Uniform Boundary Harnack Principle for Rotationally Symmetric Lévy processes in General Open Sets

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Abstract In this paper we prove the uniform boundary Harnack principle in general open sets for harmonic functions with respect to a large class of rotationally symmetric purely discontinuous Lévy processes.

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1 Introduction

The boundary Harnack principle for classical harmonic functions is a very deep result in potential theory and has many important applications in probability theory and analysis.

In the late nineties Bogdan [3] established the boundary Harnack principle for harmonic functions of rotationally symmetric α-stable processes, $\alpha \in (0, 2)$, in Lipschitz domains. This was the first time that the boundary Harnack principle was established for harmonic functions with respect to non-local operators (or, equivalently, discontinuous Markov processes). Since then the result has been generalized in various directions. In [18] Song and Wu extended the boundary Harnack principle to harmonic functions with respect to rotationally symmetric stable processes in κ-fat open sets, with the constant depending on the local geometry near the boundary. The definitive result in the case of rotationally symmetric stable processes was obtained in [4] by Bogdan, Kulczycki and Kwaśnicki who established the boundary Harnack principle in arbitrary opens sets with the constant not depending on the open set itself. This type of result is known as the uniform boundary Harnack principle. Note that the uniform boundary Harnack principle is not true for Brownian motion.

In another direction, the boundary Harnack principle has been generalized to different classes of discontinuous processes. In [8] the boundary Harnack principle was established for harmonic functions with respect to a wide class of purely discontinuous subordinate Brownian motions in κ-fat open sets, with an extension obtained in [10]. In [11] (see also, [6, 9]) the boundary

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Harnack inequality was established for harmonic functions of subordinate Brownian motions with Gaussian components.

The purpose of this paper is to generalize the main results from [4, 8, 10] and prove the uniform boundary Harnack principle for harmonic functions with respect to a large class of rotationally symmetric purely discontinuous Lévy processes in arbitrary open sets. The class of processes treated in this paper is larger than the class of processes treated in [8, 10]. The processes considered in this paper need not be subordinate Brownian motions. Even when restricted to subordinate Brownian motions, the assumptions on the subordinate Brownian motions in this paper are slightly weaker than those in [8, 10].

To be more precise, let $S = (S_t : t \geqslant 0)$ be a subordinator with Laplace exponent ϕ. We assume that ϕ is a complete Bernstein function satisfying the following *upper and lower scaling conditions* (see [22]):

(H): There exist constants $\delta_1, \delta_2 \in (0,1)$, $a_1, a_2 > 0$ and $R_0 > 0$ such that

$$\text{(LSC)} \qquad \phi(\lambda r) \geqslant a_1 \lambda^{\delta_1} \phi(r), \qquad \lambda \geqslant 1, r \geqslant 1/R_0^2$$

$$\text{(USC)} \qquad \phi(\lambda r) \leqslant a_2 \lambda^{\delta_2} \phi(r), \qquad \lambda \geqslant 1, r \geqslant 1/R_0^2.$$

Note that it follows from (USC) that ϕ has no drift.

Let $W = (W_t : t \geqslant 0)$ be a Brownian motion in $\mathbb{R}^d$, $d \geqslant 1$, independent of the subordinator S. The subordinate Brownian motion $Y = (Y_t : t \geqslant 0)$ is defined by $Y_t := W_{S_t}$. The Lévy measure of the process Y has a density given by $J(x) = j(|x|)$ where

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt, \qquad r > 0 \tag{1.1}$$

and $\mu(t)$ is the Lévy density of S. Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$.

We will assume that X is a purely discontinuous rotationally symmetric Lévy process with Lévy exponent $\Psi(\xi)$. Because of rotational symmetry, the function Ψ depends on $|\xi|$ only, and by a slight abuse of notation we write $\Psi(\xi) = \Psi(|\xi|)$. We further assume that the Lévy measure of X has a density J_X. Then

$$\mathbb{E}_x \left[e^{i\xi \cdot (X_t - X_0)} \right] = e^{-t\Psi(|\xi|)}, \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

with

$$\Psi(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) J_X(y) dy. \tag{1.2}$$

We assume that J_X is continuous on $\mathbb{R}^d \setminus \{0\}$ and that there is a constant $\gamma > 1$ such that

$$\gamma^{-1} j(|y|) \leqslant J_X(y) \leqslant \gamma j(|y|), \quad \text{for all } y \in \mathbb{R}^d. \tag{1.3}$$

Clearly (1.3) implies that

$$\gamma^{-1} \phi(|\xi|^2) \leqslant \Psi(|\xi|) \leqslant \gamma \phi(|\xi|^2), \quad \text{for all } \xi \in \mathbb{R}^d. \tag{1.4}$$

For a Greenian open set $D \subset \mathbb{R}^d$, we will use K_D to denote the Poisson kernel of X in $D \times \overline{D}^c$ (see (4.6) below). The goal of this paper is to establish the following result:

Theorem 1.1 *Let X be a purely discontinuous rotationally symmetric Lévy process with a continuous Lévy density J_X satisfying* (1.3) *where the complete Bernstein function ϕ satisfies* **(H)**. *There exists a constant $c = c(\phi, \gamma, d) > 0$ such that*

(i) For every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r \in (0,1)$ and for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to X and vanish in $D^c \cap B(z_0, r)$, we have

$$\frac{u(x)}{v(x)} \leqslant c\,\frac{u(y)}{v(y)}$$

for all $x, y \in D \cap B(z_0, r/2)$.

(ii) For every $z_0 \in \mathbb{R}^d$, every Greenian open set $D \subset \mathbb{R}^d$, every $r \in (0,1)$, we have

$$K_D(x_1, y_1)K_D(x_2, y_2) \leqslant cK_D(x_1, y_2)K_D(x_2, y_1)$$

for all $x_1, x_2 \in D \cap B(z_0, r/2)$ and all $y_1, y_2 \in \overline{D}^c \cap B(z_0, r)^c$.

The proof of the above theorem uses some results developed in [10] and several ideas from [4]. In the next section we recall some necessary definitions and results from [10]. In Section 3 we prove several results about one-dimensional symmetric Lévy processes that will be needed in the proof of Theorem 1.1. In Section 4, we present some estimates on the Poisson kernel K_D that are essential for the proof of Theorem 1.1. The proof of Theorem 1.1 is given in Section 5, where we also give an approximate factorization of the Poisson kernel, see Corollary 5.6. In the last section we relate the assumption **(H)** with the class OR of O-regularly varying functions and sketch the construction of an example of a complete Bernstein function which satisfies **(H)** but not the assumptions in [10].

At the meeting "Foundations of Stochastic Analysis" held in Banff from September 18 to 23, 2011, M. Kwaśnicki announced that, in a forthcoming joint paper with K. Bogdan and T. Kumagai, they have obtained a version of the boundary Harnack principle for Hunt processes in metric measure spaces under rather general conditions.

In this paper we always assume $d \geqslant 1$. We use the following convention: The value of the constant C will remain the same throughout this paper, while $c, c_1, c_2, \cdots$ stand for constants whose values are unimportant and which may change from location to location. The dependence of the lower case constants on the dimension d will not be mentioned explicitly. The labeling of the constants $c_1, c_2, \cdots$ starts anew in the proof of each result. The notation $f(t) \asymp g(t)$, $t \to 0$ (respectively $f(t) \asymp g(t)$, $t \to \infty$) means that the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to 0$ (respectively $t \to \infty$).

2 Preliminaries

Suppose that $S = (S_t : t \geqslant 0)$ is a subordinator with Laplace exponent ϕ, that is, S is a nonnegative Lévy process with $S_0 = 0$ and

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \qquad \forall\ t, \lambda > 0.$$

The function ϕ can be written in the form

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt) \tag{2.1}$$

where $b \geqslant 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$. b is called the drift of the subordinator and μ the Lévy measure of the subordinator. The function ϕ is a Bernstein function, i.e., it is C^∞, positive and $(-1)^{n-1}D^n\phi \geqslant 0$ for all $n \geqslant 1$.

Note that, by using (2.1) and the elementary inequality $1 - e^{-ty} \leqslant t(1 - e^{-y})$ valid for all $t \geqslant 1$ and all $y > 0$, we see that the Bernstein function ϕ satisfies

$$\phi(t\lambda) \leqslant \lambda\phi(t) \qquad \text{for all } \lambda \geqslant 1, t > 0. \tag{2.2}$$

In this paper we will always assume that ϕ is a complete Bernstein function, that is, the Lévy measure μ of S has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \geqslant 0$ for every non-negative integer n. For basic results on complete Bernstein functions, we refer our readers to [17]. It follows from [10, Lemma 2.1] that there exists $c > 1$ such that

$$\mu(t) \leqslant c\mu(t + 1), \qquad t > 1. \tag{2.3}$$

The next result will be used to obtain the asymptotic behavior of $\mu(t)$ near the origin.

Proposition 2.1 ([22, Theorem 7]) *Suppose that w is a completely monotone function given by $w(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$, where f is a strictly positive decreasing function. Then*

$$f(t) \leqslant \left(1 - e^{-1}\right)^{-1} t^{-1} w(t^{-1}), \quad t > 0.$$

If, furthermore, there exist $\delta \in (0, 1)$ and $a, t_0 > 0$ such that

$$w(\lambda t) \leqslant a\lambda^{-\delta} w(t), \quad \lambda \geqslant 1, t \geqslant 1/t_0, \tag{2.4}$$

then there exists $c = c(f, a, t_0, \delta) > 0$ such that

$$f(t) \geqslant c t^{-1} w(t^{-1}), \quad t \leqslant t_0.$$

From now on we will always assume that the Laplace exponent ϕ of S is a complete Bernstein function satisfying **(H)**.

Theorem 2.2 *For every $M > 0$, there exists $c = c(M, \phi) > 1$ such that the Lévy density μ of S satisfies*

$$c^{-1} t^{-1} \phi(t^{-1}) \leqslant \mu(t) \leqslant c\, t^{-1} \phi(t^{-1}) \quad \text{and} \quad c^{-1} \phi(t^{-1}) \leqslant \mu(t, \infty) \leqslant c\, \phi(t^{-1}), \quad \forall t \leqslant M \tag{2.5}$$

where $\mu(t, \infty) = \int_t^\infty \mu(s)\, ds$ is the tail of the Lévy measure μ.

Proof. Let $w(\lambda) := \lambda^{-1}\phi(\lambda) = \int_0^\infty e^{-\lambda t} \mu(t, \infty)\, dt$. The upper scaling condition (USC) implies that w satisfies (2.4) with $\delta = 1 - \delta_2$ and $t_0 = R_0^2$. Hence by Proposition 2.1, there exists a constant $c_1 > 1$ such that

$$c_1^{-1} t^{-1} w(t^{-1}) \leqslant \mu(t, \infty) \leqslant c_1 t^{-1} w(t^{-1})\,, \qquad t \leqslant R_0^2\,,$$

which immediately implies

$$c_1^{-1}\phi(t^{-1}) \leqslant \mu(t,\infty) \leqslant c_1\phi(t^{-1})\,, \quad t \leqslant R_0^2\,. \tag{2.6}$$

We proceed to prove the first inequality. Since $\mu(t/2,\infty) \geqslant \int_{t/2}^t \mu(s)\,ds \geqslant (t/2)\mu(t)$ by (2.2) and (2.6), for all $t \in (0, R_0^2]$,

$$\mu(t) \leqslant 2t^{-1}\mu(t/2,\infty) \leqslant 2c_1 t^{-1}\phi((t/2)^{-1}) \leqslant 4c_1 t^{-1}\phi(t^{-1}).$$

Using (LSC) we get that for every $\lambda \geqslant 1$

$$\phi(s^{-1}) = \phi(\lambda(\lambda s)^{-1}) \geqslant a_1\lambda^{\delta_1}\phi((\lambda s)^{-1})\,, \quad s \leqslant \frac{R_0^2}{\lambda}\,. \tag{2.7}$$

Fix $\lambda_1 := 2^{1/\delta_1}((c_1^2 a_1^{-1}) \vee 1)^{1/\delta_1} \geqslant 1$. Then, by (2.6) and (2.7), for $s \leqslant (R_0^2 \wedge 1)/\lambda_1$,

$$\mu(\lambda_1 s,\infty) \leqslant c_1\phi((\lambda_1 s)^{-1}) \leqslant c_1 a_1^{-1}\lambda_1{}^{-\delta_1}\phi(s^{-1}) \leqslant c_1^2 a_1^{-1}\lambda_1{}^{-\delta_1}\mu(s,\infty) \leqslant \frac{1}{2}\mu(s,\infty)$$

by our choice of λ_1. Further,

$$(\lambda_1 - 1)s\mu(s) \geqslant \int_s^{\lambda_1 s} \mu(t)\,dt = \mu(s,\infty) - \mu(\lambda_1 s,\infty) \geqslant \mu(s,\infty) - \frac{1}{2}\mu(s,\infty) = \frac{1}{2}\mu(s,\infty)\,.$$

This implies that for all $t \leqslant (R_0^2 \wedge 1)/\lambda_1$,

$$\mu(t) \geqslant \frac{1}{2(\lambda_1 - 1)}\,t^{-1}\mu(t,\infty) \geqslant \frac{1}{2c_1(\lambda_1 - 1)}\,t^{-1}\phi(t^{-1}).$$

The case $(R_0^2 \wedge 1)/\lambda_1 \leqslant t \leqslant M$ is clear since the functions we consider are all positive and continuous on $(0,\infty)$. The proof is now complete. □

A consequence of (2.5) and (USC) is that for any $K > 0$ there exists $c = c(K) > 1$ such that

$$\mu(t) \leqslant c\mu(2t), \qquad t \in (0, K). \tag{2.8}$$

Suppose that $W = (W_t : t \geqslant 0)$ is a Brownian motion in $\mathbb{R}^d$ with

$$\mathbb{E}\left[e^{i\xi\cdot(W_t - W_0)}\right] = e^{-t|\xi|^2}, \qquad \forall\, \xi \in \mathbb{R}^d, t > 0\,,$$

and that W is independent of S. The process $Y = (Y_t : t \geqslant 0)$ defined by $Y_t = W_{S_t}$ is called a subordinate Brownian motion. It is a rotationally symmetric Lévy process with characteristic exponent $\Phi_Y(\xi) = \phi(|\xi|^2)$, $\xi \in \mathbb{R}^d$. Recall that the Lévy measure of Y has a density $J(x) = j(|x|)$ with j given by (1.1) and that $r \mapsto j(r)$ is continuous and decreasing on $(0,\infty)$.

The following theorem establishes the asymptotic behavior of j near the origin.

Theorem 2.3 *It holds that*

$$j(|x|) \asymp \frac{\phi(|x|^{-2})}{|x|^d} \qquad |x| \to 0. \tag{2.9}$$

Proof. By (2.2),

$$\frac{\phi(v)}{v} \leqslant \frac{\phi(u)}{u}, \quad 0 < u \leqslant v\,. \tag{2.10}$$

To obtain the upper bound in (2.9) we write

$$j(r) = \int_0^{r^2} (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t)\, dt + \int_{r^2}^{\infty} (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t)\, dt := J_1 + J_2\,.$$

For $r \leqslant 1$, by using (2.5) in the first inequality and (2.10) in the second, we have

$$\begin{aligned} J_1 &\leqslant c_1 \int_0^{r^2} (4\pi t)^{-d/2} e^{-r^2/(4t)} t^{-1} \phi(t^{-1})\, dt \leqslant c_1 \int_0^{r^2} (4\pi t)^{-d/2} e^{-r^2/(4t)} t^{-1} t^{-1} \frac{\phi(r^{-2})}{r^{-2}}\, dt \\ &\leqslant c_2 r^2 \phi(r^{-2}) \int_0^{\infty} t^{-d/2-2} e^{-r^2/(4t)}\, dt = c_3 r^2 \phi(r^{-2}) r^{-d-2} = c_3 r^{-d} \phi(r^{-2})\,. \end{aligned}$$

Next,

$$\begin{aligned} J_2 &\leqslant c_4 \int_{r^2}^{\infty} t^{-d/2} \mu(t)\, dt = c_4 \int_{r^2}^{\infty} \left(\frac{d}{2} \int_t^{\infty} s^{-d/2-1}\, ds \right) \mu(t)\, dt \\ &= c_5 \int_{r^2}^{\infty} \left(\int_{r^2}^{s} \mu(t)\, dt \right) s^{-d/2-1}\, ds \leqslant c_5 \mu(r^2, \infty) \int_{r^2}^{\infty} s^{-d/2-1}\, ds \leqslant c_6 r^{-d} \phi(r^{-2}) \end{aligned}$$

where the last inequality follows from Theorem 2.2. The last two displays show that $j(r) \leqslant c_7 r^{-d} \phi(r^{-2})$, for r small. To prove the converse inequality, we also use Theorem 2.2 and get that for $r \leqslant 1$,

$$\begin{aligned} j(r) &\geqslant \int_0^1 (4\pi s)^{-d/2} e^{-r^2/(4s)} \mu(s)\, ds = (4\pi)^{-d/2} \int_0^{1/r^2} (tr^2)^{-d/2} e^{-r^2/(4tr^2)} \mu(r^2 t) r^2\, dt \\ &\geqslant c_8 r^{2-d} \int_0^1 t^{-d/2} e^{-1/(4t)} \mu(r^2 t)\, dt \geqslant c_9 r^{2-d} \int_0^1 t^{-d/2} e^{-1/(4t)} r^{-2} t^{-1} \phi(r^{-2} t^{-1})\, dt \\ &\geqslant c_{10} r^{-d} \int_0^1 t^{-d/2-1} e^{-1/(4t)} \phi(r^{-2})\, dt = c_{11} r^{-d} \phi(r^{-2})\,, \end{aligned}$$

where the last inequality follows because $r^{-2} t^{-1} \geqslant r^{-2}$ and ϕ is increasing. □

Using (2.3) and (2.8), we can easily show (see [10, Proposition 3.5] or [15, Lemma 4.2]) that

(1) For any $M > 0$, there exists $c = c(M, \phi) > 0$ such that

$$j(r) \leqslant c j(2r), \qquad \forall r \in (0, M)\,. \tag{2.11}$$

(2) There exists $c = c(\phi) > 0$ such that

$$j(r) \leqslant c j(r+1), \qquad \forall r > 1. \tag{2.12}$$

3 Some results on symmetric Lévy process in $\mathbb{R}$

In this section we assume that $d = 1$ and denote the process X by Z. That is, $(Z_t, \mathbb{P}_x)$ is a purely discontinuous symmetric Lévy process in $\mathbb{R}$ such that

$$\mathbb{E}_x\left[e^{i\xi\cdot(Z_t-Z_0)}\right] = e^{-t\Psi(|\theta|)}, \qquad \text{for every } x \in \mathbb{R} \text{ and } \theta \in \mathbb{R}.$$

We assume that (1.4) holds with a complete Bernstein function ϕ satisfying **(H)**, that is, $\gamma^{-1}\phi(\theta^2) \leqslant \Psi(|\theta|) \leqslant \gamma\phi(\theta^2)$ for all $\theta \in \mathbb{R}$, but we do not assume the assumption (1.3) concerning the Lévy measure of Z. As a consequence of **(H)**, (1.4) and [16, Proposition 28.1] we know that for any $t > 0$, Z_t has a density $p_t(x, y) = p_t(y - x)$ which is smooth.

Let χ (κ, respectively) be the Laplace exponent of the ladder height process of Z (Y, respectively). It follows from [7, Corollary 9.7] that

$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\Psi(\lambda\theta))}{1+\theta^2}d\theta\right), \quad \kappa(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\phi(\lambda^2\theta^2))}{1+\theta^2}d\theta\right), \quad \forall\lambda > 0. \tag{3.1}$$

It follows immediately from these two equations and (1.4) that $\gamma^{-1/2}\kappa(\lambda) \leqslant \chi(\lambda) \leqslant \gamma^{1/2}\kappa(\lambda)$, i.e., that χ is comparable to κ. From **(H)** and [10, Propsoition 3.7] or [12, Proposition 2.1] we conclude that the ladder height process of Y has no drift and is not compound Poisson, thus the ladder height process of Z has no drift and is not compound Poisson. Thus the process Z does not creep upwards. Since Z is symmetric, we know that Z also does not creep downwards. Thus if, for any $a \in \mathbb{R}$, we define

$$\tau_a = \inf\{t > 0 : Z_t < a\}, \quad \sigma_a = \inf\{t > 0 : Z_t \leqslant a\},$$

then we have

$$\mathbb{P}_x(\tau_a = \sigma_a) = 1, \quad x > a\,. \tag{3.2}$$

Let $Z^{(0,\infty)}$ be the process Z killed upon exiting $(0, \infty)$. Since Z has a smooth density, we can easily show that $Z^{(0,\infty)}$ has a density $p^{(0,\infty)}(t, x, y)$. Let $G^{(0,\infty)}(x, y) = \int_0^\infty p^{(0,\infty)}(t, x, y)\,dt$ be the Green function of $Z^{(0,\infty)}$. If we use V to denote the potential measure of the ladder height process of Z, then using the symmetry of Z and [1, Theorem 20, page 176] we have that for any $x \in (0, \infty)$ and any nonnegative function f on $(0, \infty)$

$$\int_0^\infty f(y)G^{(0,\infty)}(x, y)dy = \int_0^\infty V(dy)\int_0^x V(dz)f(x + y - z). \tag{3.3}$$

In the following, we will also use V to denote the renewal function of the ladder height process of Z: $V(t) := V((0, t))$. For any $r > 0$, let $G^{(0,r)}$ be the Green function of Z in $(0, r)$. Then we have the following result.

Proposition 3.1 *For all $r > 0$ and all $x \in (0, r)$*

$$\int_0^r G^{(0,r)}(x, y)\,dy \leqslant 2V(r)\big(V(x) \wedge V(r - x)\big)\,.$$

Proof. Since
$$\int_0^r G^{(0,r)}(x,y)dy \leqslant \int_0^r G^{(0,\infty)}(x,y)dy,$$
we can apply (3.3) with f being the indicator function of $(0,r)$ to immediately get the conclusion of the proposition. □

The following result will play an important role in this paper.

Proposition 3.2 *There exists a constant $c = c(\gamma) > 1$ such that for all $r > 0$*
$$c^{-1}\frac{1}{\sqrt{\phi(r^{-2})}} \leqslant V(r) \leqslant c\frac{1}{\sqrt{\phi(r^{-2})}}.$$

Proof. The proof is a simple modification of [12, Theorem 4.4]. By [10, Proposition 3.7] (or [12, Proposition 2.1]) we have that $c_1^{-1}\sqrt{\phi(\lambda)} \leqslant \kappa(\lambda) \leqslant c_1\sqrt{\phi(\lambda)}$ for a constant $c_1 > 1$. Hence $c_2^{-1}\sqrt{\phi(\lambda)} \leqslant \chi(\lambda) \leqslant c_2\sqrt{\phi(\lambda)}$, $c_2 > 1$, implying that
$$c_3^{-1}\frac{1}{r\sqrt{\phi(r^2)}} \leqslant \mathcal{L}V(r) \leqslant c_3\frac{1}{r\sqrt{\phi(r^2)}}$$
(where $\mathcal{L}V(r)$ denotes the Laplace transform of the *function* V). The claim now follows by repeating the second part of the proof of [12, Theorem 4.4]. □

4 Poisson Kernel Estimates

Recall that Y is a subordinate Brownian motion in $\mathbb{R}^d$ with Lévy exponent $\phi(|\xi|^2)$, X is a purely discontinuous rotationally symmetric Lévy process in $\mathbb{R}^d$ with Lévy exponent $\Psi(\xi) = \Psi(|\xi|)$ and Lévy density J_X, i.e.,
$$\mathbb{E}_x\left[e^{i\xi\cdot(X_t-X_0)}\right] = e^{-t\Psi(|\xi|)}, \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d$$
and $\Psi(|\xi|) = \int_{\mathbb{R}^d}(1-\cos(\xi\cdot y))J_X(y)dy$. Recall that we assume that (1.3) holds. As a consequence of **(H)**, (1.4) and [16, Proposition 28.1] we know that for any $t > 0$, X_t has a density $p_t(x,y) = p_t(y-x)$ which is smooth.

The infinitesimal generator $\mathbf{L}$ of X is given by
$$\mathbf{L}f(x) = \int_{\mathbb{R}^d}\left(f(x+y) - f(x) - y\cdot\nabla f(x)\mathbf{1}_{\{|y|\leqslant 1\}}\right) J_X(y)dy \tag{4.1}$$
for $f \in C_b^2(\mathbb{R}^d)$. Moreover, for every $f \in C_b^2(\mathbb{R}^d)$, $f(X_t) - f(X_0) - \int_0^t \mathbf{L}f(X_s)\,ds$ is a $\mathbb{P}_x$-martingale for every $x \in \mathbb{R}^d$.

First we record several inequalities that will be needed in the remainder of the paper.

Lemma 4.1 *There exists a constant $c = c(\phi) > 0$ such that*
$$\int_0^{\lambda^{-1}} \phi(r^{-2})^{1/2}dr \leqslant c\,\lambda^{-1}\phi(\lambda^2)^{1/2}, \quad \forall\lambda \geqslant 1/R_0, \tag{4.2}$$

$$\lambda^2 \int_0^{\lambda^{-1}} r\phi(r^{-2})dr + \int_{\lambda^{-1}}^{R_0} r^{-1}\phi(r^{-2})dr \;\leqslant\; c\,\phi(\lambda^2), \quad \forall \lambda \geqslant 1/R_0, \tag{4.3}$$

and

$$\lambda^2 \int_0^{\lambda^{-1}} r\phi(r^{-2})^{1/2}dr + \int_{\lambda^{-1}}^{R_0} r^{-1}\phi(r^{-2})^{1/2}dr \;\leqslant\; c\,\phi(\lambda^2)^{1/2}, \quad \forall \lambda \geqslant 1/R_0. \tag{4.4}$$

Proof. Assume $\lambda \geqslant 1/R_0$. By (USC), $\phi(r^{-2}) \leqslant c_1 r^{-2\delta_2}\lambda^{-2\delta_2}\phi(\lambda^2)$ for $r \leqslant \lambda^{-1}$. On the other hand, by (LSC), $\phi(r^{-2}) \leqslant c_2 r^{-2\delta_1}\lambda^{-2\delta_1}\phi(\lambda^2)$ for $\lambda^{-1} \leqslant r \leqslant R_0$. Thus

$$\int_0^{\lambda^{-1}} \phi(r^{-2})^{1/2}dr \leqslant c_1^{1/2}\phi(\lambda^2)^{1/2}\lambda^{-\delta_2}\int_0^{\lambda^{-1}} r^{-\delta_2}dr \leqslant c_3\lambda^{-1}\phi(\lambda^2)^{1/2}\frac{1}{1-\delta_2},$$

$$\begin{aligned}
&\lambda^2 \int_0^{\lambda^{-1}} r\phi(r^{-2})dr + \int_{\lambda^{-1}}^{R_0} r^{-1}\phi(r^{-2})dr \\
&\leqslant c_4\phi(\lambda^2)\left(\lambda^{2-2\delta_2}\int_0^{\lambda^{-1}} r^{1-2\delta_2}dr + \lambda^{-2\delta_1}\int_{\lambda^{-1}}^{R_0} r^{-1-2\delta_1}dr\right) \;\leqslant\; c_5\phi(\lambda^2)\left(\frac{1}{2(1-\delta_2)} + \frac{1}{2\delta_1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda^2 \int_0^{\lambda^{-1}} r\phi(r^{-2})^{1/2}dr + \int_{\lambda^{-1}}^{R_0} r^{-1}\phi(r^{-2})^{1/2}dr \\
&\leqslant c_6\phi(\lambda^2)^{1/2}\left(\lambda^{2-\delta_2}\int_0^{\lambda^{-1}} r^{1-\delta_2}dr + \lambda^{-\delta_1}\int_{\lambda^{-1}}^{R_0} r^{-1-\delta_1}dr\right) \;\leqslant\; c_7\phi(\lambda^2)^{1/2}\left(\frac{1}{2-\delta_2} + \frac{1}{\delta_1}\right).
\end{aligned}$$

□

Lemma 4.2 *There exists a constant $c = c(\phi, \gamma) > 0$ such that for every $f \in C_b^2(\mathbb{R}^d)$ with $0 \leqslant f \leqslant 1$,*

$$|\mathbf{L}f_r(x)| \leqslant c\,\phi(r^{-2})\left(2 + \frac{1}{2}\sup_y \sum_{j,k} |(\partial^2/\partial y_j \partial y_k) f(y)|\right) + b_0, \quad \textit{for every } x \in \mathbb{R}^d, r \leqslant R_0$$

where $f_r(y) := f(y/r)$ and $b_0 := 2\int_{|z|>R_0} J_X(z)dz < \infty$.

Proof. Let $L_1 = \sup_y \sum_{j,k} |(\partial^2/\partial y_j \partial y_k) f(y)|$. Then $|f(z+y) - f(z) - y \cdot \nabla f(z)| \leqslant \frac{1}{2}L_1|y|^2$. For $r \in (0, R_0]$, let $f_r(y) = f(y/r)$. Then the following estimate is valid:

$$|f_r(z+y) - f_r(z) - y \cdot \nabla f_r(z)\mathbf{1}_{\{|y|\leqslant r\}}| \leqslant \frac{L_1}{2}\frac{|y|^2}{r^2}\mathbf{1}_{\{|y|\leqslant r\}} + 2 \cdot \mathbf{1}_{\{|y|\geqslant r\}}\ .$$

Now, by using **(H)**, (2.9) and (4.3), we get

$$
\begin{aligned}
|\mathbf{L} f_r(z)| &\leqslant \int_{\mathbb{R}^d} |f_r(z+y) - f_r(z) - y\cdot\nabla f_r(z)\mathbf{1}_{\{|y|\leqslant r\}}|\, J_X(y)dy \\
&\leqslant \frac{L_1}{2}\int_{\mathbb{R}^d} \mathbf{1}_{\{|y|\leqslant r\}}\frac{|y|^2}{r^2}J_X(y)dy + 2\int_{\mathbb{R}^d}\mathbf{1}_{\{r\leqslant|y|\leqslant R_0\}}J_X(y)dy \\
&\quad +2\int_{\mathbb{R}^d}\mathbf{1}_{\{|y|\geqslant R_0\}}J_X(y)dy \\
&\leqslant \frac{\gamma L_1}{2}\int_{\mathbb{R}^d}\mathbf{1}_{\{|y|\leqslant r\}}\frac{|y|^2}{r^2}j(|y|)dy + 2\gamma\int_{\mathbb{R}^d}\mathbf{1}_{\{r\leqslant|y|\leqslant R_0\}}j(|y|)dy \\
&\quad +2\int_{\mathbb{R}^d}\mathbf{1}_{\{|y|\geqslant R_0\}}J_X(y)dy \\
&\leqslant c\phi(r^{-2})\left(2+\frac{L_1}{2}\right) + 2\int_{\{|y|\geqslant R_0\}} J_X(y)dy\,,
\end{aligned}
$$

where the constant c is independent of $r \in (0, R_0]$. □

For any open set D, we use τ_D to denote the first exit time of D, i.e., $\tau_D = \inf\{t>0:\ X_t \notin D\}$.

Using Lemma 4.2, the proof of the next result is the same as those of [10, Lemmas 4.1 and 4.2]. Thus we skip the proof.

Lemma 4.3 *There exists a constant $c = c(\phi,\gamma) > 0$ such that for every $r \in (0,1]$, and every $x \in \mathbb{R}^d$,*

$$\inf_{z\in B(x,r/2)} \mathbb{E}_z\left[\tau_{B(x,r)}\right] \geqslant \frac{c}{\phi((r/2)^{-2})}.$$

The idea of the proof of the following proposition comes from [20].

Lemma 4.4 *There exists $c = c(\gamma) > 0$ such that for any $r \in (0,\infty)$ and $x_0 \in \mathbb{R}^d$,*

$$\mathbb{E}_x[\tau_{B(x_0,r)}] \leqslant c\left(\phi(r^{-2})\phi((r-|x-x_0|)^{-2})\right)^{-1/2} \qquad x \in B(x_0,r).$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. We fix $x \neq 0$ and put $Z_t = \frac{X_t\cdot x}{|x|}$. Then, using the fact that Ψ is a radial function, Z_t is a Lévy process on $\mathbb{R}$ with

$$\mathbb{E}[e^{i\theta Z_t}] = \mathbb{E}(e^{i\theta\frac{x}{|x|}\cdot X_t}) = e^{-t\Psi(\theta\frac{x}{|x|})} = e^{-t\Psi(\theta)}, \qquad \theta\in\mathbb{R}.$$

Clearly, $\gamma^{-1}\phi(\theta^2) \leqslant \Psi(\theta) \leqslant \gamma\phi(\theta^2)$. Thus Z_t is of the type of one-dimensional symmetric Lévy processes studied in Section 3.

It is easy to see that, if $X_t \in B(0,r)$, then $|Z_t| < r$, hence $\mathbb{E}_x[\tau_{B(0,r)}] \leqslant \mathbb{E}_{|x|}[\tilde\tau]$, where $\tilde\tau = \inf\{t>0 : |Z_t| \geqslant r\}$. Thus, applying Proposition 3.1, we obtain $\mathbb{E}_x[\tau_{B(0,r)}] \leqslant 2V(2r)V(r-|x|)$. Now, by Proposition 3.2 and **(H)**, we have proved the lemma. □

We now recall the definition of harmonic functions with respect to X.

Definition 4.5 *Let D be an open subset of $\mathbb{R}^d$. A nonnegative function u on $\mathbb{R}^d$ is said to be*

(1) *harmonic in D with respect to X if*

$$u(x) = \mathbb{E}_x\left[u(X_{\tau_B})\right], \qquad x \in B,$$

for every open set B whose closure is a compact subset of D;

(2) *regular harmonic in D with respect to X if for each $x \in D$,*

$$u(x) = \mathbb{E}_x\left[u(X_{\tau_D})\right].$$

Since our X satisfies [5, (1.6), **(UJS)**], by [5, Theorem 1.4] and using the standard chain argument one have the following form of Harnack inequality.

Theorem 4.6 *For every $a \in (0,1)$, there exists $c = c(a,\phi,\gamma) > 0$ such that for every $r \in (0,1)$, $x_0 \in \mathbb{R}^d$, and any function u which is nonnegative on $\mathbb{R}^d$ and harmonic with respect to X in $B(x_0, r)$, we have*

$$u(x) \leqslant c\, u(y), \quad \textit{for all } x, y \in B(x_0, ar).$$

Given an open set $D \subset \mathbb{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geqslant \tau_D(\omega)$, where ∂ is a cemetery state. A subset D of $\mathbb{R}^d$ is said to be Greenian (for X) if X^D is transient. When $d \geqslant 3$, any non-empty open set $D \subset \mathbb{R}^d$ is Greenian. An open set $D \subset \mathbb{R}^d$ is Greenian if and only if D^c is non-polar for X (or equivalently, has positive capacity with respect to X). In particular, every bounded open set is Greenian.

Since X has a smooth density, using the strong Markov property, it is standard to show that, for every Greeninan open set D, X_t^D has a density $p_D(t,x,y)$. For any Greeninan open set D in $\mathbb{R}^d$ let $G_D(x,y) = \int_0^\infty p_D(t,x,y)$ be the Green function of X^D. Using the Lévy system for X, we know that for every Greeninan open subset D and every $f \geqslant 0$ and $x \in D$,

$$\mathbb{E}_x\left[f(X_{\tau_D});\, X_{\tau_D -} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} \int_D G_D(x,z) J_X(z-y)\,dz f(y)\,dy. \tag{4.5}$$

We define the Poisson kernel

$$K_D(x,y) := \int_D G_D(x,z) J_X(z-y)\,dz, \qquad (x,y) \in D \times \overline{D}^c. \tag{4.6}$$

Thus (4.5) can be simply written as

$$\mathbb{E}_x\left[f(X_{\tau_D});\, X_{\tau_D -} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} K_D(x,y) f(y)\,dy.$$

Using continuity of J_X, one can easily check that $K_D(x,\cdot)$ is continuous on $\overline{D}^c$ for every $x \in D$.

Proposition 4.7 *There exist $c_1 = c_1(\phi,\gamma) > 0$ and $c_2 = c_2(\phi,\gamma) > 0$ such that for every $r \in (0,1]$ and $x_0 \in \mathbb{R}^d$,*

$$\begin{aligned} K_{B(x_0,r)}(x,y) &\leqslant c_1\, j(|y-x_0| - r)\left(\phi(r^{-2})\phi((r-|x-x_0|)^{-2})\right)^{-1/2} &(4.7)\\ &\leqslant c_1\, j(|y-x_0|-r)\phi(r^{-2})^{-1} &(4.8)\end{aligned}$$

for all $(x,y) \in B(x_0,r) \times \overline{B(x_0,r)}^c$ and

$$K_{B(x_0,r)}(x_0,y) \geqslant c_2\, j(|y-x_0|)\phi(r^{-2})^{-1}, \qquad \textit{for all } y \in \overline{B(x_0,r)}^c. \tag{4.9}$$

Proof. Using (1.3) and (2.11)–(2.12), the proof of (4.7) and (4.9) is exactly the same as that of [10, Proposition 4.10] (using **(H)**), while (4.8) follows from (4.7) and the fact that ϕ is increasing. □

The next result is a direct consequence of Theorem 4.6 and the continuity of $K_{B(x_0,r)}(x,\cdot)$ on $\overline{B(x_0,r)}^c$ for every $x \in B(x_0,r)$ (see [10, Proposition 1.4.11] for more details).

Proposition 4.8 *For every $a \in (0,1)$, there exists $c = c(\phi,\gamma,a) > 0$ such that for every $r \in (0,1]$, $x_0 \in \mathbb{R}^d$ and $x_1, x_2 \in B(x_0, ar)$,*

$$K_{B(x_0,r)}(x_1,y) \leqslant cK_{B(x_0,r)}(x_2,y), \qquad y \in \overline{B(x_0,r)}^c.$$

Proposition 4.9 *For every $a \in (0,1)$, there exists $c = c(\phi,\gamma,a) > 0$ such that for every $r \in (0,1]$ and $x_0 \in \mathbb{R}^d$,*

$$K_{B(x_0,r)}(x,y) \leqslant c\, r^{-d} \left(\frac{\phi((|y-x_0|-r)^{-2})}{\phi(r^{-2})} \right)^{1/2}$$

for all $x \in B(x_0,ar)$ and all y such that $r < |x_0-y| < 2r$.

Proof. By Proposition 4.8,

$$K_{B(x_0,r)}(x,y) \leqslant \frac{c_1}{r^d} \int_{B(x_0,ar)} K_{B(x_0,r)}(w,y)dw$$

for some constant $c_1 = c_1(\phi,\gamma,a) > 0$. Thus from Lemma 4.4, (4.6) and Theorem 2.3 we have that

$$\begin{aligned} K_{B(x_0,r)}(x,y) &\leqslant \frac{c_1}{r^d} \int_{B(x_0,r)} \int_{B(x_0,r)} G_{B(x_0,r)}(w,z) J_X(z-y)dzdw \\ &= \frac{c_1}{r^d} \int_{B(x_0,r)} \mathbb{E}_z[\tau_{B(x_0,r)}] J_X(z-y)dz \\ &\leqslant \frac{c_2}{r^d(\phi(r^{-2}))^{1/2}} \int_{B(x_0,r)} \frac{\phi(|z-y|^{-2})}{(\phi((r-|z-x_0|)^{-2}))^{1/2}} |z-y|^{-d}dz \end{aligned}$$

for some constant $c_2 = c_2(\phi,\gamma,a) > 0$. Since $r - |z-x_0| \leqslant |y-z|$, we have

$$\begin{aligned} K_{B(x_0,r)}(x,y) &\leqslant \frac{c_2}{r^d(\phi(r^{-2}))^{1/2}} \int_{B(x_0,r)} \frac{(\phi(|z-y|^{-2}))^{1/2}}{|z-y|^d} dz \\ &\leqslant \frac{c_2}{r^d(\phi(r^{-2}))^{1/2}} \int_{B(y,3r)\setminus B(y,|y-x_0|-r)} \frac{(\phi(|z-y|^{-2}))^{1/2}}{|z-y|^d} dz \\ &\leqslant \frac{c_3}{r^d(\phi(r^{-2}))^{1/2}} \int_{|y-x_0|-r}^{3r} \frac{\phi(s^{-2})^{1/2}}{s} ds. \end{aligned}$$

Now, using (4.4) in the last integral (considering the cases $r < R_0/3$ and $1 \geqslant r \geqslant R_0/3$ separately), we arrive at the conclusion of the proposition. □

Lemma 4.10 *For every $a \in (0,1)$, there exists a positive constant $c = c(\phi, \gamma, a) > 0$ such that for any $r \in (0,1)$ and any open set D with $D \subset B(0,r)$ we have*

$$\mathbb{P}_x\left(X_{\tau_D} \in B(0,r)^c\right) \leqslant c\,\phi(r^{-2}) \int_D G_D(x,y)dy, \qquad x \in D \cap B(0,ar)\,.$$

Proof. The proof of the lemma is similar to that of [10, Lemma 4.15]. Transience was used in the proof [10, Lemma 4.15] in order to derive equation [10, (4.29)]. By noting that [10, (4.29)] in the proof of [10, Lemma 4.15] follows immediately from Dynkin's formula, using our Lemma 4.2, we can follow the rest of the proof [10, Lemma 4.15] (which does not use transience) to get the conclusion of the lemma here. We omit the details. □

5 Uniform Boundary Harnack Principle

In this section, we give the proof of the main result of this paper. Let $A(x,a,b) := \{y \in \mathbb{R}^d : a \leqslant |y-x| < b\}$.

Lemma 5.1 *For every $p \in (0,1)$, there exists $c = c(\phi,\gamma,p) > 0$ such that for every $r \in (0,1)$,*

$$\int_{r(1+p)/2}^{|y|} K_{B(0,s)}(x,y)ds \leqslant c\,\frac{r}{\phi(r^{-2})} j(|y|) \qquad \forall x \in B(0,pr),\, y \in A(0, r(1+p)/2, r).$$

Proof. Let $0 < p < 1$ and $q = (1+p)/2$. Note that the functions $r \mapsto r^{-d+1}$ and $r \mapsto r^{-1}(\phi(r^{-2}))^{-1/2}$ are decreasing, see (2.10). Using Proposition 4.9 we get

$$\begin{aligned}\int_{qr}^{|y|} K_{B(0,s)}(x,y)ds &\leqslant c_1 \int_{qr}^{|y|} \frac{s^{-d}}{(\phi(s^{-2}))^{1/2}} (\phi((|y|-s)^{-2}))^{1/2} ds \\ &\leqslant c_2\, \frac{r^{-d}}{(\phi((qr)^{-2}))^{1/2}} \int_{qr}^{|y|} (\phi((|y|-s)^{-2}))^{1/2} ds\end{aligned}$$

for some constants $c_1(p,\phi) > 0$ and $c_2(p,\phi) > 0$. Note that by (4.2) (considering the cases $|y| - qr < R_0$ and $1 \geqslant |y| - qr \geqslant R_0$ separately) and the fact that $r \mapsto r(\phi(r^{-2}))^{1/2}$ is increasing,

$$\begin{aligned}\int_{qr}^{|y|} (\phi((|y|-s)^{-2}))^{1/2} ds &= \int_0^{|y|-qr} (\phi(s^{-2}))^{1/2} ds \\ &\leqslant c_3(|y|-qr)(\phi((|y|-qr)^{-2}))^{1/2} \leqslant c_3 r(\phi(r^{-2}))^{1/2}\end{aligned}$$

for some constant $c_3 > 0$. Thus, by **(H)**, Theorem 2.3 and the fact that $r \to j(r)$ is decreasing, we have

$$\int_{qr}^{|y|} K_{B(0,s)}(x,y)ds \leqslant \frac{c_4}{r^{d-1}} \leqslant c_5 \frac{r}{\phi(r^{-2})} j(r) \leqslant c_6 \frac{r}{\phi(r^{-2})} j(|y|).$$

□

From the strong Markov property, it is well known and easy to see that for all Greenian open sets U and D with $U \subset D$, $G_D(x,y) = G_U(x,y) + \mathbb{E}_x\left[G_D(X_{\tau_U}, y)\right]$ for every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. Thus, for all Greenian open sets U and D with $U \subset D$,

$$K_D(x,z) = K_U(x,z) + \mathbb{E}_x\left[K_D(X_{\tau_U}, z)\right], \qquad (x,z) \in U \times \overline{D}^c \tag{5.1}$$

and

$$\mathbb{E}_x[\tau_D] = \mathbb{E}_x[\tau_U] + \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_U}}[\tau_D]\right], \qquad x \in U. \tag{5.2}$$

Lemma 5.2 *For every $p \in (0,1)$, there exists $c = c(\phi, \gamma, p) > 0$ such that for every $r \in (0,1)$, for every $z_0 \in \mathbb{R}^d$, $U \subset B(z_0, r)$ and for any $(x,y) \in (U \cap B(z_0, pr)) \times B(z_0, r)^c$,*

$$K_U(x,y) \leqslant c\frac{1}{\phi(r^{-2})}\left(\int_{U \setminus B(z_0,(1+p)r/2)} j(|z-z_0|)K_U(z,y)dz + j(|y-z_0|)\right).$$

Proof. Without loss of generality, we assume $z_0 = 0$. Let $0 < p < 1$, $q_1 := (1+p)/2$ and $q_2 := (3+2p)/5$. For every $s \in [q_1 r, q_2 r]$ and $x \in U \cap B(0, pr)$, by (5.1) we have

$$\begin{aligned}
K_U(x,y) &= \mathbb{E}_x[K_U(X_{\tau_{U\cap B(0,s)}}, y)] + K_{U \cap B(0,s)}(x,y) \\
&= \int_{U \setminus B(0,s)} K_U(z,y) K_{U\cap B(0,s)}(x,z)dz + K_{U \cap B(0,s)}(x,y) \\
&\leqslant \int_{U \setminus B(0,s)} K_U(z,y) K_{B(0,s)}(x,z)dz + K_{B(0,s)}(x,y).
\end{aligned}$$

Thus

$$\begin{aligned}
K_U(x,y) \leqslant& \frac{1}{r(q_2-q_1)}\int_{q_1 r}^{q_2 r}\int_{U\setminus B(0,s)} K_{B(0,s)}(x,z)K_U(z,y)dzds \\
&+ \frac{1}{r(q_2-q_1)}\int_{q_1 r}^{q_2 r} K_{B(0,s)}(x,y)ds \\
=:& I + II.
\end{aligned}$$

By Tonelli's theorem, we have

$$\begin{aligned}
I &= \frac{10}{r(1-p)}\int_{q_1 r}^{q_2 r}\int_{\{z\in U; |z| \geqslant q_1 r\}} 1_{\{|z|\geqslant s\}} K_{B(0,s)}(x,z)K_U(z,y)dzds \\
&\leqslant \frac{10}{r(1-p)}\int_{(U\setminus B(0,q_1 r))}\left(\int_{q_1 r}^{|z|} K_{B(0,s)}(x,z)ds\right)K_U(z,y)dz.
\end{aligned}$$

Applying Lemma 5.1 to the inner integral above, we get that

$$I \leqslant \frac{c_1}{(1-p)}\frac{1}{\phi(r^{-2})}\int_{(U\setminus B(0,q_1 r))} j(|z|)K_U(z,y)dz. \tag{5.3}$$

One the other hand, for any $s \in [q_1 r, q_2 r]$, by Proposition 4.7,

$$K_{B(0,s)}(x,y) \leqslant c_2 j(|y|-s)\frac{1}{(\phi(s^{-2}))^{1/2}}\frac{1}{(\phi((s-|x|)^{-2}))^{1/2}}.$$

When $y \in A(0,r,4)$ we have $(1-q_2)|y| \leqslant |y|-s$, while when $|y| \geqslant 4$ we have $|y|-s \geqslant |y|-1$. Since $s-|x| \leqslant s \leqslant q_2 r$, we have by the monotonicity of j,

$$j(|y|-s)\frac{1}{(\phi(s^{-2}))^{1/2}}\frac{1}{(\phi((s-|x|)^{-2}))^{1/2}} \leqslant c_3 j((1-q_2)|y|)\frac{1}{\phi(r^{-2})}, \quad y \in A(0,r,4)$$

and

$$j(|y|-s)\frac{1}{(\phi(s^{-2}))^{1/2}}\frac{1}{(\phi((s-|x|)^{-2}))^{1/2}} \leqslant c_3 j(|y|-1)\frac{1}{\phi(r^{-2})}, \quad |y| \geqslant 4$$

for some constant $c_3 > 0$. Thus by applying (2.11) and (2.12), we get

$$II \leqslant c_4(1-p)^{-1}\int_{q_1 r}^{q_2 r} j(|y|-s)\frac{1}{(\phi(s^{-2}))^{1/2}}\frac{1}{(\phi((s-|x|)^{-2}))^{1/2}}ds \leqslant c_5 j(|y|)\frac{1}{\phi(r^{-2})}. \tag{5.4}$$

Combining (5.3)-(5.4), we conclude that

$$K_U(x,y) \leqslant c_6\frac{1}{\phi(r^{-2})}\int_{(U\setminus B(0,q_1 r))} j(|z|)K_U(z,y)dz + c_6 j(|y|)\frac{1}{\phi(r^{-2})}.$$

□

Since X is a purely discontinuous rotationally symmetric Lévy process, it follows from [14, Proposition 4.1] (see also [19, Theorem 1]) that if V is a Lipschitz open set and $U \subset V$,

$$\mathbb{P}_x(X_{\tau_U} \in \partial V) = 0 \qquad \text{and} \qquad \mathbb{P}_x(X_{\tau_U} \in dz) = K_U(x,z)dz \quad \text{on } V^c. \tag{5.5}$$

Lemma 5.3 *For every $p \in (0,1)$, there exists $c = c(\phi,\gamma,p) > 0$ such that for every $r \in (0,1)$, for every $z_0 \in \mathbb{R}^d$, $U \subset B(z_0,r)$ and any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes in $U^c \cap B(z_0,r)$ we have*

$$u(x) \leqslant c\frac{1}{\phi(r^{-2})}\int_{(U\setminus B(z_0,(1+p)r/2))\cup B(z_0,r)^c} j(|y-z_0|)u(y)dy, \quad x \in U \cap B(z_0,pr).$$

Proof. Without loss of generality, we assume $z_0 = 0$. Let $0 < p < 1$ and set $q = (1+p)/2$. Note that the part of boundary of U belonging to $U^c \cap B(z_0,r)$ need not be Lipschitz, but here u vanishes. The other part of boundary of U is a part of the boundary of the ball $B(z_0,r)$. Thus, since u is regular harmonic in U with respect to X and vanishes in $U^c \cap B(z_0,r)$, by Lemma 5.2 and (5.5) we have

$$\begin{aligned} u(x) &= \mathbb{E}_x[u(X_{\tau_U})] = \int_{U^c} K_U(x,y)u(y)dy \\ &\leqslant c\frac{1}{\phi(r^{-2})}\left(\int_{U^c}\int_{U\setminus B(0,qr)} j(|z|)K_U(z,y)dz u(y)dy + \int_{B(0,r)^c} j(|y|)u(y)dy\right). \end{aligned}$$

Since $\int_{U^c} K_U(z,y)u(y)dy = u(z)$ on $U \setminus B(0,qr)$, by Tonelli's theorem, we have

$$u(x) \leqslant c\frac{1}{\phi(r^{-2})}\left(\int_{U\setminus B(0,qr)} j(|z|)u(z)dz + \int_{B(0,r)^c} j(|y|)u(y)dy\right).$$

□

Lemma 5.4 *There exists $C = C(\phi,\gamma) > 1$ such that for every $r \in (0,1)$, for every $z_0 \in \mathbb{R}^d$, $U \subset B(z_0,r)$ and for any $(x,y) \in (U \cap B(z_0,r/2)) \times (B(z_0,r)^c \cap \overline{U}^c)$,*

$$C^{-1}\,\mathbb{E}_x[\tau_U]\left(\int_{U\setminus B(z_0,r/2)} j(|z-z_0|)K_U(z,y)dz + j(|y-z_0|)\right)$$
$$\leqslant \quad K_U(x,y) \leqslant C\,\mathbb{E}_x[\tau_U]\left(\int_{U\setminus B(z_0,r/2)} j(|z-z_0|)K_U(z,y)dz + j(|y-z_0|)\right).$$

Proof. Without loss of generality, we assume $z_0 = 0$. Fix $r \in (0,1)$ and let $B := B(0,r)$, $U_1 := U \cap B(0,\frac{1}{2}r)$, $U_2 := U \cap B(0,\frac{2}{3}r)$ and $U_3 := U \cap B(0,\frac{3}{4}r)$. Let $x \in U \cap B(0,r/2)$, $y \in B(0,r)^c \cap \overline{U}^c$. By (5.1),

$$\begin{aligned}
K_U(x,y) &= \mathbb{E}_x[K_U(X_{\tau_{U_2}},y)] + K_{U_2}(x,y)\\
&= \int_{U_3\setminus U_2} K_U(z,y)\mathbb{P}_x(X_{\tau_{U_2}} \in dz) + \int_{U\setminus U_3} K_U(z,y)K_{U_2}(x,z)dz + K_{U_2}(x,y)\\
&= \int_{U_3\setminus U_2} K_U(z,y)\mathbb{P}_x(X_{\tau_{U_2}} \in dz) + \int_{U\setminus U_3} K_U(z,y)\int_{U_2} G_{U_2}(x,w)j(|z-w|)dwdz\\
&\qquad + \int_{U_2} G_{U_2}(x,w)j(|y-w|)dw =: I + II + III.
\end{aligned}$$

From Lemmas 4.10 and 5.2, we see that there exist $c_i = c_i(\phi,\gamma) > 0, i = 1,2$, such that I is less than or equal to

$$c_1\left(\sup_{z\in U_3} K_U(z,y)\right)\phi(r^{-2})\mathbb{E}_x[\tau_{U_2}] \;\leqslant\; c_2\mathbb{E}_x[\tau_{U_2}]\left(\int_{U\setminus U_3} j(|z|)K_U(z,y)dz + j(|y|)\right). \qquad (5.6)$$

Note that if $4 > |z| \geqslant \frac{3}{4}r$ and $|w| < \frac{2}{3}r$, then

$$\frac{1}{9}|z| = |z| - \frac{8}{9}|y| \leqslant |z| - \frac{2}{3}r \leqslant |z| - |w| \leqslant |z-w| \leqslant |z| + |w| \leqslant |z| + \frac{2}{3}r \leqslant 2|z|.$$

Thus, since j is monotonely decreasing, by (2.11)

$$c_3^{-1}j(|z|) \;\leqslant\; j(2|z|) \;\leqslant\; j(|z-w|) \;\leqslant\; j(\frac{1}{9}|z|) \;\leqslant\; c_3 j(|z|), \qquad \text{if } 4 > |z| \geqslant \frac{3}{4}r, |w| < \frac{2}{3}r$$

for some constant $c_3 > 0$. If $4 \leqslant |z|$ and $|w| < \frac{2}{3}r$, then $|z| - \frac{2}{3}r \leqslant |z-w| \leqslant |z| + \frac{2}{3}r$ and by (2.12)

$$j(|z-w|) \;\geqslant\; j(|z| + \frac{2}{3}r) \;\geqslant\; c_4^{-1}j(|z| + \frac{2}{3}r - 1) \;\geqslant\; c_4^{-1}j(|z|)$$

and

$$j(|z-w|) \;\leqslant\; j(|z| - \frac{2}{3}r) \;\leqslant\; c_4 j(|z| - \frac{2}{3}r + 1) \;\leqslant\; c_4 j(|z|)$$

for some constant $c_4 > 0$. Thus there exists $c_5 = c_5(\phi,\gamma) > 1$ such that

$$c_5^{-1}\mathbb{E}_x[\tau_{U_2}]\int_{U\setminus U_3} j(|z|)K_U(z,y)dz \leqslant II \leqslant c_5\mathbb{E}_x[\tau_{U_2}]\int_{U\setminus U_3} j(|z|)K_U(z,y)dz \qquad (5.7)$$

and

$$c_5^{-1}\mathbb{E}_x[\tau_{U_2}]j(|y|) \leqslant III \leqslant c_5\mathbb{E}_x[\tau_{U_2}]j(|y|)\,. \tag{5.8}$$

Now the upper bound follows from (5.6)–(5.8). To prove the lower bound we can neglect I. Further, by using Lemmas 4.4 and 4.10 in the third line, from (5.2) we get

$$\begin{aligned}
\mathbb{E}_x[\tau_U] &= \mathbb{E}_x[\tau_{U_2}] + \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_{U_2}}}[\tau_U]\right] \\
&\leqslant \mathbb{E}_x[\tau_{U_2}] + \left(\sup_{z\in U}\mathbb{E}_z[\tau_U]\right)\mathbb{P}_x\big(X_{\tau_{U_2}}\in B(0,2r/3)^c\big) \\
&\leqslant \mathbb{E}_x[\tau_{U_2}] + c_6\phi(r^{-2})^{-1}\, c_7\phi((2r/3)^{-2})\mathbb{E}_x[\tau_{U_2}] \;\leqslant\; c_8\mathbb{E}_x[\tau_{U_2}]
\end{aligned}$$

for some constants $c_8 > 0$. In the last inequality above we have used **(H)**. Since

$$\begin{aligned}
\int_{U\setminus U_1} j(|z|)K_U(z,y)dz &= \int_{U\setminus U_3} j(|z|)K_U(z,y)dz + \int_{U_3\setminus U_1} j(|z|)K_U(z,y)dz \\
&\leqslant \int_{U\setminus U_3} j(|z|)K_U(z,y)dz + \left(\sup_{z\in U_3} K_U(z,y)\right)\int_{A(0,r/2,3r/4)} j(|z|)dz,
\end{aligned}$$

by Theorem 2.3 and Lemma 5.2,

$$\begin{aligned}
\int_{U\setminus U_1} j(|z|)K_U(z,y)dz &\leqslant \left(1+\frac{c_9}{\phi(r^{-2})}\int_{r/2}^{3r/4} s^{-1}\phi(s^{-2})ds\right) \\
&\qquad \times\left(\int_{U\setminus U_3} j(|z|)K_U(z,y)dz + j(|y|)\right).
\end{aligned}$$

Applying (4.3) (considering the cases $r < 4R_0/3$ and $1 \geqslant r \geqslant 4R_0/3$ separately) and **(H)**, we obtain

$$\int_{U\setminus U_1} j(|z|)K_U(z,y)dz \leqslant c_{10}\left(\int_{U\setminus U_3} j(|z|)K_U(z,y)dz + j(|y|)\right). \tag{5.9}$$

Combining (5.7)-(5.9), we have proved the lower bound. □

Lemma 5.5 *For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes a.e. in $U^c \cap B(z_0, r)$ it holds that*

$$C^{-1}\mathbb{E}_x[\tau_U]\int_{B(z_0,r/2)^c} j(|y-z_0|)u(y)dy \;\leqslant\; u(x) \;\leqslant\; C\,\mathbb{E}_x[\tau_U]\int_{B(z_0,r/2)^c} j(|y-z_0|)u(y)dy$$

for every $x \in U \cap B(z_0, r/2)$ (where C is the constant from Lemma 5.4).

Proof. Without loss of generality we may take $z_0 = 0$. By the argument in the proof of Lemma 5.3 and by the assumption that u vanishes a.e. on $U^c \cap B(0,r)$ we have that

$$u(x) = \int_{U^c} K_U(x,y)u(y)\,dy = \int_{B(0,r)^c} K_U(x,y)u(y)\,dy\,.$$

Now the claim follows from Lemma 5.4. Indeed, by Tonelli's theorem we get

$$\begin{aligned} u(x) &\leqslant C\,\mathbb{E}_x[\tau_U]\left(\int_{U\setminus B(0,r/2)} j(|z|)\left(\int_{B(0,r)^c} K_U(z,y)u(y)\,dy\right)dz + \int_{B(0,r)^c} j(|y|)u(y)\,dy\right)\\ &= C\,\mathbb{E}_x[\tau_U]\left(\int_{U\setminus B(0,r/2)} j(|z|)u(z)\,dz + \int_{B(0,r)^c} j(|z|)u(z)\,dz\right)\\ &= C\,\mathbb{E}_x[\tau_U]\int_{B(0,r/2)^c} j(|z|)u(z)\,dz, \end{aligned}$$

where for the last line we used that u vanishes a.e. on $U^c\cap B(0,r)$. The lower bound follows in the same way. □

We remark that in the statements of Lemmas 5.1–5.5, by using the assumption (1.3), we could have replaced the density j with the density J_X of the process X (with a different constant). We will do this in the next corollary which gives an approximate factorization of the Poisson kernel. It is an immediate consequence of the last two lemmas.

Corollary 5.6 *Let $z_0\in\mathbb{R}^d$, $D\subset\mathbb{R}^d$ be Greenian open set and denote $U:=D\cap B(z_0,r)$. Then for every $r\in(0,1)$ and all $(x,y)\in(D\cap B(z_0,r/2))\times(D^c\cap B(z_0,r)^c)$ it holds that*

$$C^{-1}\mathbb{E}_x[\tau_U]A(y)\leqslant K_D(x,y)\leqslant C\,\mathbb{E}_x[\tau_U]A(y)\,, \tag{5.10}$$

where

$$\begin{aligned} A(y) &:= \int_{U\setminus B(z_0,r/2)} \big(J_X(z-z_0)K_U(z,y)\,dz + J_X(y-z_0)\big)\\ &\quad + \int_{B(z_0,r/2)^c} J_X(z-z_0)\mathbb{E}_z\left[K_D(X_{\tau_U},y)\right]\,dz\,. \end{aligned}$$

Proof. Without loss of generality, we assume $z_0=0$ and $D\cap B(0,r/2)\neq\emptyset$. We first note that by (5.1) and (5.5), for every $(x,y)\in(D\cap B(0,r))\times(D^c\cap B(0,r)^c)$,

$$K_D(x,y)=K_U(x,y)+\mathbb{E}_x\left[K_D(X_{\tau_U},y)\right].$$

The function $x\mapsto\mathbb{E}_x\left[K_D(X_{\tau_U},y)\right]$ is regular harmonic in U with respect to X and vanishes a.e. in $U^c\cap B(0,r)$. By using Lemma 5.5 for this function, and Lemma 5.4 for $K_U(x,y)$ we immediately obtain required inequalities. □

We can now easily prove Theorem 1.1.

Proof of Theorem 1.1. (i) This follows immediately from Lemma 5.5 with $c:=C^4$.
(ii) Let $x_1,x_2\in D\cap B(z_0,r/2)$, $y_1,y_2\in D^c\cap B(z_0,r)^c$ and let $U:=D\cap B(z_0,r)$. Then by (5.10)

$$\begin{aligned} K_D(x_1,y_1)K_D(x_2,y_2) &\leqslant (C\mathbb{E}_{x_1}[\tau_U]A(y_1))\,(C\mathbb{E}_{x_2}[\tau_U]A(y_2))\\ &= (C\mathbb{E}_{x_1}[\tau_U]A(y_2))\,(C\mathbb{E}_{x_2}[\tau_U]A(y_1))\\ &\leqslant C^4K_D(x_1,y_2)K_D(x_2,y_1)\,. \end{aligned}$$

The lower bound is proved in the same way. □

6 Remarks on (H)

In this section we point out the relationship between the assumption **(H)** and the class OR of O-regularly varying functions, and sketch the construction of a complete Bernstein function ϕ which satisfies **(H)** but not the assumption in [10] that ϕ is comparable to a regularly varying function. Using the idea in the construction below, one can come up with a complete Bernstein function that is bounded between any two regularly varying complete Bernstein functions.

It follows from the definitions on [2, page 65 and page 68] and [2, Proposition 2.2.1] that the assumption **(H)** is equivalent to that ϕ is in OR with its Matuszewska indices contained in $(0, 1)$.

Here is a sketch of the construction. For $x \in (0, 2]$, define

$$f(x) = x^{1/2}.$$

Then we define

$$f(x) = x^{1/3} + f(2) - 2^{1/3}, \qquad x \in (2, a_1]$$

for some large constant $a_1 > 2$. The constant a_1 is chosen so that for large values of x in $(2, a_1]$, the function f behaves like $x^{1/3}$, that is $f(\lambda x)/f(x)$ is close to $\lambda^{1/3}$ uniformly for $\lambda \in [1, 2]$. Then we define

$$f(x) = x^{1/2} + f(a_1) - a_1^{1/2}, \qquad x \in (a_1, a_2]$$

for some large constant $a_2 > a_1$. The constant a_2 is chosen so that for large values of x in $(a_1, a_2]$, the function f behaves like $x^{1/2}$, that is $f(\lambda x)/f(x)$ is close to $\lambda^{1/2}$ uniformly for $\lambda \in [1, 3]$. Then we define

$$f(x) = x^{1/3} + f(a_2) - a_2^{1/3}, \qquad x \in (a_2, a_3]$$

for some large constant $a_3 > a_2$. The constant a_3 is chosen so that for large values of x in $(a_2, a_3]$, the function f behaves like $x^{1/3}$, that is $f(\lambda x)/f(x)$ is close to $\lambda^{1/3}$ uniformly for $\lambda \in [1, 4]$. We repeat this procedure to define this function inductively.

The function f is an increasing function in OR with upper Matuszewska index 1/2 and lower Matuszewska index 1/3.

Let σ be the measure with distribution function f. Since $\int_{(0,\infty)}(1+t)^{-1}\sigma(dt) < \infty$, σ is a Stieltjes measure. Let

$$g(\lambda) := \int_{(0,\infty)} \frac{1}{\lambda + t}\, \sigma(dt)$$

be the corresponding Stieltjes function. It follows from integration by parts that

$$g(\lambda) = \int_0^\infty \frac{f(\xi)}{(\lambda + \xi)^2} d\xi.$$

Using our construction of f we know that

$$\int_2^\infty \frac{\xi^{1/3}}{(\lambda+\xi)^2} d\xi \leqslant \int_2^\infty \frac{f(\xi)}{(\lambda+\xi)^2} d\xi \leqslant \int_2^\infty \frac{\xi^{1/2}}{(\lambda+\xi)^2} d\xi.$$

Thus it follows from [21, Lemma 6.3] that

$$c_1\lambda^{-2/3} \leqslant g(\lambda) \leqslant c_2\lambda^{-1/2}, \quad \lambda \geqslant 2$$

for some positive constants $c_1 < c_2$.

Modifying the argument of the proof of the de Haan–Stadtmüller theorem ([2, Theorem 2.10.2]) one can show that g is in OR with upper Matuszewska index $-2/3$ and lower Matuszewska index $-1/2$. It follows from [17, Theorem 7.3] that $\phi(x) := 1/f(x)$ is a complete Bernstein function. Thus ϕ is a complete Bernstein function in OR with upper Matuszewska index $1/2$ and lower Matuszewska index $1/3$.

It follows from [21, Lemma 6.3] that g cannot be comparable with any regularly varying function at infinity, and therefore ϕ cannot be comparable with any regularly varying function at infinity.

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