Potential Theory of Subordinate Brownian Motion

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1 Introduction

Let $X = (X_t : t \geq 0)$ be a d-dimensional Brownian motion. Subordination of Brownian motion consists of time-changing the paths of X by an independent subordinator. To be more precise, let $S = (S_t : t \geq 0)$ be a subordinator (i.e., a nonnegative, increasing Lévy process) independent of X. The process $Y = (Y_t : t \geq 0)$ defined by $Y_t = X(S_t)$ is called a subordinate Brownian motion. The process Y is an example of a rotationally invariant d-dimensional Lévy process. A general Lévy process in $\mathbb{R}^d$ is completely characterized by its characteristic triple (b, A, π), where $b \in \mathbb{R}^d$, A is a nonnegative definite $d \times d$ matrix, and π is a measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int (1 \wedge |x|^2)\, \pi(dx) < \infty$, called the Lévy measure of the process. Its characteristic exponent Φ, defined by $\mathbb{E}[\exp\{i\langle x, Y_t\rangle\}] = \exp\{-t\Phi(x)\}$, $x \in \mathbb{R}^d$, is given by the Lévy-Khintchine formula involving the characteristic triple (b, A, π). The main difficulty in studying general Lévy processes stems from the fact that the Lévy measure π can be quite complicated.

The situation simplifies immensely in the case of subordinate Brownian motions. If we take the Brownian motion X as given, then Y is completely determined by the subordinator S. Hence, one can deduce properties of Y from properties of the subordinator S. On the analytic level this translates to the following: Let ϕ denote the Laplace exponent of the subordinator S. That is, $\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}$, $\lambda > 0$. Then the characteristic exponent Φ of the subordinate Brownian motion Y takes on the very simple form $\Phi(x) = \phi(|x|^2)$ (our Brownian motion X runs at twice the usual speed). Hence, properties of Y should follow from properties of the Laplace exponent ϕ. This will be the main theme of these lecture notes – we will study potential-theoretic properties of Y by using information given by ϕ. Two main instances of this approach are explicit formulae for the Green function of Y and the Lévy measure of Y. Let $p(t, x, y)$, $x, y \in \mathbb{R}^d$, $t > 0$, denote the transition densities of the Brownian motion X, and let μ, respectively U, denote the Lévy measure, respectively the potential measure, of the subordinator S. Then the Lévy measure π of Y is given by $\pi(dx) = J(x)\, dx$ where

$$J(x) = \int_0^\infty p(t, 0, x)\, \mu(dt)\,,$$

while, when Y is transient, the Green function $G(x, y)$, $x, y \in \mathbb{R}^d$, of Y is given by

$$G(x, y) = \int_0^\infty p(t, x, y)\, U(dt)\,.$$

Let us consider the second formula (same reasoning also applies to the first one). This formula suggests that the asymptotic behavior of $G(x, y)$ when $|x - y| \to 0$ (respectively, when $|x - y| \to \infty$) should follow from the asymptotic behavior of the potential measure U at ∞ (respectively at 0). The latter can be studied in the case when the potential measure has a monotone density u with respect to the Lebesgue measure. Indeed, the Laplace transform of U is given by $\mathcal{L}U(\lambda) = 1/\phi(\lambda)$, hence one can invoke the Tauberian and monotone density theorems to obtain the asymptotic behavior of u from the asymptotic behavior of ϕ. We will be mainly interested in the behavior of the Green function $G(x, y)$ and the jumping function $J(x)$ near zero, hence the reasonable assumption on ϕ will be that it is regularly varying at infinity with index $\alpha \in [0, 2]$. This includes subordinators having a drift, as well as subordinators with slowly varying Laplace exponent at infinity, for example, a gamma subordinator.

The materials covered in these lecture notes are based on several recent papers, primarily [47], [52], [59] and [57]. The main effort here was given to unify the exposition of those results, and in doing so we also eradicated the typos in these papers. Some new materials and generalizations are also included. Here is the outline of the notes.

In Section 2 we recall some basic facts about subordinators and give a list of examples that will be useful later on. This list contains stable subordinators, relativistic stable subordinators, subordinators which are sums of stable subordinators and a drift, gamma subordinators, geometric stable subordinators, iterated geometric stable subordinators and Bessel subordinators. All of these subordinators belong to the class of special subordinators (even complete Bernstein subordinators). Special subordinators are important to our approach because they are precisely the ones whose potential measure restricted to $(0, \infty)$ has a decreasing density u. In fact, for all of the listed subordinators the potential measure has a decreasing density u. In the last part of the section we study asymptotic behaviors of the potential density u and the Lévy density of subordinators by use of Karamata's and de Haan's Tauberian and monotone density theorems.

In Section 3 we derive asymptotic properties of the Green function and the jumping function of subordinate Brownian motion. These results follow from the technical Lemma 3.3 upon checking its conditions for particular subordinators. Of special interest is the order of singularities of the Green function near zero, starting from the Newtonian kernel at the one end, and singularities on the brink of integrability on the other end obtained for iterated geometric stable subordinators. The results for the asymptotic behavior of the jumping function are less complete, but are substituted by results on the decay at zero and at infinity. Finally, we discuss transition densities for symmetric geometric stable processes which exhibit unusual behavior on the diagonal for small (as well as large) times.

The original motivation for deriving the results in Sections 2 and 3 was an attempt to obtain the Harnack inequality for subordinate Brownian motions with subordinators whose Laplace exponent $\phi(\lambda)$ has the asymptotic behavior at infinity of one of the following two forms: (i) $\phi(\lambda) \sim \lambda$, or (ii) logarithmic behavior at ∞. A typical example of the first case is the process Y which is a sum of Brownian motion and an independent rotationally invariant α-stable process. This situation was studied in [47]. A typical example of the second case is a geometric stable process – a subordinate Brownian motion via a geometric stable subordinator. In this case, $\phi(\lambda) \sim \log \lambda$ as $\lambda \to \infty$. This was studied in [52]. Section 4 contains an exposition of these results and some generalizations, and is partially based on the general approach to Harnack inequality from [56]. After obtaining some potential-theoretic results for a class of radial Lévy processes, we derive Krylov-Safonov-type estimates for the hitting probabilities involving capacities. Similar estimates involving Lebesgue measure were obtained in [56] based on the work of Bass and Levin [4]. These estimates are crucial in proving two types of Harnack inequalities for small balls - scale invariant ones, and the weak ones in which the constant might depend on the radius of a ball. In fact, we give a full proof of the Harnack inequality only for iterated geometric stable processes, and refer the reader to the original papers for the other cases.

Finally, in Section 5 we replace the underlying Brownian motion by the Brownian motion killed upon exiting a Lipschitz domain D. The resulting process is denoted by X^D. We are interested in the potential theory of the process $Y_t^D = X^D(S_t)$ where S is a special subordinator with infinite Lévy measure or positive drift. Such questions were first studied for stable subordinators in [31], and the final solution in this case was given in [30]. The general case for special subordinators appeared in [59]. Surprisingly, it turns out that the potential theory of Y^D is in a one-to-one and onto correspondence with the potential theory of X^D. More precisely, there is a bijection (realized by the potential operator of the subordinate process $Z_t^D = X^D(T_t)$ where T is the subordinator conjugate to S) from the cone $\mathcal{S}(Y^D)$ of excessive functions of Y^D onto the cone $\mathcal{S}(X^D)$ of excessive functions for X^D which preserves nonnegative harmonic functions. This bijection makes it possible to essentially transfer the potential theory of X^D to the potential theory of Y^D. In this way we obtain the Martin kernel and the Martin representation for Y^D which immediately leads to a proof of the boundary Harnack principle for nonnegative harmonic functions of Y^D. In the case of a $C^{1,1}$ domain we obtain sharp bounds for the transition densities of the subordinate process Y^D.

The materials covered in these lecture notes by no means include all that can be said about the potential theory of subordinate Brownian motions. One of the omissions is the Green function estimates for killed subordinate Brownian motions and the boundary Harnack inequality for the positive harmonic functions of subordinate Brownian motions. By using ideas from [18] or [48] one can easily extend the Green function estimates of [17] and [37] for killed symmetric stable processes to more general killed subordinate Brownian motions under certain conditions, and then use these estimates to extend arguments in [14] and [61] to establish the boundary Harnack inequality for general subordinate Brownian motions under certain conditions. In the case when the Laplace exponent ϕ is regularly varying at infinity, this is done in [35]. Another notable omission is the spectral theory for such processes together with implications to spectral theory of killed subordinate Brownian motion. We refer the reader to [19], [20] and [21]. Related to this is the general discussion on the exact difference between subordinate killed Brownian motions and the killed subordinate Brownian motions and its consequences. This was discussed in [55] and [54]. See also [32] and the forthcoming [60].

We end this introduction with few words on the notations. For functions f and g we write $f \sim g$ if the quotient f/g converges to 1, and $f \asymp g$ if the quotient f/g stays bounded between two positive constants.

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2 Subordinators

2.1 Special subordinators and complete Bernstein functions

Let $S = (S_t : t \geq 0)$ be a subordinator, that is, an increasing Lévy process taking values in $[0, \infty]$ with $S_0 = 0$. We remark that our subordinators are what some authors call killed subordinators. The Laplace transform of the law of S_t is given by the formula

$$\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)) , \quad \lambda > 0. \tag{2.1}$$

The function $\phi : (0, \infty) \to \mathbb{R}$ is called the Laplace exponent of S, and it can be written in the form

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt) . \tag{2.2}$$

Here $a, b \geq 0$, and μ is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1)\, \mu(dt) < \infty . \tag{2.3}$$

The constant a is called the killing rate, b the drift, and μ the Lévy measure of the subordinator S. By using condition (2.3) above one can easily check that

$$\lim_{t\to 0} t\, \mu(t, \infty) = 0, \tag{2.4}$$

$$\int_0^1 \mu(t, \infty)\, dt < \infty . \tag{2.5}$$

For $t \geq 0$, let η_t be the distribution of S_t. To be more precise, for a Borel set $A \subset [0, \infty)$, $\eta_t(A) = \mathbb{P}(S_t \in A)$. The family of measures $(\eta_t : t \geq 0)$ form a convolution semigroup of measures on $[0,\infty)$. Clearly, the formula (2.1) reads $\exp(-t\phi(\lambda)) = \mathcal{L}\eta_t(\lambda)$, the Laplace transform of the measure η_t. We refer the reader to [7] for much more detailed exposition on subordinators.

Recall that a C^∞ function $\phi : (0,\infty) \to [0,\infty)$ is called a Bernstein function if $(-1)^n D^n \phi \leq 0$ for every $n \in \mathbb{N}$. It is well known (see, e.g., [6]) that a function $\phi : (0,\infty) \to \mathbb{R}$ is a Bernstein function if and only if it has the representation given by (2.2).

We now introduce the concepts of special Bernstein functions and special subordinators.

Definition 2.1 *A Bernstein function ϕ is called a special Bernstein function if $\psi(\lambda) := \lambda/\phi(\lambda)$ is also a Bernstein function. A subordinator S is called a special subordinator if its Laplace exponent is a special Bernstein function.*

We will call the function ψ in the definition above the Bernstein function *conjugate* to ϕ.

Special subordinators occur naturally in various situations. For instance, they appear as the ladder time process for a Lévy process which is not a compound Poisson process, see page 166 of [7]. Yet another situation in which they appear naturally is in connection with the exponential functional of subordinators (see [9]).

The most common examples of special Bernstein functions are complete Bernstein functions, also called operator monotone functions in some literature. A function $\phi : (0,\infty) \to \mathbb{R}$ is called a complete Bernstein function if there exists a Bernstein function η such that

$$\phi(\lambda) = \lambda^2 \mathcal{L}\eta(\lambda), \quad \lambda > 0,$$

where $\mathcal{L}$ stands for the Laplace transform of the function η: $\mathcal{L}\eta(\lambda) = \int_0^\infty e^{-\lambda t}\eta(t)\,dt$. It is known (see, for instance, Remark 3.9.28 and Theorem 3.9.29 of [34]) that every complete Bernstein function is a Bernstein function and that the following three conditions are equivalent:

(i) ϕ is a complete Bernstein function;

(ii) $\psi(\lambda) := \lambda/\phi(\lambda)$ is a complete Bernstein function;

(iii) ϕ is a Bernstein function whose Lévy measure μ is given by

$$\mu(dt) = \int_0^\infty e^{-st}\gamma(ds)dt$$

where γ is a measure on $(0,\infty)$ satisfying

$$\int_0^1 \frac{1}{s}\gamma(ds) + \int_1^\infty \frac{1}{s^2}\gamma(ds) < \infty.$$

The equivalence of (i) and (ii) says that every complete Bernstein function is a special Bernstein function. Note also that it follows from the condition (iii) above that being a complete Bernstein function only depends on the Lévy measure and that the Lévy measure $\mu(dt)$ of any complete Bernstein function has a completely monotone density. We also note that the tail $t \to \mu(t, \infty)$ of the Lévy measure μ is a completely monotone function. Indeed, by Fubini's theorem

$$\mu(x, \infty) = \int_x^\infty \int_0^\infty e^{-st} \gamma(ds)\, dt = \int_0^\infty e^{-xs} \frac{\gamma(ds)}{s} .$$

A similar argument shows that the converse is also true, namely, if the tail of the Lévy measure μ is a completely monotone function, then μ has a completely monotone density. The density of the Lévy measure with respect to the Lebesgue measure (when it exists) will be called the Lévy density.

The family of all complete Bernstein functions is a closed convex cone containing positive constants. The following properties of complete Bernstein functions are well known, see, for instance, [42]: (i) If ϕ is a nonzero complete Bernstein function, then so are $\phi(\lambda^{-1})^{-1}$ and $\lambda\phi(\lambda^{-1})$; (ii) if ϕ_1 and ϕ_2 are nonzero complete Bernstein functions and $\beta \in (0, 1)$, then $\phi_1^\beta(\lambda)\phi_2^{1-\beta}(\lambda)$ is also a complete Bernstein function; (iii) if ϕ_1 and ϕ_2 are nonzero complete Bernstein functions and $\beta \in (-1, 0) \cup (0, 1)$, then $(\phi_1^\beta(\lambda) + \phi_2^\beta(\lambda))^{1/\beta}$ is also a complete Bernstein function.

Most of the familiar Bernstein functions are complete Bernstein functions. The following are some examples of complete Bernstein functions ([34]): (i) $\lambda^\alpha, \alpha \in (0, 1]$; (ii) $(\lambda + 1)^\alpha - 1, \alpha \in (0, 1)$; (iii) $\log(1+\lambda)$; (iv) $\frac{\lambda}{\lambda+1}$. The first family corresponds to α-stable subordinators $(0 < \alpha < 1)$ and a pure drift $(\alpha = 1)$, the second family corresponds to relativistic α-stable subordinators, the third Bernstein function corresponds to the gamma subordinator, and the fourth corresponds to the compound Poisson process with rate 1 and exponential jumps. An example of a Bernstein function which is not a complete Bernstein function is $1 - e^{-\lambda}$. One can also check that $1 - e^{-\lambda}$ is not a special Bernstein function as well.

The potential measure of the subordinator S is defined by

$$U(A) = \mathbb{E} \int_0^\infty 1_{(S_t \in A)}\, dt = \int_0^\infty \eta_t(A)\, dt, \quad A \subset [0, \infty). \tag{2.6}$$

Note that $U(A)$ is the expected time the subordinator S spends in the set A. The Laplace transform of the measure U is given by

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda t}\, dU(t) = \mathbb{E} \int_0^\infty \exp(-\lambda S_t)\, dt = \frac{1}{\phi(\lambda)} . \tag{2.7}$$

We are going to derive a characterization of special subordinators in terms of their potential measures. Roughly, a subordinator S is special if and only if its potential measure

U restricted to $(0,\infty)$ has a decreasing density. To be more precise, let S be a special subordinator with the Laplace exponent ϕ given by

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt)\,.$$

Then

$$\begin{aligned}
\lim_{\lambda\to 0} \frac{\lambda}{\phi(\lambda)} &= \begin{cases} 0\,, & a>0, \\ \frac{1}{b+\int_0^\infty t\,\mu(dt)}\,, & a=0, \end{cases} \\
\lim_{\lambda\to \infty} \frac{1}{\phi(\lambda)} &= \begin{cases} 0\,, & b>0 \text{ or } \mu(0,\infty)=\infty, \\ \frac{1}{a+\mu(0,\infty)}\,, & b=0 \text{ and } \mu(0,\infty)<\infty\,. \end{cases}
\end{aligned}$$

Since $\lambda/\phi(\lambda)$ is a Bernstein function, we must have

$$\frac{\lambda}{\phi(\lambda)} = \tilde{a} + \tilde{b}\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \nu(dt) \tag{2.8}$$

for some Lévy measure ν, and

$$\tilde{a} = \begin{cases} 0\,, & a>0, \\ \frac{1}{b+\int_0^\infty t\,\mu(dt)}\,, & a=0, \end{cases} \tag{2.9}$$

$$\tilde{b} = \begin{cases} 0\,, & b>0 \text{ or } \mu(0,\infty)=\infty, \\ \frac{1}{a+\mu(0,\infty)}\,, & b=0 \text{ and } \mu(0,\infty)<\infty\,. \end{cases} \tag{2.10}$$

Equivalently,

$$\frac{1}{\phi(\lambda)} = \tilde{b} + \int_0^\infty e^{-\lambda t}\tilde{\Pi}(t)\, dt \tag{2.11}$$

with

$$\tilde{\Pi}(t) = \tilde{a} + \nu(t,\infty)\,, \quad t>0\,.$$

Let $\tau(dt) := \tilde{b}\epsilon_0(dt) + \tilde{\Pi}(t)\,dt$. Then the right-hand side in (2.11) is the Laplace transform of the measure τ. Since $1/\phi(\lambda) = \mathcal{L}U(\lambda)$, the Laplace transform of the potential measure U of S, we have that $\mathcal{L}U(\lambda) = \mathcal{L}\tau(\lambda)$. Therefore,

$$U(dt) = \tilde{b}\epsilon_0(dt) + u(t)\,dt\,,$$

with a decreasing function $u(t) = \tilde{\Pi}(t)$.

Conversely, suppose that S is a subordinator with potential measure given by

$$U(dt) = c\epsilon_0(dt) + u(t)\,dt\,,$$

for some $c \geq 0$ and some decreasing function $u : (0,\infty) \to (0,\infty)$ satisfying $\int_0^1 u(t)\,dt < \infty$. Then

$$\frac{1}{\phi(\lambda)} = \mathcal{L}U(\lambda) = c + \int_0^\infty e^{-\lambda t} u(t)\,dt\,.$$

It follows that

$$\begin{aligned}
\frac{\lambda}{\phi(\lambda)} &= c\lambda + \int_0^\infty u(t)\, d(1-e^{-\lambda t}) \\
&= c\lambda + u(t)(1-e^{-\lambda t})\,|_0^\infty - \int_0^\infty (1-e^{-\lambda t})\, u(dt) \\
&= c\lambda + u(\infty) + \int_0^\infty (1-e^{-\lambda t})\, \gamma(dt)\,, \qquad (2.12)
\end{aligned}$$

with $\gamma(dt) = -u(dt)$. In the last equality we used that $\lim_{t\to 0} u(t)(1-e^{-\lambda t}) = 0$. This is a consequence of the assumption $\int_0^1 u(t)\, dt < \infty$. It is easy to check, by using the same integrability condition on u, that $\int_0^\infty (1\wedge t)\, \gamma(dt) < \infty$, so that γ is a Lévy measure. Therefore, $\lambda/\phi(\lambda)$ is a Bernstein function, implying that S is a special subordinator.

In this way we have proved the following

Theorem 2.1 *Let S be a subordinator with the potential measure U. Then S is special if and only if*

$$U(dt) = c\epsilon_0(dt) + u(t)\, dt$$

for some $c \ge 0$ and some decreasing function $u : (0,\infty) \to (0,\infty)$ satisfying $\int_0^1 u(t)\, dt < \infty$.

Remark 2.2 *The above result appeared in [8] as Corollaries 1 and 2 and was possibly known even before. The above presentation is taken from [59]. In case $c = 0$, we will call u the potential density of the subordinator S (or of the Laplace exponent ϕ).*

Corollary 2.3 *Let S be a subordinator with the Laplace exponent ϕ and the potential measure U. Then ϕ is a complete Bernstein function if and only if U restricted to $(0,\infty)$ has a completely monotone density u.*

Proof. Note that from the proof of Theorem 2.1 we have the explicit form of the density u: $u(t) = \tilde{\Pi}(t)$ where $\tilde{\Pi}(t) = \tilde{a} + \nu(t,\infty)$. Here ν is the Lévy measure of $\lambda/\phi(\lambda)$. If ϕ is complete Bernstein, then $\lambda/\phi(\lambda)$ is complete Bernstein, and hence it follows from the property (iii) of complete Bernstein function that $u(t) = \tilde{a} + \nu(t,\infty)$ is a completely monotone function. Conversely, if u is completely monotone, then clearly the tail $t \to \nu(t,\infty)$ is completely monotone, which implies that $\lambda/\phi(\lambda)$ is complete Bernstein. Therefore, ϕ is also a complete Bernstein function. $\square$

Note that by comparing expressions (2.8) and (2.12) for $\lambda/\phi(\lambda)$, and by using formulae (2.9) and (2.10), it immediately follows that

$$\begin{aligned}
c &= \tilde{b} = \begin{cases} 0\,, & b > 0 \text{ or } \mu(0,\infty) = \infty, \\ \frac{1}{a+\mu(0,\infty)}\,, & b = 0 \text{ and } \mu(0,\infty) < \infty, \end{cases} \\
u(\infty) &= \tilde{a} = \begin{cases} 0\,, & a > 0, \\ \frac{1}{b+\int_0^\infty t\,\mu(dt)}\,, & a = 0, \end{cases} \\
u(t) &= \tilde{a} + \nu(t,\infty)\,.
\end{aligned}$$

In particular, it cannot happen that both a and $\tilde{a}$ are positive, and similarly, that both b and $\tilde{b}$ are positive. Moreover, it is clear from the definition of $\tilde{b}$ that $\tilde{b} > 0$ if and only if $b = 0$ and $\mu(0,\infty) < \infty$.

We record now some consequences of Theorem 2.1 and the formulae above.

Corollary 2.4 *Suppose that $S = (S_t : t \geq 0)$ is a subordinator whose Laplace exponent*

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt)$$

is a special Bernstein function with $b > 0$ or $\mu(0,\infty) = \infty$. Then the potential measure U of S has a decreasing density u satisfying

$$\lim_{t\to 0} t\, u(t) = 0, \tag{2.13}$$

$$\lim_{t\to 0} \int_0^t s\, du(s) = 0\,. \tag{2.14}$$

Proof. The formulae follow immediately from $u(t) = \tilde{a} + \nu(t,\infty)$ and (2.4)–(2.5) applied to ν. □

Corollary 2.5 *Suppose that $S = (S_t : t \geq 0)$ is a special subordinator with the Laplace exponent given by*

$$\phi(\lambda) = a + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt)$$

where μ satisfies $\mu(0,\infty) = \infty$. Then

$$\psi(\lambda) := \frac{\lambda}{\phi(\lambda)} = \tilde{a} + \int_0^\infty (1 - e^{-\lambda t})\, \nu(dt) \tag{2.15}$$

where the Lévy measure ν satisfies $\nu(0,\infty) = \infty$.

Let T be the subordinator with the Laplace exponent ψ. If u and v denote the potential density of S and T respectively, then

$$v(t) = a + \mu(t,\infty)\,. \tag{2.16}$$

In particular, $a = v(\infty)$ and $\tilde{a} = u(\infty)$. Moreover, a and $\tilde{a}$ cannot be both positive.

Assume that ϕ is a special Bernstein function with the representation (2.2) where $b > 0$ or $\mu(0,\infty) = \infty$. Let S be a subordinator with the Laplace exponent ϕ, and let U denote its potential measure. By Corollary 2.4, U has a decreasing density $u : (0,\infty) \to (0,\infty)$. Let T be a subordinator with the Laplace exponent $\psi(\lambda) = \lambda/\phi(\lambda)$ and let V denote its potential measure. Then $V(dt) = b\epsilon_0(dt) + v(t)\,dt$ where $v : (0,\infty) \to (0,\infty)$ is a decreasing function. If $b > 0$, the potential measure V has an atom at zero, and hence the subordinator T is a compound Poisson process (this can be also seen as follows: since $b > 0$, we have $u(0+) < \infty$, and hence $\nu(0,\infty) = u(0+) - \tilde{a} < \infty$). Note that in case $b > 0$, the Lévy measure μ can be finite. If $b = 0$, we require that $\mu(0,\infty) = \infty$, and then, by Corollary 2.5, $\psi(\lambda) = \lambda/\phi(\lambda)$ has the same form as ϕ, namely $\tilde{b} = 0$ and $\nu(0,\infty) = \infty$. In this case, subordinators S and T play symmetric roles.

The following result will be crucial for the developments in Section 5 of these notes.

Theorem 2.6 *Let ϕ be a special Bernstein function with representation (2.2) satisfying $b > 0$ or $\mu(0,\infty) = \infty$. Then*

$$b\,u(t) + \int_0^t u(s)v(t-s)\,ds = b\,u(t) + \int_0^t v(s)u(t-s)\,ds = 1, \quad t > 0. \tag{2.17}$$

Proof. Since for all $\lambda > 0$ we have

$$\frac{1}{\phi(\lambda)} = \mathcal{L}u(\lambda), \quad \frac{\phi(\lambda)}{\lambda} = b + \mathcal{L}v(\lambda)\,,$$

after multiplying we get

$$\begin{aligned} \frac{1}{\lambda} &= b\mathcal{L}u(\lambda) + \mathcal{L}u(\lambda)\mathcal{L}v(\lambda) \\ &= b\mathcal{L}u(\lambda) + \mathcal{L}(u * v)(\lambda)\,. \end{aligned}$$

Inverting this equality gives

$$1 = b\,u(t) + \int_0^t u(s)v(t-s)\,ds\,, \quad t > 0.$$

□

Theorem 2.6 has an amusing consequence related to the first passage of the subordinator S. Let $\sigma_t = \inf\{s > 0 : \, S_s > t\}$ be the first passage time across the level $t > 0$. By the first passage formula (see, e.g., [7], p.76), we have

$$\mathbb{P}(S_{\sigma_{t-}} \in ds, S_{\sigma_t} \in dx) = u(s)\mu(x-s)\,ds\,dx\,,$$

for $0 \le s \le t$, and $x > t$. Since $\mu(x,\infty) = v(x)$, by use of Fubini's theorem this implies

$$\begin{aligned} \mathbb{P}(S_{\sigma_t} > t) &= \int_t^\infty \int_0^t u(s)\mu(x-s)\,ds\,dx = \int_0^t u(s)\int_t^\infty \mu(x-s)\,dx\,ds \\ &= \int_0^t u(s)\mu(t-s,\infty)\,ds = \int_0^t u(s)v(t-s)\,ds\,. \end{aligned}$$

Since $\mathbb{P}(S_{\sigma_t} \geq t) = 1$, by comparing with (2.17) we see that $\mathbb{P}(S_{\sigma_t} = t) = b\,u(t)$. This provides a simple proof in case of special subordinators of the well-known fact true for general subordinators (see [7], pp.77-79).

In the sequel we will also need the following result on potential density that is valid for subordinators that are not necessarily special.

Proposition 2.7 *Let $S = (S_t : t \geq 0)$ be a subordinator with drift $b > 0$. Then its potential measure U has a density u continuous on $(0, \infty)$ satisfying $u(0+) = 1/b$ and $u(t) \leq u(0+)$ for every $t > 0$.*

Proof. For the proof of existence of continuous u and the fact that $u(0+) = 1/b$ see, e.g., [7], p.79. That $u(t) \leq u(0+)$ for every $t > 0$ follows from the subadditivity of the function $t \mapsto U([0,t])$ (see, e.g., [47]). □

2.2 Examples of subordinators

In this subsection we give a list of subordinators that will be relevant in the sequel and describe some of their properties.

Example 2.8 (**Stable subordinators**) Our first example covers the family of well-known stable subordinators. For $0 < \alpha < 2$, let $\phi(\lambda) = \lambda^{\alpha/2}$. By integration

$$\lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{-\lambda t})\, t^{-1-\alpha/2}\, dt\,,$$

i.e, the Lévy measure $\mu(dt)$ of ϕ has a density given by $(\alpha/2)/\Gamma(1-\alpha/2)\, t^{-1-\alpha/2}$. Since $t^{-1-\alpha/2} = \int_0^\infty e^{-ts} s^{\alpha/2}/\Gamma(1+\alpha/2)\, ds$, it follows that ϕ is a complete Bernstein function. The tail of the Lévy measure μ is equal to

$$\mu(t, \infty) = \frac{t^{-\alpha/2}}{\Gamma(1-\alpha/2)}\,.$$

The conjugate Bernstein function is $\psi(\lambda) = \lambda^{1-\alpha/2}$, hence its tail is $\nu(t, \infty) = t^{\alpha/2-1}/\Gamma(\alpha/2)$. This shows that the potential density of $\phi(\lambda) = \lambda^{\alpha/2}$ is equal to

$$u(t) = \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}\,.$$

The subordinator S corresponding to ϕ is called an $\alpha/2$-*stable subordinator*.

It is known that the distribution $\eta_1(ds)$ of the $\alpha/2$-stable subordinator has a density $\eta_1(s)$ with respect to the Lebesgue measure. Moreover, by [53],

$$\eta_1(s) \sim 2\pi\Gamma\left(1 + \frac{\alpha}{2}\right) \sin\left(\frac{\alpha\pi}{4}\right) s^{-1-\alpha/2}\,, \quad s \to \infty\,, \tag{2.18}$$

and

$$\eta_1(s) \le c(1 \wedge s^{-1-\alpha/2}), \quad s>0, \tag{2.19}$$

for some positive constant $c>0$.

Example 2.9 (**Relativistic stable subordinators**) For $0<\alpha<2$ and $m>0$, let $\phi(\lambda)=(\lambda+m^{2/\alpha})^{\alpha/2}-m$. By integration

$$(\lambda+m^{2/\alpha})^{\alpha/2}-m=\frac{\alpha/2}{\Gamma(1-\alpha/2)}\int_0^\infty (1-e^{-\lambda t})\, e^{-m^{2/\alpha}t}\, t^{-1-\alpha/2}\, dt\,,$$

i.e., the Lévy measure $\mu(dt)$ of ϕ has a density given by $(\alpha/2)/\Gamma(1-\alpha/2)\, e^{-m^{2/\alpha}t}\, t^{-1-\alpha/2}$. This Bernstein function appeared in [39] in the study of the stability of relativistic matter, and so we call the corresponding subordinator S a *relativistic $\alpha/2$-stable subordinator.* Since the Lévy density of ϕ is completely monotone, we know that ϕ is a complete Bernstein function. The explicit form of potential density u of S can be computed as follows (see [33] for this calculation): For $\gamma,\beta>0$ let

$$E_{\gamma,\beta}(t)=\sum_{n=0}^\infty \frac{t^n}{\Gamma(\beta+\gamma n)}\,, \quad t>0\,,$$

be the two parameter Mittag-Leffler function. By integrating term by term it follows that

$$\int_0^\infty e^{-\lambda t}e^{-m^{2/\alpha}t}\, t^{-1+\alpha/2}E_{\alpha/2,\alpha/2}(mt^{\alpha/2})\, dt=\frac{1}{(\lambda+m^{2/\alpha})^{\alpha/2}-m}\,.$$

Therefore,

$$u(t)=e^{-m^{2/\alpha}t}\, t^{-1+\alpha/2}E_{\alpha/2,\alpha/2}(mt^{\alpha/2})\,.$$

The subordinator $\tilde{S}$ corresponding to the complete Bernstein function $m+\phi(\lambda)=(\lambda+m^{2/\alpha})^{\alpha/2}$ is obtained by killing S at an independent exponential time with parameter m. By checking tables of Laplace transforms ([27]) we see that

$$\frac{1}{m+\phi(\lambda)}=\int_0^\infty e^{-\lambda t}\, \frac{1}{\Gamma(\alpha/2)}\, e^{-m^{2/\alpha}t}\, t^{-1+\alpha/2}\, dt\,,$$

implying that the potential measure $\tilde{U}$ of the subordinator $\tilde{S}$ has the density $\tilde{u}$ given by

$$\tilde{u}(t)=\frac{1}{\Gamma(\alpha/2)}\, e^{-m^{2/\alpha}t}\, t^{-1+\alpha/2}\,.$$

Example 2.10 (**Gamma subordinator**) Let $\phi(\lambda)=\log(1+\lambda)$. By use of Frullani's integral it follows that

$$\log(1+\lambda)=\int_0^\infty (1-e^{-\lambda t})\frac{e^{-t}}{t}\, dt\,,$$

i.e., the Lévy measure of ϕ has a density given by e^{-t}/t. Note that $e^{-t}/t = \int_0^\infty e^{-st} 1_{(1,\infty)}(s)\, ds$, implying that the density of the Lévy measure μ is completely monotone. Therefore, ϕ is a complete Bernstein function. The corresponding subordinator S is called a *gamma subordinator*. The explicit form of the potential density u is not known. In the next section we will derive the asymptotic behavior of u at 0 and at $+\infty$. On the other hand, the distribution $\eta_t(ds)$, $t > 0$, is well known and given by

$$\eta_t(ds) = \frac{1}{\Gamma(t)}\, s^{t-1} e^{-s}\, ds\,, \quad s > 0\,. \tag{2.20}$$

Before proceeding to the next two examples, let us briefly discuss composition of subordinators. Suppose that $S^1 = (S^1_t : t \geq 0)$ and $S^2 = (S^2_t : t \geq 0)$ are two independent subordinators with Laplace exponents ϕ^1, respectively ϕ^2, and convolution semigroups $(\eta^1_t : t \geq 0)$, respectively $(\eta^2_t : t \geq 0)$. Define the new process $S = (S_t : t \geq 0)$ by $S_t = S^1(S^2_t)$, subordination of S^1 by S^2. Subordinating a Lévy process by an independent subordinator always yields a Lévy process (e.g. [49], p. 197). Hence, S is another subordinator. The distribution η_t of S_t is given by

$$\eta_t(ds) = \int_0^\infty \eta^2_t(du)\eta^1_u(ds)\,. \tag{2.21}$$

Therefore, for any $\lambda > 0$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda s}\, \eta_t(ds) &= \int_0^\infty e^{-\lambda s} \int_0^\infty \eta^2_t(du)\eta^1_u(ds) \\
&= \int_0^\infty \eta^2_t(du) \int_0^\infty e^{-\lambda s}\, \eta^1_u(ds) \\
&= \int_0^\infty \eta^2_t(du) e^{-u\phi^1(\lambda)} = \phi^2(\phi^1(\lambda))\,,
\end{aligned}$$

showing that the Laplace exponent ϕ of S is given by $\phi(\lambda) = \phi^2(\phi^1(\lambda))$.

Example 2.11 (**Geometric stable subordinators**) For $0 < \alpha < 2$, let $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$. Since ϕ is a composition of the complete Bernstein functions from Examples 2.8 and 2.10, it is itself a complete Bernstein function. The corresponding subordinator S is called a *geometric $\alpha/2$-stable subordinator*. Note that this subordinator may be obtained by subordinating an $\alpha/2$-stable subordinator by a gamma subordinator. The concept of geometric stable distributions was first introduced in [36]. We will now compute the Lévy measure μ of S. Define

$$E_{\alpha/2}(t) := \sum_{n=0}^\infty (-1)^n \frac{t^{n\alpha/2}}{\Gamma(1+n\alpha/2)}\,, \quad t > 0\,.$$

By checking tables of Laplace transforms (or by computing term by term), we see that

$$\int_0^\infty e^{-\lambda t} E_{\alpha/2}(t)\, dt = \frac{1}{\lambda(1+\lambda^{-\alpha/2})} = \frac{\lambda^{\alpha/2-1}}{1+\lambda^{\alpha/2}}\,. \tag{2.22}$$

Further, since $\phi(0+) = 0$ and $\lim_{\lambda\to\infty} \phi(\lambda)/\lambda = 0$, we have that $\phi(\lambda) = \int_0^\infty (1-e^{-\lambda t})\, \mu(dt)$. By differentiating this expression for ϕ and the explicit form of ϕ we obtain that

$$\phi'(\lambda) = \int_0^\infty te^{-\lambda t}\, \mu(dt) = \frac{\alpha}{2} \frac{\lambda^{\alpha/2-1}}{1+\lambda^{\alpha/2}}\,. \tag{2.23}$$

By comparing (2.22) and (2.23) we see that the Lévy measure $\mu(dt)$ has a density given by

$$\mu(t) = \frac{\alpha}{2} \frac{E_{\alpha/2}(t)}{t}\,. \tag{2.24}$$

The explicit form of the potential density u is not known. In the next section we will derive the asymptotic behavior of u at $0+$ and at ∞.

We will now show that the distribution function of S_1 is given by

$$F(s) = 1 - E_{\alpha/2}(s) = \sum_{n=1}^\infty (-1)^{n-1} \frac{s^{n\alpha/2}}{\Gamma(1+n\alpha/2)}\,, \quad s > 0\,. \tag{2.25}$$

Indeed, for $\lambda > 0$,

$$\begin{aligned} \mathcal{L}F(\lambda) &= \int_0^\infty e^{-\lambda t}\, F(dt) = \lambda \int_0^\infty e^{-\lambda t}(1 - E_{\alpha/2}(t))\, dt \\ &= \lambda \left(\frac{1}{\lambda} - \frac{\lambda^{\alpha/2-1}}{1+\lambda^{\alpha/2}} \right) = \exp\{-\log(1+\lambda^{\alpha/2})\}\,. \end{aligned}$$

Since the function $\lambda \mapsto 1 + \lambda^{\alpha/2}$ is a complete Bernstein function, its reciprocal function, $\lambda \mapsto 1/(1+\lambda^{\alpha/2})$ is a Stieltjes function (see [34] for more details about Stieltjes functions). Moreover, since $\lim_{t\to\infty} 1/(1+\lambda^{\alpha/2}) = 0$, it follows that there exists a measure σ on $(0,\infty)$ such that

$$\frac{1}{1+\lambda^{\alpha/2}} = \mathcal{L}(\mathcal{L}\sigma)(\lambda)\,.$$

But this means that the function F has a completely monotone density f given by $f(t) = \mathcal{L}\sigma(t)$. It is shown in [43] that the distribution function of S_t, $t > 0$, is equal to

$$\sum_{n=1}^\infty (-1)^{n-1} \frac{\Gamma(t+n-1)s^{(t+n-1)\alpha/2}}{\Gamma(t)(n-1)!\Gamma(1+(t+n-1)\alpha/2)}\,.$$

Note that the case of the gamma subordinator may be subsumed under the case of geometric $\alpha/2$-stable subordinator by taking $\alpha = 2$ in the definition.

Example 2.12 (**Iterated geometric stable subordinators**) Let $0 < \alpha \le 2$. Define,

$$\phi^{(1)}(\lambda) = \phi(\lambda) = \log(1+\lambda^{\alpha/2})\,, \quad \phi^{(n)}(\lambda) = \phi(\phi^{(n-1)}(\lambda))\,, \quad n \ge 2\,.$$

Since $\phi^{(n)}$ is a complete Bernstein function, we have that $\phi^{(n)}(\lambda) = \int_0^\infty (1-e^{-\lambda t})\mu^{(n)}(t)\,dt$ for a completely monotone Lévy density $\mu^{(n)}(t)$. The exact form of this density is not known.

Let $S^{(n)} = (S_t^{(n)} : t \ge 0)$ be the corresponding (iterated) subordinator, and let $U^{(n)}$ denote the potential measure of $S^{(n)}$. Since $\phi^{(n)}$ is a complete Bernstein function, $U^{(n)}$ admits a completely monotone density $u^{(n)}$. The explicit form of the potential density $u^{(n)}$ is not known. In the next section we will derive the asymptotic behavior of u at 0 and at $+\infty$.

Example 2.13 (**Stable subordinators with drifts**) For $0 < \alpha < 2$ and $b > 0$, let $\phi(\lambda) = b\lambda + \lambda^{\alpha/2}$. Since $\lambda \mapsto \lambda^{\alpha/2}$ is complete Bernstein, it follow that ϕ is also a complete Bernstein function. The corresponding subordinator $S = (S_t : t \ge 0)$ is a sum of the pure drift subordinator $t \mapsto bt$ and the $\alpha/2$-stable subordinator. Its Lévy measure is the same as the Lévy measure of the $\alpha/2$-stable subordinator. In order to compute the potential density u of the subordinator S, we first note that, similarly as in (2.22),

$$\int_0^\infty e^{-\lambda t} b E_{\alpha/2}(b^{-2/\alpha}t)\,dt = \frac{1}{b\lambda + \lambda^{\alpha/2}} = \frac{1}{\phi(\lambda)}\,.$$

Therefore, $u(t) = bE_{\alpha/2}(b^{-2/\alpha}t)$ for $t > 0$.

Example 2.14 (**Bessel subordinators**) The two subordinators in this example are taken from [41]. The Bessel subordinator $S_I = (S_I(t) : t \ge 0)$ is a subordinator with no drift, no killing and Lévy density

$$\mu_I(t) = \frac{1}{t}\, I_0(t)\, e^{-t},$$

where for any real number ν, I_ν is the modified Bessel function. Since μ_I is the Laplace transform of the function $\gamma(t) = \int_0^t g(s)\,ds$ with

$$g(s) = \begin{cases} \pi^{-1}(2s - s^2)^{-1/2}\,, & s \in (0,2), \\ 0\,, & s \ge 2, \end{cases}$$

the Laplace exponent of S_I is a complete Bernstein function. The Laplace exponent of S_I is given by

$$\phi_I(\lambda) = \log((1+\lambda) + \sqrt{(1+\lambda)^2 - 1}).$$

For any $t > 0$, the density of $S_I(t)$ is given by

$$f_t(x) = \frac{t}{x} I_t(x) e^{-x}.$$

The Bessel subordinator $S_K = (S_K(t) : t \geq 0)$ is a subordinator with no drift, no killing and Lévy density

$$\mu_K(t) = \frac{1}{t} K_0(t) e^{-t},$$

where for any real number ν, K_ν is the modified Bessel function. Since μ_K is the Laplace transform of the function

$$\gamma(t) = \begin{cases} 0, & t \in (0,2], \\ \log(t - 1 + \sqrt{(t-1)^2 + 1}), & t > 2\,, \end{cases}$$

the Laplace exponent of S_K is a complete Bernstein function. The Laplace exponent of S_K is given by

$$\phi_K(\lambda) = \frac{1}{2}\left(\log((1+\lambda) + \sqrt{(1+\lambda)^2 - 1})\right)^2 .$$

For any $t > 0$, the density of $S_K(t)$ is given by

$$f_t(x) = \sqrt{\frac{2\pi}{t}}\, \vartheta_x(\frac{1}{t}) \frac{e^{-x}}{x},$$

where

$$\vartheta_v(t) = \frac{v}{\sqrt{2\pi^3 t}} \int_0^\infty \exp(\frac{\pi^2 - \xi^2}{2t}) \exp(-v\cosh(\xi)) \sinh(\xi) \sin(\frac{\pi\xi}{t}) d\xi.$$

Example 2.15 For any $\alpha \in (0,2)$ and $\beta \in (0, 2-\alpha)$, it follows from the properties of complete Bernstein functions that

$$\phi(\lambda) = \lambda^{\alpha/2} (\log(1+\lambda))^{\beta/2}$$

is a complete Bernstein function.

Example 2.16 For any $\alpha \in (0,2)$ and $\beta \in (0,\alpha)$, it follows from the properties of complete Bernstein functions that

$$\phi(\lambda) = \lambda^{\alpha/2} (\log(1+\lambda))^{-\beta/2}$$

is a complete Bernstein function.

2.3 Asymptotic behavior of the potential, Lévy and transition densities

Recall the formula (2.7) relating the Laplace exponent ϕ of the subordinator S with the Laplace transform of its potential measure U. In the case U has a density u, this formula reads

$$\mathcal{L}u(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt = \frac{1}{\phi(\lambda)} .$$

The asymptotic behavior of ϕ at ∞ (resp. at 0) determines, by use of Tauberian and the monotone density theorems, the asymptotic behavior of the potential density u at 0 (resp. at ∞). We first recall Karamata's version of these theorems from [10].

Theorem 2.17 *(a) (Karamata's Tauberian theorem) Let $U:(0,\infty)\to(0,\infty)$ be an increasing function. If ℓ is slowly varying at ∞ (resp. at $0+$), $\rho\geq 0$, the following are equivalent:*
(i) As $t\to\infty$ (resp. $t\to 0+$)

$$U(t)\sim\frac{t^{\rho}\ell(t)}{\Gamma(1+\rho)}\,.$$

(ii) As $\lambda\to 0$ (resp. $\lambda\to\infty$)

$$\mathcal{L}U(\lambda)\sim\lambda^{-\rho}\ell(1/\lambda)\,.$$

(b)(Karamata's monotone density theorem) If additionally $U(dx)=u(x)\,dx$, where u is monotone and nonnegative, and $\rho>0$, then (i) and (ii) are equivalent to:
(iii) As $t\to\infty$ (resp. $t\to 0+$)

$$u(t)\sim\frac{\rho t^{\rho-1}\ell(t)}{\Gamma(1+\rho)}\,.$$

We are going to use Theorem 2.17 for Laplace exponents that are regularly varying at ∞ (resp. at 0). To be more specific we will assume that either (i)

$$\phi(\lambda)\sim\lambda^{\alpha/2}\ell(\lambda)\,,\quad\lambda\to\infty\,,\tag{2.26}$$

where $0<\alpha\leq 2$, and ℓ is slowly varying at ∞, or (ii)

$$\phi(\lambda)\sim\lambda^{\alpha/2}\ell(\lambda)\,,\quad\lambda\to 0,\tag{2.27}$$

where $0<\alpha\leq 2$, and ℓ is slowly varying at 0. In case (i), (2.26) implies $b>0$ or $\mu(0,\infty)=\infty$. If ϕ is a special Bernstein function, then the corresponding subordinator S has a decreasing potential density u whose asymptotic behavior at 0 is then given by

$$u(t)\sim\frac{1}{\Gamma(\alpha/2)}\frac{t^{\alpha/2-1}}{\ell(1/t)}\,,\quad t\to 0+\,.\tag{2.28}$$

In case (ii), if ϕ is a special Bernstein function with $\lim_{\lambda\to\infty}\phi(\lambda)=\infty$, then the corresponding subordinator S has a decreasing potential density u whose asymptotic behavior at ∞ is then given by

$$u(t)\sim\frac{1}{\Gamma(\alpha/2)}\frac{t^{\alpha/2-1}}{\ell(1/t)}\,,\quad t\to\infty\,.\tag{2.29}$$

As consequences of the above, we immediately get the following: (1) for $\alpha \in (0,2)$, the potential density of the relativistic $\alpha/2$-stable subordinator satisfies

$$u(t) \sim \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}, \quad t \to 0+, \tag{2.30}$$

$$u(t) \sim \frac{\alpha}{2} m^{1-2/\alpha}, \quad t \to \infty; \tag{2.31}$$

(2) for $\alpha \in (0,2)$, $\beta \in (0, 2-\alpha)$, the potential density of the subordinator corresponding to Example 2.15 satisfies

$$u(t) \sim \frac{1}{\Gamma(\alpha/2)} \frac{1}{t^{1-\alpha/2}|\log t|^{\beta/2}}, \quad t \to 0+, \tag{2.32}$$

$$u(t) \sim \frac{1}{\Gamma(\alpha/2+\beta/2)} \frac{1}{t^{1-(\alpha+\beta)/2}}, \quad t \to \infty; \tag{2.33}$$

(3) for $\alpha \in (0,2)$, $\beta \in (0,\alpha)$, the potential density of the subordinator corresponding to Example 2.16 satisfies

$$u(t) \sim \frac{\alpha}{2\Gamma(1+\alpha/2)} \frac{|\log t|^{\beta/2}}{t^{1-\alpha/2}}, \quad t \to 0+, \tag{2.34}$$

$$u(t) \sim \frac{\alpha-\beta}{2\Gamma(1+(\alpha-\beta)/2)} \frac{1}{t^{1-(\alpha-\beta)/2}}, \quad t \to \infty. \tag{2.35}$$

In the case when the subordinator has a positive drift $b > 0$, the potential density u always exists, it is continuous, and $u(0+) = b$. For example, this will be the case when $\phi(\lambda) = b\lambda + \lambda^{\alpha/2}$. Recall (see Example 2.13) that the potential density is given by the rather explicit formula $u(t) = bE_{\alpha/2}(b^{-2/\alpha}t)$. The asymptotic behavior of $u(t)$ as $t \to \infty$ is not easily derived from this formula. On the other hand, since $\phi(\lambda) \sim \lambda^{\alpha/2}$ as $\lambda \to 0$, it follows from (2.29) that $u(t) \sim t^{\alpha/2-1}/\Gamma(\alpha/2)$ as $t \to \infty$.

Note that the gamma subordinator, geometric $\alpha/2$-stable subordinators, iterated geometric stable subordinators and Bessel subordinators have Laplace exponents that are not regularly varying with strictly positive exponent at ∞, but are rather slowly varying at ∞. In this case, Karamata's monotone density theorem cannot be used, and we need more refined versions of both Tauberian and monotone density theorems. The results are also taken from [10].

Theorem 2.18 *(a) (de Haan's Tauberian Theorem) Let $U : (0,\infty) \to (0,\infty)$ be an increasing function. If ℓ is slowly varying at ∞ (resp. at $0+$), $c \geq 0$, the following are equivalent:*

(i) As $t \to \infty$ (resp. $t \to 0+$)

$$\frac{U(\lambda t) - U(t)}{\ell(t)} \to c \log \lambda, \quad \forall \lambda > 0.$$

(ii) As $t \to \infty$ (resp. $t \to 0+$)

$$\frac{\mathcal{L}U(\frac{1}{\lambda t}) - \mathcal{L}U(\frac{1}{t})}{\ell(t)} \to c \log \lambda, \quad \forall \lambda > 0.$$

(b) (de Haan's Monotone Density Theorem) If additionally $U(dx) = u(x)\,dx$, where u is monotone and nonnegative, and $c > 0$, then (i) and (ii) are equivalent to:

(iii) As $t \to \infty$ (resp. $t \to 0+$)

$$u(t) \sim ct^{-1}\ell(t).$$

We are going to apply this result to establish the asymptotic behaviors of the potential density of geometric stable subordinators, iterated geometric stable subordinators and Bessel subordinators at zero.

Proposition 2.19 *For any $\alpha \in (0,2]$, let $\phi(\lambda) = \log(1+\lambda^{\alpha/2})$, and let u be the potential density of the corresponding subordinator. Then*

$$\begin{aligned} u(t) &\sim \frac{2}{\alpha t(\log t)^2}, \quad t \to 0+, \\ u(t) &\sim \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}, \quad t \to \infty. \end{aligned}$$

Proof. Recall that

$$\mathcal{L}U(\lambda) = 1/\phi(\lambda) = 1/\log(1+\lambda^{\alpha/2}).$$

Since

$$\frac{\mathcal{L}U(\frac{1}{x\lambda}) - \mathcal{L}U(\frac{1}{\lambda})}{(\log \lambda)^{-2}} \to \frac{2}{\alpha}\log x, \quad \forall x > 0,$$

as $\lambda \to 0+$, we have by (the 0+ version of) Theorem 2.18 (a) that

$$\frac{U(xt) - U(t)}{(\log t)^{-2}} \to \frac{2}{\alpha}\log x, \quad x > 0,$$

as $t \to 0+$. Now we can apply (the 0+ version of) Theorem 2.18 (b) to get that

$$u(t) \sim \frac{2}{\alpha t(\log t)^2}$$

as $t \to 0+$. The asymptotic behavior of $u(t)$ as $t \to \infty$ follows from Theorem 2.17. □

In order to deal with the iterated geometric stable subordinators, let $e_0 = 0$, and inductively, $e_n = e^{e_{n-1}}$, $n \geq 1$. For $n \geq 1$ define $l_n : (e_n, \infty) \to (0, \infty)$ by

$$l_n(y) = \log\log\ldots\log y, \quad n \text{ times}. \tag{2.36}$$

Further, let $L_0(y) = 1$, and for $n \in \mathbb{N}$, define $L_n : (e_n, \infty) \to (0, \infty)$ by

$$L_n(y) = l_1(y) l_2(y) \dots l_n(y)\,. \tag{2.37}$$

Note that $l_n'(y) = 1/(y L_{n-1}(y))$ for every $n \geq 1$. Let $\alpha \in (0, 2]$ and recall from Example 2.12 that $\phi^{(1)}(y) := \log(1 + y^{\alpha/2})$, and for $n \geq 1$, $\phi^{(n)}(y) := \phi(\phi^{(n-1)}(y))$. Let $k_n(y) := 1/\phi^{(n)}(y)$.

Lemma 2.20 *Let $t > 0$. For every $n \in \mathbb{N}$,*

$$\lim_{y\to\infty} (k_n(ty) - k_n(y)) L_{n-1}(y) l_n(y)^2 = -\frac{2}{\alpha} \log t\,.$$

Proof. The proof for $n = 1$ is straightforward and is implicit in the proof of Proposition 2.19. We only give the proof for $n = 2$, the proof for general n is similar. Using the fact that

$$\log(1 + y) \sim y, \quad y \to 0+, \tag{2.38}$$

we can easily get that

$$\lim_{y\to\infty} \left(\log \frac{\log y}{\log(yt)}\right) \log y = -\lim_{y\to\infty} \left(\log \frac{\log y + \log t}{\log y}\right) \log y = -\log t. \tag{2.39}$$

Using (2.38) and the elementary fact that $\log(1 + y) \sim \log y$ as $y \to \infty$ we get that

$$\begin{aligned}
&\lim_{y\to\infty} (k_2(ty) - k_2(y)) L_1(y) l_2(y)^2 \\
&= \frac{\alpha}{2} \lim_{y\to\infty} \left(\log \frac{\log(1 + y^{\alpha/2})}{\log(1 + (ty)^{\alpha/2})}\right) \frac{\log y (\log\log y)^2}{(\alpha/2)^2 \log(\log(1 + y^{\alpha/2})) \log(\log(1 + (ty)^{\alpha/2}))} \\
&= \frac{2}{\alpha} \lim_{y\to\infty} \left(\log \frac{\log y}{\log(yt)}\right) \log y = -\frac{2}{\alpha} \log t\,.
\end{aligned}$$

□

Recall that $U^{(n)}$ denotes the potential measure and $u^{(n)}(t)$ the potential density of the iterated geometric stable subordinator $S^{(n)}$ with the Laplace exponent $\phi^{(n)}$.

Proposition 2.21 *For any $\alpha \in (0, 2]$, we have*

$$u^{(n)}(t) \sim \frac{2}{\alpha t L_{n-1}(\frac{1}{t}) l_n(\frac{1}{t})^2}\,, \quad t \to 0+\,, \tag{2.40}$$

$$u^{(n)}(t) \sim \frac{t^{(\alpha/2)^n - 1}}{\Gamma((\alpha/2)^n)}\,, \quad t \to 0+\ . \tag{2.41}$$

Proof. Using Lemma 2.20 we can easily see that

$$\frac{\mathcal{L}U^{(n)}(\frac{1}{x\lambda}) - \mathcal{L}U^{(n)}(\frac{1}{\lambda})}{(L_{n-1}(\frac{1}{\lambda})l_n(\frac{1}{\lambda})^2)^{-1}} \to \frac{2}{\alpha}\log x, \quad \forall x > 0\,,$$

as $\lambda \to 0+$. Therefore, by (the 0+ version of) Theorem 2.18 (a) we have that

$$\frac{U^{(n)}(xt) - U^{(n)}(t)}{(L_{n-1}(\frac{1}{t})l_n(\frac{1}{t})^2)^{-1}} \to \frac{2}{\alpha}\log x, \quad x > 0\,,$$

as $t \to 0+$. Now we can apply (the 0+ version of) Theorem 2.18 (b) to get that

$$u^{(n)}(t) \sim \frac{2}{\alpha t L_{n-1}(\frac{1}{t})l_n(\frac{1}{t})^2}$$

as $t \to 0+$. The asymptotic behavior of $u^{(n)}(t)$ at ∞ follows easily from Theorem 2.17. □

Let u_I and u_K be the potential densities of the Bessel subordinators I and K respectively. Then we have the following result.

Proposition 2.22 *The potential densities of the Bessel subordinators satisfy the following asymptotics*

$$\begin{aligned}
u_I(t) &\sim \frac{1}{t(\log t)^2}, \quad t \to 0+\,, \\
u_K(t) &\sim \frac{1}{t|\log t|^3}, \quad t \to 0+\,, \\
u_I(t) &\sim \frac{1}{\sqrt{2\pi}}\,t^{-1/2}, \quad t \to \infty\,, \\
u_K(t) &\sim 1, \quad t \to \infty\,.
\end{aligned}$$

Proof. The proofs of first two relations are direct applications of de Haan's Tauberian and monotone density theorems and the proofs of the last two are direct applications of Karamata's Tauberian and monotone density theorems. We omit the details. □

We now discuss the asymptotic behavior of the Lévy density of a subordinator.

Proposition 2.23 *Assume that the Laplace exponent ϕ of the subordinator S is a complete Bernstein function and let $\mu(t)$ denote the density of its Lévy measure.*

(i) Let $0 < \alpha < 2$. If $\phi(\lambda) \sim \lambda^{\alpha/2}\ell(\lambda)$, $\lambda \to \infty$, and ℓ is a slowly varying function at ∞, then

$$\mu(t) \sim \frac{\alpha/2}{\Gamma(1-\alpha/2)}\,t^{-1-\alpha/2}\,\ell(1/t)\,, \quad t \to 0+\ . \tag{2.42}$$

(ii) Let $0 < \alpha \le 2$. If $\phi(\lambda) \sim \lambda^{\alpha/2}\ell(\lambda)$, $\lambda \to 0$, and ℓ is a slowly varying function at 0, then

$$\mu(t) \sim \frac{\alpha/2}{\Gamma(1-\alpha/2)}\, t^{-1-\alpha/2}\, \ell(1/t)\,, \quad t \to \infty\,. \tag{2.43}$$

Proof. (i) The assumption implies that there is no drift, $b = 0$, and hence by integration by parts,

$$\phi(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\mu(t,\infty)\, dt\,.$$

Thus, $\int_0^\infty e^{-\lambda t}\mu(t,\infty)\, dt \sim \lambda^{\alpha/2-1}\ell(\lambda)$ as $\lambda \to \infty$, and (2.42) follows by first using Karamata's Tauberian theorem and then Karamata's monotone density theorem.

(ii) In this case it is possible that the drift b is strictly positive, and thus

$$\phi(\lambda) = \lambda \left(b + \int_0^\infty e^{-\lambda t}\mu(t,\infty)\, dt \right)\,.$$

This implies that $\int_0^\infty e^{-\lambda t}\mu(t)\, dt \sim \lambda^{\alpha/2-1}\ell(\lambda)$ as $\lambda \to 0$, and (2.43) holds by Theorem 2.17. □

Note that if $\phi(\lambda) \sim b\lambda$ as $\lambda \to \infty$ and $b > 0$, nothing can be inferred about the behavior of the density $\mu(t)$ near zero. Next we record the asymptotic behavior of the Lévy density of the geometric stable subordinator. The first claim follows from (2.24), and the second from the previous proposition.

Proposition 2.24 *Let $\mu(dt) = \mu(t)\, dt$ be the Lévy measure of a geometrically $\alpha/2$-stable subordinator. Then*

(i) For $0 < \alpha \le 2$, $\mu(t) \sim \frac{\alpha}{2t}$, $t \to 0+$.

(ii) For $0 < \alpha < 2$, $\mu(t) \sim \frac{\alpha/2}{\Gamma(1-\alpha/2)}\, t^{-\alpha/2-1}$, $t \to \infty$. For $\alpha = 2$, $\mu(t) = \frac{e^{-t}}{t}$.

In the case of iterated geometric stable subordinators, we have only partial result for the asymptotic behavior of the density $\mu^{(n)}$ which follows from Proposition 2.43 (ii).

Proposition 2.25 *For any $\alpha \in (0,2)$,*

$$\mu^{(n)}(t) \sim \frac{(\alpha/2)^n}{\Gamma(1-(\alpha/2)^n)}\, t^{-1-(\alpha/2)^n},\ t \to \infty\,.$$

Remark 2.26 *Note that we do not give the asymptotic behavior of $\mu^{(n)}(t)$ as $t \to \infty$ for $\alpha = 2$ (iterated gamma subordinator), and the asymptotic behavior of $\mu^{(n)}(t)$ as $t \to 0+$ for all $\alpha \in (0,2]$. It is an open problem to determine the correct asymptotic behavior.*

The following results are immediate consequences of Proposition 2.23.

Proposition 2.27 *Suppose that $\alpha \in (0,2)$ and $\beta \in (0, 2-\alpha)$. Let $\mu(t)$ be the Lévy density of the subordinator corresponding to Example 2.15. Then*

$$\begin{aligned}
\mu(t) &\sim \frac{\alpha}{2\Gamma(1-\alpha/2)} t^{-1-\alpha/2} (\log(1/t))^{\beta/2}, \quad t \to 0+\,, \\
\mu(t) &\sim \frac{\alpha+\beta}{2\Gamma(1-(\alpha+\beta)/2)}\, t^{-1-(\alpha+\beta)/2}, \quad t \to \infty\,.
\end{aligned}$$

Proposition 2.28 *Suppose that $\alpha \in (0,2)$ and $\beta \in (0, \alpha)$. Let $\mu(t)$ be the Lévy density of the subordinator corresponding to Example 2.16. Then*

$$\begin{aligned}
\mu(t) &\sim \frac{\alpha}{2\Gamma(1-\alpha/2)} \frac{1}{t^{1+\alpha/2}(\log(1/t))^{\beta/2}}, \quad t \to 0+\,, \\
\mu(t) &\sim \frac{\alpha-\beta}{2\Gamma(1-(\alpha-\beta)/2)}\, \frac{1}{t^{1+(\alpha-\beta)/2}}, \quad t \to \infty\,.
\end{aligned}$$

We conclude this section with a discussion of the asymptotic behavior of transition densities of geometric stable subordinators. Let $S = (S_t : t \geq 0)$ be a geometric $\alpha/2$-stable subordinator, and let $(\eta_s : s \geq 0)$ be the corresponding convolution semigroup. Further, let $(\rho_s : s \geq 0)$ be the convolution semigroup corresponding to an $\alpha/2$-stable subordinator, and by abuse of notation, let ρ_s denote the corresponding density. Then by (2.21) and the explicit formula (2.20), we see that η_t has a density

$$f_s(t) = \int_0^\infty \rho_u(t) \frac{1}{\Gamma(s)}\, u^{s-1} e^{-u}\, du\,.$$

For $s = 1$, this formula reads

$$f_1(t) = \int_0^\infty \rho_u(t) e^{-u}\, du\,.$$

Moreover, we have shown in Example 2.11 that $f_1(t)$ is completely monotone. To be more precise, $f_1(t)$ is the density of the distribution function $F(t) = 1 - E_{\alpha/2}(t)$ of the probability measure η_1 (see (2.25)).

Proposition 2.29 *For any $\alpha \in (0,2)$,*

$$f_1(t) \sim \frac{1}{\Gamma(\alpha/2)}\, t^{\alpha/2-1}\,, \quad t \to 0+\,, \tag{2.44}$$

$$f_1(t) \sim 2\pi\Gamma\left(1+\frac{\alpha}{2}\right) \sin\left(\frac{\alpha\pi}{4}\right) t^{-1-\frac{\alpha}{2}}\,, \quad t \to \infty\,. \tag{2.45}$$

Proof. The first relation follows from the explicit form of the distribution function $F(t) = 1 - E_{\alpha/2}(t)$ and Karamata's monotone density theorem. For the second relation, use the scaling property of stable distribution, $\rho_u(t) = u^{-2/\alpha}\rho_1(u^{-2/\alpha}t)$, to get

$$f_1(t) = \int_0^\infty e^{-u}u^{-2/\alpha}\rho_1(u^{-2/\alpha}t)\,du.$$

Now use (2.18), (2.19) and dominated convergence theorem to obtain the required asymptotic behavior. $\square$

3 Subordinate Brownian motion

3.1 Definitions and technical lemma

Let $X = (X_t, \mathbb{P}^x)$ be a d-dimensional Brownian motion. The transition densities $p(t,x,y) = p(t, y-x)$, $x, y \in \mathbb{R}^d, t > 0$, of X are given by

$$p(t,x) = (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

The semigroup $(P_t : t \geq 0)$ of X is defined by $P_t f(x) = \mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy$, where f is a nonnegative Borel function on $\mathbb{R}^d$. Recall that if $d \geq 3$, the Green function $G^{(2)}(x,y) = G^{(2)}(x-y)$, $x, y \in \mathbb{R}^d$, of X is well defined and is equal to

$$G^{(2)}(x) = \int_0^\infty p(t,x)\,dt = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\,|x|^{-d+2}.$$

Let $S = (S_t : t \geq 0)$ be a subordinator independent of X, with Laplace exponent $\phi(\lambda)$, Lévy measure μ, drift $b \geq 0$, no killing, potential measure U, and convolution semigroup $(\eta_t : t \geq 0)$. We define a new process $Y = (Y_t : t \geq 0)$ by $Y_t := X(S_t)$. Then Y is a Lévy process with characteristic exponent $\Phi(x) = \phi(|x|^2)$ (see e.g. [49], pp.197–198) called a *subordinate Brownian motion*. The semigroup $(Q_t : t \geq 0)$ of the process Y is given by

$$Q_t f(x) = \mathbb{E}^x[f(Y_t)] = \mathbb{E}^x[f(X(S_t))] = \int_0^\infty P_s f(x)\,\eta_t(ds).$$

If the subordinator S is not a compound Poisson process, then Q_t has a density $q(t,x,y) = q(t, x-y)$ given by $q(t,x) = \int_0^\infty p(s,x)\,\eta_t(ds)$.

From now on we assume that the subordinate process Y is transient. According to the criterion due to Port and Stone ([45]), Y is transient if and only if for some small $r > 0$, $\int_{|x|<r} \Re(\frac{1}{\Phi(x)})\,dx < \infty$. Since $\Phi(x) = \phi(|x|^2)$ is real, it follows that Y is transient if and only if

$$\int_{0+} \frac{\lambda^{d/2-1}}{\phi(\lambda)}\,d\lambda < \infty. \tag{3.1}$$

This is always true if $d \geq 3$, and, depending on the subordinator, may be true for $d=1$ or $d=2$. For $x \in \mathbb{R}^d$ and A Borel subset of $\mathbb{R}^d$, the occupation measure is given by

$$\begin{aligned} G(x,A) &= \mathbb{E}^x \int_0^\infty 1_{(Y_t \in A)} = \int_0^\infty Q_t 1_A(x)\, dt = \int_0^\infty \int_0^\infty P_s 1_A(x) \eta_t(ds)\, dt \\ &= \int_0^\infty P_s 1_A\, U(ds) = \int_A \int_0^\infty p(s,x,y)\, U(ds)\, dy\,, \end{aligned}$$

where the second line follows from (2.6). If A is bounded, then by the transience of Y, $G(x,A) < \infty$ for every $x \in \mathbb{R}^d$. Let $G(x,y)$ denote the density of the occupation measure $G(x,\cdot)$. Clearly, $G(x,y) = G(y-x)$ where

$$G(x) = \int_0^\infty p(t,x)\, U(dt) = \int_0^\infty p(t,x) u(t)\, dt\,, \tag{3.2}$$

and the last equality holds in case when U has a potential density u.

The Lévy measure π of Y is given by (see e.g. [49], pp. 197–198)

$$\pi(A) = \int_A \int_0^\infty p(t,x)\, \mu(dt)\, dx = \int_A J(x)\, dx\,, \quad A \subset \mathbb{R}^d\,,$$

where

$$J(x) := \int_0^\infty p(t,x)\, \mu(dt) = \int_0^\infty p(t,x) \mu(t) dt\,, \tag{3.3}$$

is called the jumping function of Y. The last equality is valid in the case when $\mu(dt)$ has a density $\mu(t)$. Define the function $j : (0,\infty) \to (0,\infty)$ by

$$j(r) := \int_0^\infty (4\pi)^{-d/2} t^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(dt)\,, \quad r > 0\,, \tag{3.4}$$

and note that by (3.3), $J(x) = j(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$. We state the following well-known conditions describing when a Lévy process is a subordinate Brownian motion (for a proof, see e.g. [34], pp. 190–192).

Proposition 3.1 *Let $Y = (Y_t : t \geq 0)$ be a d-dimensional Lévy process with the characteristic triple (b, A, π). Then Y is a subordinate Brownian motion if and only if π has a rotationally invariant density $x \mapsto j(|x|)$ such that $r \mapsto j(\sqrt{r})$ is a completely monotone function on $(0,\infty)$, $A = cI_d$ with $c \geq 0$, and $b = 0$.*

Example 3.2 (i) Let $\phi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha < 2$, and let S be the corresponding $\alpha/2$-stable subordinator. The characteristic exponent of the subordinate process Y is equal to $\Phi(x) = \phi(|x|) = |x|^\alpha$. Hence Y is a rotationally invariant α-stable process. From now on we will (imprecisely) refer to this process as a symmetric α-stable process. Y is transient if and only if $d > \alpha$. The jumping function of Y is given by

$$J(x) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}\,|x|^{-\alpha-d}, \quad x \in \mathbb{R}^d\,,$$

and when $d \geq 3$, the Green function of Y is given by the Riesz kernel

$$G(x) = \frac{1}{\pi^{d/2}2^{\alpha}}\frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}|x|^{\alpha-d}, \quad x \in \mathbb{R}^d\,.$$

(ii) For $0 < \alpha < 2$ and $m > 0$, let $\phi(\lambda) = (\lambda + m^{\alpha/2})^{2/\alpha} - m$, and let S be the corresponding relativistic $\alpha/2$-stable subordinator. The characteristic exponent of the subordinate process Y is equal to $\Phi(x) = \phi(|x|) = (|x|^2 + m^{\alpha/2})^{2/\alpha} - m$. The process Y is called the *symmetric relativistic α-stable process*. Y is transient if and only if $d > 2$.
(iii) Let $\phi(\lambda) = \log(1+\lambda)$, and let S be the corresponding gamma subordinator. The characteristic exponent of the subordinate process Y is given by $\Phi(x) = \log(1 + |x|^2)$. The process Y is known in some finance literature (see [40] and [29]) as a *variance gamma* process (at least for $d = 1$). Y is transient if and only if $d > 2$.
(iv) For $0 < \alpha < 2$, let $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$, and let S be the corresponding subordinator. The characteristic exponent of the subordinate process Y is given by $\Phi(x) = \log(1 + |x|^{\alpha})$. The process Y is known as a rotationally invariant *geometric α-stable process*. From now on we will (imprecisely) refer to this process as a symmetric geometric α-stable process. Y is transient if and only if $d > \alpha$.
(v) For $0 < \alpha < 2$, let $\phi^{(1)}(\lambda) = \log(1+\lambda^{\alpha/2})$, and for $n > 1$, let $\phi^{(n)}(\lambda) = \phi^{(1)}(\phi^{(n-1)}(\lambda)$. Let $S^{(n)}$ be the corresponding iterated geometric stable subordinator. Denote $Y^{(n)}_t = X(S^{(n)}_t)$. $Y^{(n)}$ is transient if and only if $d > 2(\alpha/2)^n$.
(vi) For $0 < \alpha < 2$ and let $\phi(\lambda) = b\lambda + \lambda^{\alpha/2}$, and let S be the corresponding subordinator. The characteristic exponent of the subordinate process Y is $\Phi(x) = b|x|^2 + |x|^{\alpha}$. Hence Y is the sum of a (multiple of) Brownian motion and an independent α-stable process. Similarly, we can realize the sum of an α-stable and an independent β-stable processes by subordinating Brownian motion X with a subordinator having the Laplace exponent $\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}$.
(vii) The characteristic exponent of the subordinate Brownian motion with the Bessel subordinator S_I is $\log((1 + |x|^2) + \sqrt{(1 + |x|^2)^2 - 1})$ and so this process is transient if and only if $d > 1$. The characteristic exponent of the subordinate Brownian motion with the Bessel subordinator S_K is $\frac{1}{2}(\log((1 + |x|^2) + \sqrt{(1 + |x|^2)^2 - 1}))^2$ and so this process is transient if and only if $d > 2$.
(viii) For $\alpha \in (0, 2), \beta \in (0, 2 - \alpha)$, let S be the subordinator with Laplace exponent $\phi(\lambda) = \lambda^{\alpha/2}(\log(1 + \lambda))^{\beta/2}$. The characteristic exponent of the subordinate process Y is $\Phi(x) = |x|^{\alpha}(\log(1 + |x|^2))^{\beta/2}$. Y is transient if and only if $d > \alpha + \beta$.

(ix) For $\alpha \in (0,2), \beta \in (0,\alpha)$, let S be the subordinator with Laplace exponent $\phi(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{-\beta/2}$. The characteristic exponent of the subordinate process Y is $\Phi(x) = |x|^\alpha(\log(1+|x|^2))^{-\beta/2}$. Y is transient if and only if $d > \alpha-\beta$.

In order to establish the asymptotic behaviors of the Green function G and the jumping function J of the subordinate Brownian motion Y, we start by defining an auxiliary function. For any slowly varying function ℓ at infinity and any $\xi > 0$, let

$$f_{\ell,\xi}(y,t) := \begin{cases} \frac{\ell(1/y)}{\ell(4t/y)}, & y < \frac{t}{\xi}\,, \\ 0, & y \geq \frac{t}{\xi}\,. \end{cases}$$

Now we state and prove the key technical lemma.

Lemma 3.3 *Suppose that $w : (0,\infty) \to (0,\infty)$ is a decreasing function satisfying the following two assumptions:*

(i) There exist constants $c_0 > 0$ and $\beta \in [0,2]$ with $\beta > 1-d/2$, and a continuous functions $\ell : (0,\infty) \to (0,\infty)$ slowly varying at ∞ such that

$$w(t) \sim \frac{c_0}{t^\beta \ell(1/t)}\,, \quad t \to 0+\,. \tag{3.5}$$

(ii) If $d=1$ or $d=2$, then there exist a constant $c_\infty > 0$ and a constant $\gamma < d/2$ such that

$$w(t) \sim c_\infty t^{\gamma-1}\,, \quad t \to +\infty\,. \tag{3.6}$$

Let $g : (0,\infty) \to (0,\infty)$ be a function such that

$$\int_0^\infty t^{d/2-2+\beta}\, e^{-t} g(t)\, dt < \infty\,.$$

If there is $\xi > 0$ such that $f_{\ell,\xi}(y,t) \leq g(t)$ for all $y,t > 0$, then

$$I(x) := \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} w(t)\, dt \sim \frac{c_0 \Gamma(d/2+\beta-1)}{4^{1-\beta}\pi^{d/2}} \frac{1}{|x|^{d+2\beta-2}\,\ell(\frac{1}{|x|^2})}\,, \quad |x| \to 0\,.$$

Proof. Let us first note that the assumptions of the lemma guarantee that $I(x) < \infty$ for every $x \neq 0$. By a change of variable we get

$$\begin{aligned} \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} w(t)\, dt &= \frac{|x|^{-d+2}}{4\pi^{d/2}} \int_0^\infty t^{d/2-2} e^{-t} w\left(\frac{|x|^2}{4t}\right) dt \\ &= \frac{1}{4\pi^{d/2}} \left(|x|^{-d+2} \int_0^{\xi|x|^2} + |x|^{-d+2} \int_{\xi|x|^2}^\infty \right) \\ &= \frac{1}{4\pi^{d/2}} \left(|x|^{-d+2} I_1 + |x|^{-d+2} I_2 \right)\,. \end{aligned}$$

We first consider I_1 for the case $d = 1$ or $d = 2$. It follows from the assumptions that there exists a positive constant c_1 such that $w(s) \le c_1 s^{\gamma-1}$ for all $s \ge 1/(4\xi)$. Thus

$$\begin{aligned} I_1 &\le \int_0^{\xi|x|^2} t^{d/2-2} e^{-t} c_1 \left(\frac{|x|^2}{4t}\right)^{\gamma-1} dt \\ &\le c_2 |x|^{2\gamma-2} \int_0^{\xi|x|^2} t^{d/2-\gamma-1}\, dt = c_3 |x|^{d-2} . \end{aligned}$$

It follows that

$$\lim_{|x|\to 0} \frac{|x|^{-d+2} I_1}{\frac{1}{|x|^{d+2\beta-2}\,\ell(\frac{1}{|x|^2})}} = 0 . \tag{3.7}$$

In the case $d \ge 3$, we proceed similarly, using the bound $w(s) \le w(1/(4\xi))$ for $s \ge 1/(4\xi)$.

Now we consider I_2:

$$\begin{aligned} |x|^{-d+2} I_2 &= \frac{1}{|x|^{d-2}} \int_{\xi|x|^2}^{\infty} t^{d/2-2} e^{-t} w\left(\frac{|x|^2}{4t}\right) dt \\ &= \frac{4^\beta}{|x|^{d+2\beta-2}\,\ell(\frac{1}{|x|^2})} \int_{\xi|x|^2}^{\infty} t^{d/2-2+\beta} e^{-t} \frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\left(\frac{|x|^2}{4t}\right)^\beta \ell(\frac{4t}{|x|^2})}} \frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})}\, dt . \end{aligned}$$

Using the assumption (3.5), we can see that there is a constant $c > 0$ such that

$$\frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\left(\frac{|x|^2}{4t}\right)^\beta \ell(\frac{4t}{|x|^2})}} < c$$

for all t and x satisfying $|x|^2/(4t) \le 1/(4\xi)$. Since ℓ is slowly varying at infinity,

$$\lim_{|x|\to 0} \frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})} = 1$$

for all $t > 0$. Note that

$$\frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})} = f_{\ell,\xi}(|x|^2, t) .$$

It follows from the assumption that

$$t^{d/2-2+\beta} e^{-t} \frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\left(\frac{|x|^2}{4t}\right)^\beta \ell(\frac{4t}{|x|^2})}} \frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})} \le c t^{d/2-2+\beta} e^{-t} g(t) .$$

Therefore, by the dominated convergence theorem we have

$$\lim_{|x|\to 0}\int_{\xi|x|^2}^{\infty} t^{d/2-2+\beta}e^{-t}\frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\left(\frac{|x|^2}{4t}\right)^{\beta}\ell(\frac{4t}{|x|^2})}}\frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})}\,dt=\int_0^{\infty} c_0t^{d/2-2+\beta}e^{-t}\,dt=c_0\Gamma(d/2+\beta-1)\,.$$

Hence,

$$\lim_{|x|\to 0}\frac{|x|^{-d+2}I_2}{\frac{4^{\beta}}{|x|^{d+2\beta-2}\ell(\frac{1}{|x|^2})}}=c_0\Gamma(d/2+\beta-1)\,. \tag{3.8}$$

Finally, combining (3.7) and (3.8) we get

$$\lim_{|x|\to 0}\frac{I(x)}{\frac{1}{|x|^{d+2\beta-2}\ell(\frac{1}{|x|^2})}}=\frac{c_0\Gamma(d/2+\beta-1)}{4^{1-\beta}\pi^{d/2}}\,.$$

□

Remark 3.4 *Note that if in (3.5) we have that $\ell=1$, then $f_{\ell,\xi}=1$, hence $f_{\ell,\xi}(y,t)\le g(y)$ and $\int_0^\infty t^{d/2+\beta-2}e^{-t}g(t)\,dt<\infty$ with $g=1$ provided $\beta>1-d/2$.*

3.2 Asymptotic behavior of the Green function

The goal of this subsection is to establish the asymptotic behavior of the Green function $G(x)$ of the subordinate process Y under certain assumptions on the Laplace exponent of the subordinator S. We start with the asymptotic behavior when $|x|\to 0$ for the following cases: (1) $\phi(\lambda)$ has a power law behavior at ∞, (2) S is a geometric $\alpha/2$-stable subordinator, $0<\alpha\le 2$, (3) S is an iterated geometric stable subordinator, (4) S is a Bessel subordinator, and (v) S is the subordinator corresponding to Example 2.15 or Example 2.16.

Theorem 3.5 *Suppose that $S=(S_t:t\ge 0)$ is a subordinator whose Laplace exponent $\phi(\lambda)=b\lambda+\int_0^\infty(1-e^{-\lambda t})\,\mu(dt)$ satisfies one of the following two assumptions:*

(i) $b>0$,

(ii) S is a special subordinator and $\phi(\lambda)\sim\gamma^{-1}\lambda^{\alpha/2}$ as $\lambda\to\infty$, for $0<\alpha<2$.

If Y is transient, then

$$G(x)\sim\frac{\gamma}{\pi^{d/2}2^{\alpha}}\frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}|x|^{\alpha-d},\quad |x|\to 0, \tag{3.9}$$

(where in case (i), $\gamma^{-1}=b$ and $\alpha=2$).

Proof. (i) In this case, $\phi(\lambda) \sim b\lambda$, $\lambda \to \infty$. By Proposition 2.7, the potential measure U has a continuous density u satisfying $u(0+) = 1/b = \gamma$ and $u(t) \le u(0+)$ for all $t > 0$. Note first that by change of variables

$$\int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) u(t)\, dt = \frac{|x|^{-d+2}}{4\pi^{d/2}} \int_0^\infty s^{d/2-2} e^{-s} u\left(\frac{|x|^2}{4s}\right) ds\,. \tag{3.10}$$

By Proposition 2.7, $\lim_{x\to 0} u(|x|^2/(4s)) = u(0+) = \gamma$ for all $s > 0$ and $u(|x|^2/(4s))$ is bounded by $u(0+)$. Hence, by the bounded convergence theorem,

$$\lim_{x\to 0} \frac{1}{|x|^{-d+2}} \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) u(t)\, dt = \frac{\gamma\Gamma(d/2-1)}{4\pi^{d/2}}\,. \tag{3.11}$$

(ii) In this case the potential measure U has a decreasing density u which by (2.28) satisfies

$$u(t) \sim \frac{\gamma}{\Gamma(\alpha/2)} \frac{1}{t^{1-\alpha/2}}\,, \quad t \to 0+\,.$$

By recalling Remark 3.4, we can now apply Lemma 3.3 with $\beta = 1 - \alpha/2$ to obtain the required asymptotic behavior. □

Theorem 3.6 *For any $\alpha \in (0, 2]$, let $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ and let S be the corresponding geometric $\alpha/2$-stable subordinator. If $d > \alpha$, then the Green function of the subordinate process Y satisfies*

$$G(x) \sim \frac{\Gamma(d/2)}{2\alpha\pi^{d/2}|x|^d \log^2 \frac{1}{|x|}}\,, \quad |x| \to 0. \tag{3.12}$$

Proof. We apply Lemma 3.3 with $w(t) = u(t)$, the potential density of S. By Proposition 2.19, $u(t) \sim \frac{2}{\alpha t \log^2 t}$ as $t \to 0+$, so we take $c_0 = 2/\alpha$, $\beta = 1$ and $\ell(t) = \log^2 t$. Moreover, by the second part of Proposition 2.19, $u(t) \sim t^{\alpha/2-1}/(\Gamma(\alpha)/2)$ as $t \to +\infty$, so we can take $\gamma = \alpha/2 < d/2$. Choose $\xi = 1/2$. Let

$$f(y,t) := f_{\ell,1/2}(y,t) = \begin{cases} \frac{\log^2 y}{\log^2 \frac{y}{4t}}\,, & y < 2t\,, \\ 0\,, & y \ge 2t\,. \end{cases}$$

Define

$$g(t) := \begin{cases} \frac{\log^2 2t}{\log^2 2}\,, & t < \frac{1}{4}\,, \\ 1\,, & t \ge \frac{1}{4}\,. \end{cases}$$

In order to show that $f(y,t) \le g(t)$, first let $t < 1/4$. Then $y \mapsto f(y,t)$ is an increasing function for $0 < y < 2t$. Hence,

$$\sup_{0<y<2t} f(y,t) = f(2t,t) = \frac{\log^2 2t}{\log^2 2}\,.$$

Clearly, $f(y,1/4)=1$. For $t>1/4$, $y\mapsto f(y,t)$ is a decreasing function for $0<y<1$. Hence

$$\sup_{0<y<(2t)\wedge 1} f(y,t)=f(0,t):=\lim_{y\to 0} f(y,t)=1\,.$$

For $t>1/2$, elementary consideration gives that

$$\sup_{1<y<2t} f(y,t)\le \frac{\log^2 2t}{\log^2 2}\,.$$

Clearly,

$$\int_0^\infty t^{d/2-1}e^{-t}g(t)\,dt<\infty\,,$$

and the required asymptotic behavior follows from Lemma 3.3. $\square$

For $n\ge 1$, let $S^{(n)}$ be the iterated geometric stable subordinator with the Laplace exponent $\phi^{(n)}$. Recall that $\phi^{(1)}(\lambda)=\log(1+\lambda^{\alpha/2})$, $0<\alpha\le 2$, and $\phi^{(n)}=\phi^{(1)}\circ\phi^{(n-1)}$. Let $Y_t^{(n)}=X(S_t^{(n)})$ be the subordinate process and assume that $d>2(\alpha/2)^n$. Denote the Green function of $Y^{(n)}$ by $G^{(n)}$. We want to study the asymptotic behavior of $G^{(n)}$ using Lemma 3.3. In order to check the conditions of that lemma, we need some preparations.

For $n\in\mathbb{N}$, define $f_n:(0,1/e_n)\times(0,\infty)\to[0,\infty)$ by

$$f_n(y,t):=\begin{cases}\frac{L_{n-1}(\frac{1}{y})l_n(\frac{1}{y})^2}{L_{n-1}(\frac{4t}{y})l_n(\frac{4t}{y})^2}\,, & y<\frac{2t}{e_n}\,,\\ 0\,, & y\ge\frac{2t}{e_n}\,.\end{cases}$$

Note that f_n is equal to the function $f_{\ell,\xi}$, defined before Lemma 3.3, with $\ell(y)=L_{n-1}(y)l_n(y)^2$ and $\xi=e_n/2$. Also, for $n\in\mathbb{N}$, let

$$g_n(t):=\begin{cases} f_n(\frac{2t}{e_n},t)\,, & t<1/4\,,\\ 1\,, & t\ge 1/4\,.\end{cases}$$

Moreover, for $n\in\mathbb{N}$, define $h_n:(0,1/e_n)\times(0,\infty)\to(0,\infty)$ by

$$h_n(y,t):=\frac{l_n(\frac{1}{y})}{l_n(\frac{4t}{y})}\,.$$

Clearly, for $0<y<\frac{2t}{e_n}\wedge\frac{1}{e_n}$ we have that

$$f_n(y,t)=h_1(y,t)\dots h_{n-1}(y,t)h_n(y,t)^2\,. \tag{3.13}$$

Lemma 3.7 *For all $y\in(0,1/e_n)$ and all $t>0$ we have $f_n(y,t)\le g_n(t)$. Moreover, $\int_0^\infty t^{d/2-1}e^{-t}g_n(t)\,dt<\infty$.*

Proof. A direct calculation of partial derivative gives

$$\frac{\partial h_n}{\partial y}(y,t) = \frac{L_n(\frac{1}{y}) - L_n(\frac{4t}{y})}{yL_{n-1}(\frac{1}{y})L_{n-1}(\frac{4t}{y})l_n(\frac{4t}{y})^2}\,.$$

The denominator is always positive. Clearly, the numerator is positive if and only if $t > 1/4$. Therefore, for $t < 1/4$, $y \mapsto h_n(y,t)$ is increasing on $(0, 2t/e_n)$, while for $t > 1/4$ it is decreasing on $(0, 2t/e_n)$.

Let $t < 1/4$. It follows from (3.13) and the fact that $y \mapsto h_n(y,t)$ is increasing on $(0, 2t/e_n)$ that $y \mapsto f_n(y,t)$ is increasing for $0 < y < 2t/e_n$. Therefore,

$$\sup_{0<y<2t/e_n} f_n(y,t) \le f_n(2t/e_n, t) = g_n(t)\,.$$

Clearly, $f_n(y, 1/4) = 1$. For $y \ge 1/4$, it follows from (3.13) and the fact that $y \mapsto h_n(y,t)$ is decreasing on $(0, 2t/e_n)$ that $y \mapsto f_n(y,t)$ is decreasing for $0 < y < 1/e_n$. Hence

$$\sup_{0<y<\frac{2t}{e_n}\wedge\frac{1}{e_n}} f_n(y,t) = f(0,t) := \lim_{y\to 0} f_n(y,t) = 1\,.$$

For $t > 1/2$, elementary consideration gives that

$$\sup_{\frac{1}{e_n}<y<\frac{2t}{e_n}\wedge\frac{1}{e_n}} f_n(y,t) \le g_n(t).$$

The integrability statement of the lemma is obvious. □

Theorem 3.8 *If $d > 2(\alpha/2)^n$, we have*

$$G^{(n)}(x) \sim \frac{\Gamma(d/2)}{2\alpha\pi^{d/2}|x|^d L_{n-1}(1/|x|^2)l_n(1/|x|^2)^2}, \quad |x| \to 0.$$

Proof. We apply Lemma 3.3 with $w(t) = u^{(n)}(t)$, the potential density of $S^{(n)}$. By Proposition 2.21,

$$u^{(n)}(t) \sim \frac{2}{\alpha t L_{n-1}(1/t)l_n(1/t)^2}\,, \quad t \to 0+,$$

so we take $c_0 = 2/\alpha$, $\beta = 1$ and $\ell(t) = L_{n-1}(t)l_n(t)^2$. By the second part of Proposition 2.21, $u^{(n)}(t)$ is of order $t^{(\alpha/2)^n-1}$ as $t \to \infty$, so we may take $\gamma = (\alpha/2)^n < d/2$. Choose $\xi = e_n/2$. The result follows from Lemma 3.3 and Lemma 3.7 □

Using arguments similar to that used in the proof of Theorem 3.6, together with Proposition 2.22, (2.32) and (2.34), we can easily get the following two results.

Theorem 3.9 *(i) Suppose $d > 1$. Let G_I be the Green function of the subordinate Brownian motion via the Bessel subordinator S_I. Then*

$$G_I(x) \sim \frac{\Gamma(d/2)}{4\pi^{d/2}|x|^d \log^2 \frac{1}{|x|}}, \quad |x| \to 0.$$

(ii) Suppose $d > 2$. Let G_K be the Green function of the subordinate Brownian motion via the Bessel subordinator S_K. Then

$$G_K(x) \sim \frac{\Gamma(d/2)}{4\pi^{d/2}|x|^d \log^3 \frac{1}{|x|}}, \quad |x| \to 0.$$

Theorem 3.10 *Suppose $\alpha \in (0,2), \beta \in (0, 2-\alpha)$ and that S is the subordinator corresponding Example 2.15. If $d > \alpha+\beta$, the Green function of the subordinate Brownian motion via S satisfies*

$$G(x) \sim \frac{\alpha\Gamma((d-\alpha)/2}{2^{\alpha+1}\pi^{d/2}\Gamma(1+\alpha/2)} \frac{1}{|x|^{d-\alpha}(\log(1/|x|^2))^{\beta/2}}, \quad |x| \to 0.$$

Theorem 3.11 *Suppose $\alpha \in (0,2)$, $\beta \in (0,\alpha)$ and that S is the subordinator corresponding Example 2.16. If $d > \alpha$, the Green function of the subordinate Brownian motion via S satisfies*

$$G(x) \sim \frac{\alpha\Gamma((d-\alpha)/2}{2^{\alpha+1}\pi^{d/2}\Gamma(1+\alpha/2)} \frac{(\log(1/|x|^2))^{\beta/2}}{|x|^{d-\alpha}}, \quad |x| \to 0.$$

Proof. The proof of this theorem is similar to that of Theorem 3.6, the only difference is that in this case when applying Lemma 3.3 we take the slowly varying function ℓ to be

$$\ell(t) = \begin{cases} (\log^2 t)^{-\beta/4}, & t \geq 2, \\ (\log^2 2)^{-\beta/4}, & t \leq 2. \end{cases}$$

Then using argument similar to that in the proof of Theorem 3.6 we can show that with the functions defined by

$$f(y,t) = \begin{cases} \frac{\ell(1/y)}{\ell(4t/y)}, & y < 2t, \\ 0, & y \geq 2t \end{cases} = \begin{cases} \left(\frac{\log^2(4t/y)}{\log^2(1/y)}\right)^{\beta/4}, & t < 1/4,\ y < 2t, \\ \left(\frac{\log^2(4t/y)}{\log^2(1/y)}\right)^{\beta/4}, & t \geq 1/4,\ y < 1/2, \\ \left(\frac{\log^2(4t/y)}{\log^2 2}\right)^{\beta/4}, & t < 1/4,\ 1/2 < y < 2t, \\ 0, & y \geq 2t, \end{cases}$$

and

$$g(t) = \begin{cases} \left(\frac{\log^2(8t)}{\log^2 2}\right)^{\beta/4}, & t > 1/4, \\ 1, & t \leq 1/4. \end{cases}$$

we have $f(y,t) \le g(t)$ for all $y>0$ and $t>0$. The rest of the proof is exactly the same as that of Theorem 3.6. $\square$

By using results and methods developed so far, we can obtain the following table of the asymptotic behavior of the Green function of the subordinate Brownian motion depending on the Laplace exponent of the subordinator. The left column contains Laplace exponents, while the right column describes the asymptotic behavior of $G(x)$ as $|x| \to 0$, up to a constant.

Laplace exponent ϕ	Green function $G \sim c\cdot$
λ	$\|x\|^{-d}\ \|x\|^2$
$\int_0^1 \lambda^{1-\beta}\beta^\eta d\beta \quad (\eta > -1)$	$\|x\|^{-d}\ \|x\|^2 \log(\frac{1}{\|x\|^2})^{\eta+1}$
$\lambda^{\alpha/2}(\log(1+\lambda))^{\beta/2}, 0<\alpha<2, 0<\beta<2-\alpha,$	$\|x\|^{-d}\ \|x\|^\alpha \frac{1}{(\log(1/\|x\|^2))^{\beta/2}}$
$\lambda^{\alpha/2}, 0<\alpha<2$	$\|x\|^{-d}\ \|x\|^\alpha$
$\lambda^{\alpha/2}(\log(1+\lambda))^{-\beta/2}, 0<\alpha<2, 0<\beta<\alpha,$	$\|x\|^{-d}\ \|x\|^\alpha(\log(1/\|x\|^2))^{\beta/2}$
$\log(1+\lambda^{\alpha/2}), 0<\alpha\le 2$	$\|x\|^{-d}\ \frac{1}{\log^2 \frac{1}{\|x\|^2}}$
$\phi^{(n)}(\lambda)$	$\|x\|^{-d}\ \frac{1}{L_{n-1}(\frac{1}{x}) l_n(\frac{1}{x})^2}$

Notice that the singularity of the Green function increases from top to bottom. This is, of course, a consequence of the fact that the corresponding subordinator becomes slower and slower, hence the subordinate process Y moves also more slowly for small times.

We look now at the asymptotic behavior of the Green function $G(x)$ for $|x| \to \infty$.

Theorem 3.12 *Suppose that $S=(S_t : t\ge 0)$ is a subordinator whose Laplace exponent*

$$\phi(\lambda) = b\lambda + \int_0^\infty (1-e^{-\lambda t})\,\mu(dt)$$

is a special Bernstein function such that $\lim_{\lambda\to\infty}\phi(\lambda)=\infty$. If $\phi(\lambda) \sim \gamma^{-1}\lambda^{\alpha/2}$ as $\lambda \to 0+$ for $\alpha \in (0,2]$ with $\alpha < d$ and a positive constant γ, then

$$G(x) \sim \frac{\gamma}{\pi^{d/2}2^\alpha}\frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}|x|^{\alpha-d}$$

as $|x| \to \infty$.

Proof. By Theorem 2.1 the potential measure of the subordinator has a decreasing density. By use of Theorem 2.17, the assumption $\phi(\lambda) \sim \gamma^{-1}\lambda^{\alpha/2}$ as $\lambda \to 0+$ implies that

$$u(t) \sim \frac{\gamma}{\Gamma(\alpha/2)}\, t^{\alpha/2-1}, \quad t\to\infty\,.$$

Since u is decreasing and integrable near 0, it is easy to show that there exists $t_0 > 0$ such that $u(t) \le t^{-1}$ for all $t \in (0, t_0)$. Hence, we can find a positive constant C such that

$$u(t) \le C(t^{-1} \vee t^{\alpha/2-1}) \,. \tag{3.14}$$

By change of variables we have

$$\begin{aligned}
&\int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) u(t)\, dt \\
= \quad & \frac{1}{4\pi^{d/2}} |x|^{-d+2} \int_0^\infty s^{d/2-2} e^{-s} u\left(\frac{|x|^2}{4s}\right) ds \\
= \quad & \frac{\gamma}{4\pi^{d/2}\Gamma(\alpha/2)} |x|^{-d+\alpha} \int_0^\infty s^{d/2-2} e^{-s} \frac{u\left(\frac{|x|^2}{4s}\right)}{\frac{\gamma}{\Gamma(\alpha/2)}\left(\frac{|x|^2}{4s}\right)^{\alpha/2-1}} \left(\frac{1}{4s}\right)^{\alpha/2-1} ds \\
= \quad & \frac{\gamma}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)} |x|^{-d+\alpha} \int_0^\infty s^{d/2-\alpha/2-1} e^{-s} \frac{u\left(\frac{|x|^2}{4s}\right)}{\frac{\gamma}{\Gamma(\alpha/2)}\left(\frac{|x|^2}{4s}\right)^{\alpha/2-1}} ds \,.
\end{aligned}$$

Let $|x| \ge 2$. Then by (3.14),

$$\frac{u\left(\frac{|x|^2}{4s}\right)}{\left(\frac{|x|^2}{4s}\right)^{\alpha/2-1}} \le C\left(\left(\frac{|x|^2}{4s}\right)^{-\alpha/2} \vee 1\right) \le C(s^{\alpha/2} \vee 1) \,.$$

It follows that the integrand in the last formula above is bounded by an integrable function, so we may use the dominated convergence theorem to obtain

$$\lim_{|x|\to\infty} \frac{1}{|x|^{-d+\alpha}} \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) u(t)\, dt = \frac{\gamma}{2^\alpha \pi^{d/2}} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \,,$$

which proves the result. □

Examples of subordinators that satisfy the assumptions of the last theorem are relativistic $\beta/2$-stable subordinators (with α in the theorem equal to 2), gamma subordinator ($\alpha = 2$), geometric $\beta/2$-stable subordinators ($\alpha = \beta$), iterated geometric stable subordinators, Bessel subordinators S_I, $\alpha = 1$, and S_K, $\alpha = 2$, and also subordinators corresponding to Examples 2.15 and 2.16.

Remark 3.13 *Suppose that $S_t = bt + \tilde{S}_t$ where b is positive and $\tilde{S}_t$ is a pure jump special subordinator with finite expectation. Then $\phi(\lambda) \sim b\lambda$, $\lambda \to \infty$, and $\phi(\lambda) \sim \phi'(0+)\lambda$, $\lambda \to 0$. This implies that, when $d \ge 3$, the Green function of the subordinate process Y satisfies $G(x) \asymp G^{(2)}(x)$ for all $x \in \mathbb{R}^d$.*

3.3 Asymptotic behavior of the jumping function

The goal of this subsection is to establish results on the asymptotic behavior of the jumping function near zero, and results about the rate of decay of the jumping function near zero and near infinity. We start by stating two theorems on the asymptotic behavior of the jumping functions at zero for subordinate Brownian motions via subordinators corresponding to Examples 2.15 and 2.16. We omit the proofs which rely on Lemma 3.3 and are similar to proofs of Theorems 3.10 and 3.11.

Theorem 3.14 *Suppose $\alpha \in (0, 2), \beta \in (0, 2-\alpha)$ and that S is the subordinator corresponding to Example 2.15. Then the jumping function of the subordinate Brownian motion Y via S satisfies*

$$J(x) \sim \frac{\alpha\Gamma((d+\alpha)/2)}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\alpha/2)} \frac{(\log(1/|x|^2))^{\beta/2}}{|x|^{d+\alpha}}, \quad |x| \to 0.$$

Theorem 3.15 *Suppose $\alpha \in (0, 2), \beta \in (0, \alpha)$ and that S is the subordinator corresponding to Example 2.16. Then the jumping function of the subordinate Brownian motion Y via S satisfies*

$$J(x) \sim \frac{\alpha\Gamma((d+\alpha)/2)}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\alpha/2)} \frac{1}{|x|^{d+\alpha}(\log(1/|x|^2))^{\beta/2}}, \quad |x| \to 0.$$

We continue by establishing the asymptotic behavior of the jumping function for the geometric stable processes. More precisely, for $0 < \alpha \le 2$, let $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$, S the corresponding geometric $\alpha/2$-stable subordinator, $Y_t = X(S_t)$ the subordinate process and J the jumping function of Y.

Theorem 3.16 *For every $\alpha \in (0, 2]$, it holds that*

$$J(x) \sim \frac{\alpha\Gamma(d/2)}{2|x|^d}, \quad |x| \to 0.$$

Proof. We again apply Lemma 3.3, this time with $w(t) = \mu(t)$, the density of the Lévy measure of S. By Proposition 2.24 (i), $\mu(t) \sim \frac{\alpha}{2t}$ as $t \to 0+$, so we take $c_0 = \alpha/2$, $\beta = 1$ and $\ell(t) = 1$. By Proposition 2.24 (ii), $\mu(t)$ is of the order $t^{-\alpha/2-1}$ as $t \to +\infty$, so we may take $\gamma = -\alpha/2$. Choose $\xi = 1/2$ and let $g = 1$. □

Theorem 3.17 *For every $\alpha \in (0, 2)$ we have*

$$J(x) \sim \frac{\alpha}{2^{\alpha+1}\pi^{d/2}} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}|x|^{-d-\alpha}, \quad |x| \to \infty.$$

Proof. By Proposition 2.24 (ii),

$$\mu(t) \sim \frac{\alpha}{2\Gamma(1-\alpha/2)}\, t^{-\alpha/2-1}\,, \quad t \to \infty\,.$$

Now combine this with Proposition 2.24 (i) to get that

$$\mu(t) \le C(t^{-1} \vee t^{-\alpha/2-1})\,, \quad t > 0\,. \tag{3.15}$$

By change of variables we have

$$\begin{aligned}
&\int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) \mu(t)\, dt \\
&= \frac{1}{4\pi^{d/2}} |x|^{-d+2} \int_0^\infty s^{d/2-2} e^{-s} \mu\left(\frac{|x|^2}{4s}\right) ds \\
&= \frac{\alpha}{8\pi^{d/2}\Gamma(1-\alpha/2)} |x|^{-d-\alpha} \int_0^\infty s^{d/2-2} e^{-s} \frac{\mu\left(\frac{|x|^2}{4s}\right)}{\frac{\alpha}{2\Gamma(1-\alpha/2)}\left(\frac{|x|^2}{4s}\right)^{-\alpha/2-1}} \left(\frac{1}{4s}\right)^{-\alpha/2-1} ds \\
&= \frac{\alpha}{2^{\alpha+1}\pi^{d/2}\Gamma(1-\alpha/2)} |x|^{-d-\alpha} \int_0^\infty s^{d/2+\alpha/2-1} e^{-s} \frac{\mu\left(\frac{|x|^2}{4s}\right)}{\frac{\alpha}{a\Gamma(1-\alpha/2)}\left(\frac{|x|^2}{4s}\right)^{-\alpha/2-1}} ds\,.
\end{aligned}$$

Let $|x| \ge 2$. Then by (3.15),

$$\frac{u\left(\frac{|x|^2}{4s}\right)}{\left(\frac{|x|^2}{4s}\right)^{-\alpha/2-1}} \le C\left(\left(\frac{|x|^2}{4s}\right)^{\alpha/2} \vee 1\right) \le C(s^{-\alpha/2} \vee 1)\,.$$

It follows that the integrand in the last display above is bounded by an integrable function, so we may use the dominated convergence theorem to obtain

$$\lim_{|x|\to\infty} \frac{1}{|x|^{-d-\alpha}} \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) \mu(t)\, dt = \frac{\alpha}{2^{\alpha+1}\pi^{d/2}} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\,, \tag{3.16}$$

which proves the result. □

In the case $\alpha = 2$, the behavior of J at ∞ is different and is given in the following result.

Theorem 3.18 *When $\alpha = 2$, we have*

$$J(x) \sim 2^{-d/2}\pi^{-\frac{d-1}{2}} \frac{e^{-|x|}}{|x|^{\frac{d+1}{2}}}, \quad |x| \to \infty.$$

Proof. By change of variables we get that

$$\begin{aligned} J(x) &= \frac{1}{2}\int_0^\infty t^{-1}e^{-t}(4\pi t)^{-d/2}\exp(-\frac{|x|^2}{2})dt \\ &= 2^{-d-1}\pi^{-d/2}|x|^{-d}\int_0^\infty s^{\frac{d}{2}-1}e^{-\frac{s}{4}-\frac{|x|^2}{s}}ds \\ &= 2^{-d-1}\pi^{-d/2}|x|^{-d}I(|x|), \end{aligned}$$

where

$$I(r) = \int_0^\infty s^{\frac{d}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}ds.$$

Using the change of variable $u = \frac{\sqrt{s}}{2} - \frac{r}{\sqrt{s}}$ we get

$$\begin{aligned} I(r) &= e^{-r}\int_0^\infty s^{\frac{d}{2}-1}e^{-(\frac{\sqrt{s}}{2}-\frac{r}{\sqrt{s}})^2}ds \\ &= e^{-r}\int_{-\infty}^\infty \frac{2(u+\sqrt{u^2+2r})^d}{\sqrt{u^2+2r}}e^{-u^2}du \\ &= 2e^{-r}r^{\frac{d-1}{2}}\int_{-\infty}^\infty \frac{u+\sqrt{u^2+2r}}{\sqrt{u^2+2r}}(\frac{u}{\sqrt{r}}+\sqrt{\frac{u^2}{r}+2})^{d-1}e^{-u^2}du\,. \end{aligned}$$

Therefore by the dominated convergence theorem we obtain

$$I(r) \sim 2^{\frac{d}{2}+1}\sqrt{\pi}e^{-r}r^{\frac{d-1}{2}}, \quad r\to\infty.$$

Now the assertion of the theorem follows immediately. □

Let $Y_t^{(n)} = X(S_t^{(n)})$ be Brownian motion subordinate by the iterated geometric subordinator $S^{(n)}$, and let $J^{(n)}$ be the corresponding jumping function. Because of Remark 2.26, we were unable to determine the asymptotic behavior of $J^{(n)}$.

Assume now that $\phi(\lambda)$ is a complete Bernstein function which asymptotically behaves as $\lambda^{\alpha/2}$ as $\lambda \to 0+$ (resp. as $\lambda\to\infty$). Similar arguments as in Theorems 3.16 and 3.17 would yield that the jumping function J of the corresponding subordinate Brownian motion behaves (up to a constant) as $|x|^{-\alpha-d}$ as $|x|\to\infty$ (resp. as $|x|^{-\alpha-d}$ as $|x|\to 0$). We are not going to pursue this here, because, firstly, such behavior of the jumping kernel is known from the case of α-stable processes, and secondly, in the sequel we will not be interested in precise asymptotics of J, but rather in the rate of decay near zero and near infinity. Recall that $\mu(t)$ denotes the decreasing density of the Lévy measure of the subordinator S (which exists since ϕ is assumed to be complete Bernstein), and recall that the function $j:(0,\infty)\to(0,\infty)$ was defined by

$$j(r) := \int_0^\infty (4\pi)^{-d/2}t^{-d/2}\exp\left(-\frac{r^2}{4t}\right)\mu(t)\,dt\,, \quad r>0\,, \tag{3.17}$$

and that $J(x) = j(|x|)$, $x\in\mathbb{R}^d\setminus\{0\}$.

Proposition 3.19 *Suppose that there exists a positive constant $c_1 > 0$ such that*

$$\mu(t) \leq c_1\,\mu(2t) \quad \text{for all } t \in (0,8)\,, \tag{3.18}$$
$$\mu(t) \leq c_1\,\mu(t+1) \quad \text{for all } t > 1\,. \tag{3.19}$$

Then there exists a positive constant c_2 such that

$$j(r) \leq c_2 j(2r) \quad \text{for all } r \in (0,2)\,, \tag{3.20}$$
$$j(r) \leq c_2 j(r+1) \quad \text{for all } r > 1\,. \tag{3.21}$$

Also, $r \mapsto j(r)$ is decreasing on $(0,\infty)$.

Proof. For simplicity we redefine in this proof the function j by dropping the factor $(4\pi)^{-d/2}$ from its definition. This does not effect (3.20) and (3.21).

Let $0 < r < 2$. We have

$$\begin{aligned}
j(2r) &= \int_0^\infty t^{-d/2}\exp(-r^2/t)\mu(t)\,dt \\
&= \frac{1}{2}\left(\int_0^{1/2} t^{-d/2}\exp(-r^2/t)\mu(t)\,dt + \int_{1/2}^\infty t^{-d/2}\exp(-r^2/t)\mu(t)\,dt\right. \\
&\quad \left. + \int_0^2 t^{-d/2}\exp(-r^2/t)\mu(t)\,dt + \int_2^\infty t^{-d/2}\exp(-r^2/t)\mu(t)\,dt\right) \\
&\geq \frac{1}{2}\left(\int_{1/2}^\infty t^{-d/2}\exp(-r^2/t)\mu(t)\,dt + \int_0^2 t^{-d/2}\exp(-r^2/t)\mu(t)\,dt\right) \\
&= \frac{1}{2}(I_1 + I_2).
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 &= \int_{1/2}^\infty t^{-d/2}\exp(-\frac{r^2}{t})\mu(t)\,dt = \int_{1/2}^\infty t^{-d/2}\exp(-\frac{r^2}{4t})\exp(-\frac{3r^2}{4t})\mu(t)\,dt \\
&\geq \int_{1/2}^\infty t^{-d/2}\exp(-\frac{r^2}{4t})\exp(-\frac{3r^2}{2})\mu(t)\,dt \geq e^{-6}\int_{1/2}^\infty t^{-d/2}\exp(-\frac{r^2}{4t})\mu(t)\,dt\,, \\
I_2 &= \int_0^2 t^{-d/2}\exp(-\frac{r^2}{t})\mu(t)\,dt = 4^{-d/2+1}\int_0^{1/2} s^{-d/2}\exp(-\frac{r^2}{4s})\mu(4s)\,ds \\
&\geq c_1^{-2}4^{-d/2+1}\int_0^{1/2} s^{-d/2}\exp(-\frac{r^2}{4s})\mu(s)\,ds.
\end{aligned}$$

Combining the three displays above we get that $j(2r) \geq c_3\,j(r)$ for all $r \in (0,2)$.

To prove (3.21) we first note that for all $t \geq 2$ and all $r \geq 1$ it holds that

$$\frac{(r+1)^2}{t} - \frac{r^2}{t-1} \leq 1\,.$$

This implies that

$$\exp(-\frac{(r+1)^2}{4t}) \geq e^{-1/4}\exp(-\frac{r^2}{4(t-1)}), \qquad \text{for all } r>1, t>2\,. \tag{3.22}$$

Now we have

$$\begin{aligned} j(r+1) &= \int_0^\infty t^{-d/2}\exp(-\frac{(r+1)^2}{4t})\mu(t)\,dt \\ &\geq \frac{1}{2}\Big(\int_0^8 t^{-d/2}\exp(-\frac{(r+1)^2}{4t})\mu(t)\,dt + \int_3^\infty t^{-d/2}\exp(-\frac{(r+1)^2}{4t})\mu(t)\,dt\Big) \\ &= \frac{1}{2}(I_3+I_4). \end{aligned}$$

For I_3 note that $(r+1)^2 \leq 4r^2$ for all $r>1$. Thus

$$\begin{aligned} I_3 &= \int_0^8 t^{-d/2}\exp(-\frac{(r+1)^2}{4t})\mu(t)\,dt \geq \int_0^8 t^{-d/2}\exp(-r^2/t)\mu(t)\,dt \\ &= 4^{-d/2+1}\int_0^2 s^{-d/2}\exp(-\frac{r^2}{4s})\mu(4s)\,ds \geq c_1^{-2}4^{-d/2+1}\int_0^2 s^{-d/2}\exp(-\frac{r^2}{4s})\mu(s)\,ds\,, \\ I_4 &= \int_3^\infty t^{-d/2}\exp(-\frac{(r+1)^2}{4t})\mu(t)\,dt \geq \int_3^\infty t^{-d/2}\exp\{-1/4\}\exp(-\frac{r^2}{4(t-1)})\,\mu(t)\,dt \\ &= e^{-1/4}\int_2^\infty (s-1)^{-d/2}\exp(-\frac{r^2}{4s})\,\mu(s+1)\,ds \geq c_1^{-1}e^{-1/4}\int_2^\infty s^{-d/2}\exp(-\frac{r^2}{4s})\mu(s)\,ds\,. \end{aligned}$$

Combining the three displays above we get that $j(r+1) \geq c_4\, j(r)$ for all $r>1$. □

Suppose that $S=(S_t : t\geq 0)$ is an $\alpha/2$-stable subordinator, or a relativistic $\alpha/2$-stable subordinator, or a gamma subordinator. By the explicit forms of the Lévy densities given in Examples 2.8, 2.9 and 2.10 it is straightforward to verify that in all three cases $\mu(t)$ satisfies (3.18) and (3.19). For the Bessel subordinators, by use of asymptotic behavior of modified Bessel functions I_0 and K_0, one obtains that $\mu_I(t) \sim e^{-t}/t$, $t\to 0+$, $\mu_I(t) \sim (1/\sqrt{2\pi})\,t^{-3/2}$, $t\to\infty$, $\mu_K(t)\sim \log(1/t)/t$, $t\to 0+$, and $\mu_K(t)\sim \sqrt{\pi/2}\,e^{-2t}\,t^{-3/2}$, $t\to\infty$. From Propositions 2.27 and 2.28, it is easy to see that corresponding Lévy densities satisfy (3.18) and (3.19). In the case when S is a geometric $\alpha/2$-stable subordinator or when S is the subordinator corresponding to Example 2.15, respectively Example 2.16, these two properties follow from Proposition 2.24, and Proposition 2.27, respectively Proposition 2.28. In the case of an iterated geometric stable subordinator with $0 < \alpha < 2$, (3.19) is a consequence of Proposition 2.25, but we do not know whether (3.18) holds true. By using a different approach, we will show that if $j^{(n)} : (0,\infty) \to (0,\infty)$ is such that $J^{(n)}(x) = j^{(n)}(|x|)$, then (3.20) and (3.21) are still true.

We first observe that symmetric geometric α-stable process Y can be obtained by subordinating a symmetric α-stable process X^α via a gamma subordinator S. Indeed, the characteristic exponent of X^α being equal to $|x|^\alpha$, and the Laplace exponent of S being equal to $\log(1+\lambda)$, the composition of these two gives the characteristic exponent $\log(1+|x|^\alpha)$ of a symmetric geometric α-stable process. Let $p_\alpha(t,x,y) = p_\alpha(t,x-y)$ denote the transition densities of the symmetric α-stable process, and let $q_\alpha(t,x,y) = q_\alpha(t,x-y)$ denote the transition densities of the symmetric geometric α-stable process, $x,y \in \mathbb{R}^d$, $t \ge 0$. Then

$$q_\alpha(t,x) = \int_0^\infty p_\alpha(s,x)\frac{1}{\Gamma(t)}s^{t-1}e^{-s}ds\,. \tag{3.23}$$

Also, similarly as in (3.3), the jumping function of Y can be written as

$$J(x) = \int_0^\infty p_\alpha(t,x)t^{-1}e^{-t}\,dt\,, \quad x \in \mathbb{R}^d \setminus \{0\}\,. \tag{3.24}$$

Define functions $j^{(n)} : (0,\infty) \to (0,\infty)$ by

$$j^{(n)}(r) := \int_0^\infty t^{-d/2}\exp\left(-\frac{r^2}{4t}\right)\mu^{(n)}(t)\,dt\,, \quad r > 0\,, \tag{3.25}$$

where $\mu^{(n)}$ denotes the Lévy density of the iterated geometric subordinator, and note that by (3.3), $J^{(n)}(x) = (4\pi)^{-d/2}j^{(n)}(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$.

Proposition 3.20 *For any $\alpha \in (0,2)$ and $n \ge 1$, there exists a positive constant c such that*

$$j^{(n)}(r) \le cj^{(n)}(2r), \quad \text{for all } r > 0 \tag{3.26}$$

and

$$j^{(n)}(r) \le cj^{(n)}(r+1), \quad \text{for all } r > 1. \tag{3.27}$$

Proof. The inequality (3.27) follows from Proposition 3.19. Now we prove (3.26). It is known (see Theorem 2.1 of [12]) that there exist positive constants C_1 and C_2 such that for all $t > 0$ and all $x \in \mathbb{R}^d$,

$$C_1\min(t^{-d/\alpha}, t\,|x|^{-d-\alpha}) \le p_\alpha(t,x) \le C_2\min(t^{-d/\alpha}, t\,|x|^{-d-\alpha})\,. \tag{3.28}$$

Using these estimates one can easily see that there exists $C_3 > 0$ such that

$$p_\alpha(t,x) \le C_3 p_\alpha(t,2x), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{3.29}$$

Let $J^{(1)}(x) = J(x)$ and $q_\alpha^{(1)}(t,x) = q_\alpha(t,x)$. By use of (3.29), it follows from (3.23) and (3.24) that $J^{(1)}(x) \le C_3 J^{(1)}(2x)$, for all $x \in \mathbb{R}^d \setminus \{0\}$, and $q_\alpha^{(1)}(t,x) \le C_3 q_\alpha^{(1)}(t,2x)$, for all $t > 0$ and $x \in \mathbb{R}^d$. Further, $Y^{(2)}$ is obtained by subordinating $Y^{(1)}$ by a geometrically $\alpha/2$-stable subordinator S. Therefore,

$$J^{(2)}(x) = \frac{1}{2}\int_0^\infty q_\alpha^{(1)}(s,x)\mu_{\alpha/2}(s)\,ds\,, \quad q_\alpha^{(2)}(t,x) = \int_0^\infty q_\alpha^{(1)}(s,x) f_{\alpha/2}(t,s)\,ds\,, \tag{3.30}$$

where $\mu(s)$ is the Lévy density of S and $f_{\alpha/2}(t,s)$ the density of $\mathbb{P}(S_t \in ds)$. By use of $q_\alpha^{(1)}(s,x) \le C_3 q_\alpha^{(1)}(s,2x)$, it follows $J^{(2)}(x) \le C_3 J^{(2)}(2x)$ and $q_\alpha^{(2)}(t,x) \le C_3 q_\alpha^{(2)}(t,2x)$ for all $t > 0$ and $x \in \mathbb{R}^d$. The proof is completed by induction. □

We conclude this section with a result that is essential in proving the Harnack inequality for jump processes, and was the motivation behind Propositions 3.19 and 3.20.

Proposition 3.21 *Let Y be a subordinate Brownian motion such that the function j defined in (3.17) satisfies conditions (3.20) and (3.21). There exist positive constants C_4 and C_5 such that if $r \in (0,1)$, $x \in B(0,r)$, and H is a nonnegative function with support in $B(0,2r)^c$, then*

$$\mathbb{E}^x H(Y(\tau_{B(0,r)})) \le C_4(\mathbb{E}^x \tau_{B(0,r)}) \int H(z)J(z)\,dz$$

and

$$\mathbb{E}^x H(Y(\tau_{B(0,r)})) \ge C_5(\mathbb{E}^x \tau_{B(0,r)}) \int H(z)J(z)\,dz.$$

Proof. Let $y \in B(0,r)$ and $z \in B(0,2r)^c$. If $z \in B(0,2)$ we use the estimates

$$2^{-1}|z| \le |z-y| \le 2|z|, \tag{3.31}$$

while if $z \notin B(0,2)$ we use

$$|z| - 1 \le |z-y| \le |z| + 1. \tag{3.32}$$

Let $B \subset B(0,2r)^c$. Then by using the Lévy system we get

$$\mathbb{E}^x 1_B(Y(\tau_{B(0,r)})) = \mathbb{E}^x \int_0^{\tau_{B(0,r)}} \int_B J(z-Y_s)\,dz\,ds = \mathbb{E}^x \int_0^{\tau_{B(0,r)}} \int_B j(|z-Y_s|)\,dz\,ds\,.$$

By use of (3.20), (3.21), (3.31), and (3.32), the inner integral is estimated as follows:

$$\begin{aligned}
\int_B j(|z-Y_s|)\,dz &= \int_{B\cap B(0,2)} j(|z-Y_s|)\,dz + \int_{B\cap B(0,2)^c} j(|z-Y_s|)\,dz \\
&\le \int_{B\cap B(0,2)} j(2^{-1}|z|)\,dz + \int_{B\cap B(0,2)^c} j(|z|-1)\,dz \\
&\le \int_{B\cap B(0,2)} c_2 j(|z|)\,dz + \int_{B\cap B(0,2)^c} c_2 j(|z|)\,dz \\
&= c_2 \int_B J(z)\,dz\,.
\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}^x 1_B(Y(\tau_{B(0,r)})) &\le \mathbb{E}^x \int_0^{\tau_{B(0,r)}} c_2 \int_B J(z)\,dz \\ &= c_2\,\mathbb{E}^x(\tau_{B(0,r)}) \int 1_B(z)J(z)\,dz\,.\end{aligned}$$

Using linearity we get the above inequality when 1_B is replaced by a simple function. Approximating H by simple functions and taking limits we have the first inequality in the statement of the lemma.

The second inequality is proved in the same way. □

3.4 Transition densities of symmetric geometric stable processes

Recall that for $0 < \alpha \le 2$, $q_\alpha(t,x)$ denotes the transition density of the symmetric geometric α-stable process. The asymptotic behavior of $q_\alpha(1,x)$ as $|x| \to \infty$ is given in the following result.

Proposition 3.22 *For $\alpha \in (0,2)$ we have*

$$q_\alpha(1,x) \sim \frac{\alpha 2^{\alpha-1}\sin\frac{\alpha\pi}{2}\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{\alpha}{2})}{\pi^{\frac{d}{2}+1}|x|^{d+\alpha}}, \quad |x| \to \infty.$$

For $\alpha = 2$ we have

$$q_2(1,x) \sim 2^{-\frac{d}{2}}\pi^{-\frac{d-1}{2}}\frac{e^{-|x|}}{|x|^{\frac{d-1}{2}}}, \quad |x| \to \infty.$$

Proof. The proof of the case $\alpha < 2$ is similar to the proof of Proposition 2.29 and uses (3.28), while the proof of the case $\alpha = 2$ is similar to the proof of Theorem 3.18. We omit the details. □

The following theorem from [21] provides the sharp estimate for $q_\alpha(t,x)$ for small time t in case $0 < \alpha < 2$.

Theorem 3.23 *Let $\alpha \in (0,2)$. There are positive constants $C_1 < C_2$ such that for all $x \in \mathbb{R}^d$ and $0 < t < 1 \wedge \frac{d}{2\alpha}$,*

$$C_1 t \min(|x|^{-d-\alpha}, |x|^{-d+t\alpha}) \le q_\alpha(t,x) \le C_2 t \min(|x|^{-d-\alpha}, |x|^{-d+t\alpha})\,.$$

Proof. The following sharp estimates for the stable densities (3.28) is well known (see, for instance, [12])

$$p_\alpha(s,x) \asymp s^{-\frac{d}{\alpha}}\left(1 \wedge \frac{s^{\frac{d+\alpha}{\alpha}}}{|x|^{d+\alpha}}\right), \quad \forall s > 0 \text{ and } x \in \mathbb{R}^d\,.$$

Hence, by (3.23) it follows that $q_\alpha(t,x) \asymp \frac{1}{\Gamma(t)}\, I(t,|x|)$ where

$$\begin{aligned} I(t,r) &:= \int_0^\infty s^{-\frac{d}{\alpha}} \left(1 \wedge \frac{s^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}}\right) s^{t-1} e^{-s}\, ds \\ &= \frac{1}{r^{d+\alpha}} \int_0^{r^\alpha} s^t e^{-s}\, ds + \int_{r^\alpha}^\infty s^{t-1-d/\alpha} e^{-s}\, ds\,. \end{aligned}$$

From now on assume that $0 < t \le 1 \wedge \frac{d}{2\alpha}$. Then for $0 < r \le 1$,

$$\begin{aligned} I(t,r) &\asymp \frac{1}{r^{d+\alpha}} \int_0^{r^\alpha} s^t\, ds + \int_{r^\alpha}^1 s^{t-1-d/\alpha} e^{-s}\, ds + \int_1^\infty s^{t-1-d/\alpha} e^{-s}\, ds \\ &= \frac{1}{t+1} r^{\alpha t-d} + \frac{1}{d/\alpha - t}(r^{\alpha t-d} - 1) + \int_1^\infty s^{t-1-d/\alpha} e^{-s}\, ds \asymp r^{\alpha t-d}\,. \end{aligned}$$

We also have,

$$\begin{aligned} I(t,r) &\le \frac{1}{r^{d+\alpha}} \int_0^\infty s^t e^{-s}\, ds = \frac{\Gamma(t+1)}{r^{d+\alpha}} \le \frac{1}{r^{d+\alpha}}\,, \quad r > 0\,, \\ I(t,r) &\ge \frac{1}{r^{d+\alpha}} \int_0^1 s^t e^{-s}\, ds \ge \frac{1}{r^{d+\alpha}} \frac{1}{(1+t)e} \ge \frac{1}{2er^{d+\alpha}}\,, \quad r > 1\,. \end{aligned}$$

Note that

$$\frac{1}{r^{d+\alpha}} \wedge \frac{1}{r^{d-t\alpha}} = \begin{cases} \frac{1}{r^{d-t\alpha}}, & 0 < r \le 1 \\ \frac{1}{r^{d+\alpha}}, & r > 1. \end{cases}$$

Therefore $I(t,r) \asymp \frac{1}{r^{d+\alpha}} \wedge \frac{1}{r^{d-t\alpha}}$. This implies that

$$q_\alpha(t,x) \asymp \frac{1}{\Gamma(t)} I(t,|x|) \asymp \frac{1}{\Gamma(t)} \left(\frac{1}{|x|^{d+\alpha}} \wedge \frac{1}{|x|^{d-t\alpha}}\right) \asymp t \left(\frac{1}{|x|^{d+\alpha}} \wedge \frac{1}{|x|^{d-t\alpha}}\right),$$

since for $0 < t \le 1$, $\Gamma(t) \asymp t^{-1}$. $\square$

Note that by taking $x = 0$, one obtains that $q_\alpha(t,0) = \infty$ for $0 < t < 1 \wedge \frac{d}{2\alpha}$. This somewhat unusual feature of the transition density is easier to show when $\alpha = 2$, i.e., in the case of a gamma subordinator. Indeed, then

$$q_2(t,x) := \int_0^\infty (4\pi s)^{-d/2} e^{-|x|^2/(4t)} \frac{1}{\Gamma(t)} s^{t-1} e^{-s}\, ds\,,$$

and therefore

$$q_2(t,0) = \frac{(4\pi)^{-d/2}}{\Gamma(t)} \int_0^\infty s^{-d/2+t-1} e^{-s}\, ds = \begin{cases} +\infty\,, & t \le d/2, \\ \frac{\Gamma(t-d/2)}{(4\pi)^{d/2}\Gamma(t)}\,, & t > d/2\,. \end{cases}$$

Assume now that $S^{(2)}$ is an iterated geometric stable subordinator with the Laplace exponent $\phi(\lambda) = \log(1 + \log(1 + \lambda))$, and let $q_2^{(2)}(t, x)$ be the transition density of the process $Y_t^{(2)} = X(S_t^{(2)})$. Then by (3.30),

$$q_2^{(2)}(t, 0) = \int_0^\infty q_2(s, 0)\frac{1}{\Gamma(t)}s^{t-1}e^{-s}\, ds = \infty$$

for all $t > 0$.

4 Harnack inequality for subordinate Brownian motion

4.1 Capacity and exit time estimates for some symmetric Lévy processes

The purpose of this subsection is to establish lower and upper estimates for the capacity of balls and the exit time from balls, with respect to a class of radially symmetric Lévy processes.

Suppose that $Y = (Y_t, \mathbb{P}^x)$ is a transient radially symmetric Lévy process on $\mathbb{R}^d$. We will assume that the potential kernel of Y is absolutely continuous with a density $G(x, y) = G(|y - x|)$ with respect to the Lebesgue measure. Let us assume the following condition: $G : [0, \infty) \to (0, \infty]$ *is a positive and decreasing function satisfying* $G(0) = \infty$. We will have need of the following elementary lemma.

Lemma 4.1 *There exists a positive constant $C_1 = C_1(d)$ such that for every $r > 0$ and all $x \in \overline{B(0, r)}$,*

$$C_1 \int_{B(0,r)} G(|y|)\, dy \le \int_{B(0,r)} G(x, y)\, dy \le \int_{B(0,r)} G(|y|)\, dy\, .$$

Moreover, the supremum of $\int_{B(0,r)} G(x, y)\, dy$ is attained at $x = 0$, while the infimum is attained at any point on the boundary of $B(0, r)$.

Proof. The proof is elementary. We only present the proof of the left-hand side inequality for $d \ge 2$. Consider the intersection of $B(0, r)$ and $B(x, r)$. This intersection contains the intersection of $B(x, r)$ and the cone with vertex x of aperture equal to $\pi/3$ pointing towards the origin. Let $C(x)$ be the latter intersection. Then

$$\int_{B(0,r)} G(|y - x|)\, dy \ge \int_{C(x)} G(|y - x|)\, dy \ge c_1 \int_{B(x,r)} G(|y - x|)\, dy = c_1 \int_{B(0,r)} G(|y|)\, dy\, ,$$

where the constant c_1 depends only on the dimension d. It is easy to see that the infimum of $\int_{B(0,r)} G(x, y)\, dy$ is attained at any point on the boundary of $B(0, r)$. $\square$

Let Cap denote the (0-order) capacity with respect to X (for the definition of capacity see e.g. [13] or [49]). For a measure μ we define

$$G\mu(x) := \int G(x,y)\,\mu(dy)\,.$$

For any compact subset K of $\mathbb{R}^d$, let $\mathcal{P}_K$ be the set of probability measures supported by K. Define

$$e(K) := \inf_{\mu\in\mathcal{P}_K} \int G\mu(x)\,\mu(dx)\,.$$

Since the kernel G satisfies the maximum principle (see, for example, Theorem 5.2.2 in [22]), it follows from ([28], page 159) that for any compact subset K of $\mathbb{R}^d$

$$\mathrm{Cap}(K) = \frac{1}{\inf_{\mu\in\mathcal{P}_K}\sup_{x\in\mathrm{Supp}(\mu)} G\mu(x)} = \frac{1}{e(K)}\,. \tag{4.1}$$

Furthermore, the infimum is attained at the capacitary measure μ_K. The following lemma is essentially proved in [38].

Lemma 4.2 *Let K be a compact subset of $\mathbb{R}^d$. For any probability measure μ on K, it holds that*

$$\inf_{x\in\mathrm{Supp}(\mu)} G\mu(x) \le e(K) \le \sup_{x\in\mathrm{Supp}(\mu)} G\mu(x)\,. \tag{4.2}$$

Proof. The right-hand side inequality follows immediately from (4.1). In order to prove the left-hand side inequality, suppose that for some probability measure μ on K it holds that $e(K) < \inf_{x\in\mathrm{Supp}(\mu)} G\mu(x)$. Then $e(K)+\epsilon < \inf_{x\in\mathrm{Supp}(\mu)} G\mu(x)$ for some $\epsilon > 0$. We first have

$$\int_K G\mu(x)\,\mu_K(dx) > \int_K (e(K)+\epsilon)\,\mu_K(dx) = e(K)+\epsilon\,.$$

On the other hand,

$$\int_K G\mu(x)\,\mu_K(dx) = \int_K G\mu_K(x)\,\mu(dx) = \int_K e(K)\,\mu(dx) = e(K)\,,$$

where we have used the fact that $G\mu_K = e(K)$ quasi everywhere in K, and the measure of finite energy does not charge sets of capacity zero. This contradiction proves the lemma. □

Proposition 4.3 *There exist positive constants $C_2 < C_3$ depending only on d, such that for all $r > 0$*

$$\frac{C_2 r^d}{\int_{B(0,r)} G(|y|)\,dy} \le \mathrm{Cap}(\overline{B(0,r)}) \le \frac{C_3 r^d}{\int_{B(0,r)} G(|y|)\,dy}. \tag{4.3}$$

Proof. Let $m_r(dy)$ be the normalized Lebesgue measure on $B(0,r)$. Thus, $m_r(dy) = dy/(c_1 r^d)$, where c_1 is the volume of the unit ball. Consider $Gm_r = \sup_{x\in B(0,r)} Gm_r(x)$. By Lemma 4.1, the supremum is attained at $x=0$, and so

$$Gm_r = \frac{1}{c_1 r^d}\int_{B(0,r)} G(|y|)dy\,.$$

Therefore from Lemma 4.2

$$\mathrm{Cap}(\overline{B(0,r)}) \geq \frac{c_1 r^d}{\int_{B(0,r)} G(|y|)dy}\,. \tag{4.4}$$

For the right-hand side of (4.2), it follows from Lemma 4.1 and Lemma 4.2 that

$$\mathrm{Cap}(\overline{B(0,r)}) \leq \frac{1}{Gm_r(z)} = \frac{c_1 r^d}{\int_{B(0,r)} G(z,y)}\,dy \leq \frac{c_1 r^d}{C_1\int_{B(0,r)} G(|y|)}\,dy\,,$$

where $z\in\partial B(0,r)$. □

In the remaining part of this section we assume in addition that G satisfies the following assumption: There exist $r_0>0$ and $c_0\in(0,1)$ such that

$$c_0 G(r) \geq G(2r)\,,\quad 0<2r<r_0\,. \tag{4.5}$$

Note that if G is regularly varying at 0 with index $\delta<0$, i.e., if

$$\lim_{r\to 0}\frac{G(2r)}{G(r)} = 2^\delta\,,$$

then (4.5) is satisfied with $c_0=(2^\delta+1)/2$ for some positive r_0. Let $\tau_{B(0,r)} = \inf\{t>0: Y_t\notin B(0,r)\}$ be the first exit time of Y from the ball $B(0,r)$.

Proposition 4.4 *There exists a positive constant C_4 such that for all $r\in(0,r_0/2)$,*

$$C_4\int_{B(0,r/6)} G(|y|)\,dy \leq \inf_{x\in B(0,r/6)}\mathbb{E}^x\tau_{B(0,r)} \leq \sup_{x\in B(0,r)}\mathbb{E}^x\tau_{B(0,r)} \leq \int_{B(0,r)} G(|y|)\,dy\,. \tag{4.6}$$

Proof. Let $G_{B(0,r)}(x,y)$ denote the Green function of the process Y killed upon exiting $B(0,r)$. Clearly, $G_{B(0,r)}(x,y)\leq G(x,y)$, for $x,y\in B(0,r)$. Therefore,

$$\begin{aligned}\mathbb{E}^x\tau_{B(0,r)} &= \int_{B(0,r)} G_{B(0,r)}(x,y)\,dy\\ &\leq \int_{B(0,r)} G(x,y)\,dy \leq \int_{B(0,r)} G(|y|)\,dy\,.\end{aligned}$$

For the left-hand side inequality, let $r \in (0, r_0/2)$, and let $x, y \in B(0, r/6)$. Then,

$$\begin{aligned} G_{B(0,r)}(x,y) &= G(x,y) - \mathbb{E}^x G(Y(\tau_{B(0,r)}), y) \\ &\geq G(|y-x|) - G(2|y-x|) \,. \end{aligned}$$

The last inequality follows because $|y - Y(\tau_{B(0,r)})| \geq \frac{2}{3} r \geq 2|y-x|$. Let $c_1 = 1 - c_0 \in (0,1)$. By (4.5) we have that for all $u \in (0, r_0)$, $G(u) - G(2u) \geq c_1 G(u)$. Hence, $G(|y-x|) - G(2|y-x|) \geq c_1 G(|y-x|)$, which implies that $G_{B(0,r)}(x,y) \geq c_1 G(x,y)$ for all $x, y \in B(0, r/6)$. Now, for $x \in B(0, r/6)$,

$$\begin{aligned} \mathbb{E}^x \tau_{B(0,r)} &= \int_{B(0,r)} G_{B(0,r)}(x,y)\, dy \geq \int_{B(0,r/6)} G_{B(0,r)}(x,y)\, dy \\ &\geq c_1 \int_{B(0,r/6)} G(x,y)\, dy \geq c_1 C_1 \int_{B(0,r/6)} G(|y|)\, dy \,, \end{aligned}$$

where the last inequality follows from Lemma 4.1. □

Example 4.5 We illustrate the last two propositions by applying them to the iterated geometric stable process $Y^{(n)}$ introduced in Example 3.2 (iv) and (v). Hence, we assume that $d > 2(\alpha/2)^n$. By a slight abuse of notation we define a function $G^{(n)} : [0, \infty) \to (0, \infty]$ by $G^{(n)}(|x|) = G^{(n)}(x)$. Note that by Theorem 3.8, G is regularly varying at zero with index $\beta = -d$. Let r_0 be the constant from (4.5). Let us first look at the asymptotic behavior of $\int_{B(0,r)} G^{(n)}(|y|)\, dy$ for small r. We have

$$\begin{aligned} &\int_{B(0,r)} G^{(n)}(|y|)\, dy = c_d \int_0^r u^{d-1} G^{(n)}(u)\, du \\ &\sim \frac{c_d \Gamma(d/2)}{\alpha \pi^{d/2}} \int_0^r \frac{u^{d-1}\, du}{u^d L_{n-1}(1/u^2) l_n^2(1/u^2)} = \frac{c_d \Gamma(d/2)}{2\alpha \pi^{d/2}} \int_0^{r^2} \frac{dv}{v L_{n-1}(1/v) l_n^2(1/v)} \\ &= \frac{c_d \Gamma(d/2)}{2\alpha \pi^{d/2}} \frac{1}{l_n(1/r^2)} \sim c_{\alpha,d} \frac{1}{l_n(1/r)} \,, \quad r \to 0 \,. \end{aligned}$$

It follows from Proposition 4.3 that there exist positive constants $C_5 \leq C_6$ such that for all $r \in (0, 1/e_n)$,

$$C_5 r^d l_n(1/r) \leq \mathrm{Cap}(\overline{B(0,r)}) \leq C_6 r^d l_n(1/r) \,.$$

Similarly, it follows from Proposition 4.4 that there exist positive constants $C_7 \leq C_8$ such that for all $r \in (0, (1/e_n) \wedge (r_0/2))$,

$$\frac{C_7}{l_n(1/r)} \leq \inf_{x \in B(0,r/6)} \mathbb{E}^x \tau_{B(0,r)} \leq \sup_{x \in B(0,r)} \mathbb{E}^x \tau_{B(0,r)} \leq \frac{C_8}{l_n(1/r)} \,. \tag{4.7}$$

Here we also used the fact that l_n is slowly varying.

By use of Theorem 3.12 and Proposition 4.3, we can estimate capacity of large balls. It easily follows that as $r \to \infty$, $\mathrm{Cap}(\overline{B(0,r)}$ is of the order $r^{\alpha(\alpha/2)^{n-1}}$.

4.2 Krylov-Safonov-type estimate

In this subsection we retain the assumptions from the beginning of the previous one. Thus, $Y = (Y_t, \mathbb{P}^x)$ is a transient radially symmetric Lévy process on $\mathbb{R}^d$ with the potential kernel having the density $G(x,y) = G(|y-x|)$ which is positive, decreasing and $G(0) = \infty$. Let $r_1 \in (0,1)$ and let $\ell : (1/r_1, \infty) \to (0,\infty)$ be a slowly varying function at ∞. Let $\beta \in [0,1]$ be such that $d + 2\beta - 2 > 0$. We introduce the following additional assumption about the density G: There exists a positive constant c_1 such that

$$G(x) \sim \frac{c_1}{|x|^{d+2\beta-2}\ell(1/|x|^2)}\,, \quad |x| \to 0\,. \tag{4.8}$$

If we abuse notation and let $G(|x|) = G(x)$, then G is regularly varying at 0 with index $-d-2\beta+2 < 0$, hence satisfies the assumption (4.5) with some $r_0 > 0$. In order to simplify notations, we define the function $g : (0, r_1) \to (0,\infty)$ by

$$g(r) = \frac{1}{r^{d+2\beta-2}\ell(1/r^2)}\,.$$

Clearly, g is regularly varying at 0 with index $-d-2\beta+2 < 0$. Let $\overline{g}$ be a monotone equivalent of g at 0. More precisely, we define $\overline{g} : (0, r_1/2) \to \infty$ by

$$\overline{g}(r) := \sup\{g(\rho) : r \le \rho \le r_1\}\,.$$

By the 0-version of Theorem 1.5.3. in [10], $\overline{g}(r) \sim g(r)$ as $r \to 0$. Moreover, $\overline{g}(r) \ge g(r)$, and $\overline{g}$ is decreasing. Let $r_2 = \min(r_0, r_1)$. There exist positive constants $C_9 < C_{10}$ such that

$$C_9\overline{g}(r) \le G(r) \le C_{10}\overline{g}(r)\,, \quad r < r_2\,. \tag{4.9}$$

We define

$$c = \max\left\{\frac{1}{3}\left(\frac{4C_{10}}{C_9}\right)^{\frac{1}{d+2\beta-2}}, 1\right\}\,. \tag{4.10}$$

Since $\overline{g}$ is regularly varying at 0 with index $-d-2\beta+2$, there exists $r_3 > 0$ such that

$$\frac{1}{2}\left(\frac{1}{3c}\right)^{d+2\beta-2} \le \frac{\overline{g}(6cr)}{\overline{g}(2r)} \le 2\left(\frac{1}{3c}\right)^{d+2\beta-2}\,, \quad r < r_3\,. \tag{4.11}$$

Finally, let

$$R = \min(r_2, r_3, 1) = \min(r_0, r_1, r_3, 1)\,. \tag{4.12}$$

Lemma 4.6 *There exists $C_{11} > 0$ such that for any $r \in (0, (7c)^{-1}R)$, any closed subset A of $B(0,r)$, and any $y \in B(0,r)$,*

$$\mathbb{P}^y(T_A < \tau_{B(0,7cr)}) \ge C_{11}\overline{\kappa}(r)\,\frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(0,r)})}\,,$$

where

$$\overline{\kappa}(r) = \frac{r^d \overline{g}(r)}{\int_0^r \rho^{d-1}\overline{g}(\rho)\, d\rho}\,. \tag{4.13}$$

Proof. Without loss of generality we may assume that $\mathrm{Cap}(A) > 0$. Let $G_{B(0,7cr)}$ be the Green function of the process obtained by killing Y upon exiting from $B(0, 7cr)$. If ν is the capacitary measure of A with respect to Y, then we have for all $y \in B(0, r)$,

$$\begin{aligned} G_{B(0,7cr)}\nu(y) &= \mathbb{E}^y[G_{B(0,7cr)}\nu(Y_{T_A}) : T_A < \tau_{B(0,7cr)}] \\ &\le \sup_{z\in\mathbb{R}^d} G_{B(0,7cr)}\nu(z)\mathbb{P}^y(T_A < \tau_{B(0,7cr)}) \\ &\le \mathbb{P}^y(T_A < \tau_{B(0,7cr)}). \end{aligned}$$

On the other hand we have for all $y \in B(0, r)$,

$$\begin{aligned} G_{B(0,7cr)}\nu(y) &= \int G_{B(0,7cr)}(y, z)\nu(dz) \ge \nu(A) \inf_{z\in B(0,r)} G_{B(0,7cr)}(y, z) \\ &= \mathrm{Cap}(A) \inf_{z\in B(0,r)} G_{B(0,7cr)}(y, z)\,. \end{aligned}$$

In order to estimate the infimum in the last display, note that $G_{B(0,7cr)}(y, z) = G(y, z) - \mathbb{E}^y[G(Y_{\tau_{B(0,7cr)}}, z)]$. Since $|y - z| < 2r < R$, it follows by (4.9) and the monotonicity of $\overline{g}$ that

$$G(y, z) \ge C_9\overline{g}(|z - y|) \ge C_9\overline{g}(2r)\,. \tag{4.14}$$

Now we consider $G(Y_{\tau_{B(0,7cr)}}, z)$. First note that $|Y_{\tau_{B(0,7cr)}} - z| \ge 7cr - r \ge 7cr - cr \ge 6cr$. If $|Y_{\tau_{B(0,7cr)}} - z| \le R$, then by (4.9) and the monotonicity of $\overline{g}$,

$$G(Y_{\tau_{B(0,7cr)}}, z) \le C_{10}\overline{g}(|z - Y_{\tau_{B(0,7cr)}}|) \le C_{10}\overline{g}(6cr)\,.$$

If, on the other hand, $|Y_{\tau_{B(0,7cr)}} - z| \ge R$, then $G(Y_{\tau_{B(0,7cr)}}, z) \le G(w)$, where $w \in \mathbb{R}^d$ is any point such that $|w| = R$. Here we have used the monotonicity of G. For $|w| = R$ we have that $G(w) \le C_{10}\overline{g}(|w|) = C_{10}\overline{g}(R) \le C_{10}\overline{g}(6cr)$. Therefore

$$\mathbb{E}^y[G(Y_{\tau_{B(0,7cr)}}, z)] \le C_{10}\overline{g}(6cr)\,. \tag{4.15}$$

By use of (4.14) and (4.15) we obtain

$$\begin{aligned} G_{B(0,7cr)}(y, z) &\ge C_9\overline{g}(2r) - C_{10}\overline{g}(6cr) \\ &= \overline{g}(2r)\left(C_9 - C_{10}\,\frac{\overline{g}(6cr)}{\overline{g}(2r)}\right) \\ &\ge \overline{g}(2r)\left(C_9 - 2C_{10}\left(\frac{1}{3c}\right)^{d+2\beta-2}\right) \\ &\ge \overline{g}(2r)\left(C_9 - 2C_{10}\frac{C_9}{4C_{10}}\right) = \frac{C_9}{2}\,\overline{g}(2r)\,, \end{aligned}$$

where the next to last line follows from (4.11) and the last from definition (4.10). By using one more time that $\overline{g}$ is regularly varying at 0, we conclude that there exists a constant $C_{12} > 0$ such that for all $y, z \in B(0, r)$,

$$G_{B(0,7cr)}(y, z) \geq C_{12}\overline{g}(r) .$$

Further, it follows from Proposition 4.3 that there exists a constant $C_{13} > 0$, such that

$$\frac{C_{13}}{\mathrm{Cap}(\overline{B(0,r)})} \frac{r^d}{\int_0^r \rho^{d-1}\overline{g}(\rho)\, d\rho} \leq 1 . \tag{4.16}$$

Hence

$$\begin{aligned} G_{B(0,7cr)}(y, z) &\geq C_{12}C_{13}\frac{1}{\mathrm{Cap}(\overline{B(0,r)})} \frac{r^d\overline{g}(r)}{\int_0^r \rho^{d-1}\overline{g}(\rho)\, d\rho} \\ &\geq C_{14}\frac{1}{\mathrm{Cap}(\overline{B(0,r)})}\, \overline{\kappa}(r) . \end{aligned}$$

To finish the proof, note that

$$\mathbb{P}^y(T_A < \tau_{B(0,7cr)}) \geq G_{B(0,7cr)}\nu(y) \geq C_{14}\overline{\kappa}(r)\frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(0,r)})} .$$

□

Remark 4.7 *Note that in the estimate (4.16) we could use g instead of $\overline{g}$. Together with the fact that $\overline{g}(r) \geq g(r)$ this would lead to the hitting time estimate*

$$\mathbb{P}^y(T_A < \tau_{B(0,7cr)}) \geq C_{11}\kappa(r)\, \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(0,r)})} ,$$

where

$$\kappa(r) = \frac{r^d g(r)}{\int_0^r \rho^{d-1} g(\rho)\, d\rho} . \tag{4.17}$$

We will apply the above lemma to subordinate Brownian motions. Assume, first, that $Y_t = X(S_t)$ where $S = (S_t : t \geq 0)$ is the special subordinator with the Laplace exponent ϕ satisfying $\phi(\lambda) \sim \lambda^{\alpha/2}\ell(\lambda)$, $\lambda \to \infty$, where $0 < \alpha < 2 \wedge d$, and ℓ is slowly varying at ∞. Then the Green function of Y satisfies all assumptions of this subsection, in particular (4.8) with $\beta = 1 - \alpha/2$, see (2.28) and Lemma (3.3). Define c as in (4.10) for appropriate C_9 and C_{10} and $\beta = 1 - \alpha/2$, and let R be as in (4.12).

Proposition 4.8 *Assume that $Y_t = X(S_t)$ where $S = (S_t : t \geq 0)$ is the special subordinator with the Laplace exponent ϕ satisfying one of the following two conditions: (i) $\phi(\lambda) \sim \lambda^{\alpha/2}\ell(\lambda)$, $\lambda \to \infty$, where $0 < \alpha < 2$, and ℓ is slowly varying at ∞, or (ii) $\phi(\lambda) \sim \lambda$, $\lambda \to \infty$. If Y is transient, the following statements are true:*
(a) There exists a constant $C_{15} > 0$ such that for any $r \in (0, (7c)^{-1}R)$, any closed subset A of $B(0, r)$, and any $y \in B(0, r)$,

$$\mathbb{P}^y(T_A < \tau_{B(0,7cr)}) \geq C_{15} \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(0,r)})} .$$

(b) There exists a constant $C_{16} > 0$ such that for any $r \in (0, R)$ we have

$$\sup_{y\in B(0,r)} \mathbb{E}^y \tau_{B(0,r)} \leq C_{16} \inf_{y\in B(0,r/6)} \mathbb{E}^y \tau_{B(0,r)} .$$

Proof. We give the proof for case (i), case (ii) being simpler.
(a) It suffices to show that $\overline{\kappa}(r)$, $r < (7c)^{-1}R$, is bounded from below by a positive constant. Note that $\overline{g}$ is regularly varying at 0 with index $-d + \alpha$. Hence there is a slowly varying function $\overline{\ell}$ such that $\overline{g}(r) = r^{-d+\alpha}\overline{\ell}(r)$. By Karamata's monotone density theorem one can conclude that

$$\int_0^r \rho^{d-1}\overline{g}(\rho)\, d\rho = \int_0^r \rho^{\alpha-1}\overline{\ell}(\rho)\, d\rho \sim \frac{1}{\alpha}\, r^{\alpha}\overline{\ell}(r) = \frac{1}{\alpha}\, r^{d}\overline{g}(r) , \quad r \to 0 .$$

Therefore,

$$\overline{\kappa}(r) = \frac{r^d\overline{g}(r)}{\int_0^r \rho^{d-1}\overline{g}(\rho)\, d\rho} \sim \frac{1}{\alpha} .$$

(b) By Proposition 4.4 it suffices to show that $\int_{B(0,r)} G(|y|)\, dy \leq c \int_{B(0,r/6)} G(|y|)\, dy$ for some positive constant c. But, by the proof of part (a), $\int_{B(0,r)} G(|y|)\, dy \asymp r^d\overline{g}(r)$, while $\int_{B(0,r/6)} G(|y|)\, dy \asymp (r/6)^d\overline{g}(r/6)$. Since $\overline{g}$ is regularly varying, the claim follows. □

Proposition 4.9 *Let $S^{(n)}$ be the iterated geometric stable subordinator and let $Y_t^{(n)} = X(S_t^{(n)})$ be the corresponding subordinate process. Assume that $d > 2(\alpha/2)^n$.*
(a) Let $\gamma > 0$. There exists a constant $C_{17} > 0$ such that for any $r \in (0, (7c)^{-1}R)$, any closed subset A of $B(0, r)$, and any $y \in B(0, r)$

$$\mathbb{P}^y(T_A < \tau_{B(0,7cr)}) \geq C_{17}\, r^{\gamma} \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(0,r)})} .$$

(b) There exists a constant $C_{18} > 0$ such that for any $r \in (0, R)$ we have

$$\sup_{y\in B(0,r)} \mathbb{E}^y \tau_{B(0,r)} \leq C_{18} \inf_{y\in B(0,r/6)} \mathbb{E}^y \tau_{B(0,r)} .$$

Proof. (a) By Proposition 2.21 we take

$$g(r) = \frac{1}{r^d L_{n-1}(1/r^2) l_n(1/r^2)^2} \, .$$

Recall that the functions l_n, respectively L_n, were defined in (2.36), respectively (2.37). Integration gives that

$$\int_0^r \rho^{d-1} g(\rho)\, d\rho = \int_0^r \frac{1}{\rho L_{n-1}(1/\rho^2) l_n(1/\rho^2)^2}\, d\rho = \frac{2}{l_n(1/r^2)} \, .$$

Therefore,

$$\kappa(r) = \frac{1}{L_n(1/r^2)} \geq \tilde{c} r^{\gamma} \, ,$$

and the claim follows from Remark 4.7.
(b) This was shown in Example 4.5. □

Remark 4.10 *We note that part (b) of both Propositions 4.8 and 4.9 are true for every pure jump process. This was proved in [46], and later also in [51].*

In the remainder of this subsection we discuss briefly the Krylov-Safonov type estimate involving the Lebesgue measure instead of the capacity. This type of estimate turns out to be very useful in case of a pure jump Lévy process. The method of proof comes from [4], while our exposition follows [56].

Assume that $Y = (Y_t : t \geq 0)$ is a subordinate Brownian motion via a subordinator with no drift. We retain the notation $j(|x|) = J(x)$, introduce functions

$$\eta_1(r) = r^{-2} \int_0^r \rho^{d+1} j(\rho)\, d\rho \, , \quad \eta_2(r) = \int_r^\infty \rho^{d-1} j(\rho)\, d\rho \, ,$$

and let $\eta(r) = \eta_1(r) + \eta_2(r)$. The proof of the following result can be found in [56].

Lemma 4.11 *There exists a constant $C_{19} > 0$ such that for every $r \in (0,1)$, every $A \subset B(0,r)$ and any $y \in B(0,2r)$,*

$$\mathbb{P}^y(T_A < \tau_{B(0,3r)}) \geq C_{19} \frac{r^d j(4r)}{\eta(r)} \frac{|A|}{|B(0,r)|} \, ,$$

where $|\cdot|$ denotes the Lebesgue measure.

Proposition 4.12 *Assume that $Y_t = X(S_t)$ where $S = (S_t : t \geq 0)$ is a pure jump subordinator, and the jumping function $J(x) = j(|x|)$ of Y is such that j satisfies $j(r) \sim r^{-d-\alpha}\ell(r)$, $r \to 0+$, with $0 < \alpha < 2$ and ℓ slowly varying at 0. Then there exists a constant $C_{20} > 0$ such that for every $r \in (0,1)$, every $A \subset B(0,r)$ and any $y \in B(0,2r)$,*

$$\mathbb{P}^y(T_A < \tau_{B(0,3r)}) \geq C_{20} \frac{|A|}{|B(0,r)|} \, .$$

Proof. It suffices to prove that $r^d j(4r)/\eta(r)$ is bounded from below by a positive constant. This is accomplished along the lines of the proof of Proposition 4.8. $\square$

Note that the assumptions of Proposition 4.12 are satisfied for subordinate Brownian motions via $\alpha/2$-stable subordinators, relativistic α-stable subordinators and the subordinators corresponding to Examples 2.15 and 2.16 (see Theorems 3.14 and 3.15).

In the case of, say, a geometric stable process Y, one obtains from Lemma 4.11 a weak form of the hitting time estimate: There exists $C_{21} > 0$ such that for every $r \in (0, 1/2)$, every $A \subset B(0,r)$ and any $y \in B(0, 2r)$,

$$\mathbb{P}^y(T_A < \tau_{B(0,3r)}) \geq C_{21} \frac{1}{\log(1/r)} \frac{|A|}{|B(0,r)|}\,. \tag{4.18}$$

4.3 Proof of Harnack inequality

Let $Y = (Y_t : t \geq 0)$ be a subordinate Brownian motion in $\mathbb{R}^d$ and let D be an open subset of $\mathbb{R}^d$. A function $h : \mathbb{R}^d \to [0, +\infty]$ is said to be harmonic in D with respect to the process Y if for every bounded open set $B \subset \overline{B} \subset D$,

$$h(x) = \mathbb{E}^x[h(Y_{\tau_B})]\,, \quad \forall x \in B\,,$$

where $\tau_B = \inf\{t > 0 : Y_t \notin B\}$ is the exit time of Y from B. Harnack inequality is a statement about the growth rate of nonnegative harmonic functions in compact subsets of D. We will first discuss two proofs of a scale invariant Harnack inequality for small balls. Next, we will give a proof of a weak form of Harnack inequality for small balls for the iterated geometric stable process. All discussed forms of the inequality lead to the following Harnack inequality: For any compact set $K \subset D$, there exists a constant $C > 0$, depending only on D and K, such that for every nonnegative harmonic function h with respect to Y in D, it holds that

$$\sup_{x \in K} h(x) \leq C \inf_{x \in K} h(x)\,.$$

The general methodology of proving Harnack inequality for jump processes is explained in [56] following the pioneering work [4] (for an alternative approach see [15]). The same method was also used in [5] and [16] to prove a parabolic Harnack inequality. There are two essential ingredients: The first one is a Krylov-Safonov-type estimate for the hitting probability discussed in the previous subsection. The form given in Lemma 4.11 and Proposition 4.12 can be used in the case of pure jump processes for which one has good control of the behavior of the jumping function J at zero. More precisely, one needs that $j(r)$ is a regularly varying function of index $-d-\alpha$ for $0 < \alpha < 2$ when $r \to 0+$. This, as shown in Proposition 4.12, implies that the function of r on the right-hand side of the estimate can be replaced by a constant, which is desirable to obtaining the scale invariant form of Harnack inequality for small balls. In the case of a geometric stable process the behavior of J near zero is known (see Theorem 3.16), but leads to the inequality (4.18) having the factor $1/\log(1/r)$ on the right-hand side. This yields a weak type of Harnack inequality for balls. In the case of the iterated geometric stable processes, no information about the behavior of J near zero is available, and hence one does not have any control on the factor $r^d j(r)/\eta(r)$ in Lemma 4.11. In the case where Y has a continuous component (i.e, the subordinator S has a drift), or the case when information on the behavior of J near zero is missing, one can use the form of Krylov-Safonov inequality described in Propositions 4.8 and 4.9.

The second ingredient in the proof is the following result which can be considered as a very weak form of Harnack inequality (more precisely, Harnack inequality for harmonic measures of sets away from the ball). Recall that $R > 0$ was defined in (4.12).

Proposition 4.13 *Let Y be a subordinate Brownian motion such that the function j defined in (3.17) satisfies conditions (3.20) and (3.21). There exists a positive constant $C_{22} > 0$ such that for any $r \in (0, R)$, any $y, z \in B(0, r/2)$ and any nonnegative function H supported on $B(0, 2r)^c$ it holds that*

$$\mathbb{E}^z H(Y(\tau_{B(0,r)})) \leq C_{22} \mathbb{E}^y H(Y(\tau_{B(0,r)}))\,. \tag{4.19}$$

Proof. This is an immediate consequence of Proposition 3.21 and the comparison results for the mean exit times explained in Remark 4.10 (see also Propositions 4.8 and 4.9). □

We are now ready to state Harnack inequality under two different set of conditions.

Theorem 4.14 *Let Y be a subordinate Brownian motion such that the function j defined in (3.17) satisfies conditions (3.20) and (3.21) and is further regularly varying at zero with index $-d-\alpha$ where $0 < \alpha < 2$. Then there exists a constant $C > 0$ such that, for any $r \in (0, 1/4)$, and any function h which is nonnegative, bounded on $\mathbb{R}^d$, and harmonic with respect to Y in $B(0, 16r)$, we have*

$$h(x) \leq C h(y), \quad \forall x, y \in B(0, r).$$

Proof of this Harnack inequality follows from [56] and uses Proposition 4.12. The second set of conditions for Harnack inequality uses Proposition 4.8. Recall the constant c defined in (4.10).

Theorem 4.15 *Let Y be a transient subordinate Brownian motion such that the function j defined in (3.17) satisfies conditions (3.20) and (3.21), and assume further that the subordinator S is special and its Laplace exponent ϕ satisfies $\phi(\lambda) \sim b\lambda^{\alpha/2}$, $\lambda \to \infty$, with $\alpha \in (0, 2]$ and $b > 0$. Then there exists a constant $C > 0$ such that, for any $r \in (0, (14c)^{-1}R)$, and any function h which is nonnegative, bounded on $\mathbb{R}^d$, and harmonic with respect to Y in $B(0, 14cr)$, we have*

$$h(x) \leq C h(y), \quad \forall x, y \in B(0, r/2).$$

Under these conditions, Harnack inequality was proved in [47]. Unfortunately, despite the fact that Proposition 4.8 holds under weaker conditions for ϕ than the ones stated in the theorem above, we were unable to carry out a proof in this more general case.

Now we are going to present a proof of a weak form of Harnack inequality for iterated geometric stable processes. Let $S^{(n)}$ be the iterated geometric stable subordinator and let $Y_t^{(n)} = X(S_t^{(n)})$ be the corresponding subordinate process. We assume that $d > 2(\alpha/2)^n$. For simplicity we write Y instead of $Y^{(n)}$. We state again Propositions 4.9 (a) and 4.13: Let $\gamma > 0$. There exists a constant $C_{17} > 0$ such that for any $r \in (0, (7c)^{-1}R)$, any closed subset A of $B(0,r)$, and any $y \in B(0,r)$

$$\mathbb{P}^y(T_A < \tau_{B(0,7cr)}) \geq C_{17}\, r^{\gamma} \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(0,r)})}\,, \tag{4.20}$$

There exists a positive constant $C_{22} > 0$ such that for any $r \in (0,R)$, any $y, z \in B(0,r/2)$ and any nonnegative function H supported on $B(0,r)^c$ it holds that

$$\mathbb{E}^z H(Y(\tau_{B(0,r)})) \leq C_{22} \mathbb{E}^y H(Y(\tau_{B(0,r)}))\,. \tag{4.21}$$

We will also need the following lemma.

Lemma 4.16 *There exists a positive constant C_{23} such that for all $0 < \rho < r < 1/e_{n+1}$,*

$$\frac{\mathrm{Cap}(\overline{B(0,\rho)})}{\mathrm{Cap}(\overline{B(0,r)})} \geq C_{23} \left(\frac{\rho}{r}\right)^d .$$

Proof. By Example 4.5,

$$C_5 r^d l_n(1/r) \leq \mathrm{Cap}(\overline{B(0,r)}) \leq C_6 r^d l_n(1/r)$$

for every $r < 1/e_{n+1}$. Therefore,

$$\frac{\mathrm{Cap}(\overline{B(0,\rho)})}{\mathrm{Cap}(\overline{B(0,r)})} \geq \frac{C_5 \rho^d l_n(1/\rho)}{C_6 r^d l_n(1/r)} \geq \frac{C_5}{C_6} \left(\frac{\rho}{r}\right)^d ,$$

where the last inequality follows from the fact that l_n is increasing at infinity. □

Theorem 4.17 *Let R and c be defined by (4.12) and (4.10) respectively. Let $r \in (0, (14c)^{-1}R)$. There exists a constant $C > 0$ such that for every nonnegative bounded function h in $\mathbb{R}^d$ which is harmonic with respect to Y in $B(0,14cr)$ it holds*

$$h(x) \leq C h(y), \quad x, y \in B(0, r/2).$$

Remark 4.18 *Note that the constant C in the theorem may depend on the radius r. This is why the above Harnack inequality is weak. A version of a weak Harnack inequality appeared in [3], and our proof follows the arguments there. A similar proof, in a somewhat different context, was given in [58].*

Proof. We fix $\gamma \in (0,1)$. Suppose that h is nonnegative and bounded in $\mathbb{R}^d$ and harmonic with respect to Y in $B(0,14cr)$. By looking at $h+\epsilon$ and letting $\epsilon \downarrow 0$, we may suppose that h is bounded from below by a positive constant. By looking at ah for a suitable $a>0$, we may suppose that $\inf_{B(0,r/2)} h = 1/2$. We want to bound h from above in $B(0,r/2)$ by a constant depending only on r, d and γ. Choose $z_1 \in B(0,r/2)$ such that $h(z_1) \le 1$. Choose $\rho \in (1,\gamma^{-1})$. For $i \ge 1$ let

$$r_i = \frac{c_1 r}{i^\rho},$$

where c_1 is a constant to be determined later. We require first of all that c_1 is small enough so that

$$\sum_{i=1}^{\infty} r_i \le \frac{r}{8}. \tag{4.22}$$

Recall that there exists $c_2 := C_{17} > 0$ such that for any $s \in (0,(7c)^{-1}R)$, any closed subset $A \subset B(0,s)$ and any $y \in B(0,s)$,

$$\mathbb{P}^y(T_A < \tau_{B(0,7cs)}) \ge c_2 s^\gamma \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(0,s)})}\,. \tag{4.23}$$

Let c_3 be a constant such that

$$c_3 \le c_2 2^{-4-\gamma+\rho\gamma}\,.$$

Denote the constant C_8 from Lemma 4.16 by c_4. Once c_1 and c_3 have been chosen, choose K_1 sufficiently large so that

$$\frac{1}{4}(7c)^{-d-\gamma} c_2 c_4 K_1 \exp((14c)^{-\gamma} r^\gamma c_1 c_3 i^{1-\rho\gamma})\, c_1^{4\gamma+d} r^{4\gamma} \ge 2 i^{4\rho\gamma+\rho d} \tag{4.24}$$

for all $i \ge 1$. Such a choice is possible since $\rho\gamma < 1$. Note that K_1 will depend on r, d and γ as well as constants c, c_1, c_2, c_3 and c_4. Suppose now that there exists $x_1 \in B(0,r/2)$ with $h(x_1) \ge K_1$. We will show that in this case there exists a sequence $\{(x_j,K_j) : j \ge 1\}$ with $x_{j+1} \in B(x_j, 2r_j) \subset B(0,3r/4)$, $K_j = h(x_j)$, and

$$K_j \ge K_1 \exp((14c)^{-\gamma} r^\gamma c_1 c_3 j^{1-\rho\gamma}). \tag{4.25}$$

Since $1-\rho\gamma > 0$, we have $K_j \to \infty$, a contradiction to the assumption that h is bounded. We can then conclude that h must be bounded by K_1 on $B(0,r/2)$, and hence $h(x) \le 2K_1 h(y)$ if $x,y \in B(0,r/2)$.

Suppose that $x_1, x_2, \dots, x_i$ have been selected and that (4.25) holds for $j = 1, \dots, i$. We will show that there exists $x_{i+1} \in B(x_i, 2r_i)$ such that if $K_{i+1} = h(x_{i+1})$, then (4.25) holds for $j = i+1$; we then use induction to conclude that (4.25) holds for all j.

Let
$$A_i = \{y \in B(x_i, (14c)^{-1}r_i) : h(y) \geq K_i r_i^{2\gamma}\}.$$
First we prove that
$$\frac{\mathrm{Cap}(A_i)}{\mathrm{Cap}(\overline{B(x_i, (14c)^{-1}r_i)})} \leq \frac{1}{4}. \tag{4.26}$$
To prove this claim, we suppose to the contrary that $\mathrm{Cap}(A_i)/\mathrm{Cap}(\overline{B(x_i, (14c)^{-1}r_i)}) > 1/4$. Let F be a compact subset of A_i with $\mathrm{Cap}(F)/\mathrm{Cap}(\overline{B(x_i, (14c)^{-1}r_i)}) > 1/4$. Recall that $r \geq 8r_i$. Now we have

$$\begin{aligned}
1 &\geq h(z_1) \geq \mathbb{E}^{z_1}[h(Y_{T_F \wedge \tau_{B(0,7cr)}}); T_F < \tau_{B(0,7cr)}] \\
&\geq K_i r_i^{2\gamma} \mathbb{P}^{z_1}(T_F < \tau_{B(0,7cr)}) \\
&\geq c_2 K_i r_i^{2\gamma} r^\gamma \frac{\mathrm{Cap}(F)}{\mathrm{Cap}(\overline{B(0,r)})} \\
&= c_2 K_i r_i^{2\gamma} r^\gamma \frac{\mathrm{Cap}(F)}{\mathrm{Cap}(\overline{B(x_i, (7c)^{-1}r_i)})} \frac{\mathrm{Cap}(\overline{B(x_i, (7c)^{-1}r_i)})}{\mathrm{Cap}(\overline{B(0,r)})} \\
&\geq \frac{1}{4} c_2 K_i r_i^{2\gamma} r^\gamma \frac{\mathrm{Cap}\overline{B(0, (7c)^{-1}r_i)})}{\mathrm{Cap}(B(0,r))} \\
&\geq \frac{1}{4} c_2 K_i r_i^{2\gamma} r^\gamma c_4 \left(\frac{(7c)^{-1}r_i}{r}\right)^d \\
&= \frac{1}{4} c_2 c_4 (7c)^{-d} K_i r_i^{2\gamma} r^\gamma \left(\frac{r_i}{r}\right)^d \\
&\geq \frac{1}{4} c_2 c_4 (7c)^{-d-\gamma} K_i r_i^{4\gamma} \left(\frac{r_i}{r}\right)^d \\
&\geq \frac{1}{4} c_2 c_4 (7c)^{-d-\gamma} K_1 \exp((14c)^{-\gamma} r^\gamma c_1 c_3 i^{1-\rho\gamma}) r_i^{4\gamma} \left(\frac{r_i}{r}\right)^d \\
&\geq \frac{1}{4} c_2 c_4 (7c)^{-d-\gamma} K_1 \exp((14c)^{-\gamma} r^\gamma c_1 c_3 i^{1-\rho\gamma}) \left(\frac{c_1 r}{i^\rho}\right)^{4\gamma} \left(\frac{c_1}{i^\rho}\right)^d \\
&\geq \frac{1}{4} c_2 c_4 (7c)^{-d-\gamma} K_1 \exp((14c)^{-\gamma} r^\gamma c_1 c_3 j^{1-\rho\gamma}) c_1^{4\gamma+d} r^{4\gamma} i^{-4\gamma\rho-\rho d} \\
&\geq 2 i^{4\gamma\rho+\rho d} i^{-4\gamma\rho-\rho d} = 2\,.
\end{aligned}$$

We used the definition of harmonicity in the first line, (4.23) in the third, Lemma 4.16 in the sixth, (4.25) in the ninth, and (4.24) in the last line. This is a contradiction, and therefore (4.26) is valid.

By subadditivity of the capacity and by (4.26) it follows that there exists $E_i \subset B(x_i,(14c)^{-1}c)\setminus A_i$ such that

$$\frac{\mathrm{Cap}(E_i)}{\mathrm{Cap}(\overline{B(x_i,(14c)^{-1}r_i)})} \geq \frac{1}{2}\,.$$

Write τ_i for $\tau_{B(x_i,r_i/2)}$ and let $p_i := \mathbb{P}^{x_i}(T_{E_i} < \tau_i)$. It follows from (4.23) that

$$\begin{aligned} p_i &\geq c_2\left(\frac{r_i}{14c}\right)^{\gamma}\frac{\mathrm{Cap}(E_i)}{\mathrm{Cap}(\overline{B(x_i,(14c)^{-1})})} \\ &\geq \frac{c_2}{2}\left(\frac{r_i}{14c}\right)^{\gamma}\,. \end{aligned} \tag{4.27}$$

Set $M_i = \sup_{B(x_i,r_i)} h$. Then

$$\begin{aligned} K_i = h(x_i) &= \mathbb{E}^{x_i}[h(Y_{T_{E_i}\wedge\tau_i}); T_{E_i} < \tau_i] \\ &+ \mathbb{E}^{x_i}[h(Y_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, Y_{\tau_i} \in B(x_i,r_i)] \\ &+ \mathbb{E}^{x_i}[h(Y_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, Y_{\tau_i} \notin B(x_i,r_i)]. \end{aligned} \tag{4.28}$$

We are going to estimate each term separately. Since E_i is compact, we have

$$\mathbb{E}^{x_i}[h(Y_{T_{E_i}\wedge\tau_i}); T_{E_i} < \tau_i] \leq K_i r_i^{2\gamma}\mathbb{P}^{x_i}(T_{E_i} < \tau_i) \leq K_i r_i^{2\gamma}.$$

Further,

$$\mathbb{E}^{x_i}[h(Y_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, Y_{\tau_i} \in B(x_i,r_i)] \leq M_i(1-p_i).$$

Inequality (4.26) implies in particular that there exists $y_i \in B(x_i,(14c)^{-1}r_i)$ with $h(y_i) \leq K_i r_i^{2\gamma}$. We then have, by (4.21) and with $c_5 = C_{22}$

$$\begin{aligned} K_i r_i^{2\gamma} &\geq h(y_i) \geq \mathbb{E}^{y_i}[h(Y_{\tau_i}) : Y_{\tau_i} \notin B(x_i,r_i)] \\ &\geq c_5\mathbb{E}^{x_i}[h(Y_{\tau_i}) : Y_{\tau_i} \notin B(x_i,r_i)]\,. \end{aligned} \tag{4.29}$$

Therefore

$$\mathbb{E}^{x_i}[h(Y_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, Y_{\tau_i} \notin B(x_i,r_i)] \leq c_6 K_i r_i^{2\gamma}$$

for the positive constant $c_6 = 1/c_5$. Consequently we have

$$K_i \leq (1+c_6)K_i r_i^{2\gamma} + M_i(1-p_i). \tag{4.30}$$

Rearranging, we get

$$M_i \geq K_i\left(\frac{1-(1+c_6)r_i^{2\gamma}}{1-p_i}\right). \tag{4.31}$$

Now choose

$$c_1 \leq \min\{\frac{1}{14c}\frac{1}{r}\left(\frac{1}{4}\frac{c_2}{1+c_6}\right)^{1/\gamma}, 1\}\,.$$

This choice of c_1 implies that

$$2(1+c_6)r_i^{2\gamma} \leq \frac{c_2}{2}\left(\frac{r_i}{14c}\right)^\gamma \leq p_i\,,$$

where the second inequality follows from (4.27). Therefore, $1-(1+c_6)r_i^{2\gamma} \geq 1-p_i/2$, and hence by use of (4.31)

$$M_i \geq K_i\left(\frac{1-\frac{1}{2}p_i}{1-p_i}\right) > (1+\frac{p_i}{2})K_i.$$

Using the definition of M_i and (4.21), there exists a point $x_{i+1} \in \overline{B(x_i,r_i)} \subset B(x_i,2r_i)$ such that

$$K_{i+1} = h(x_{i+1}) \geq K_i\left(1+\frac{c_2}{4}\left(\frac{r_i}{14c}\right)^\gamma\right)\,.$$

Taking logarithms and writing

$$\log K_{i+1} = \log K_i + \sum_{j=1}^{i}[\log K_{j+1} - \log K_j],$$

we have

$$\begin{aligned}
\log K_{i+1} &\geq \log K_1 + \sum_{j=1}^{i}\log\left(1+\frac{c_2}{4}\left(\frac{r_j}{14c}\right)^\gamma\right)\\
&\geq \log K_1 + \sum_{j=1}^{i}\frac{c_2}{4}\frac{r_j^\gamma}{(14c)^\gamma}\\
&= \log K_1 + \frac{c_2}{4}\frac{1}{(14c)^\gamma}\sum_{j=1}^{i}\left(\frac{c_1 r}{j^\rho}\right)^\gamma\\
&\geq \log K_1 + \frac{c_2}{4}\frac{1}{(14c)^\gamma}r^\gamma c_1^\gamma\sum_{j=1}^{i}j^{-\rho\gamma}\\
&\geq \log K_1 + \frac{c_2}{4}\frac{1}{(14c)^\gamma}r^\gamma c_1 i^{1-\rho\gamma}\\
&\geq \log K_1 + \frac{1}{(14c)^\gamma}r^\gamma c_1 c_3(i+1)^{1-\rho\gamma}\,.
\end{aligned}$$

In the fifth line we used the fact that $c_1 < 1$. For the last line recall that

$$c_3 \leq c_2 2^{-4-\gamma+\rho\gamma} = \frac{c_2}{2^{3+\gamma}}\left(\frac{1}{2}\right)^{1-\rho\gamma} \leq \frac{c_2}{2^{3+\gamma}}\left(\frac{i}{i+1}\right)^{1-\rho\gamma}\,,$$

implying that

$$\frac{c_2}{4}i^{1-\rho\gamma} \geq 2^{1+\gamma}c_3(1+i)^{1-\rho\gamma}\,.$$

Therefore we have obtained that

$$K_{i+1} \geq K_1 \exp((14c)^{-\gamma} r^{\gamma} c_1 c_3 (i+1)^{1-\rho\gamma})$$

which is (4.25) for $i+1$. The proof is now finished. $\square$

Remark 4.19 *The proof given above can be easily modified to provide a proof of Theorem 4.15. Indeed, one can modify slightly Lemma 4.16, take $\gamma = 0$ and choose any $\rho > 1$ in the proof. The choice of K_1 in (4.24) and K_j in (4.25) will not depend on $r > 0$, thus giving a strong form of Harnack inequality.*

5 Subordinate killed Brownian motion

5.1 Definitions

Let $X = (X_t, \mathbb{P}^x)$ be a d-dimensional Brownian motion. Let D be a bounded connected open set in $\mathbb{R}^d$, and let $\tau_D = \inf\{t > 0 : X_t \notin D\}$ be the exit time of X from D. Define

$$X_t^D = \begin{cases} X_t, & t < \tau_D\,, \\ \partial\,, & t \geq \tau_D\,, \end{cases}$$

where ∂ is the cemetery. We call X^D a Brownian motion killed upon exiting D, or simply, a killed Brownian motion. The semigroup of X^D will be denoted by $(P_t^D : t \geq 0)$, and its transition density by $p^D(t,x,y)$, $t > 0$, $x, y \in D$. The transition density $p^D(t,x,y)$ is strictly positive, and hence the eigenfunction φ_0 of the operator $-\Delta|_D$ corresponding to the smallest eigenvalue λ_0 can be chosen to be strictly positive, see, for instance, [24]. The potential operator of X^D is given by

$$G^D f(x) = \int_0^\infty P_t^D f(x)\, dt$$

and has a density $G^D(x,y)$, $x, y \in D$. Here, and further below, f denotes a nonnegative Borel function on D. We recall the following well-known facts: If h is a nonnegative harmonic function for X^D (i.e., harmonic for Δ in D), then both h and $P_t^D h$ are continuous functions in D.

In this section we always assume that $(P_t^D : t \geq 0)$ is *intrinsically ultracontractive*, that is, for each $t > 0$ there exists a constant c_t such that

$$p^D(t,x,y) \leq c_t \varphi_0(x) \varphi_0(y), \quad x, y \in D, \tag{5.1}$$

where φ_0 is the positive eigenfunction corresponding to the smallest eigenvalue λ_0 of the Dirichlet Laplacian $-\Delta|_D$. It is well known that (see, for instance, [25]) when $(P_t^D; t \geq 0)$ is intrinsically ultracontractive there is $\tilde{c}_t > 0$ such that

$$p^D(t,x,y) \geq \tilde{c}_t \varphi_0(x) \varphi_0(y), \quad x, y \in D.$$

Intrinsic ultracontractivity was introduced by Davies and Simon in [25]. It is well known that (see, for instance, [1]) $(P_t^D : t \geq 0)$ is intrinsically ultracontractive when D is a bounded Lipschitz domain, or a Hölder domain of order 0, or a uniformly Hölder domain of order $\beta \in (0, 2)$.

Let $S = (S_t : t \geq 0)$ and $T = (T_t : t \geq 0)$ be two special subordinators. Suppose that X, S and T are independent. We assume that the Laplace exponents of S and T, denoted by ϕ and ψ respectively, are conjugate, i.e., $\lambda = \phi(\lambda)\psi(\lambda)$. We also assume that ϕ has the representation (2.2) with $b > 0$ or $\mu(0, \infty) = \infty$. We define two subordinate processes Y^D and Z^D by

$$\begin{aligned} Y_t^D &= X^D(S_t), \quad t \geq 0 \\ Z_t^D &= X^D(T_t), \quad t \geq 0. \end{aligned}$$

Then $Y^D = (Y_t^D : t \geq 0)$ and $Z^D = (Z_t^D : t \geq 0)$ are strong Markov processes on D. We call Y^D (resp. Z^D) a subordinate killed Brownian motion. If we use $\eta_t(ds)$ and $\theta_t(ds)$ to denote the distributions of S_t and T_t respectively, the semigroups of Y^D and Z^D are given by

$$\begin{aligned} Q_t^D f(x) &= \int_0^\infty P_s^D f(x) \eta_t(ds), \\ R_t^D f(x) &= \int_0^\infty P_s^D f(x) \theta_t(ds), \end{aligned}$$

respectively. The semigroup Q_t^D has a density given by

$$q^D(t, x, y) = \int_0^\infty p^D(s, x, y) \eta_t(ds) \, .$$

The semigroup R_t^D will have a density

$$r^D(t, x, y) = \int_0^\infty p^D(s, x, y) \theta_t(ds)$$

in the case $b = 0$, while for $b > 0$, R_t^D is not absolutely continuous with respect to the Lebesgue measure. Let U and V denote the potential measures of S and T, respectively. Then there are decreasing functions u and v defined on $(0, \infty)$ such that $U(dt) = u(t)\, dt$ and $V(dt) = b\epsilon_0(dt) + v(t)\, dt$. The potential kernels of Y^D and Z^D are given by

$$\begin{aligned} U^D f(x) &= \int_0^\infty P_t^D f(x) \, U(dt) = \int_0^\infty P_t^D f(x) \, u(t) \, dt, \\ V^D f(x) &= \int_0^\infty P_t^D f(x) \, V(dt) = b f(x) + \int_0^\infty P_t^D f(x) \, v(t) \, dt \, , \end{aligned}$$

respectively. The potential kernel U^D has a density given by

$$U^D(x, y) = \int_0^\infty p^D(t, x, y) \, u(t) \, dt \, ,$$

while V^D needs not be absolutely continuous with respect to the Lebesgue measure. Note that $U^D(x,y)$ is the Green function of the process Y^D. For the process Y^D we define the potential of a Borel measure m on D by

$$U^D m(x) := \int_D U^D(x,y)\, m(dy) = \int_0^\infty P_t^D m(x)\, u(t)\, dt\,.$$

Let $(U_\lambda^D, \lambda > 0)$ be the resolvent of the semigroup $(Q_t^D,\, t \geq 0)$. Then U_λ^D is given by a kernel which is absolutely continuous with respect to the Lebesgue measure. Moreover, one can easily show that for a bounded Borel function f vanishing outside a compact subset of D, the functions $x \mapsto U_\lambda^D f(x)$, $\lambda > 0$, and $x \mapsto U^D f(x)$ are continuous. This implies (e.g., [13], p.266) that excessive functions of Y^D are lower semicontinuous.

Recall that a measurable function $s : D \to [0, \infty]$ is excessive for Y^D (or Q_t^D), if $Q_t^D s \leq s$ for all $t \geq 0$ and $s = \lim_{t\to 0} Q_t^D s$. We will denote the family of all excessive function for Y^D by $\mathcal{S}(Y^D)$. The notation $\mathcal{S}(X^D)$ and $\mathcal{S}(Z^D)$ are now self-explanatory.

A measurable function $h : D \to [0, \infty]$ is harmonic for Y^D if h is not identically infinite in D and if for every relatively compact open subset $U \subset \overline{U} \subset D$,

$$h(x) = \mathbb{E}^x[h(Y^D(\tau_U^Y))], \quad \forall x \in U,$$

where $\tau_U^Y = \inf\{t : Y_t^D \notin U\}$ is the first exit time of Y^D from U. We will denote the family of all excessive function for Y^D by $\mathcal{H}^+(Y^D)$. Similarly, $\mathcal{H}^+(X^D)$ will denote the family of all nonnegative harmonic functions for X^D. It is well known that $\mathcal{H}^+(\cdot) \subset \mathcal{S}(\cdot)$.

5.2 Representation of excessive and harmonic functions of subordinate process

The factorization in the next proposition is similar in spirit to Theorem 4.1 (5) in [50].

Proposition 5.1 *(a) For any nonnegative Borel function f on D we have*

$$U^D V^D f(x) = V^D U^D f(x) = G^D f(x), \quad x \in D.$$

(b) For any Borel measure m on D we have

$$V^D U^D m(x) = G^D m(x)\,.$$

Proof. (a) We are only going to show that $U^D V^D f(x) = G^D f(x)$ for all $x \in D$. For the proof of $V^D U^D f(x) = G^D f(x)$ see part (b). For any nonnegative Borel function f on D, by using the Markov property and Theorem 2.6 we get that

$$\begin{aligned}
U^D V^D f(x) &= \int_0^\infty P_t^D V^D f(x) u(t) dt \\
&= \int_0^\infty P_t^D \left(bf(x) + \int_0^\infty P_s^D f(x) v(s) ds \right) u(t) dt \\
&= bU^D f(x) + \int_0^\infty P_t^D \left(\int_0^\infty P_s^D f(x) v(s) ds \right) u(t) dt \\
&= bU^D f(x) + \int_0^\infty \int_0^\infty P_{t+s}^D f(x) v(s) ds\, u(t) dt \\
&= bU^D f(x) + \int_0^\infty \int_t^\infty P_r^D f(x) v(r-t) dr\, u(t) dt \\
&= bU^D f(x) + \int_0^\infty \left(\int_0^r u(t) v(r-t) dt \right) P_r^D f(x) dr \\
&= \int_0^\infty \left(bu(r) + \int_0^r u(t) v(r-t) dt \right) P_r^D f(x) dr \\
&= \int_0^\infty P_r^D f(x) dr = G^D f(x).
\end{aligned}$$

(b) Similarly as above,

$$\begin{aligned}
V^D U^D m(x) &= bU^D m(x) + \int_0^\infty P_t^D U^D m(x) v(t)\, dt \\
&= bU^D m(x) + \int_0^\infty P_t^D \left(\int_0^\infty P_s^D m(x) u(s)\, ds \right) v(t)\, dt \\
&= bU^D m(x) + \int_0^\infty \int_0^\infty P_{t+s}^D m(x) u(s)\, ds\, v(t)\, dt \\
&= bU^D m(x) + \int_0^\infty \int_r^\infty P_r^D m(x) u(r-t)\, dr\, v(t)\, dt \\
&= bU^D m(x) + \int_0^\infty \left(\int_0^r u(r-t) v(t)\, dt \right) P_r^D m(x)\, dr \\
&= \int_0^\infty \left(b + \int_0^r u(r-t) v(t)\, dt \right) P_r^D m(x)\, dr \\
&= \int_0^\infty P_r^D m(x)\, dr = G^D m(x)\,.
\end{aligned}$$

□

Proposition 5.2 *Let g be an excessive function for Y^D. Then $V^D g$ is excessive for X^D.*

Proof. We first observe that if g is excessive with respect to Y^D, then g is the increasing limit of $U^D f_n$ for some f_n. Hence it follows from Proposition 5.1 that

$$V^D g = \lim_{n\to\infty} V^D U^D f_n = \lim_{n\to\infty} G^D f_n,$$

which implies that $V^D g$ is either identically infinite or excessive with respect to X^D. We prove now that $V^D g$ is not identically infinite. In fact, since g is excessive with respect to Y^D, there exists $x_0 \in D$ such that for every $t > 0$,

$$\infty > g(x_0) \geq Q_t^D g(x_0) = \int_0^\infty P_s^D g(x_0)\rho_t(ds).$$

Thus there is $s > 0$ such that $P_s^D g(x_0)$ is finite. Hence

$$\infty > P_s^D g(x_0) = \int_D p^D(s, x_0, y)g(y)\, dy \geq \tilde{c}_s \varphi_0(x_0) \int_D \varphi_0(y)g(y)\, dy,$$

so we have $\int_D \varphi_0(y)g(y)\, dy < \infty$. Consequently

$$\begin{aligned}
&\int_D V^D g(x)\varphi_0(x)\, dx = \int_D g(x) V^D \varphi_0(x)\, dx \\
&= \int_D g(x) \left(b\varphi_0(x) + \int_0^\infty P_t^D \varphi_0(x) v(t)\, dt \right) dx \\
&= \int_D g(x) \left(b\varphi_0(x) + \int_0^\infty e^{-\lambda_0 t} \varphi_0(x) v(t)\, dt \right) dx \\
&= \int_D \varphi_0(x) g(x)\, dx \left(b + \int_0^\infty e^{-\lambda_0 t} v(t)\, dt \right) < \infty .
\end{aligned}$$

Therefore $s = V^D g$ is not identically infinite in D. □

Remark 5.3 *Note that the proposition above is valid with Y^D and Z^D interchanged: If g is excessive for Z^D, then $U^D g$ is excessive for X^D. Using this we can easily get the following simple fact: If f and g are two nonnegative Borel functions on D such that $V^D f$ and $V^D g$ are not identically infinite, and such that $V^D f = V^D g$ a.e., then $f = g$ a.e. In fact, since $V^D f$ and $V^D g$ are excessive for Z^D, we know that $G^D f = U^D V^D f$ and $G^D g = U^D V^D g$ are excessive for X^D. Moreover, by the absolute continuity of U^D, we have that $G^D f = G^D g$. The a.e. equality of f and g follows from the uniqueness principle for G^D.*

The second part of Proposition 5.1 shows that if $s = G^D m$ is the potential of a measure, then $s = V^D g$ where $g = U^D m$ is excessive for Y^D. The function g can be written in the following way:

$$\begin{aligned}
g(x) &= \int_0^\infty P_s^D m(x)u(s)\, ds \\
&= \int_0^\infty P_s^D m(x) \left(u(\infty) + \int_s^\infty -du(t) \right) ds \\
&= \int_0^\infty P_s^D m(x)u(\infty)\, ds + \int_0^\infty P_s^D m(x) \left(\int_s^\infty -du(t) \right) ds \\
&= u(\infty)s(x) + \int_0^\infty \left(\int_0^t P_s^D m(x)\, ds \right) (-du(t)) \\
&= u(\infty)s(x) + \int_0^\infty (P_t^D s(x) - s(x))\, du(t)\,. \qquad (5.2)
\end{aligned}$$

In the next proposition we will show that every excessive function s for X^D can be represented as a potential $V^D g$, where g, given by (5.2), is excessive for Y^D. We need the following important lemma.

Lemma 5.4 *Let h be a nonnegative harmonic function for X^D, and let*

$$g(x) = u(\infty)h(x) + \int_0^\infty (P_t^D h(x) - h(x))\, du(t)\,. \qquad (5.3)$$

Then g is continuous.

Proof. For any $\epsilon > 0$ it holds that $|\int_\epsilon^\infty du(t)| \le u(\epsilon)$. Hence from the continuity of h and $P_t^D h$ it follows by the dominated convergence theorem that the function

$$x \mapsto \int_\epsilon^\infty (P_t^D h(x) - h(x))\, du(t), \quad x \in D,$$

is continuous. Therefore we only need to prove that the function

$$x \mapsto \int_0^\epsilon (P_t^D h(x) - h(x))\, du(t), \quad x \in D,$$

is continuous. For any $x_0 \in D$ choose $r > 0$ such that $B(x_0, 2r) \subset D$, and let $B = B(x_0, r)$. It is enough to show that

$$\lim_{\epsilon \downarrow 0} \int_0^\epsilon (P_t^D h(x) - h(x))\, du(t) = 0$$

uniformly on $\overline{B}$, the closure of B. For any $x \in B$, $h(X_{t \wedge \tau_B})$ is a $\mathbb{P}^x$-martingale. Therefore,

$$\begin{aligned}
0 &\le h(x) - P_t^D h(x) = \mathbb{E}^x[h(X_{t\wedge\tau_B})] - \mathbb{E}^x[h(X_t), t < \tau_D] \\
&= \mathbb{E}^x[h(X_t), t < \tau_B] + \mathbb{E}^x[h(X_{\tau_B}), \tau_B \le t] \\
&\quad -\mathbb{E}^x[h(X_t), t < \tau_B] - \mathbb{E}^x[h(X_t), \tau_B \le t < \tau_D] \\
&= \mathbb{E}^x[h(X_{\tau_B}), \tau_B \le t] - \mathbb{E}^x[h(X_t), \tau_B \le t < \tau_D] \\
&\le \mathbb{E}^x[h(X_{\tau_B}), \tau_B \le t] \le M\mathbb{P}^x(\tau_B \le t)\,, \qquad (5.4)
\end{aligned}$$

where M is a constant such that $h(y) \le M$ for all $y \in \overline{B}$. It is a standard fact that there exists a constant $c > 0$ such that for every $x \in \overline{B}$ it holds that $\mathbb{P}^x(\tau_B \le t) \le ct$, for all $t > 0$. Therefore, $0 \le h(x) - P_t^D h(x) \le Mct$, for all $x \in \overline{B}$ and all $t > 0$. It follows that for every $x \in \overline{B}$,

$$\left| \int_0^\epsilon (P_t^D h - h)(x)\, du(t) \right| \le Mc \left| \int_0^\epsilon t\, du(t) \right| .$$

By use of (2.14) we get that

$$\lim_{\epsilon \downarrow 0} \int_0^\epsilon (P_t^D h(x) - h(x))\, du(t) = 0$$

uniformly on $\overline{B}$. The proof is now complete. $\square$

Proposition 5.5 *If s is an excessive function with respect to X^D, then*

$$s(x) = V^D g(x), \quad x \in D\,,$$

where g is the excessive function for Y^D given by the formula

$$\begin{aligned} g(x) &= u(\infty)s(x) + \int_0^\infty (P_t^D s(x) - s(x))\, du(t) & (5.5)\\ &= \psi(0)s(x) + \int_0^\infty (s(x) - P_t^D s(x))\, d\nu(t)\,. & (5.6) \end{aligned}$$

Proof. We know that the result is true when s is the potential of a measure. Let s be an arbitrary excessive function of X^D. By the Riesz decomposition theorem (see, for instance, Chapter 6 of [13]), $s = G^D m + h$, where m is a measure on D, and h is a nonnegative harmonic function for X^D. By linearity, it suffices to prove the result for nonnegative harmonic functions.

In the rest of the proof we assume therefore that s is a nonnegative harmonic function for X^D. Define the function g by formula (5.5). We have to prove that g is excessive for Y^D and $s = V^D g$. By Lemma 5.4, we know that g is continuous.

Further, since s is excessive, there exists a sequence of nonnegative functions f_n such that $s_n := G^D f_n$ increases to s. Then also $P_t^D s_n \uparrow P_t^D s$, implying $s_n - P_t^D s_n \to s - P_t^D s$. If

$$g_n = u(\infty)s_n + \int_0^\infty (s_n - P_t^D s_n)(-du(t))\,,$$

then we know that $s_n = V^D g_n$ and g_n is excessive for Y^D. By use of Fatou's lemma we get that

$$
\begin{aligned}
g &= u(\infty)s + \int_0^\infty (s - P_t^D s)(-du(t)) \\
&= \lim_n u(\infty)s_n + \int_0^\infty \lim_n (s_n - P_t^D s_n)(-du(t)) \\
&\le \liminf_n \left(u(\infty)s_n + \int_0^\infty (s_n - P_t^D s_n)(-du(t)) \right) \\
&= \liminf_n g_n\,.
\end{aligned}
$$

This implies (again by Fatou's lemma) that

$$
\begin{aligned}
V^D g &\le V^D(\liminf_n g_n) \\
&\le \liminf_n V^D g_n = \liminf_n s_n = s\,.
\end{aligned}
\tag{5.7}
$$

For any nonnegative function f, put $G_1^D f(x) := \int_0^\infty e^{-t} P_t^D f(x)\, dt$, and define $s^1 := s - G_1^D s$. Using an argument similar to that of the proof of Proposition 5.2 we can show that $G^D s$ is not identically infinite. Thus by the resolvent equation we get $G^D s^1 = G^D s - G^D G_1^D s = G_1^D s$, or equivalently,

$$
s(x) = s^1(x) + G_1^D s(x) = s^1(x) + G^D s^1(x), \quad x \in D,
$$

By use of formula (5.2) for the potential $G^D s_1$, Fubini's theorem and the easy fact that V^D and G_1^D commute, we have

$$
\begin{aligned}
G_1^D s &= G^D s^1 = V^D \left(u(\infty) G^D s^1 + \int_0^\infty (P_t^D G^D s^1 - G^D s^1)\, du(t) \right) \\
&= V^D \left(u(\infty) G_1^D s + \int_0^\infty (P_t^D G_1^D s - G_1^D s)\, du(t) \right) \\
&= G_1^D V^D \left(u(\infty) s + \int_0^\infty (P_t^D s - s)\, du(t) \right).
\end{aligned}
$$

By the uniqueness principle it follows that

$$
s = V^D \left(u(\infty)s + \int_0^\infty (P_t^D s - s)\, du(t) \right) = V^D g \quad \text{a.e. in } D\,.
$$

Together with (5.7), this implies that $V^D g = V^D(\liminf_n g_n)$ a.e. From Remark 5.3 it follows that

$$
g = \liminf_n g_n \quad \text{a.e.} \tag{5.8}
$$

By Fatou's lemma and the Y^D-excessiveness of g_n we get that,

$$\lambda U_\lambda^D g = \lambda U_\lambda^D(\liminf g_n) \le \liminf_n \lambda U_\lambda^D g_n \le \liminf g_n = g \quad \text{a.e.}$$

We want to show that, in fact, $\lambda U_\lambda^D g \le g$ everywhere, i.e., that g is supermedian. In order to do this we define $\tilde{g} := \sup_{n\in\mathbb{N}} nU_n^D g$. Then $\tilde{g} \le g$ a.e., hence, by the absolute continuity of U_n^D, $nU_n^D \tilde{g} \le nU_n^D g \le \tilde{g}$ everywhere. This implies that $\lambda \mapsto \lambda U_\lambda^D \tilde{g}$ is increasing (see, e.g., Lemma 3.6 in [11]), hence $\tilde{g}$ is supermedian. The same argument gives that $n \mapsto nU_n^D g$ is increasing a.e. Define

$$\tilde{\tilde{g}} := \sup_{\lambda>0} \lambda U_\lambda^D \tilde{g} = \sup_n nU_n^D \tilde{g}\,.$$

Then $\tilde{\tilde{g}}$ is excessive, and therefore lower semicontinuous. Moreover,

$$\tilde{\tilde{g}} = \sup_n nU_n^D \tilde{g} \le \tilde{g} \le g \quad \text{a.e.}$$

Combining this with the continuity of g and the lower semicontinuity of $\tilde{\tilde{g}}$, we can get that $\tilde{\tilde{g}} \le g$ everywhere. Further, for $x \in D$ such that $\tilde{g}(x) < \infty$, we have by the monotone convergence theorem and the resolvent equation

$$\begin{aligned}
\lambda U_\lambda^D \tilde{g}(x) &= \lim_{n\to\infty} \lambda U_\lambda^D (nU_n^D) g(x) \\
&= \lim_{n\to\infty} \frac{n\lambda}{n-\lambda}(U_\lambda^D g(x) - U_n^D g(x) \\
&= \lambda U_\lambda^D g(x)\,.
\end{aligned}$$

Since $\tilde{g} < \infty$ a.e., we have

$$\lambda U_\lambda^D \tilde{g} = \lambda U_\lambda^D g \quad \text{a.e.}$$

Together with the definition of $\tilde{g}$ this implies that

$$\tilde{\tilde{g}} = \tilde{g} \quad \text{a.e.} \tag{5.9}$$

By the continuity of g and the fact that the measures $nU_n^D(x,\cdot)$ converge weakly to the point mass at x, we have that for every $x \in D$

$$g(x) \le \liminf_{n\to\infty} g(x) \le \tilde{g}(x)\,.$$

Hence, by using (5.9), it follows that $g \le \tilde{\tilde{g}}$ a.e. Since we already proved that $\tilde{\tilde{g}} \le g$, it holds that $g = \tilde{\tilde{g}}$ a.e. By the absolute continuity of U_λ^D, $g \ge \tilde{\tilde{g}} \ge \lambda U_\lambda^D \tilde{\tilde{g}} = \lambda U_\lambda^D g$ everywhere, i.e., g is supermedian.

Since it is well known (see e.g. [22]) that a supermedian function which is lower semicontinuous is in fact excessive, this proves that g is excessive for Y^D. By Proposition 5.2 we then have that $V^D g \le s$ is excessive for X^D. Moreover, $V^D g = s$ a.e., and both functions being excessive for X^D, they are equal everywhere.

It remains to notice that the formula (5.6) follows immediately from (5.5) by noting that $u(\infty) = \psi(0)$ and $du(t) = -d\nu(t)$. □

Propositions 5.1 and 5.5 can be combined in the following theorem containing additional information on harmonic functions.

Theorem 5.6 *If s is excessive with respect to X^D, then there is a function g excessive with respect to Y^D such that $s = V^D g$. The function g is given by the formula (5.2). Furthermore, if s is harmonic with respect to X^D, then g is harmonic with respect to Y^D.*

Conversely, if g is excessive with respect to Y^D, then the function s defined by $s = V^D g$ is excessive with respect to X^D. If, moreover, g is harmonic with respect to Y^D, then s is harmonic with respect to X^D.

Every nonnegative harmonic function for Y^D is continuous.

Proof. It remains to show the statements about harmonic functions. First note that every excessive functions g for Y^D admits the Riesz decomposition $g = U^D m + h$ where m is a Borel measure on D and h is harmonic function of Y^D (see Chapter 6 of [13] and note that the assumptions on pp. 265, 266 are satisfied). We have already mentioned that excessive functions of X^D admit such decomposition. Since excessive functions of X^D and Y^D are in 1-1 correspondence, and since potentials of measures of X^D and Y^D are in 1-1 correspondence, the same must hold for nonnegative harmonic functions of X^D and Y^D.

The continuity of nonnegative harmonic functions for Y^D follows from Lemma 5.4 and Proposition 5.5. □

It follows from the theorem above that V^D is a bijection from $\mathcal{S}(Y^D)$ to $\mathcal{S}(X^D)$, and is also a bijection from $\mathcal{H}^+(Y^D)$ to $\mathcal{H}^+(X^D)$. We are going to use $(V^D)^{-1}$ to denote the inverse map and so we have for any $s \in \mathcal{S}(Y^D)$,

$$\begin{aligned} (V^D)^{-1}s(x) &= u(\infty)s(x) + \int_0^\infty (P_t^D s(x) - s(x))\, du(t) \\ &= \psi(0)s(x) + \int_0^\infty (s(x) - P_t^D s(x))\, d\nu(t)\,. \end{aligned} \tag{5.10}$$

Although the map V^D is order preserving, we do not know if the inverse map $(V^D)^{-1}$ is order preserving on $\mathcal{S}(X^D)$. However from the formula above we can see that $(V^D)^{-1}$ is order preserving on $\mathcal{H}^+(X^D)$.

By combining Proposition 5.1 and Theorem 5.6 we get the following relation which we are going to use later.

Proposition 5.7 *For any $x, y \in D$, we have*

$$U^D(x,y) = (V^D)^{-1}(G^D(\cdot,y))(x).$$

5.3 Harnack inequality for subordinate process

In this subsection we are going to prove the Harnack inequality for positive harmonic functions for the process Y^D under the assumption that D is a bounded domain such that (P_t^D) is intrinsic ultracontractive. The proof we offer uses the intrinsic ultracontractivity in an essential way, and differs from the existing proofs of Harnack inequalities in other settings.

We first recall that since $(P_t^D : t \geq 0)$ is intrinsic ultracontractive, by Theorem 4.2.5 of [24] there exists $T > 0$ such that

$$\frac{1}{2}e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y) \leq p^D(t,x,y) \leq \frac{3}{2}e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y), \quad t \geq T,\ x,y \in D. \tag{5.11}$$

Lemma 5.8 *Suppose that D is a bounded domain such that (P_t^D) is intrinsic ultracontractive. There exists a constant $C > 0$ such that*

$$V^D g \leq Cg, \quad \forall g \in \mathcal{S}(Y^D)\,. \tag{5.12}$$

Proof. Let T be the constant from (5.11). For any nonnegative function f,

$$U^D f(x) = \left(\int_0^T P_t^D f(x)u(t)\,dt + \int_T^\infty P_t^D f(x)u(t)\,dt\right).$$

We obviously have

$$\int_0^T P_t^D f(x)u(t)\,dt \geq u(T)\int_0^T P_t^D f(x)\,dt.$$

By using (5.11) we see that

$$\begin{aligned}\int_T^\infty P_t^D f(x)u(t)\,dt &\geq \left(\frac{1}{2}\int_T^\infty e^{-\lambda_0 t}u(t)\,dt\right)\int_D \varphi_0(x)\varphi_0(y)f(y)\,dy\\ &= c_1\int_D \varphi_0(x)\varphi_0(y)f(y)\,dy\end{aligned}$$

and

$$\begin{aligned}\int_T^\infty P_t^D f(x)\,dt &\leq \left(\frac{3}{2}\int_T^\infty e^{-\lambda_0 t}dt\right)\int_D \varphi_0(x)\varphi_0(y)f(y)dy\\ &= c_2\int_D \varphi_0(x)\varphi_0(y)f(y)dy\,.\end{aligned}$$

The last two displays imply that

$$\int_T^\infty P_t^D f(x)u(t)\,dt \geq \frac{c_1}{c_2}\int_T^\infty P_t^D f(x)\,dt\,.$$

Therefore,

$$\begin{aligned} U^D f(x) &\geq u(T)\int_0^T P_t^D f(x)\,dt + \frac{c_1}{c_2}\int_T^\infty P_t^D f(x)\,dt \\ &\geq C\int_0^\infty P_t^D f(x)\,dt = CG^D f(x)\,. \end{aligned}$$

From $G^D f(x) = V^D U^D f(x)$, we obtain $V^D U^D f(x) \leq CU^D f(x)$. Since every $g \in \mathcal{S}(Y^D)$ is an increasing limit of potentials $U^D f(x)$, the claim follows. □

Lemma 5.9 *Suppose D is a bounded domain such that (P_t^D) is intrinsic ultracontractive. If $g \in \mathcal{S}(Y^D)$, then for any $x \in D$,*

$$g(x) \geq \frac{1}{2C}e^{-\lambda_0 T}\frac{1}{\psi(\lambda_0)}\varphi_0(x)\int_D g(y)\varphi_0(y)\,dy,$$

where T is the constant in (5.11) and C is the constant in (5.12).

Proof. From the lemma above we know that, for every $x \in D$, $V^D g(x) \leq Cg(x)$, where C is the constant in (5.12). Since $V^D g$ is in $\mathcal{S}(X^D)$, we have

$$\begin{aligned} V^D g(x) &\geq \int_D p^D(T,x,y)V^D g(y)\,dy \\ &\geq \frac{1}{2}e^{-\lambda_0 T}\varphi_0(x)\int_D \varphi_0(y)V^D g(y)\,dy. \end{aligned}$$

Hence

$$\begin{aligned} Cg(x) &\geq V^D g(x) \geq \frac{1}{2}e^{-\lambda_0 T}\varphi_0(x)\int_D \varphi_0(y)V^D g(y)\,dy \\ &= \frac{1}{2}e^{-\lambda_0 T}\varphi_0(x)\int_D g(y)V^D\varphi_0(y)\,dy \\ &= \frac{1}{2}e^{-\lambda_0 T}\frac{1}{\psi(\lambda_0)}\varphi_0(x)\int_D g(y)\varphi_0(y)\,dy\,, \end{aligned}$$

where the last line follows from

$$\begin{aligned} V^D\varphi_0(y) &= \int_0^\infty P_t^D\varphi_0(y)\,V(dt) = \int_0^\infty e^{-\lambda_0 t}\varphi_0(y)\,V(dt) \\ &= \varphi_0(y)\mathcal{L}V(\lambda_0) = \frac{\varphi_0(y)}{\psi(\lambda_0)}\,. \end{aligned}$$

□

In particular, it follows from the lemma that if $g \in \mathcal{S}(Y^D)$ is not identically infinite, then $\int_D \varphi_0(y)g(y)\,dy < \infty$.

Theorem 5.10 *Suppose D is a bounded domain such that (P_t^D) is intrinsic ultracontractive. For any compact subset K of D, there exists a constant C depending on K and D such that for any $h \in \mathcal{H}^+(Y^D)$,*

$$\sup_{x\in K} h(x) \le C \inf_{x\in K} h(x).$$

Proof. If the conclusion of the theorem were not true, for any $n \ge 1$, there would exist $h_n \in \mathcal{H}^+(Y^D)$ such that

$$\sup_{x\in K} h_n(x) \ge n2^n \inf_{x\in K} h_n(x). \tag{5.13}$$

By the lemma above, we may assume without loss of generality that

$$\int_D h_n(y)\varphi_0(y)dy = 1, \quad n \ge 1.$$

Define

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x), \quad x \in D.$$

Then

$$\int_D h(y)\varphi_0(y)dy = 1\,,$$

and so $h \in \mathcal{H}^+(Y^D)$. By (5.13) and the lemma above, for every $n \ge 1$, there exists $x_n \in K$ such that $h_n(x_n) \ge n2^n c_1$ where

$$c_1 = \frac{1}{2C} e^{-\lambda_0 T} \frac{1}{\psi(\lambda_0)} \inf_{x\in K} \varphi_0(x)$$

with T as in (5.11) and C in (5.12). Therefore we have $h(x_n) \ge nc_1$. Since K is compact, there is a convergent subsequence of x_n. Let x_0 be the limit of this convergent subsequence. Theorem 5.6 implies that h is continuous, and so we have $h(x_0) = \infty$. This is a contradiction. So the conclusion of the theorem is valid. □

5.4 Martin boundary of subordinate process

In this subsection we assume that $d \ge 3$ and that D is a bounded Lipschitz domain in $\mathbb{R}^d$. Fix a point $x_0 \in D$ and set

$$M^D(x,y) = \frac{G^D(x,y)}{G^D(x_0,y)}, \quad x, y \in D.$$

It is well known that the limit $\lim_{D\ni y\to z} M^D(x,y)$ exists for every $x\in D$ and $z\in\partial D$. The function $M^D(x,z) := \lim_{D\ni y\to z} M^D(x,y)$ on $D\times\partial D$ defined above is called the Martin kernel of X^D based at x_0. The Martin boundary and minimal Martin boundary of X^D both coincide with the Euclidean boundary ∂D. For these and other results about the Martin boundary of X^D one can see [2]. One of the goals of this section is to determine the Martin boundary of Y^D.

By using the Harnack inequality, one can easily show that (see, for instance, pages 17–18 of [26]), if (h_j) is a sequence of functions in $\mathcal{H}^+(X^D)$ converging pointwise to a function $h\in\mathcal{H}^+(X^D)$, then (h_j) is locally uniformly bounded in D and equicontinuous at every point in D. Using this, one can get that, if (h_j) is a sequence of functions in $\mathcal{H}^+(X^D)$ converging pointwise to a function $h\in\mathcal{H}^+(X^D)$, then (h_j) converges to h uniformly on compact subsets of D. We are going to use this fact below.

Lemma 5.11 *Suppose that $x_0\in D$ is a fixed point.*

(a) Let $(x_j : j\geq 1)$ be a sequence of points in D converging to $x\in D$ and let (h_j) be a sequence of functions in $\mathcal{H}^+(X^D)$ with $h_j(x_0)=1$ for all j. If the sequence (h_j) converges to a function $h\in\mathcal{H}^+(X^D)$, then for each $t>0$

$$\lim_{j\to\infty} P_t^D h_j(x_j) = P_t^D h(x)\,.$$

(b) If $(y_j : j\geq 1)$ is a sequence of points in D such that $\lim_j y_j = z\in\partial D$, then for each $t>0$ and for each $x\in D$

$$\lim_{j\to\infty} P_t^D\left(\frac{G^D(\cdot,y_j)}{G^D(x_0,y_j)}\right)(x) = P_t^D(M^D(\cdot,z))(x)\,.$$

Proof. (a) For each $j\in\mathbb{N}$, since $h_j(x_0)=1$, there exists a probability measure μ_j on ∂D such that

$$h_j(x) = \int_{\partial D} M^D(x,z)\mu_j(dz),\quad x\in D.$$

Similarly, there exists a probability measure μ on ∂D such that

$$h(x) = \int_{\partial D} M^D(x,z)\mu(dz),\quad x\in D.$$

Let D_0 be a relatively compact open subset of D such that $x_0\in D_0$, and also $x, x_j\in D_0$.

Then

$$\begin{aligned}
&|P_t^D h_j(x_j) - P_t^D h(x)| \\
&\quad = \left| \int_D p^D(t, x_j, y) h_j(y) dy - \int_D p^D(t, x, y) h(y) dy \right| \\
&\quad \le \left| \int_{D_0} p^D(t, x_j, y) h_j(y) dy - \int_{D_0} p^D(t, x, y) h(y) dy \right| \\
&\qquad + \int_{D \setminus D_0} p^D(t, x_j, y) h_j(y)\, dy + \int_{D \setminus D_0} p^D(t, x, y) h(y)\, dy\,.
\end{aligned}$$

Recall that (see Section 6.2 of [23], for instance) there exists a constant $c > 0$ such that

$$\frac{G^D(x,y)G^D(y,w)}{G^D(x,w)} \le c \left(\frac{1}{|x-y|^{d-2}} + \frac{1}{|y-w|^{d-2}} \right), \quad x, y, w \in D. \tag{5.14}$$

From this and the definition of the Martin kernel we immediately get

$$G^D(x_0, y) M^D(y, z) \le c \left(\frac{1}{|x_0 - y|^{d-2}} + \frac{1}{|y-z|^{d-2}} \right), \quad y \in D, z \in \partial D. \tag{5.15}$$

Recall (see [24], p.131, Theorem 4.6.11) that there is a constant $c > 0$ such that

$$\varphi_0(x_0)\varphi_0(y) \le c\, G^D(x_0, y), \quad y \in D\,.$$

By the boundedness of φ_0 we have that $\varphi_0(u) \le c_1 \varphi_0(x_0)$ for every $u \in D$. Hence it follows from the last display that

$$\varphi_0(u)\varphi_0(y) \le c\, G^D(x_0, y), \quad u, y \in D\,, \tag{5.16}$$

with a possibly different constant $c > 0$. Now using (5.1), (5.15) and (5.16) we get that for any $u \in D$,

$$\begin{aligned}
&\int_{D \setminus D_0} p^D(t, u, y) h(y)\, dy \le c_t \varphi_0(u) \int_{D \setminus D_0} \varphi_0(y) h(y)\, dy \\
&= c_t \varphi_0(u) \int_{D \setminus D_0} dy\, \varphi_0(y) \int_{\partial D} M^D(y, z) \mu(dz) \\
&= c_t \varphi_0(u) \int_{\partial D} \mu(dz) \int_{D \setminus D_0} \varphi_0(y) M^D(y, z)\, dy \\
&\le c c_t \int_{\partial D} \mu(dz) \int_{D \setminus D_0} G^D(x_0, y) M^D(y, z)\, dy \\
&\le c c_t \int_{\partial D} \mu(dz) \int_{D \setminus D_0} \left(\frac{1}{|y-z|^{d-2}} + \frac{1}{|x_0 - y|^{d-2}} \right) dy \\
&\le c c_t \int_{\partial D} \mu(dz) \int_{D \setminus D_0} \sup_{z \in \partial D} \left(\frac{1}{|y-z|^{d-2}} + \frac{1}{|x_0 - y|^{d-2}} \right) dy \\
&= c c_t \int_{D \setminus D_0} \sup_{z \in \partial D} \left(\frac{1}{|y-z|^{d-2}} + \frac{1}{|x_0 - y|^{d-2}} \right) dy\,.
\end{aligned}$$

The same estimate holds with h_j instead of h. For a given $\epsilon > 0$ choose D_0 large enough so that the last line in the display above is less than ϵ. Put $A = \sup_{D_0} h$. Take $j_0 \in \mathbb{N}$ large enough so that for all $j \geq j_0$ we have

$$|p^D(t, x_j, y) - p^D(t, x, y)| \leq \epsilon \quad \text{and} \quad |h_j(y) - h(y)| < \epsilon$$

for all $y \in D_0$. Then

$$\begin{aligned}
&\left| \int_{D_0} p^D(t, x_j, y) h_j(y) dy - \int_{D_0} p^D(t, x, y) h(y) dy \right| \\
&\quad \leq \int_{D_0} p^D(t, x_j, y) |h_j(y) - h(y)| dy + \int_{D_0} |p^D(t, x_j, y) - p^D(t, x, y)| h(y) dy \\
&\quad \leq \epsilon + A|D_0|\epsilon,
\end{aligned}$$

where $|D_0|$ stands for the Lebesgue measure of D_0. This proves the first part.
(b) We proceed similarly as in the proof of the first part. The only difference is that we use (5.14) to get the following estimate:

$$\begin{aligned}
&\int_{D \setminus D_0} p^D(t, x, y) \frac{G^D(y, y_j)}{G^D(x_0, y_j)} \, dy \\
&\quad \leq c_t \varphi_0(x) \int_{D \setminus D_0} \varphi_0(y) \frac{G^D(y, y_j)}{G^D(x_0, y_j)} \, dy \\
&\quad \leq c c_t \int_{D \setminus D_0} \frac{G^D(x_0, y) G^D(y, y_j)}{G^D(x_0, y_j)} \, dy \\
&\quad \leq c c_t \int_{D \setminus D_0} (|x_0 - y|^{2-d} + |y - y_j|^{2-d}) \, dy \\
&\quad \leq c c_t \sup_j \int_{D \setminus D_0} (|x_0 - y|^{2-d} + |y - y_j|^{2-d}) \, dy \, .
\end{aligned}$$

The corresponding estimate for $M^D(\cdot, z)$ is given in part (a) of the lemma. For a given $\epsilon > 0$ find D_0 large enough so that the last line in the display above is less than ϵ. Then find $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\left| \frac{G^D(y, y_j)}{G^D(x_0, y_j)} - M^D(y, z) \right| < \epsilon, \quad y \in D_0.$$

Then

$$\int_{D_0} p^D(t, x, y) \left| \frac{G^D(y, y_n)}{G^D(x_0, y_j)} - M^D(y, z) \right| dy < \epsilon \quad \text{for all} \quad j \geq j_0 \, .$$

This proves the second part. $\square$

Theorem 5.12 *Suppose that $D \subset \mathbb{R}^d$, $d \geq 3$ is a bounded Lipschitz domain and let $x_0 \in D$ be a fixed point.*

(a) If (x_j) is a sequence of points in D converging to $x \in D$ and (h_j) is a sequence of functions in $\mathcal{H}^+(X^D)$ converging to a function $h \in \mathcal{H}^+(X^D)$, then

$$\lim_j (V^D)^{-1} h_j(x_j) = (V^D)^{-1} h(x)\,.$$

(b) If (y_j) is a sequence of points in D converging to $z \in \partial D$, then for every $x \in D$,

$$\lim_j (V^D)^{-1}\Big(\frac{G^D(\cdot, y_j)}{G^D(x_0, y_j)}\Big)(x) = \lim_j \frac{(V^D)^{-1}(G^D(\cdot, y_j))(x)}{G^D(x_0, y_j)} = (V^D)^{-1} M^D(\cdot, z)(x)\,.$$

Proof. (a) Normalizing by $h_j(x_0)$ if necessary, we may assume without loss of generality that $h_j(x_0) = 1$ for all $j \geq 1$. Let $\epsilon > 0$. By (5.10) we have

$$\begin{aligned}
&|(V^D)^{-1} h_j(x_j) - (V^D)^{-1} h(x)| \\
&= \left| \int_0^\infty (P_t^D h_j(x_j) - h_j(x_j)\, du(t) - \int_0^\infty (P_t^D h(x) - h(x))\, du(t) + u(\infty)(h_j(x_j) - h(x)) \right| \\
&\leq \int_0^\epsilon (P_t^D h_j(x_j) - h_j(x_j))\, du(t) + \int_0^\epsilon (P_t^D h(x) - h(x))\, du(t) \\
&\quad + \left| \int_\epsilon^\infty (P_t^D h_j(x_j) - h_j(x_j))\, du(t) - \int_\epsilon^\infty (P_t^D h(x) - h(x))\, du(t) \right| \\
&\quad + u(\infty) |h_j(x_j) - h(x)|\,.
\end{aligned}$$

The last term clearly converges to zero as $j \to \infty$.

For any $x \in D$ choose $r > 0$ such that $B(x, 2r) \subset D$ and put $B = B(x, r)$. Without loss of generality we may and do assume that $x_j \in B$ for all $j \geq 1$. Since h and h_j are continuous in D and (h_j) is locally uniformly bounded in D, there is a constant $M > 0$ such that h and h_j, $j = 1, 2, \dots$, are all bounded from above by M on $\overline{B}$. Now from the proof of Lemma 5.4, more precisely from display (5.4), it follows that there is a constant $c_1 > 0$ such that

$$0 \leq h(y) - P_t^D h(y) \leq c_1 t, \quad y \in \overline{B}\,,$$

and

$$0 \leq h_j(y) - P_t^D h_j(y) \leq c_1 t, \quad y \in \overline{B}, j \geq 1\,.$$

Therefore we have,

$$\left| \int_0^\epsilon (P_t^D h - h)(y)\, du(t) \right| \leq c_1 \left| \int_0^\epsilon t\, du(t) \right|, \quad y \in \overline{B}$$

and
$$\left|\int_0^\epsilon (P_t^D h_j - h_j)(y)\, du(t)\right| \le c_1 \left|\int_0^\epsilon t\, du(t)\right|, \quad y \in \overline{B}\,, j \ge 1.$$
Using (2.14) we get that
$$\lim_{\epsilon \downarrow 0} \int_0^\epsilon (P_t^D h(x) - h(x))\, du(t) = 0\,,$$
and
$$\lim_{\epsilon \downarrow 0} \int_0^\epsilon (P_t^D h_j(x_j) - h_j(x_j))\, du(t) = 0\,.$$
Further,
$$\begin{aligned} &\left|\int_\epsilon^\infty (P_t^D h_j(x_j) - h_j(x_j))\, du(t) - \int_\epsilon^\infty (P_t^D h(x) - h(x))\, du(t)\right| \\ \le\; & \int_\epsilon^\infty (|h_j(x_j) - h(x_j)| + |h(x_j) - h(x)|)\, du(t) + \int_\epsilon^\infty |P_t^D h_j(x_j) - P_t^D h(x)|\, du(t)\,. \end{aligned}$$
Since $|h_j(x_j) - h(x_j)| + |h(x_j) - h(x)| \le 2M$ and $|P_t^D h_j(x_j) - P_t^D h(x)| \le M$ for all $j \ge 1$ and all $x \in \overline{B}$, we can apply Lemma 5.11(a) and the dominated convergence theorem to get
$$\lim_{j \to \infty} \int_\epsilon^\infty (|h_j(x_j) - h(x_j)| + |h(x_j) - h(x)|)\, du(t) = 0$$
and
$$\lim_{j \to \infty} \int_\epsilon^\infty |P_t^D h_j(x_j) - P_t^D h(x)| du(t) = 0.$$
The proof of (a) is now complete.

(b) The proof of (b) is similar to (a). The only difference is that we use 5.11(b) in this case. We omit the details. □

Let us define the function $K_Y^D(x,z) := (V^D)^{-1} M^D(\cdot, z)(x)$ on $D \times \partial D$. For each fixed $z \in \partial D$, $K_Y^D(\cdot, z) \in \mathcal{H}^+(Y^D)$. By the first part of Theorem 5.12, we know that $K_Y^D(x,z)$ is continuous on $D \times \partial D$. Let (y_j) be a sequence of points in D converging to $z \in \partial D$, then from Theorem 5.12(b) we get that
$$\begin{aligned} K_Y^D(x,z) &= \lim_{j \to \infty} (V^D)^{-1} \left(\frac{G^D(\cdot, y_j)}{G^D(x_0, y_j)}\right)(x) \\ &= \lim_{j \to \infty} \frac{(V^D)^{-1}(G^D(\cdot, y_j))(x)}{G^D(x_0, y_j)} \\ &= \lim_{j \to \infty} \frac{U^D(x, y_j)}{G^D(x_0, y_j)}\,, \end{aligned} \tag{5.17}$$

where the last line follows from Proposition 5.7. In particular, there exists the limit

$$\lim_{j\to\infty} \frac{U^D(x_0, y_j)}{G^D(x_0, y_j)} = K_Z^D(x_0, z)\,. \tag{5.18}$$

Now we define a function M_Y^D on $D \times \partial D$ by

$$M_Y^D(x, z) := \frac{K_Y^D(x, z)}{K_Y^D(x_0, z)}, \quad x \in D, z \in \partial D. \tag{5.19}$$

For each $z \in \partial D$, $M_Y^D(\cdot, z) \in \mathcal{H}_+(Y^D)$. Moreover, M_Y^D is jointly continuous on $D \times \partial D$. From the definition above and (5.17) we can easily see that

$$\lim_{D\ni y\to z} \frac{U^D(x, y)}{U^D(x_0, y)} = M_Y^D(x, z), \quad x \in D, z \in \partial D. \tag{5.20}$$

Theorem 5.13 *Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. The Martin boundary and the minimal Martin boundary of Y^D both coincide with the Euclidean boundary ∂D, and the Martin kernel based at x_0 is given by the function M_Y^D.*

Proof. The fact that M_Y^D is the Martin kernel of Y^D based at x_0 has been proven in the paragraph above. It follows from Theorem 5.6 that when z_1 and z_2 are two distinct points on ∂D, the functions $M_Y^D(\cdot, z_1)$ and $M_Y^D(\cdot, z_2)$ are not identical. Therefore the Martin boundary of Y^D coincides with the Euclidean boundary ∂D. Since $M^D(\cdot, z) \in \mathcal{H}^+(X^D)$ is minimal, by the order preserving property of $(V^D)^{-1}$ we know that $M_Y^D(\cdot, z) \in \mathcal{H}^+(Y^D)$ is also minimal. Therefore the minimal Martin boundary of Y_D also coincides with the Euclidean boundary ∂D. □

It follows from Theorem 5.13 and the general theory of Martin boundary that for any $g \in \mathcal{H}^+(Y^D)$ there exists a finite measure n on ∂D such that

$$g(x) = \int_{\partial D} M_Y^D(x, z) n(dz), \quad x \in D.$$

The measure n is sometimes called the Martin measure of g. The following result gives the relation between the Martin measure of $h \in \mathcal{H}^+(X^D)$ and the Martin measure of $(V^D)^{-1}h \in \mathcal{H}^+(Y^D)$.

Proposition 5.14 *If $h \in \mathcal{H}^+(X^D)$ has the representation*

$$h(x) = \int_{\partial D} M^D(x, z)\, m(dz), \quad x \in D,$$

then

$$(V^D)^{-1}h(x) = \int_{\partial D} M_Y^D(x, z)\, n(dz), \quad x \in D$$

with $n(dz) = K_Y^D(x_0, z)\, m(dz)$.

Proof. By assumption we have

$$h(x) = \int_{\partial D} M^D(x,z)\, m(dz), \quad x \in D.$$

Using (5.5) and Fubini's theorem we get

$$\begin{aligned}(V^D)^{-1}h(x) &= \int_{\partial D} (V^D)^{-1}(M^D(\cdot, z))(x)\, m(dz) \\ &= \int_{\partial D} M_Y^D(x,z) K_Y^D(x_0,z)\, m(dz) = \int_{\partial D} M_Y^D(x,z)\, n(dz)\,,\end{aligned}$$

with $n(dz) = K_Y^D(x_0,z)m(dz)$. The proof is now complete. □

From Theorem 5.12 we know that $(V^D)^{-1} : \mathcal{H}^+(X^D) \to \mathcal{H}^+(Y^D)$ is continuous with respect to topologies of locally uniform convergence. In the next result we show that $V^D : \mathcal{H}^+(Y^D) \to \mathcal{H}^+(X^D)$ is also continuous.

Proposition 5.15 *Let $(g_j,\, j \ge 0)$ be a sequence of functions in $\mathcal{H}^+(Y^D)$ converging pointwise to the function $g \in \mathcal{H}^+(Y^D)$. Then $\lim_{j\to\infty} V^D g_j(x) = V^D g(x)$ for every $x \in D$.*

Proof. Without loss of generality we may assume that $g_j(x_0) = 1$ for all $j \in \mathbb{N}$. Then there exist probability measures n_j, $j \in \mathbb{N}$, and n on ∂D such that $g_j(x) = \int_{\partial D} M_Y^D(x,z)n_j(dz)$, $j \in \mathbb{N}$, and $g(x) = \int_{\partial D} M_Y^D(x,z)n(dz)$. It is easy to show that the convergence of the harmonic functions h_j implies that $n_j \to n$ weakly. Let $V^D g_j(x) = \int_{\partial D} M^D(x,z)m_j(dz)$ and $V^D g(x) = \int_{\partial D} M^D(x,z)m(dz)$. Then $n_j(dz) = K_Y^D(x_0,z)m_j(dz)$ and $n(dz) = K_Y^D(x_0,z)m(dz)$. Since the density $K_Y^D(x_0,\cdot)$ is bounded away from zero and bounded from above, it follows that $m_j \to m$ weakly. From this the claim of proposition follows immediately. □

5.5 Boundary Harnack principle for subordinate process

The boundary Harnack principle is a very important result in potential theory and harmonic analysis. For example, it is usually used to prove that, when D is a bounded Lipschitz domain, both the Martin boundary and the minimal Martin boundary of X^D coincide with the Euclidean boundary ∂D. We have already proved in Theorem 5.13 that for Y^D, both the Martin boundary and the minimal Martin boundary coincide with the Euclidean boundary ∂D. By using this we are going to prove a boundary Harnack principle for functions in $\mathcal{H}^+(Y^D)$.

In this subsection we will always assume that $D \subset \mathbb{R}^d$, $d \ge 3$, is a bounded Lipschitz domain and $x_0 \in D$ is fixed. Recall that φ_0 is the eigenfunction corresponding to the smallest eigenvalue λ_0 of $-\Delta|_D$. Also recall that the potential operator V^D is not absolutely continuous in case $b > 0$ and is given by

$$V^D f(x) = bf(x) + \int_0^\infty P_t^D f(x) v(t)\, dt\,.$$

Define

$$\tilde{V}^D(x,y) = \int_0^\infty p^D(t,x,y) v(t)\, dt\,.$$

Then

$$V^D f(x) = bf(x) + \int_D \tilde{V}^D(x,y) f(y)\, dy\,.$$

Proposition 5.16 *Suppose that D is a bounded Lipschitz domain. There exist $c > 0$ and $k > d$ such that*

$$\begin{aligned} U^D(x,y) &\le c\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^k}, \\ \tilde{V}^D(x,y) &\le c\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^k}, \end{aligned}$$

for all $x, y \in D$.

Proof. We give a proof of the second estimate, the proof of the first being exactly the same. Note that similarly as in (2.13)

$$\lim_{t\to 0} tv(t) = 0\,. \tag{5.21}$$

It follows from Theorem 4.6.9 of [24] that the density p^D of the killed Brownian motion on D satisfies the following estimate

$$p^D(t,x,y) \le c_1 t^{-k/2} \varphi_0(x)\varphi_0(y) e^{-\frac{|x-y|^2}{6t}}, \quad t > 0,\ x,y \in D,$$

for some $k > d$ and $c_2 > 0$. Recall that v is a decreasing function. From (5.21) it follows that there exists a $t_0 > 0$ such that $v(t) \le \frac{1}{t}$ for $t \le t_0$. Consequently,

$$v(t) \le M + \frac{1}{t}, \quad t > 0,$$

for some $M > 0$. Now we have

$$\begin{aligned} \tilde{V}^D(x,y) &= \int_0^\infty p^D(t,x,y)v(t)dt \le c_1 \int_0^\infty t^{-k/2}\varphi_0(x)\varphi_0(y)e^{-\frac{|x-y|^2}{6t}} v(t)dt \\ &\le c_1 \int_0^\infty t^{-k/2-1}\varphi_0(x)\varphi_0(y)e^{-\frac{|x-y|^2}{6t}} dt + Mc_1 \int_0^\infty t^{-k/2}\varphi_0(x)\varphi_0(y)e^{-\frac{|x-y|^2}{6t}} dt \\ &\le c_2 \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^k} + Mc_3 \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{k-2}} \\ &\le c_4 \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^k}. \end{aligned}$$

The proof is now finished. $\square$

Lemma 5.17 *Suppose that D is a bounded Lipschitz domain and W an open subset of $\mathbb{R}^d$ such that $W \cap \partial D$ is non-empty. If $h \in \mathcal{H}^+(Y^D)$ satisfies*

$$\lim_{x\to z} \frac{h(x)}{(V^D)^{-1}1(x)} = 0, \quad \text{for all } z \in W \cap \partial D,$$

then

$$\lim_{x\to z} V^D h(x) = 0, \quad \text{for all } z \in W \cap \partial D\,.$$

Proof. Fix $z \in W \cap \partial D$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $h(x) \le \epsilon (V^D)^{-1}1(x)$ for $x \in B(z,\delta)\cap D$. Thus we have

$$V^D h(x) \le V^D(h\,1_{D\setminus B(z,\delta)})(x) + \epsilon V^D (V^D)^{-1}1(x) = V^D(h\,1_{D\setminus B(z,\delta)})(x) + \epsilon, \quad x \in D.$$

For any $x \in B(z,\delta/2)\cap D$ we have

$$\begin{aligned} V^D(h\,1_{D\setminus B(z,\delta)})(x) &= bh(x)\,1_{D\setminus B(z,\delta)}(x) + \int_{D\setminus B(z,\delta)} \tilde{V}^D(x,y)h(y)\,dy \\ &= \int_{D\setminus B(z,\delta)} \tilde{V}^D(x,y)h(y)\,dy \end{aligned}$$

since $1_{D\setminus B(z,\delta)}(x) = 0$ for $x \in B(z,\delta/2)\cap D$. By Proposition 5.16 we get that there exists $c > 0$ such that for any $x \in B(z,\delta/2)\cap D$,

$$\begin{aligned} \int_{D\setminus B(z,\delta)} \tilde{V}^D(x,y)h(y) &\le c\varphi_0(x)\int_{D\setminus B(z,\delta)} \frac{\varphi_0(y)}{|x-y|^k} h(y)\,dy \\ &\le c\varphi_0(x)\int_{D\setminus B(z,\delta)} \frac{\varphi_0(y)}{(\delta/2)^k} h(y)\,dy \le c\varphi_0(x)\int_D \varphi_0(y)h(y)\,dy\,. \end{aligned}$$

Hence,

$$V^D h(x) \le c\varphi_0(x)\int_D \varphi_0(y)h(y)dy + \epsilon\,.$$

From Lemma 5.9 we know that $\int_D \varphi_0(y)h(y)dy < \infty$. Now the conclusion of the lemma follows easily from the fact that $\lim_{x\to z}\varphi_0(x) = 0$. $\square$

Now we can prove the main result of this section: the boundary Harnack principle.

Theorem 5.18 *Suppose that $D \subset \mathbb{R}^d$, $d \ge 3$, is a bounded Lipschitz domain, W an open subset of $\mathbb{R}^d$ such that $W \cap \partial D$ is non-empty, and K a compact subset of W. There exists a constant $c > 0$ such that for any two functions h_1 and h_2 in $\mathcal{H}^+(Y^D)$ satisfying*

$$\lim_{x\to z} \frac{h_i(x)}{(V^D)^{-1}1(x)} = 0, \quad z \in W \cap \partial D,\ i = 1, 2,$$

we have

$$\frac{h_1(x)}{h_2(x)} \le c\,\frac{h_1(y)}{h_2(y)}, \quad x, y \in K \cap D.$$

Proof. By use of (5.11) and Proposition 5.16 there exist positive constants c_1 and c_2 such that

$$c_1\varphi_0(x)\varphi_0(y) \le U^D(x,y) \le c_2\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^k}, \quad x, y \in D,$$

where $k > d$ is given in Proposition 5.16. Therefore it follows from (5.20) that there exist positive constants c_3 and c_4 such that

$$c_3\varphi_0(x) \le M_Y^D(x,z) \le c_4\varphi_0(x), \quad x \in K \cap D, z \in \partial D \setminus W. \tag{5.22}$$

Suppose that h_1 and h_2 are two functions in $\mathcal{H}^+(Y^D)$ such that

$$\lim_{x\to z} \frac{h_i(x)}{(V^D)^{-1}1(x)} = 0, \quad z \in W \cap \partial D,\ i = 1, 2,$$

then by Lemma 5.17 we have

$$\lim_{x\to z} V^D h_i(x) = 0, \quad z \in W \cap \partial D,\ i = 1, 2.$$

Now by Corollary 8.1.6 of [44] we know that the Martin measures m_1 and m_2 of $V^D h_1$ and $V^D h_2$ are supported by $\partial D \setminus W$ and so we have

$$V^D h_i(x) = \int_{\partial D\setminus W} M^D(x,z)\, m_i(dz), \quad x \in D,\ i = 1, 2.$$

Using Proposition 5.14 we get that

$$h_i(x) = \int_{\partial D\setminus W} M_Y^D(x,z)\, n_i(dz), \quad x \in D,\ i = 1, 2,$$

where $n_i(dz) = K_Y^D(x_0, z)\, m_i(dz), i = 1, 2$. Now using (5.22) it follows that

$$c_3\varphi_0(x)n_i(\partial D \setminus W) \le h_i(x) \le c_4\varphi_0(x)n_i(\partial D \setminus W), \quad x \in K \cap D,\ i = 1, 2.$$

The conclusion of the theorem follows immediately. $\square$

From the proof of Theorem 5.18 we can see that the following result is true.

Proposition 5.19 *Suppose that $D \subset \mathbb{R}^d$, $d \ge 3$, is a bounded Lipschitz domain and W an open subset of $\mathbb{R}^d$ such that $W \cap \partial D$ is non-empty. If $h \in \mathcal{H}^+(Y^D)$ satisfies*

$$\lim_{x\to z} \frac{h(x)}{(V^D)^{-1}1(x)} = 0, \quad z \in W \cap \partial D,$$

then

$$\lim_{x\to z} h(x) = 0, \quad z \in W \cap \partial D.$$

Proof. From the proof of Theorem 5.18 we see that the Martin measure n of h is supported by $\partial D \setminus W$ and so we have

$$h(x) = \int_{\partial D \setminus W} M_Y^D(x,z)n(dz), \quad x \in D.$$

For any $z_0 \in W \cap \partial D$, take $\delta > 0$ small enough so that $B(z_0,\delta) \subset \overline{B(z_0,\delta)} \subset W$. Then it follows from (5.22) that

$$c_5\varphi_0(x) \le M_Y^D(x,z) \le c_6\varphi_0(x), \quad x \in B(z_0,\delta) \cap D, z \in \partial D \setminus W\,,$$

for some positive constants c_5 and c_6. Thus

$$h(x) \le c_6\varphi_0(x)n(\partial D \setminus W), \quad x \in B(z_0,\delta) \cap D,$$

from which the assertion of the proposition follows immediately. □

5.6 Sharp bounds for the Green function and the jumping function of subordinate process

In this subsection we are going to derive sharp bounds for the Green function and the jumping function of the process Y^D. The method uses the upper and lower bounds for the transition densities $p^D(t,x,y)$ of the killed Brownian motion. The lower bound that we need is available only in case when D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Therefore, throughout this subsection we assume that $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ domain. Moreover, recall the standing assumption that S is a special subordinator such that $b > 0$ or $\mu(0,\infty) = \infty$ which guarantees the existence of a decreasing potential density u.

Recall that a bounded domain $D \subset \mathbb{R}^d$, $d \ge 2$, is called a bounded $C^{1,1}$ domain if there exist positive constants r_0 and M with the following property: For every $z \in \partial D$ and every $r \in (0, r_0]$, there exist a function $\Gamma_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying the condition $|\nabla\Gamma_z(\xi) - \nabla\Gamma_z(\eta)| \le M|\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}^{d-1}$, and an orthonormal coordinate system CS_z such that if $y = (y_1, \dots, y_d)$ in CS_z coordinates, then

$$B(z,r) \cap D = B(z,r) \cap \{y :\, y_d > \Gamma_z(y_1, \dots, y_{d-1}\}\,.$$

When we speak of a bounded $C^{1,1}$ domain in $\mathbb{R}$ we mean a finite open interval.

For any $x \in D$, let $\rho(x)$ denote the distance between x and ∂D. We will use the following two bounds for transition densities $p^D(t,x,y)$: There exists a positive constant c_1 such that for all $t > 0$ and any $x, y \in D$,

$$p^D(t,x,y) \le c_1 t^{-d/2-1}\rho(x)\rho(y)\exp\left(-\frac{|x-y|^2}{6t}\right)\,. \tag{5.23}$$

This result (valid also for Lipschitz domains) can be found in [24] (see also [55]). The lower bound was obtained in [62] and [54] and states that for any $A>0$, there exist positive constants c_2 and c such that for any $t\in(0,A]$ and any $x,y\in D$,

$$p^D(t,x,y)\geq c_2\left(\frac{\rho(x)\rho(y)}{t}\wedge 1\right)t^{-d/2}\exp\left(-\frac{c|x-y|^2}{t}\right). \tag{5.24}$$

Recall that the Green function of Y^D is given by

$$U^D(x,y)=\int_0^\infty p^D(t,x,y)u(t)\,dt\,,$$

where u is the potential density of the subordinator S. Instead of assuming conditions on the asymptotic behavior of the Laplace exponent $\phi(\lambda)$ as $\lambda\to\infty$, we will directly assume the asymptotic behavior of $u(t)$ as $t\to 0+$.

Assumption A: (i) There exist constants $c_0>0$ and $\beta\in[0,1]$ with $\beta>1-d/2$, and a continuous function $\ell:(0,\infty)\to(0,\infty)$ which is slowly varying at ∞ such that

$$u(t)\sim\frac{c_0}{t^\beta\ell(1/t)}\,,\quad t\to 0+\,. \tag{5.25}$$

(ii) In the case when $d=1$ or $d=2$, there exist constants $c>0, T>0$ and $\gamma<d/2$ such that

$$u(t)\leq ct^{\gamma-1},\quad t\geq T. \tag{5.26}$$

Note that under certain assumptions on the asymptotic behavior of $\phi(\lambda)$ as $\lambda\to\infty$, one can obtain (5.25) and (5.26) for the density u.

Theorem 5.20 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and that the potential density u of the special subordinator $S=(S_t:t\geq 0)$ satisfies the Assumption A. Suppose also that there is a function $g:(0,\infty)\to(0,\infty)$ such that*

$$\int_0^\infty t^{d/2-2+\beta}e^{-t}g(t)dt<\infty$$

and $\xi>0$ such that $f_{\ell,\xi}(y,t)\leq g(t)$ for all $y,t>0$, where $f_{\ell,\xi}$ is the function defined before Lemma 3.3 using the ℓ in (5.25). Then there exist positive constants $C_1\leq C_2$ such that for all $x,y\in D$,

$$\begin{aligned}C_1\left(\frac{\rho(x)\rho(y)}{|x-y|^2}\wedge 1\right)\frac{1}{|x-y|^{d+2\beta-2}\,\ell(\frac{1}{|x-y|^2})}&\leq U^D(x,y)\\ \leq\ C_2\left(\frac{\rho(x)\rho(y)}{|x-y|^2}\wedge 1\right)\frac{1}{|x-y|^{d+2\beta-2}\,\ell(\frac{1}{|x-y|^2})}&\,. \end{aligned}\tag{5.27}$$

Proof. We start by proving the upper bound. Using the obvious upper bound $p^D(t,x,y) \le (4\pi t)^{-d/2}\exp(-|x-y|^2/4t)$ and Lemma 3.3 one can easily show that

$$U^D(x,y) \le c_1 \frac{1}{|x-y|^{d+2\beta-2}\,\ell\left(\frac{1}{|x-y|^2}\right)}\,.$$

Now note that (5.23) gives

$$U^D(x,y) \le c_2\rho(x)\rho(y)\int_0^\infty t^{-d/2-1}e^{-|x-y|^2/6t}u(t)\,dt\,.$$

Thus it follows from Lemma 3.3 that

$$U^D(x,y) \le c_3\rho(x)\rho(y)\frac{1}{|x-y|^{d+2\beta}\,\ell\left(\frac{1}{|x-y|^2}\right)}\,.$$

Now combining the two upper bounds obtained so far we arrive at the upper bound in (5.27).

In order to prove the lower bound, we first recall the following result about slowly varying functions (see [10], p. 22, Theorem 1.5.12):

$$\lim_{\lambda\to\infty}\frac{\ell(t\lambda)}{\ell(\lambda)} = 1$$

uniformly in $t \in [a,b]$ where $[a,b] \subset (0,\infty)$. Together with joint continuity of $(t,\lambda) \mapsto \ell(t\lambda)/\ell(\lambda)$, this shows that for a given $\lambda_0 > 0$ and an interval $[a,b] \subset (0,\infty)$, there exists a positive constant $c(a,b,\lambda_0)$ such that

$$\frac{\ell(t\lambda)}{\ell(\lambda)} \le c(a,b,\lambda_0)\,, \quad a \le t \le b, \lambda \ge \lambda_0\,. \tag{5.28}$$

Now, by (5.24),

$$U^D(x,y) \ge c_4\int_0^A \left(\frac{\rho(x)\rho(y)}{t}\wedge 1\right) t^{-d/2}\exp\left(-\frac{c|x-y|^2}{t}\right)\,dt\,.$$

Assume $x \ne y$. Let R be the diameter of D and assume that A has been chosen so that $A = R^2$. Then for any $x,y \in D$, $\rho(x)\rho(y) < R^2 = A$. The lower bound is proved by considering two separate cases:

(i) $|x-y|^2 < 2\rho(x)\rho(y)$. In this case we have

$$\begin{aligned} U^D(x,y) &\ge c_4\int_0^{\rho(x)\rho(y)}\left(\frac{\rho(x)\rho(y)}{t}\wedge 1\right)t^{-d/2}\exp\{-c|x-y|^2/t\}u(t)\,dt \\ &\ge c_5|x-y|^{-d+2}\int_{\frac{c|x-y|^2}{\rho(x)\rho(y)}}^\infty s^{d/2-2}e^{-s}u(c|x-y|^2/s)\,ds \\ &\ge c_5|x-y|^{-d+2}\int_{2c}^{4c} s^{d/2-2}e^{-s}u(c|x-y|^2/s)\,ds\,. \end{aligned} \tag{5.29}$$

For $2c \le s \le 4c$, we have that $1/4 \le c|x-y|^2/s \le 1/2$. Hence, by (5.25), there exists $c_6 > 0$ such that

$$u\left(\frac{c|x-y|^2}{s}\right) \ge \frac{c_6}{\left(\frac{c|x-y|^2}{s}\right)^\beta \ell\left(\frac{s}{c|x-y|^2}\right)}.$$

Further, since $1/|x-y|^2 \ge 1/R^2$ for all $x, y \in D$, we can use (5.28) to conclude that there exists $c_7 > 0$ such that

$$\frac{\ell\left(\frac{1}{|x-y|^2}\right)}{\ell\left(\frac{s}{c|x-y|^2}\right)} \ge c_7\,, \quad 2c \le s \le 4c\,,\ x, y \in D\,.$$

It follows from (5.29), that

$$\begin{aligned} U^D(x,y) &\ge c_5|x-y|^{-d+2}\int_{2c}^{4c} s^{d/2-2}e^{-s}\frac{c_6c_7}{\left(\frac{c|x-y|^2}{s}\right)^\beta \ell\left(\frac{1}{|x-y|^2}\right)}\,ds \\ &= \frac{c_4}{|x-y|^{d+2\beta-2}\ell(\frac{1}{|x-y|^2})}\int_{2c}^{4c} s^{d/2+\beta-2}e^{-s}\,ds \\ &= \frac{c_9}{|x-y|^{d+2\beta-2}\ell(\frac{1}{|x-y|^2})}. \end{aligned}$$

(ii) $|x-y|^2 \ge 2\rho(x)\rho(y)$. In this case we have

$$\begin{aligned} U^D(x,y) &\ge c_4\rho(x)\rho(y)\int_{\rho(x)\rho(y)}^{A} t^{-d/2-1}\exp\{-c|x-y|^2/t\}u(t)\,dt \\ &= c_{10}\rho(x)\rho(y)|x-y|^{-d}\int_{\frac{c|x-y|^2}{A}}^{\frac{c|x-y|^2}{\rho(x)\rho(y)}} s^{d/2-1}e^{-s}u(c|x-y|^2/s)\,ds \\ &\ge c_{10}\rho(x)\rho(y)|x-y|^{-d}\int_{c}^{2c} s^{d/2-1}e^{-s}u(c|x-y|^2/s)\,ds\,. \end{aligned}$$

The integral above is estimated in the same way as in case (i). It follows that there exists a positive constant c_{11} such that

$$\begin{aligned} U^D(x,y) &\ge c_{10}\rho(x)\rho(y)|x-y|^{-d}\frac{c_{11}}{|x-y|^{2\beta}\ell(\frac{1}{|x-y|^2})} \\ &= c_{12}\,\frac{\rho(x)\rho(y)}{|x-y|^{d+2\beta}\ell(\frac{1}{|x-y|^2})}\,. \end{aligned}$$

Combining the two cases above we arrive at the lower bound (5.27). $\square$

Suppose that the subordinator S has a strictly positive drift b and $d \ge 3$. Then we can take $\beta = 0$ and $\ell = 1$ in the Assumption A, and Theorem 5.20 implies that the Green function U^D of Y^D is comparable to the Green function of X^D. Further, if $\phi(\lambda) \sim c_0 \lambda^{\alpha/2}$, as $\lambda \to \infty$, $0 < \alpha < 2$, then by (2.28) it follows that the Assumption A holds true with $\beta = 1 - \alpha/2$ and $\ell = 1$. In this way we recover a result from [57] saying that under the stated assumption,

$$C_1 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d-\alpha}} \le U^D(x,y) \le C_2 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d-\alpha}} .$$

The jumping function $J^D(x,y)$ of the subordinate process Y^D is given by the following formula:

$$J^D(x,y) = \int_0^\infty p^D(t,x,y)\, \mu(dt) .$$

Suppose that $\mu(dt)$ has a decreasing density $\mu(t)$ which satisfies

Assumption B: There exist constants $c_0 > 0$, $\beta \in [1,2]$ and a continuous function $\ell : (0,\infty) \to (0,\infty)$ which is slowly varying at ∞ such that such that

$$\mu(t) \sim \frac{c_0}{t^\beta \ell(1/t)} , \quad t \to 0+ . \tag{5.30}$$

Then we have the following result on sharp bounds of $J^D(x,y)$. The proof is similar to the proof of Theorem 5.20, and therefore omitted.

Theorem 5.21 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and that the Lévy density $\mu(t)$ of the subordinator $S = (S_t : t \ge 0)$ exists, is decreasing and satisfies the Assumption B. Suppose also that there is a function $g : (0,\infty) \to (0,\infty)$ such that*

$$\int_0^\infty t^{d/2-2+\beta} e^{-t} g(t) dt < \infty$$

and $\xi > 0$ such that $f_{\ell,\xi}(y,t) \le g(t)$ for all $y, t > 0$, where $f_{\ell,\xi}$ is the function defined before Lemma 3.3 using the ℓ in (5.30). Then there exist positive constants $C_3 \le C_4$ such that for all $x, y \in D$

$$\begin{aligned} C_3 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d+2\beta-2} \ell(\frac{1}{|x-y|^2})} &\le J^D(x,y) \\ \le \ C_4 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d+2\beta-2} \ell(\frac{1}{|x-y|^2})} &. \end{aligned} \tag{5.31}$$

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