, |y-z| > \epsilon} \frac{(v_k * g)(y) - (v_k * g)(z)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \\
 &= \lim_{\epsilon \rightarrow 0} \int_{|u| < b^{-k}} v_k(u) \int_{\mathbb{R}_+^d, |y-z| > \epsilon} \frac{g(y-u) - g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy du \\
 &= \int_{|u| < b^{-k}} v_k(u) \left(\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}_+^d, |y-z| > \epsilon} \frac{g(y-u) - g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right) du,
 \end{aligned}$$

which is (4.5). The last statement follows from (4.6). \square

We note also that if g is continuous in $D(R, R)$, then $\lim_{k \rightarrow \infty} (v_k * g)(z) = g(z)$ for all $z \in D(R, R)$. Let $h_{p,R}(x) = x_d^p \mathbf{1}_{D(R,R)}(x)$ and $h_{p,\infty}(x) = g_p(x) = x_d^p$ for $x \in \mathbb{R}_+^d$. We also let $h_p(x) = h_{p,1}(x)$.

Lemma 4.3. *Let $p \in [\alpha - 1, \alpha + \beta_1)$ and set*

$$a_k := 2^{-k(p+1+\frac{1}{2}\beta_1)/(\alpha+1+\frac{3}{2}\beta_1)} \vee 2^{-k(2+\beta_1)/(1+\alpha+\frac{3}{2}\beta_1-p)}.$$

There exists a constant $C = C(R) > 0$ such that for any $k \geq 1$ and $z \in U_{a_k}(R)$,

$$|L_\alpha^\mathcal{B}(v_k * h_{p,R})(z) - L_\alpha^\mathcal{B}h_{p,R}(z)| \leq C \left(\frac{4}{5} \right)^k. \quad (4.7)$$

In particular, the functions $z \mapsto |L_\alpha^{\mathcal{B}}(v_k * h_{p,R})(z) - L_\alpha^{\mathcal{B}}h_{p,R}(z)|$ are all bounded by the constant C on $U(R)$, and for any $z \in U(R)$, $\lim_{k \rightarrow \infty} |L_\alpha^{\mathcal{B}}(v_k * h_{p,R})(z) - L_\alpha^{\mathcal{B}}h_{p,R}(z)| = 0$.

Proof. First note that $a_k \geq 2^{-k}$ since the first term in its definition is larger than 2^{-k} . Fix $z \in U_{a_k}(r)$. By using Lemma 4.2 with $g = h_{p,R}$ in the second line below we see that

$$\begin{aligned} & L_\alpha^{\mathcal{B}}(v_k * h_{p,R})(z) \\ &= \int_{\mathbb{R}^d} v_k(u) \left(\text{p.v.} \int_{\mathbb{R}_+^d} \frac{h_{p,R}(y-u) - h_{p,R}(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right) du \\ &= \int_{\mathbb{R}^d} v_k(u) \left(\text{p.v.} \int_{\mathbb{R}_+^d} \frac{h_{p,R}(y-u) - h_{p,R}(z-u) - (h_{p,R}(y) - h_{p,R}(z))}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right) du \\ &+ L_\alpha^{\mathcal{B}}h_{p,R}(z). \end{aligned}$$

Set $b = 10 \vee 2^{4(p-2)-+3}$. Now we write, for $u \in B(0, b^{-k})$,

$$\begin{aligned} & \int_{\mathbb{R}_+^d} \frac{h_{p,R}(y-u) - h_{p,R}(z-u) - (h_{p,R}(y) - h_{p,R}(z))}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \\ &= \int_{D(R+b^{-k}, R+b^{-k}) \setminus U(R), y_d > 5^{-k}} + \int_{D(R+b^{-k}, R+b^{-k}), y_d < 5^{-k}} \\ &+ \int_{U(R), y_d > 5^{-k}, |y-z| > 2^{-1}z_d} + \int_{U(R), y_d > 5^{-k}, B(z, z_d/2)} =: I + II + III + IV. \end{aligned}$$

We deal with I first. For $u \in B(0, b^{-k})$,

$$I = \int_{D(R+b^{-k}, R+b^{-k}) \setminus D(R-b^{-k}, R-b^{-k}), y_d > 5^{-k}} + \int_{D(R-b^{-k}, R-b^{-k}) \setminus U(R), y_d > 5^{-k}} =: I_1 + I_2.$$

Obviously, we have $|I_1| \leq c_1(R)b^{-k}$.

Let $A_k := (D(R-b^{-k}, R-b^{-k}) \setminus U(R)) \cap \{y : y_d > 5^{-k}\}$. Then, we have

$$\begin{aligned} |I_2| &= \left| \int_{A_k} \frac{(y_d - u_d)^p - y_d^p - ((z_d - u_d)^p - z_d^p)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right| \\ &= \left| \int_{A_k} \frac{p(y_d - z_d) \cdot \int_0^1 ((z_d - u_d + \theta(y_d - z_d))^{p-1} - (z_d + \theta(y_d - z_d))^{p-1}) d\theta}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right| \\ &\leq c_2(R)b^{-k}2^{3k(p-2)-} \leq c_22^{-k((p-2)-+3)}, \end{aligned}$$

where in the first inequality we use the mean value theorem inside the integral in the numerator.

For II , we have

$$|III| \leq c_3 \int_{D(R+b^{-k}, R+b^{-k}), y_d < 5^{-k}} \frac{y_d^p + z_d^p + b^{-kp}}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy.$$

Similarly as in the proof of Lemma 4.1 we first estimate

$$\begin{aligned} & \int_{D(R+b^{-k}, R+b^{-k}), y_d < 5^{-k}} \frac{y_d^p + b^{-kp}}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \\ &\leq c_4 \int_{\mathbb{R}^{d-1}} \int_0^{5^{-k}} \frac{y_d^{p+\beta_1/2} + 10^{-kp}y_d^{\beta_1/2}}{|y-z|^{d+\alpha+\frac{1}{2}\beta_1}} dy_d d\tilde{y} \\ &\leq c_5 5^{-k(p+1+\frac{1}{2}\beta_1)} \int_0^\infty \frac{t^{d-2}}{(t^2 + cz_d^2)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} dt \end{aligned}$$

$$\begin{aligned}
 &= c_6 5^{-k(p+1+\frac{1}{2}\beta_1)} z_d^{-1-\alpha-\frac{1}{2}\beta_1} \int_0^\infty \frac{s^{d-2}}{(s^2+1)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} ds \\
 &\leq c_7 (2/5)^{k(p+1+\frac{1}{2}\beta_1)} z_d^{\beta_1}.
 \end{aligned}$$

For the remaining part, we use a similar argument:

$$\begin{aligned}
 &\int_{D(R+b^{-k}, R+b^{-k}), y_d < 5^{-k}} \frac{z_d^p}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \\
 &\leq c_8 z_d^p \int_{\mathbb{R}^{d-1}} \int_0^{5^{-k}} \frac{y_d^{\beta_1/2}}{|y-z|^{d+\alpha+\frac{1}{2}\beta_1}} dy_d d\tilde{y} \\
 &\leq c_9 z_d^p 5^{-k(1+\frac{1}{2}\beta_1)} \int_0^\infty \frac{t^{d-2}}{(t^2 + cz_d^2)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} dt \\
 &= c_{10} z_d^p 5^{-k(1+\frac{1}{2}\beta_1)} z_d^{-1-\alpha-\frac{1}{2}\beta_1} \int_0^\infty \frac{s^{d-2}}{(s^2+1)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} ds \\
 &\leq c_{11} (4/5)^{k(1+\frac{1}{2}\beta_1)} 2^{-k(2+\beta_1)} z_d^{p-1-\alpha-\frac{1}{2}\beta_1} \leq c_{12} (4/5)^{k(1+\frac{1}{2}\beta_1)} z_d^{\beta_1}.
 \end{aligned}$$

Thus

$$|II| \leq c_{13} \left((2/5)^{k(p+1+\frac{1}{2}\beta_1)} + (4/5)^{k(1+\frac{1}{2}\beta_1)} \right) z_d^{\beta_1}.$$

Let $B_k := U(R) \cap \{y_d > 5^{-k}\} \cap \{y : |y-z| > 2^{-1}z_d\}$. Then, we have

$$\begin{aligned}
 |III| &= \left| \int_{B_k} \frac{(y_d - u_d)^p - y_d^p - ((z_d - u_d)^p - z_d^p)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right| \\
 &= \left| \int_{B_k} \frac{p(y_d - z_d) \cdot \int_0^1 ((z_d - u_d + \theta(y_d - z_s))^{p-1} - (z_d + \theta(y_d - z_d))^{p-1}) d\theta}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right| \\
 &\leq c_{14} b^{-k} 2^{k3(p-2)-} \int_{U(R), |y-z| > 2^{-1}z_d} \frac{1}{|y-z|^{d+\alpha-1}} dy \\
 &\leq c_{15} (4/5)^k 2^{-3k} (z_d^{1-\alpha} \vee \log \frac{1}{z_d}) \leq c_{16} (4/5)^k z_d^3 (z_d^{1-\alpha} \vee \log \frac{1}{z_d}),
 \end{aligned}$$

where in the first inequality we use mean value theorem inside the integral in the numerator and the fact the derivative of the integrand is bounded above by $c(5^{-k} - b^{-k})^{-(p-2)-} \leq c2^{3k(p-2)-}$.

Let $F(y_d, z_d, u_d) := p(p-1) \int_0^1 ((z_d - u_d + \theta(y_d - z_d))^{p-2} - (z_d + \theta(y_d - z_d))^{p-2}) (1-\theta) d\theta$. For $\theta \in [0, 1]$, $u \in B(0, b^{-k})$ and $y \in B(z, \frac{1}{2}z_d)$, $z_d - u_d + \theta(y_d - z_d)$ and $z_d + \theta(y_d - z_d)$ are both comparable with z_d . Thus, for IV , we have for large k ,

$$\begin{aligned}
 |IV| &= \left| \int_{U(R), y_d > 5^{-k}, B(z, 2^{-1}z_d)} \frac{(y_d - u_d)^p - y_d^p - ((z_d - u_d)^p - z_d^p)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right| \\
 &= \left| \int_{U(R), y_d > 5^{-k}, B(z, 2^{-1}z_d)} \frac{p(p-1)(y_d - z_d)^2 F(y_d, z_d, u_d)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right| \\
 &\leq c_{17} b^{-k} z_d^{p-3} \int_{B(z, 2^{-1}z_d)} \frac{1}{|y-z|^{d+\alpha-2}} dy = c_{18} b^{-k} z_d^{p-1-\alpha} \leq c_{18} (4/5)^k z_d^{p+1-\alpha},
 \end{aligned}$$

where in the first inequality we used the mean value theorem inside the integral in the numerator.

Combining the estimates for I , II , III and IV , we arrive at the desired assertion. \square

Lemma 4.4. (a) *There exists $C_1 > 0$ such that for every $R \geq 1$ and $z \in U(R)$,*

$$0 \geq L_\alpha^\mathcal{B} h_{\alpha-1, R}(z) \geq -C_1 z_d^{\beta_1} (|\log z_d|^{\beta_3} \vee 1) \int_{|y| \geq R} |y|^{-\beta_1-d-1} (1 + \mathbf{1}_{|y| \geq 1} (\log |y|)^{\beta_3}) dy.$$

(b) *Let $\alpha - 1 < p < \alpha + \beta_1$. There exist $r_0 \in (0, 1/2]$ and $C_2 > 0$ and $C_3 > 0$ such that for every $z \in D(\frac{1}{2}, r_0)$,*

$$C_2 z_d^{p-\alpha} \leq L_\alpha^\mathcal{B} h_p(z) \leq C_3 z_d^{p-\alpha}.$$

Proof. (a) Let $z \in U(R)$. Then by (3.6), we see that

$$L_\alpha^\mathcal{B} h_{\alpha-1, R}(z) = - \int_{D(R, R)^c \cap \mathbb{R}_+^d} \frac{y_d^{\alpha-1}}{|y-z|^{d+\alpha}} \mathcal{B}(z, y) dy,$$

which is negative. Further, if $y \in D(R, R)^c$ and $z \in U(R) = D(R/2, R/2)$, then $|y| \geq R > |z|$, $|y-z| \geq z_d$ and $|y-z| \asymp |y|$. Thus it follows from [6, Lemma 5.2] that

$$\begin{aligned} |L_\alpha^\mathcal{B} h_{\alpha-1, R}(z)| &\leq \int_{D(R, R)^c \cap \mathbb{R}_+^d} \frac{|y|^{\alpha-1}}{|y-z|^{d+\alpha}} \mathcal{B}(z, y) dy \\ &\leq c_1 \int_{y \in \mathbb{R}_+^d, |y| \geq R, |y-z| \geq z_d} |y|^{-d-1} \mathcal{B}(z, y) dy \\ &\leq c_2 z_d^{\beta_1} (|\log z_d|^{\beta_3} \vee 1) \int_{|y| \geq R} |y|^{-\beta_1-d-1} (1 + \mathbf{1}_{|y| \geq 1} (\log |y|)^{\beta_3}) dy. \end{aligned}$$

(b) Let $z \in U$. Then by using Lemma 3.1,

$$L_\alpha^\mathcal{B} h_p(z) = C(\alpha, p, \mathcal{B}) z_d^{p-\alpha} - \int_{D(1, 1)^c \cap \mathbb{R}_+^d} \frac{y_d^p}{|y-z|^{d+\alpha}} \mathcal{B}(z, y) dy.$$

Since the second term is non-negative, by removing it we obtain the upper bound. In the same way as before (this uses $p < \alpha + \beta_1$),

$$\left| \int_{D(1, 1)^c \cap \mathbb{R}_+^d} \frac{y_d^p}{|y-z|^{d+\alpha}} \mathcal{B}(z, y) dy \right| \leq c_2 z_d^{\beta_1} |\log z_d|^{\beta_3}.$$

Thus, for any $z \in U$,

$$L_\alpha^\mathcal{B} h_q(z) \geq C(\alpha, p, \mathcal{B}) z_d^{p-\alpha} - c_2 z_d^{\beta_1} |\log z_d|^{\beta_3}.$$

Since $p - \alpha < \beta_1$ and $C(\alpha, p, \mathcal{B}) > 0$, we can find $r_0 \in (0, 1/2]$ such that the function $t \mapsto C(\alpha, p, \mathcal{B}) t^{q-\alpha} - c_2 t^{\beta_1} |\log t|^{\beta_3}$ is positive and bounded away from zero for all $t \in (0, r_0)$. This concludes the proof of the lower bound. \square

In the remainder of this paper, r_0 always stands for the constant in the lemma above.

Lemma 4.5. *There exists a constant $C > 0$ such that*

$$\mathbb{E}_x \tau_U \leq C x_d^{\alpha-1}, \quad x \in U. \quad (4.8)$$

Proof. Choose $q \in (\alpha - 1, \alpha)$ and let $\eta(x) := h_{\alpha-1}(x) - h_q(x)$, $x \in \mathbb{R}_+^d$. For $x \notin D(1, 1)$, $\eta(x) = 0$, while if $x \in D(1, 1)$ we have $\eta(x) = x_d^{\alpha-1} - x_d^q > 0$. By Lemma 4.4, for all $x \in U(r_0)$ we have that $L^\mathcal{B} h_{\alpha-1}(x) \leq 0$ and $L^\mathcal{B} h_q(x) \geq c_1 x_d^{q-\alpha}$. Thus we can find $r_1 \in (0, r_0]$ such that

$$L_\alpha^\mathcal{B} \eta(x) = L_\alpha^\mathcal{B} h_{\alpha-1}(x) - L_\alpha^\mathcal{B} h_q(x) \leq -c_1 x_d^{q-\alpha} \leq -1, \quad x \in U(r_1). \quad (4.9)$$

Let $g_k = v_k * (\eta \mathbf{1}_{D(5^{-k})})$. It follows from Lemmas 4.1 and 4.3 applied to h_q and $h_{\alpha-1}$ that $L_\alpha^\mathcal{B} g_k \rightarrow L_\alpha^\mathcal{B} \eta$ on U and the sequence of functions $|L_\alpha^\mathcal{B} g_k - L_\alpha^\mathcal{B} \eta|$ is bounded by some constant $c_2 > 0$. In particular,

$$-L_\alpha^\mathcal{B} g_k(z) \geq -L_\alpha^\mathcal{B} \eta(z) - c_2 \geq 1 - c_2, \quad z \in U(r), r \leq r_1. \quad (4.10)$$

It follows from [6, Lemma 3.6] that for all $t \geq 0$,

$$\mathbb{E}_x[g_k(Y_{t \wedge \tau_{U_{a_k}(r)}})] - g_k(x) = \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L_\alpha^\mathcal{B} g_k(Y_s) ds, \quad x \in U(r), r \leq r_1.$$

As $k \rightarrow \infty$, the left-hand side converges to $\mathbb{E}_x[\eta(Y_{t \wedge \tau_{U(r)}})] - \eta(x)$. For the right-hand side we can use Fatou's lemma (justified because of (4.10)) to conclude that for $x \in U(r)$ with $r \leq r_1$,

$$\limsup_{k \rightarrow \infty} \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L_\alpha^\mathcal{B} g_k(Y_s) ds \leq \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U(r)}} L_\alpha^\mathcal{B} \eta(Y_s) ds \leq -\mathbb{E}_x(t \wedge \tau_{U(r)}).$$

Thus we get that $\mathbb{E}_x[\eta(Y_{t \wedge \tau_{U(r)}})] - \eta(x) \leq -\mathbb{E}_x(t \wedge \tau_{U(r)})$, and by letting $t \rightarrow \infty$,

$$-\eta(x) \leq \mathbb{E}_x[\eta(Y_{\tau_{U(r)}})] - \eta(x) \leq -\mathbb{E}_x \tau_{U(r)}, \quad x \in U(r), r \leq r_1.$$

Thus we get $\mathbb{E}_x \tau_{U(r)} \leq \eta(x) \leq x_d^{\alpha-1}$. By using that $U(r_1) = r_1 U$ and (2.3), for any $x \in U$,

$$\mathbb{E}_x \tau_U = r_1^{-\alpha} \mathbb{E}_{r_1 x} \tau_{r_1 U} \leq r_1^{-\alpha} (r_1 x_d)^{\alpha-1} = r_1^{-1} x_d^{\alpha-1}.$$

We have proved the claim of the lemma with $C = r_1^{-1}$. \square

Proposition 4.6. *Let $p \in [\alpha - 1, \alpha + \beta_1)$, $R \geq 1$ and $r \leq R$. For every $x \in U(r)$ it holds that*

$$\mathbb{E}_x[h_{p,R}(Y_{\tau_{U(r)}})] = h_{p,R}(x) + \mathbb{E}_x \int_0^{\tau_{U(r)}} L_\alpha^\mathcal{B} h_{p,R}(Y_s) ds. \quad (4.11)$$

Proof. Set $g_k := v_k * (h_{p,R} \mathbf{1}_{D(5^{-k})})$. Let $x \in U(r)$, $r \leq R$. There is $k_0 \geq 1$ such that $x \in U_{a_k}(r)$ for all $k \geq k_0$. Note that since $g_k \in C_c^\infty(\mathbb{R}_+^d)$, it follows from [6, Lemma 3.6] that for all $t \geq 0$,

$$\mathbb{E}_x[g_k(Y_{t \wedge \tau_{U_{a_k}(r)}})] = g_k(x) + \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L_\alpha^\mathcal{B} g_k(Y_s) ds.$$

Clearly, $\lim_{k \rightarrow \infty} \tau_{U_{a_k}(r)} = \tau_{U(r)}$. Since $g_k \rightarrow h_{p,R}$ as $k \rightarrow \infty$, we get that the left-hand side above converges to $\mathbb{E}_x[h_{p,R}(Y_{t \wedge \tau_{U(r)}})]$.

On the other hand, by combining Lemmas 4.1 and 4.3, we see that for every $z \in U(r)$ with $r \leq R$,

$$\lim_{k \rightarrow \infty} |L_\alpha^\mathcal{B} g_k(z) - L_\alpha^\mathcal{B} h_{p,R}(z)| = 0$$

and $|L_\alpha^\mathcal{B} g_k(z) - L_\alpha^\mathcal{B} h_{p,R}(z)|$ is bounded. Thus, we can use the bounded convergence theorem and get

$$\lim_{k \rightarrow \infty} \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L_\alpha^\mathcal{B} g_k(Y_s) ds = \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U(r)}} L_\alpha^\mathcal{B} h_{p,R}(Y_s) ds.$$

Therefore,

$$\mathbb{E}_x[h_{p,R}(Y_{t \wedge \tau_{U_{a_k}(r)}})] = h_{p,R}(x) + \mathbb{E}_x \int_0^{t \wedge \tau_{U(r)}} L_\alpha^\mathcal{B} h_{p,R}(Y_s) ds.$$

By letting $t \rightarrow \infty$ we get that the left-hand side above converges to $\mathbb{E}_x[h_{p,R}(Y_{\tau_{U(r)}})]$.

When $p = \alpha - 1$, by Lemma 4.4 (a), $L_\alpha^\mathcal{B} h_{p,R}(z) \leq 0$. Thus we can use the monotone convergence theorem and obtain (4.11).

When $p \in (\alpha - 1, \alpha + \beta_1)$, by Lemma 4.4 (b) and scaling $L_\alpha^\mathcal{B} h_{p,R}(z) > 0$ on $D(R/2, r_0 R) \supset D(r/2, rr_0/2)$. Thus, we can use the monotone convergence theorem and get

$$\lim_{t \rightarrow \infty} \mathbb{E}_x \int_0^{t \wedge \tau_{U(r)}} \mathbf{1}_{Y_s \in D(r/2, rr_0/2)} L_\alpha^\mathcal{B} h_{p,R}(Y_s) ds = \mathbb{E}_x \int_0^{\tau_{U(r)}} \mathbf{1}_{Y_s \in D(r/2, rr_0/2)} L_\alpha^\mathcal{B} h_{p,R}(Y_s) ds.$$

On the other hand, since

$$L_\alpha^\mathcal{B} h_{p,R}(z) = C(\alpha, p, \mathcal{B}) z_d^{p-\alpha} - \int_{D(R,R)} \frac{y_d^p}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy, \quad z \in D(R, R),$$

we know that $L_\alpha^\mathcal{B} h_{p,R}(z)$ is bounded on $U(r) \setminus D(r/2, rr_0/2)$. Thus, using Lemma 4.5 and the bounded convergence theorem, we get

$$\lim_{t \rightarrow \infty} \mathbb{E}_x \int_0^{t \wedge \tau_{U(r)}} \mathbf{1}_{Y_s \in U(r) \setminus D(r/2, rr_0/2)} L_\alpha^\mathcal{B} h_{p,R}(Y_s) ds = \mathbb{E}_x \int_0^{\tau_{U(r)}} \mathbf{1}_{Y_s \in U(r) \setminus D(r/2, rr_0/2)} L_\alpha^\mathcal{B} h_{p,R}(Y_s) ds.$$

Combining these, we obtain (4.11) for $p \in (\alpha - 1, \alpha + \beta_1)$ too. \square

5. BOUNDARY HARNACK PRINCIPLE

In this section we always assume that $\alpha > 1$. The next two results are applications of Proposition 4.6.

Lemma 5.1. *For all $r > 0$ it holds that*

$$\mathbb{E}_x[h_{\alpha-1, \infty}(Y_{\tau_{U(r)}})] = h_{\alpha-1, \infty}(x), \quad \text{for all } x \in U(r). \quad (5.1)$$

In particular, the function $h_{\alpha-1, \infty}(x) = g_{\alpha-1}(x) = x_d^{\alpha-1}$ is harmonic in \mathbb{R}_+^d with respect to Y .

Proof. Fix $r > 0$ and $x \in U(r)$. Let $R \geq 1 \vee r$. By Proposition 4.6,

$$\mathbb{E}_x[h_{\alpha-1, R}(Y_{\tau_{U(r)}})] = x_d^{\alpha-1} + \mathbb{E}_x \int_0^{\tau_{U(r)}} L_\alpha^\mathcal{B} h_{\alpha-1, R}(Y_s) ds. \quad (5.2)$$

By the monotone convergence theorem,

$$\lim_{R \rightarrow \infty} \mathbb{E}_x[h_{\alpha-1, R}(Y_{\tau_{U(r)}})] = \mathbb{E}_x[h_{\alpha-1, \infty}(Y_{\tau_{U(r)}})].$$

Using Lemmas 4.4 (a) and 4.5, we see that

$$\begin{aligned} 0 &\geq \mathbb{E}_x \int_0^{\tau_{U(r)}} L_\alpha^\mathcal{B} h_{\alpha-1, R}(Y_s) ds \\ &\geq -c_1(r) \mathbb{E}_x \tau_{U(r)} \int_{|y| \geq R} |y|^{-\beta_1 - d - 1} (1 + \mathbf{1}_{|y| \geq 1} (\log |y|)^{\beta_3}) dy \\ &\geq -c_2(r) \int_{|y| \geq R} |y|^{-\beta_1 - d - 1} (1 + \mathbf{1}_{|y| \geq 1} (\log |y|)^{\beta_3}) dy. \end{aligned}$$

Thus

$$\lim_{R \rightarrow \infty} \mathbb{E}_x \int_0^{\tau_{U(r)}} L_\alpha^\mathcal{B} h_{\alpha-1, R}(Y_s) ds = 0.$$

The proof is now complete. \square

Recall that r_0 in the constant in Lemma 4.4(b).

Lemma 5.2. *There exists $C > 0$ such that*

$$\mathbb{E}_x \tau_{U(r_0)} \leq C \mathbb{P}_x \left(Y_{\tau_{U(r_0)}} \in D(1, 1) \right) \quad \text{for all } x \in U(r_0).$$

Proof. Choose a $p \in (\alpha - 1, \alpha)$. By Propostion 4.6, for every $x \in U(r_0)$,

$$\mathbb{E}_x[h_p(Y_{\tau_{U(r_0)}})] = h_p(x) + \mathbb{E}_x \int_0^{\tau_{U(r_0)}} L_\alpha^\mathcal{B} h_p(Y_s) ds.$$

Thus, using Lemma 4.4 (b) and that h_p is bounded by 1 and supported on $D(1, 1)$, we get

$$\mathbb{P}_x \left(Y_{\tau_{U(r_0)}} \in D(1, 1) \right) \geq \mathbb{E}_x[h_p(Y_{\tau_{U(r_0)}})] \geq \mathbb{E}_x \int_0^{\tau_{U(r_0)}} L_\alpha^\mathcal{B} h_p(Y_s) ds \geq c \mathbb{E}_x \tau_{U(r_0)}.$$

□

Since $\int_{D(1,1)} y_d^{\alpha-1} |y|^{-d-\alpha-\beta_1} dy = +\infty$, by the same argument as that leading to [6, (5.10)], we also have that there exists an $r_1 \in (0, r_0)$ small enough so that for all $r \in (0, r_1]$ and $R \geq 1$,

$$\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt \leq \mathbb{E}_x[h_{\alpha-1}(Y_{\tau_{U(r)}})] \leq \mathbb{E}_x[h_{\alpha-1,R}(Y_{\tau_{U(r)}})], \quad x \in U(r). \quad (5.3)$$

Using (2.3), (5.3), Proposition 4.6 and Lemma 4.4 (a), repeating the proof of [6, Lemma 5.7], we get the following upper bound.

Lemma 5.3. *There exists a constant $C > 0$ such that for all $x \in U$,*

$$\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt \leq C x_d^{\alpha-1}. \quad (5.4)$$

Using the above lemma, [6, Lemma 5.2(b)] and the scaling propetry of Y , the proof of the next lemma is the same as that of [6, Lemma 6.1], we omit the proof.

Lemma 5.4. *There exists $C > 0$ such that for all $0 < 4r \leq R \leq 1$ and $w \in D(r, r)$,*

$$\mathbb{P}_w \left(Y_{\tau_{B(w,r) \cap \mathbb{R}_+^d}} \in A(w, R, 4) \cap \mathbb{R}_+^d \right) \leq C \frac{r^{\alpha+\beta_1}}{R^{\alpha+\beta_1}} \frac{w_d^{\alpha-1}}{r^{\alpha-1}}.$$

Recall that r_0 in the constant in Lemma 4.4(b).

Lemma 5.5. *There exists $C > 0$ such that for any $x \in U(2^{-4}r_0)$,*

$$\mathbb{P}_x \left(Y_{\tau_{U(r_0)}} \in D(1, 1) \right) \leq C \mathbb{P}_x \left(Y_{\tau_{U(r_0)}} \in D(1/2, 1) \setminus D(1/2, 3/4) \right).$$

Proof. Let $V = U(r_0)$ and

$$H_2 := \{Y_{\tau_V} \in D(1, 1)\}, \quad H_1 := \{Y_{\tau_V} \in D(1/2, 1) \setminus D(1/2, 3/4)\}.$$

Choose a $p \in (\alpha - 1, \alpha)$ and let $\kappa(x) = C(\alpha, p, \mathcal{B}) x_d^{-\alpha}$. Let Y^κ be the subprocess of Y with killing potential κ so that the corresponding Dirichlet form is $\mathcal{E}(u, v) + \int_{\mathbb{R}_+^d} u(x)v(x)\kappa(x)dx$. Either by repeating the proof of [6, (5.17)] or using [6, Theorem 1.3], we get that

$$\mathbb{P}_w \left(Y_{\tau_V}^\kappa \in D(r_0/4, 1) \setminus D(r_0/4, 3/4) \right) \geq c_1 w_d^p, \quad w \in U(r_0/2).$$

Thus,

$$\mathbb{P}_w(H_1) \geq \mathbb{P}_w \left(Y_{\tau_V}^\kappa \in D(1/2, 1) \setminus D(1/2, 3/4) \right) \geq c_1 w_d^p, \quad w \in U(r_0/2). \quad (5.5)$$

For $i \geq 1$, set

$$s_0 = s_1, \quad s_i = \frac{r_0}{8} \left(\frac{1}{2} - \frac{1}{50} \sum_{j=1}^i \frac{1}{j^2} \right) \quad \text{and} \quad J_i = D(s_i, 2^{-i-3}r_0) \setminus D(s_i, 2^{-i-4}r_0).$$

Note that $r_0/(20) < s_i < r_0/(16)$. Define for $i \geq 1$,

$$d_i = \sup_{z \in J_i} \frac{\mathbb{P}_z(H_2)}{\mathbb{P}_z(H_1)}, \quad \tilde{J}_i = D(s_{i-1}, 2^{-i-3}r_0), \quad \tau_i = \tau_{\tilde{J}_i}. \quad (5.6)$$

Repeating the argument leading to [5, (6.29)], we get that for $z \in J_i$ and $i \geq 2$,

$$\mathbb{P}_z(H_2) \leq \left(\sup_{1 \leq k \leq i-1} d_k \right) \mathbb{P}_z(H_1) + \mathbb{P}_z(Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k). \quad (5.7)$$

Recall that n_0 in the constant in Corollary 3.4 (b). For $i \geq 2$, define $\sigma_{i,0} = 0, \sigma_{i,1} = \inf\{t > 0 : |Y_t - Y_0| \geq n_0 2^{-i-2} r_0\}$ and $\sigma_{i,m+1} = \sigma_{i,m} + \sigma_{i,1} \circ \theta_{\sigma_{i,m}}$ for $m \geq 1$. By Corollary 3.4 (b), we have that

$$\mathbb{P}_w(Y_{\sigma_{i,1}} \in \tilde{J}_i) \leq 1 - \mathbb{P}_w(\sigma_{i,1} = \zeta) \leq 1 - \mathbb{P}_w(\tau_{B(w, n_0 w_d)} = \zeta) < 2^{-1}, \quad w \in \tilde{J}_i. \quad (5.8)$$

For the purpose of further estimates, we now choose a positive integer l such that $l \geq \alpha + \beta_1$. Next we choose $i_0 \geq 2$ large enough so that $n_0 2^{-i+1} < 1/(200li^3)$ for all $i \geq i_0$. Now we assume $i \geq i_0$. Using (5.8) and the strong Markov property we have that for $z \in J_i$,

$$\begin{aligned} \mathbb{P}_z(\tau_i > \sigma_{i,li}) &\leq \mathbb{P}_z(Y_{\sigma_{i,k}} \in \tilde{J}_i, 1 \leq k \leq li) \\ &= \mathbb{E}_z \left[\mathbb{P}_{Y_{\sigma_{i,li-1}}}(Y_{\sigma_{i,1}} \in \tilde{J}_i) : Y_{\sigma_{i,li-1}} \in \tilde{J}_i, Y_{\sigma_{i,k}} \in \tilde{J}_i, 1 \leq k \leq li-2 \right] \\ &\leq \mathbb{P}_z \left(Y_{\sigma_{i,k}} \in \tilde{J}_i, 1 \leq k \leq li-1 \right) 2^{-1} \leq 2^{-li}. \end{aligned} \quad (5.9)$$

Note that if $z \in J_i$ and $y \in D(1, 1) \setminus [\tilde{J}_i \cup (\cup_{k=1}^{i-1} J_k)]$, then $|y - z| \geq (s_{i-1} - s_i) \wedge (2^{-4} r_0 - 2^{-i-3} r_0) = r_0/(400i^2)$. Furthermore, since $2^{-i-2} n_0 < 1/(400i^2)$ (recall that $i \geq i_0$), if $Y_{\tau_i}(\omega) \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k$ and $\tau_i(\omega) \leq \sigma_{i,li}(\omega)$, then $\tau_i(\omega) = \sigma_{i,k}(\omega)$ for some $k = k(\omega) \leq li$. Dependence of k on ω will be omitted in the next few lines. Hence on $\{Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k, \tau_i \leq \sigma_{i,li}\}$ with $Y_0 = z \in J_i$, we have $|Y_{\sigma_{i,k}} - Y_{\sigma_{i,0}}| = |Y_{\tau_i} - Y_0| > \frac{r_0}{400i^2}$ for some $1 \leq k \leq li$. Thus for some $1 \leq k \leq li$, $\sum_{j=0}^k |Y_{\sigma_{i,j}} - Y_{\sigma_{i,j-1}}| > r_0(400i^2)^{-1}$ which implies for some $1 \leq j \leq k \leq li$, $|Y_{\sigma_{i,j}} - Y_{\sigma_{i,j-1}}| \geq r_0(k400i^2)^{-1} \geq r_0(li)^{-1}(400i^2)^{-1}$. Thus, we have

$$\begin{aligned} &\{Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k, \tau_i \leq \sigma_{i,li}\} \\ &\subset \cup_{j=1}^{li} \{|Y_{\sigma_{i,j}} - Y_{\sigma_{i,j-1}}| \geq r_0/(800li^3), Y_{\sigma_{i,j}} \in D(1, 1), Y_{\sigma_{i,j-1}} \in \tilde{J}_i\}. \end{aligned}$$

Now, using Lemma 5.4 (with $r = 2^{-i-2} n_0 r_0$ and $R = r_0/(800li^3)$) (noting that $4 \cdot 2^{-i-1} n_0 < 1/(400li^3)$ for all $i \geq i_0$), and repeating the argument leading to [5, (6.34)], we get that for $z \in J_i$,

$$\begin{aligned} \mathbb{P}_z(Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k, \tau_i \leq \sigma_{i,li}) &\leq li \sup_{w \in \tilde{J}_i} \mathbb{P}_w(|Y_{\sigma_{i,1}} - w| \geq r_0(800li^3)^{-1}, Y_{\sigma_{i,1}} \in D(1, 1)) \\ &\leq li \sup_{w \in \tilde{J}_i} \mathbb{P}_w(4 > |Y_{\sigma_{i,1}} - w| \geq r_0(800li^3)^{-1}) \leq c_{12} li \left(\frac{800li^3}{2^{i+3}} \right)^{\alpha+\beta_1}. \end{aligned}$$

By this and (5.9), we have for $z \in J_i, i \geq i_0$,

$$\mathbb{P}_z(Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k) \leq 2^{-li} + c_2 li \left(\frac{800li^3}{2^{i+3}} \right)^{\alpha+\beta_1}. \quad (5.10)$$

By our choice of l , we have

$$li \left(\frac{800li^3}{2^{i+3}} \right)^{\alpha+\beta_1} = 100^{\alpha+\beta_1} l^{1+\alpha+\beta_1} i^{1+3(\alpha+\beta_1)} (2^{-(\alpha+\beta_1)})^i \geq (2^{-(\alpha+\beta_1)})^i \geq (2^{-l})^i. \quad (5.11)$$

Thus combining (5.11) with (5.10), and then using (5.5), we get that for $z \in J_i, i \geq i_0$,

$$\frac{\mathbb{P}_z(Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k)}{\mathbb{P}_z(H_1)} \leq c_3 li 2^{ip} \left(\frac{800li^3}{2^{i+3}} \right)^{\alpha+\beta_1} \leq c_4 i^{1+3(\alpha+\beta_1)} 2^{(p-\alpha-\beta_1)i}. \quad (5.12)$$

By this and (5.7), for $z \in J_i$, $i \geq i_0$, for all $i \geq i_0$

$$\frac{\mathbb{P}_z(H_2)}{\mathbb{P}_z(H_1)} \leq \sup_{1 \leq k \leq i-1} d_k + \frac{\mathbb{P}_z(Y_{\tau_i} \in D(1,1) \setminus \cup_{k=1}^{i-1} J_k)}{\mathbb{P}_z(H_1)} \leq \sup_{1 \leq k \leq i-1} d_k + c_4 \frac{i^{1+3(\alpha+\beta_1)}}{2^{(\alpha+\beta_1-p)i}}.$$

This implies that for all $i \geq 1$

$$d_i \leq \sup_{1 \leq k \leq i_0-1} d_k + c_4 \sum_{k=1}^i \frac{i^{1+3(\alpha+\beta_1)}}{2^{(\alpha+\beta_1-p)i}} \leq \sup_{1 \leq k \leq i_0-1} d_k + c_4 \sum_{k=1}^{\infty} \frac{i^{1+3(\alpha+\beta_1)}}{2^{(\alpha+\beta_1-p)i}} =: c_5 < \infty.$$

Since $U(2^{-4}r_0) \subset \cup_{k=1}^{\infty} J_k$, the proof is now complete. \square

Using Corollary 3.4 (b), we can prove the following Carleson estimate.

Theorem 5.6 (Carleson estimate). *There exists a constant $C > 0$ such that for any $w \in \partial\mathbb{R}_+^d$, $r > 0$, and any non-negative function f in \mathbb{R}_+^d that is harmonic in $\mathbb{R}_+^d \cap B(w, r)$ with respect to Y and vanishes continuously on $\partial\mathbb{R}_+^d \cap B(w, r)$, we have*

$$f(x) \leq C f(\hat{x}) \quad \text{for all } x \in \mathbb{R}_+^d \cap B(w, r/2), \quad (5.13)$$

where $\hat{x} \in \mathbb{R}_+^d \cap B(w, r)$ with $\hat{x}_d \geq r/4$.

Proof. Recall that n_0 in the constant in Corollary 3.4 (b). By using $B_0(x) = B(x, n_0 x_d)$ instead of $B_0(x) = B(x, x_d/2)$ in the proof of [6, Theorem 1.2] and applying our Corollary 3.4 (b), the proof of the theorem is almost identical to that of [6, Theorem 1.2]. We omit the details. \square

Proof of Theorem 1.1. Recall that r_0 in the constant in Lemma 4.4(b). By scaling, it suffices to deal with the case $r = 1$. Moreover, by Theorem 2.3 (b), it suffices to prove (1.4) for $x, y \in D_{\tilde{w}}(2^{-8}r_0, 2^{-8}r_0)$. Since f is harmonic in $D_{\tilde{w}}(2, 2)$ and vanishes continuously on $B(\tilde{w}, 2) \cap \partial\mathbb{R}_+^d$, it is regular harmonic in $D_{\tilde{w}}(7/4, 7/4)$ and vanishes continuously on $B(\tilde{w}, 7/4) \cap \partial\mathbb{R}_+^d$ (see [5, Lemma 5.1] and its proof). Throughout the remainder of this proof, we assume that $x \in D_{\tilde{w}}(2^{-8}r_0, 2^{-8}r_0)$. Without loss of generality we take $\tilde{w} = 0$.

Define $x_0 = (\tilde{x}, 1/(16))$ and $V = U(r_0)$. By the Harnack inequality and Lemma 5.5, we have

$$\begin{aligned} f(x) &= \mathbb{E}_x[f(Y_{\tau_V})] \geq \mathbb{E}_x[f(Y_{\tau_V}); Y_{\tau_V} \in D(1/2, 1) \setminus D(1/2, 3/4)] \\ &\geq c_0 f(x_0) \mathbb{P}_x(Y_{\tau_V} \in D(1/2, 1) \setminus D(1/2, 3/4)) \geq c_1 f(x_0) \mathbb{P}_x(Y_{\tau_V} \in D(1, 1)). \end{aligned} \quad (5.14)$$

Set $w_0 = (\tilde{0}, 2^{-7})$. Then, using [6, Proposition 3.11 (a)], we also have [6, (6.13)-(6.14)], that is,

$$f(w_0) \geq c_2 \int_{\mathbb{R}_+^d \setminus D(1,1)} J_{\mathbb{R}_+^d}(w_0, y) f(y) dy, \quad (5.15)$$

and

$$J_{\mathbb{R}_+^d}(z, y) \leq c_3 J_{\mathbb{R}_+^d}(w_0, y), \quad \text{for any } z \in U \text{ and } y \in \mathbb{R}_+^d \setminus D(1, 1). \quad (5.16)$$

Combining (5.16) with (5.15) we now have

$$\begin{aligned} \mathbb{E}_x[f(Y_{\tau_V}); Y_{\tau_V} \notin D(1, 1)] &= \mathbb{E}_x \int_0^{\tau_V} \int_{\mathbb{R}_+^d \setminus D(1,1)} J_{\mathbb{R}_+^d}(Y_t, y) f(y) dy dt \\ &\leq c_3 \mathbb{E}_x \tau_V \int_{\mathbb{R}_+^d \setminus D(1,1)} J_{\mathbb{R}_+^d}(w_0, y) f(y) dy \leq c_4 f(w_0) \mathbb{E}_x \tau_V. \end{aligned} \quad (5.17)$$

On the other hand, by the Harnack inequality (Theorem 2.3) and Carleson's estimate (Theorem 5.6), we have

$$\mathbb{E}_x [f(Y_{\tau_V}); Y_{\tau_V} \in D(1, 1)] \leq c_5 f(x_0) \mathbb{P}_x (Y_{\tau_V} \in D(1, 1)). \quad (5.18)$$

Combining (5.17) and (5.18), and using the Harnack inequality, we get

$$\begin{aligned} f(x) &= \mathbb{E}_x [f(Y_{\tau_V}); Y_{\tau_V} \in D(1, 1)] + \mathbb{E}_x [f(Y_{\tau_V}); Y_{\tau_V} \notin D(1, 1)] \\ &\leq c_4 f(w_0) \mathbb{E}_x \tau_V + c_5 f(x_0) \mathbb{P}_x (Y_{\tau_V} \in D(1, 1)) \leq c_6 f(x_0) (\mathbb{E}_x \tau_V + \mathbb{P}_x (Y_{\tau_V} \in D(1, 1))). \end{aligned}$$

This with (5.15) and Lemma 5.2 implies that $f(x) \asymp f(x_0) \mathbb{P}_x (Y_{\tau_V} \in D(1, 1))$. For any $y \in D(2^{-8}r_0, 2^{-8}r_0)$, we have the same estimate with $f(y_0)$ instead of $f(x_0)$, where $y_0 = (\tilde{y}, 1/(16))$. By the Harnack inequality, we have $f(x_0) \asymp f(y_0)$. Thus,

$$\frac{f(x)}{f(y)} \underset{\sim}{\asymp} \frac{\mathbb{P}_x (Y_{\tau_V} \in D(1, 1))}{\mathbb{P}_y (Y_{\tau_V} \in D(1, 1))}.$$

We now apply this with $g_{\alpha-1}$ (which is harmonic by Lemma 5.1) to conclude that

$$\frac{x_d^{\alpha-1}}{y_d^{\alpha-1}} \underset{\sim}{\asymp} \frac{\mathbb{P}_x (Y_{\tau_V} \in D(1, 1))}{\mathbb{P}_y (Y_{\tau_V} \in D(1, 1))} \underset{\sim}{\asymp} \frac{f(x)}{f(y)}.$$

□

We now show that any non-negative function which is regular harmonic near a portion of boundary vanishes continuously on that portion of boundary, cf. [2, Remark 6.2] and [3, Lemma 3.2]. Thus, the above boundary Harnack principle also holds for regular harmonic functions.

Lemma 5.7. *There exists a constant $C > 0$ such that for every bounded function $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ which is regular harmonic in $U = D(1/2, 1/2)$, it holds that*

$$f(x) \leq C \|f\|_\infty x_d^{\alpha-1}, \quad x \in D(2^{-5}, 2^{-5}).$$

Proof. Let $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ be a bounded function which is regular harmonic in $U = D(1/2, 1/2)$. Then for every $x \in D(2^{-5}, 2^{-5})$,

$$f(x) = \mathbb{E}_x [f(Y_{\tau_U}), Y_{\tau_U} \in D(1, 1)] + \mathbb{E}_x [f(Y_{\tau_U}), Y_{\tau_U} \notin D(1, 1)]. \quad (5.19)$$

In the first term we use $f(Y_{\tau_U}) \leq \|f\|_\infty$ and then apply Theorem 1.1 to get

$$\mathbb{E}_x [f(Y_{\tau_U}), Y_{\tau_U} \in D(1, 1)] \leq \|f\|_\infty \mathbb{P}_x (Y_{\tau_U} \in D(1, 1)) \leq c_1 \|f\|_\infty x_d^{\alpha-1}. \quad (5.20)$$

Now we estimate the second term. For $z \in U$ and $w \in \mathbb{R}_+^d \setminus D(1, 1)$, we have $|w - z| \asymp |w|$. Thus, by using [6, Lemma 5.2(a)],

$$\begin{aligned} &\int_{\mathbb{R}_+^d \setminus D(1, 1)} f(y) \mathcal{B}(z, w) |z - w|^{-d-\alpha} dw \\ &\leq c_2 \|f\|_\infty z_d^{\beta_1} (|\log z_d|^{\beta_3} \vee 1) \int_{\mathbb{R}_+^d \setminus D(1, 1)} \frac{1}{|w|^{d+\alpha+\beta_1}} (1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3}) dw \\ &\leq c_2 \|f\|_\infty z_d^{\beta_1} |\log z_d|^{\beta_3} \int_{\mathbb{R}_+^d \setminus D(1, 1)} \frac{(1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3})}{|w|^{d+\alpha+\beta_1}} dw. \end{aligned}$$

Hence, by using the Lévy system formula and Lemma 5.3, for $x \in D(2^{-5}, 2^{-5})$,

$$\begin{aligned} &\mathbb{E}_x [f(Y_{\tau_U}); Y_{\tau_U} \notin D(1, 1)] \\ &= \mathbb{E}_x \int_0^{\tau_U} \int_{\mathbb{R}_+^d \setminus D(1, 1)} f(Y_t) J(Y_t, z) dz dt \end{aligned}$$

$$\begin{aligned}
 &\leq c_3 \|f\|_\infty x_d^{\alpha-1} \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{(1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3})}{|w|^{d+\alpha+\beta_1}} dw \\
 &\leq c_4 \|f\|_\infty x_d^{\alpha-1}.
 \end{aligned} \tag{5.21}$$

The claim of the lemma follows by using (5.20) and (5.21) in (5.19). \square

Proposition 5.8. *There exists $C > 0$ such that for every $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ which is regular harmonic in $U = D(1/2, 1/2)$, it holds that*

$$f(x) \leq C \left(\frac{f(w)}{w_d^{\alpha-1}} \right) x_d^{\alpha-1}, \quad w, x \in D(2^{-3}, 2^{-3}).$$

In particular, f vanishes continuously $\{x = (\tilde{x}, 0) \in \partial\mathbb{R}_+^d : |\tilde{x}| < 2^{-3}\}$.

Proof. For any $k \in \mathbb{N}$ define

$$f_k(x) := \mathbb{E}_x[f(Y_{\tau_U}) \wedge k] = \mathbb{E}_x[(f \wedge k)(Y_{\tau_U})].$$

Then f_k is regular harmonic in U and bounded in \mathbb{R}_+^d . By Lemma 5.7, $f_k(x) \leq c_4 \|f \wedge k\|_\infty x_d^{\alpha-1}$. Thus f_k vanishes continuously at the boundary. By the boundary Harnack principle, Theorem 1.1, there is $C > 0$ such that for every $k \in \mathbb{N}$

$$\frac{f_k(x)}{f_k(w)} \leq C \frac{x_d^{\alpha-1}}{w_d^{\alpha-1}}, \quad x, w \in D(2^{-3}, 2^{-3}).$$

By letting $k \rightarrow \infty$ we get that

$$\frac{f(x)}{f(w)} \leq C \frac{x_d^{\alpha-1}}{w_d^{\alpha-1}}, \quad x, w \in D(2^{-3}, 2^{-3}).$$

\square

6. ESTIMATES ON GREEN FUNCTIONS AND POTENTIAL

6.1. Green function estimates. By following [7, Section2] we see that there exists a symmetric function $G(x, y)$, $x, y \in \mathbb{R}^d$ such that for all Borel functions $f : \mathbb{R}^d \rightarrow [0, \infty)$ we have that

$$\mathbb{E}_x \int_0^\zeta f(Y_t) dt = \int_{\mathbb{R}_+^d} G(x, y) f(y) dy.$$

Moreover, since Y is transient, $G(x, y)$ is not identically infinite, and as in [7, Proposition 2.2], we can conclude that $G(\cdot, \cdot)$ is lower-semicontinuous in each variable and finite outside the diagonal. Further, for every $x \in \mathbb{R}_+^d$, $G(x, \cdot)$ is harmonic with respect to Y in $\mathbb{R}_+^d \setminus \{x\}$ and regular harmonic in $\mathbb{R}_+^d \setminus B(x, \epsilon)$ for every $\epsilon > 0$. The function G enjoys the following scaling property (proved as in [7, Proposition 2.4]): For all $x, y \in \mathbb{R}_+^d$, $x \neq y$,

$$G(x, y) = G\left(\frac{x}{|x-y|}, \frac{y}{|x-y|}\right) |x-y|^{\alpha-d}. \tag{6.1}$$

Choose a $p \in (\alpha - 1, \alpha)$ and let $\kappa(x) = C(\alpha, p, \mathcal{B}) x_d^{-\alpha}$. Let Y^κ be the subprocess of Y with killing potential κ so that the corresponding Dirichlet form is $\mathcal{E}^\kappa(u, v) = \mathcal{E}(u, v) + \int_{\mathbb{R}_+^d} u(x)v(x)\kappa(x)dx$. Let $G(x, y)$ and $G^\kappa(x, y)$ be the Green functions of Y and Y^κ respectively. Then $G(x, y) \geq G^\kappa(x, y)$. Now the following result follows immediately from [7, Proposition 4.1].

Proposition 6.1. *For any $C_1 > 0$, there exists a constant $C_2 > 0$ such that for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \leq C_1(x_d \wedge y_d)$, it holds that*

$$G(x, y) \geq C_2|x - y|^{-d+\alpha}.$$

From now on we assume $d > (\alpha + \beta_1 + \beta_2) \wedge 2$. In [7, Section 4.2], the killing function plays no role. Repeating the argument leading to [6, Proposition 4.6], we get

Proposition 6.2. *There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \leq x_d \wedge y_d$, it holds that*

$$G(x, y) \leq C|x - y|^{-d+\alpha}.$$

Using Proposition 5.8, we can combine Proposition 6.2 with Theorem 5.6 to get the following result, which is key for us to get sharp two-sided Green functions estimates.

Proposition 6.3. *There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}_+^d$,*

$$G(x, y) \leq C|x - y|^{-d+\alpha}. \quad (6.2)$$

Proof. It follows from Proposition 6.2 that there exists $c_1 > 0$ such that $G(x, y) \leq c_1$ for all $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$ and $x_d \wedge y_d > 1$. By Theorem 2.3, for any $c_2 > 0$, there exists $c_3 > 0$ such that $G(x, y) \leq c_3$ for all $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$ and $x_d \wedge y_d > c_2$. Now by Theorem 5.6, we see that there exists $c_4 > 0$ such that $G(x, y) \leq c_4$ for all $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$. Therefore, by (6.1), we have

$$G(x, y) \leq C|x - y|^{-d+\alpha}, \quad x, y \in \mathbb{R}_+^d.$$

□

Now we prove the two-sided Green function estimates.

Proof of Theorem 1.2. The scaling property and the invariance property of the half space under scaling imply that in order to prove Theorem 1.2 we only need to show that for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| = 1$,

$$C^{-1}(x_d \wedge 1)^{\alpha-1}(y_d \wedge 1)^{\alpha-1} \leq G(x, y) \leq C(x_d \wedge 1)^{\alpha-1}(y_d \wedge 1)^{\alpha-1}. \quad (6.3)$$

By scaling, Theorem 2.3, and Propositions 6.1 and 6.3, we only need to show (6.3) for $x_d \wedge y_d \leq 2^{-3}$ and $|x - y| = 1$.

We now assume that $|x - y| = 1$ and let $x_0 = (\tilde{x}, 2^{-3})$ and $y_0 = (\tilde{y}, 2^{-3})$. We first note that Proposition 5.8, together with scaling, clearly implies that for all $x \in \mathbb{R}_+^d$, $y \mapsto G(x, y)$ vanishes at the boundary of \mathbb{R}_+^d .

Suppose that $y_d \geq 2^{-3}$. Then, by Theorem 2.3, and Propositions 6.1 and 6.3, we have $G(x_0, y) \asymp c > 0$. Thus by Theorem 1.1,

$$G(x, y) \asymp G(x_0, y)(x_d/2^{-3})^{\alpha-1} \asymp x_d^{\alpha-1}. \quad (6.4)$$

Suppose that $y_d < 2^{-3}$. Then by Theorem 1.1 and (6.4)

$$G(x, y) \asymp G(x, y_0)(y_d/2^{-3})^{\alpha-1} \asymp x_d^{\alpha-1}y_d^{\alpha-1}. \quad (6.5)$$

(6.4) and (6.5) imply that (6.3) hold for $x_d \wedge y_d \leq 2^{-3}$ and $|x - y| = 1$. □

6.2. Estimates on Potentials. Recall that we have assumed $d > (\alpha + \beta_1 + \beta_2) \wedge 2$. Let $G^{B(w,R) \cap \mathbb{R}_+^d}(x, y)$ be the Green function of the process Y killed upon exiting $B(w, R) \cap \mathbb{R}_+^d$, $w \in \partial \mathbb{R}_+^d$.

For any $a > 0$, let $B_a^+ := B(0, a) \cap \mathbb{R}_+^d$. Using Theorem 1.2 and the formula

$$G^{B_1^+}(y, z) = G(y, z) - \mathbb{E}_y[G(Y_{\tau_{B_1^+}}, z)],$$

the proof of next result is standard. For example, see [7, Lemma 5.1]

Lemma 6.4. *For any $\varepsilon \in (0, 1)$ and $M > 1$, there exists a constant $C > 0$ such that for all $y, z \in B_{1-\varepsilon}^+$ with $|y - z| \leq M(y_d \wedge z_d)$.*

$$G^{B_1^+}(y, z) \geq C|y - z|^{-d+\alpha}.$$

By Theorems 1.1 and 2.3 we have for any $r > 0$ and $x \in \mathbb{R}_+^d$ with $x_d < r/2$,

$$\mathbb{P}_x(Y_{\tau_{D_{\tilde{x}}(r,r)}} \in D_{\tilde{x}}(r, 4r) \setminus D_{\tilde{x}}(r, 3r)) \geq cx^{\alpha-1}.$$

Using this and Lemma 6.4, the proofs of next two lemmas are identical to those of [7, Lemmas 5.2 and 5.3].

Lemma 6.5. *For every $\varepsilon \in (0, 1/4)$ and $M, N > 1$, there exists a constant $C > 0$ such that for all $x, z \in B_{1-\varepsilon}^+$ with $x_d \leq z_d$ satisfying $x_d/N \leq |x - z| \leq Mz_d$, it holds that*

$$G^{B_1^+}(x, z) \geq Cx_d^{\alpha-1}|x - z|^{-d+1}.$$

Lemma 6.6. *For every $\varepsilon \in (0, 1/4)$ and $M \geq 40/\varepsilon$, there exists a constant $C > 0$ such that for all $x, z \in B_{1-\varepsilon}^+$ with $x_d \leq z_d$ satisfying $|x - z| \geq Mz_d$, it holds that*

$$G^{B_1^+}(x, z) \geq Cx_d^{\alpha-1}z_d^{\alpha-1}|x - z|^{-d-\alpha+2}.$$

Combining the above results with scaling, we get

Theorem 6.7. *For any $\varepsilon \in (0, 1/4)$, there exists a constant $C > 0$ such that for all $w \in \partial \mathbb{R}_+^d$, $R > 0$ and $x, y \in B(w, (1 - \varepsilon)R) \cap \mathbb{R}_+^d$, it holds that*

$$G^{B(w,R) \cap \mathbb{R}_+^d}(x, y) \geq C \left(\frac{x_d}{|x - y|} \wedge 1 \right)^{\alpha-1} \left(\frac{y_d}{|x - y|} \wedge 1 \right)^{\alpha-1} \frac{1}{|x - y|^{d-\alpha}}.$$

Proposition 6.8. *For any $\tilde{w} \in \mathbb{R}^{d-1}$, any Borel set D satisfying $D_{\tilde{w}}(R/2, R/2) \subset D \subset D_{\tilde{w}}(R, R)$ and any $x = (\tilde{w}, x_d)$ with $x_d \leq R/10$,*

$$\mathbb{E}_x \int_0^{\tau_D} (Y_t^d)^\gamma dt = \int_D G^D(x, y) y_d^\gamma dy \asymp \begin{cases} R^{\gamma+1} x_d^{\alpha-1}, & \gamma > -1, \\ x_d^{\alpha-1} \log(R/x_d), & \gamma = -1, \\ x_d^{\alpha+\gamma}, & -\alpha < \gamma < -1, \end{cases} \quad (6.6)$$

where the comparison constant is independent of $\tilde{w} \in \mathbb{R}^{d-1}$, D , R and x .

Proof. Let D be a Borel set satisfying $D(R/2, R/2) \subset D \subset D(R, R)$. By Theorems 1.2 and 6.7, we have that for all $x \in D$

$$\begin{aligned} \int_D G^D(x, y) y_d^\gamma dy &\leq \int_D G(x, y) y_d^\gamma dy \\ &\leq c_1 \int_{D(R,R)} \left(\frac{x_d}{|x - y|} \wedge 1 \right)^{\alpha-1} \left(\frac{y_d}{|x - y|} \wedge 1 \right)^{\alpha-1} \frac{y_d^\gamma dy}{|x - y|^{d-\alpha}}, \end{aligned} \quad (6.7)$$

and for $x = (\tilde{0}, x_d)$ with $x_d \leq R/10$

$$\begin{aligned} \int_D G^D(x, y) y_d^\gamma dy &\geq \int_{B_{R/2}^+} y_d^\gamma G^{B_{R/2}^+}(x, y) dy \geq c_2 \int_{D(R/5, R/5)} y_d^\gamma G^{B_{R/2}^+}(x, y) dy \\ &\geq \int_{D(R/5, R/5)} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \left(\frac{y_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \frac{y_d^\gamma dy}{|x-y|^{d-\alpha}}. \end{aligned} \quad (6.8)$$

We now apply [1, Lemma 3.3 and Theorem 3.4] and their proofs to (6.7) and (6.8) and get (6.6) \square

Corollary 6.9. *For all $x \in \mathbb{R}_+^d$,*

$$\mathbb{E}_x \int_0^\zeta (Y_t^d)^\gamma dt = \int_{\mathbb{R}_+^d} G(x, y) y_d^\gamma dy \asymp \begin{cases} \infty & \gamma \geq -1 \text{ or } \gamma \leq -\alpha, \\ x_d^{\alpha+\gamma}, & -\alpha < \gamma < -1. \end{cases}$$

In particular, for all $x \in \mathbb{R}_+^d$, $\mathbb{E}_x[\zeta] = \infty$.

Proof. When $\gamma > -\alpha$, the result follows by letting $R \rightarrow \infty$ in Proposition 6.8. If $\gamma \leq -\alpha$, by Theorems 1.2, for all $x \in \mathbb{R}_+^d$

$$\begin{aligned} \int_{\mathbb{R}_+^d} G(x, y) y_d^\gamma dy &\geq c_1 \int_{D_{\tilde{x}}(x_d, x_d/2)} \left(\frac{x_d}{|x-y|} \right)^{\alpha-1} \left(\frac{y_d}{|x-y|} \right)^{\alpha-1} \frac{y_d^\gamma dy}{|x-y|^{d-\alpha}} \\ &= c_1 x_d^{\alpha-1} \int_{D_{\tilde{x}}(x_d, x_d/2)} \frac{y_d^{\gamma+\alpha-1} dy}{|x-y|^{d+\alpha-2}} \geq c_2 x_d^{1-d} \int_{D_{\tilde{x}}(x_d, x_d/2)} y_d^{\gamma+\alpha-1} dy = \infty. \end{aligned}$$

\square

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