Minimal thinness for subordinate Brownian motion in half-space

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Abstract

We study minimal thinness in the half-space $H := \{x = (\widetilde{x}, x_d) : \widetilde{x} \in \mathbb{R}^{d-1}, x_d > 0\}$ for a large class of rotationally invariant Lévy processes, including symmetric stable processes and sums of Brownian motion and independent stable processes. We show that the same test for the minimal thinness of a subset of H below the graph of a nonnegative Lipschitz function is valid for all processes in the considered class. In the classical case of Brownian motion this test was proved by Burdzy.

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1 Introduction

Minimal thinness is a notion that describes the smallness of a set at a boundary point. More precisely, let D be a domain in $\mathbb{R}^d$, $d \geq 2$, let $\partial^M D$ (respectively $\partial^m D$) denote its Martin boundary (respectively minimal Martin boundary) with respect to Brownian motion, and let $M^D(x, z)$, $x \in D$, $z \in \partial^M D$, be the corresponding Martin kernel with respect to Brownian motion. For $A \subset D$, let $\widehat{R}^A_{M^D(\cdot,z)}$ denote the balayage of $M^D(\cdot, z)$ onto A. The set A is said to be minimally thin in D at $z \in \partial^m D$ with respect to Brownian motion if $\widehat{R}^A_{M^D(\cdot,z)} \neq M^D(\cdot, z)$. The concept of minimal thinness in the context of classical potential theory was introduced and studied by Naïm in [25]; for a recent exposition see [1, Chapter 9]. A probabilistic interpretation of minimal thinness is due to Doob, see, e.g., [15]: $A \subset D$ is minimally thin in D with respect to Brownian motion at $z \in \partial^m D$ if there exists a point $x \in D$ such that with positive probability the $M^D(\cdot, z)$-conditioned Brownian motion starting from x does not hit A.

We recall now two results about minimal thinness in the half-space $H := \{x = (\widetilde{x}, x_d) : \widetilde{x} \in \mathbb{R}^{d-1}, x_d > 0\}$, $d \geq 2$, with respect to Brownian motion. The Martin boundary of H with respect

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to Brownian motion can be identified with $\partial H \cup \{\infty\}$, where $\partial H = \{(\widetilde{x}, 0) : \widetilde{x} \in \mathbb{R}^{d-1}\}$, and all boundary points are minimal. The first result is due to Beurling [3] in the case $d = 2$ and Dahlberg [14] in the case $d \geq 3$. By $B(z, r)$ we denote the ball centered at $z \in \mathbb{R}^d$ with radius $r > 0$.

Theorem 1.1 *Let A be a Borel subset of H and assume that*

$$\int_{A\cap B(0,1)} |x|^{-d}\, dx = \infty\,. \tag{1.1}$$

Then A is not minimally thin in H with respect to Brownian motion at $z = 0$.

The second result is a test for the minimal thinness of a subset of H below the graph of a nonnegative Lipschitz function originally proved by Burdzy [9] using a probabilistic approach. An alternative proof using Theorem 1.1 was given by Gardiner [17].

Theorem 1.2 *Let $f : \mathbb{R}^{d-1} \to [0, \infty)$ be a Lipschitz function with Lipschitz constant $a > 0$. The set $A := \{x = (\widetilde{x}, x_d) \in H : 0 < x_d \leq f(\widetilde{x})\}$ is minimally thin in H with respect to Brownian motion at $z = 0$ if and only if*

$$\int_{\{|\widetilde{x}|<1\}} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty\,. \tag{1.2}$$

The goal of this paper is to show that the above two theorems are still valid in exactly the same form when Brownian motion is replaced with a wide class of rotationally invariant Lévy processes (see Theorems 4.3–4.4). The precise description of this class will be given in the next section – for now it suffices to know that it includes rotationally invariant α-stable processes, $\alpha \in (0, 2)$. The Martin boundary theory for Hunt processes admitting a dual process (and satisfying an additional hypothesis) was developed by Kunita and Watanabe [24], while the concept of minimal thinness for such processes was studied by Föllmer [16]. To the best of our knowledge no concrete criteria for minimal thinness in the spirit of Theorems 1.1–1.2 have been obtained for any discontinuous processes, not even the symmetric stable ones. Time is now ripe for such results due to the recent progress in the potential theory of rotationally invariant Lévy processes, in particular subordinate Brownian motions. Our proofs of the analogs of Theorems 1.1–1.2 will heavily rely on the very recent work [19, 21, 22, 23] where a boundary Harnack principle and sharp estimates of the Green function of certain subordinate Brownian motions were obtained.

We find the conclusion of our main result, Theorem 4.4, surprising since the test (1.2) is the same for all processes in the considered class. In particular, for symmetric α-stable processes, the criterion for the minimal thinness of the set A in H at $z = 0$ does not depend on the index of stability α. This is in contrast with the following criterion for the thinness of thorns: Let $f : [0, \infty) \to [0, \infty)$ be an increasing function such that $f(r) > f(0)$ for all $r > 0$, and $f(r)/r$ is non-decreasing for sufficiently small $r > 0$. Let $A := \{x \in H : |\widetilde{x}| < f(x_d)\}$. Then A is thin in H

at 0 with respect to Brownian motion if and only if

$$\int_0^1 \left(\frac{f(r)}{r}\right)^{d-3} \frac{dr}{r} < \infty\,, \qquad d \geq 4\,,$$
$$\int_0^1 \left|\log \frac{f(r)}{r}\right|^{-1} \frac{dr}{r} < \infty\,, \qquad d = 3\,.$$

On the other hand, for $\alpha \in (0,2)$, A is thin at 0 with respect to the symmetric α-stable process if and only if

$$\int_0^1 \left(\frac{f(r)}{r}\right)^{d-\alpha-1} \frac{dr}{r} < \infty, \qquad d \geq 3.$$

The above criterion for the thinness of thorns in H at 0 with respect to α-stable processes can be proved by using Wiener's test for stable processes [5, Corollary 4.17] and by slightly modifying the proofs in [26, pp. 67–69] (by changing cylindrical surfaces to full cylinders).

This paper is organized as follows. In the next section we precisely describe the class of subordinate Brownian motions for which we will study minimal thinness in the half-space, recall from [21, 22, 23] relevant results on the boundary Harnack principle for those processes, and derive necessary estimates for the Green function of the half-space (for points close to the boundary and to each other). In Section 3 we use these Green function estimates to obtain two-sided estimates on the Martin kernel for points close to the origin and to each other. These estimates and the boundary Harnack principle suffice to identify the finite part of the minimal Martin boundary of H with ∂H. This result may be of independent interest. We then show how the studied processes fit in the framework of minimal thinness in [16] and recall both the potential-theoretic and the probabilistic definitions of minimal thinness. In the last section we state and prove Theorems 4.3–4.4, the analogs of Theorems 1.1–1.2 for our processes. The main ingredients of the proof are Lemma 4.1 and Proposition 4.2 which are generalizations of [30, Theorem 2]. Instead of the estimates of the classical Green function and Martin kernel we use our estimates from Sections 2 and 3 for points close to the origin and the boundary Harnack principle for points away from the boundary.

We will use the following conventions in this paper. The values of the constants $C_1(R), \dots, C_5(R)$, depending only on d, $R > 0$ and the Laplace exponent of the subordinator, will remain the same throughout this paper, while the constants $c, c_0, c_1, c_2, \dots$ stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. We assume $d \geq 2$ and the dependence of the constants on the dimension d may not be mentioned explicitly.

For two nonnegative functions f, g, $f(t) \sim g(t)$, $t \to 0$ ($f(t) \sim g(t)$, $t \to \infty$, respectively) means that $\lim_{t\to 0} f(t)/g(t) = 1$ ($\lim_{t\to\infty} f(t)/g(t) = 1$, respectively). On the other hand, $f(t) \asymp g(t)$, $t \to 0$ ($f(t) \asymp g(t)$, $t \to \infty$, respectively) means that the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to 0$ (as $t \to \infty$, respectively). Simply, $f \asymp g$ means that the quotient $f(t)/g(t)$ stays bounded between two positive constants on their common domain of definitions.

For any open set U, we denote by $\delta_U(x)$ the distance between x and the complement of U, i.e., $\delta_U(x) = \mathrm{dist}(x, U^c)$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure and $\mathrm{diam}(A)$ to denote the diameter of the set A. Finally, we will use ":=" to denote a definition, which is read as "is defined to be".

2 Preliminaries on subordinate Brownian motion

In this section we will first describe a class of subordinate Brownian motions and their potential theory. Recall that a subordinator $S = (S_t)_{t\ge 0}$ is simply a nonnegative Lévy process with $S_0 = 0$. The Laplace exponent of S is a function $\phi : (0,\infty) \to (0,\infty)$ having the representation

$$\phi(\lambda) = a\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \eta(dt)\,, \tag{2.1}$$

where $a \ge 0$ is the drift and η the Lévy measure of S, i.e., a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \eta(dt) < \infty$. The Laplace exponent ϕ determines the distribution of S_t through the formula $\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}$. Formula (2.1) shows that ϕ is a Bernstein function, i.e. a nonnegative C^∞ function on $(0,\infty)$ satisfying $(-1)^{n-1}\phi^{(n)} \ge 0$ for all $n \ge 1$. Since the sample paths of S are nondecreasing functions, the subordinator S can serve as a stochastic time-change. More precisely, let $Y = (Y_t, \mathbb{P}_x)_{t\ge 0, x\in\mathbb{R}^d}$ be a Brownian motion in $\mathbb{R}^d$ independent of S with

$$\mathbb{E}\left[e^{i\xi(Y_t - Y_0)}\right] = e^{-t|\xi|^2} \quad \xi \in \mathbb{R}^d, t > 0.$$

The stochastic process $X = (X_t, \mathbb{P}_x)_{t\ge 0, x\in\mathbb{R}^d}$ defined by the formula $X_t := Y_{S_t}$ is called a subordinate Brownian motion. It is a rotationally invariant Lévy process in $\mathbb{R}^d$ with characteristic exponent $\Phi(\xi) = \phi(|\xi|^2)$ and infinitesimal generator $-\phi(-\Delta)$. Here Δ denotes the Laplacian and $\phi(-\Delta)$ is defined through functional calculus.

A Bernstein function ϕ is a complete Bernstein function if its Lévy measure η has a completely monotone density, which will be denoted by $\eta(t)$. We will consider the following class of subordinate Brownian motions determined mainly by the asymptotic behavior at infinity of the Laplace exponents of the corresponding subordinators:

Hypothesis (H): $d \ge 2$, ϕ is a complete Bernstein function, and there exists $\alpha \in (0, 2]$ such that $\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda)$ as $\lambda \to \infty$, where $\ell : (0,\infty) \to (0,\infty)$ is measurable, locally bounded above and below by positive constants, and slowly varying at ∞. Additionally,

- in case $\alpha \in (0, 2)$ and $d = 2$, assume that there exists $\gamma < 1$ such that $\liminf_{\lambda\to 0} \phi(\lambda)/\lambda^\gamma > 0$;
- in case $\alpha = 2$, assume that $d \ge 3$, ϕ has a positive drift a and the Lévy density η of ϕ satisfies the following condition: for any $K > 0$, there exists $c = c(K) > 1$ such that

$$\eta(t) \le c\,\eta(2t), \qquad t \in (0, K). \tag{2.2}$$

In this case one can take $\ell \equiv 1$.

In case the above hypothesis holds true we will say that subordinate Brownian motion X satisfies **(H)**. It is easy to check that in this case X is transient. Moreover, in this case the potential measure U of the corresponding subordinator S has a density u which is also completely monotone (see, e.g., [2, III, Theorem 5], [8, Corollary 5.4 and Corollary 5.5] and [28, Remark 10.6]).

In the case $\alpha \in (0,2)$, (2.2) is a consequence of the asymptotic behavior of ϕ at infinity given in the first sentence of **(H)**, see [21, Theorem 2.10].

Subordinate Brownian motions satisfying **(H)** and with $\alpha \in (0,2)$ were studied in [19, 21, 22]. Such a subordinate Brownian motion X is a purely discontinuous Lévy process in $\mathbb{R}^d$ with characteristic exponent Φ satisfying $\Phi(\xi) \asymp |\xi|^\alpha \ell(|\xi|^2)$, $|\xi| \to \infty$, $\xi \in \mathbb{R}^d$. This class of processes includes α-stable processes, corresponding to $\phi(\lambda) = \lambda^{\alpha/2}$, relativistic α-stable processes, corresponding to $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, $m > 0$, sums of independent α-stable and β-stable processes, corresponding to $\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}$, $0 < \beta < \alpha < 2$, and many others (see [22] for further examples).

Subordinate Brownian motions satisfying **(H)** with $\alpha = 2$ and $d \geq 3$ were studied in [23]. This class of processes includes independent sums of Brownian motion and β-stable processes corresponding to $\Phi(\xi) = a|\xi|^2 + b^\beta|\xi|^\beta$ with $a, b > 0$, and independent sums of Brownian motion and relativistic β-stable processes corresponding to $\Phi(\xi) = a|\xi|^2 + (\lambda + m^{2/\beta})^{\beta/2} - m$ with $a, m > 0$, and many others.

Let us first consider one-dimensional subordinate Brownian motions. Suppose that $B = (B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}$, independent of S, with

$$\mathbb{E}\left[e^{i\theta(B_t - B_0)}\right] = e^{-t\theta^2}, \qquad \theta \in \mathbb{R},\ t > 0.$$

The subordinate Brownian motion $Z = (Z_t)_{t\geq 0}$ in $\mathbb{R}$ defined by $Z_t := B_{S_t}$ is a symmetric Lévy process with characteristic exponent $\Phi(\theta) = \phi(\theta^2)$, $\theta \in \mathbb{R}$. Define $\overline{Z}_t := \sup\{0 \vee Z_s : 0 \leq s \leq t\}$ and let $L = (L_t : t \geq 0)$ be a local time of $\overline{Z} - Z$ at 0. L is also called a local time of the process Z reflected at the supremum. Then the right continuous inverse L_t^{-1} of L is a subordinator and is called the ladder time process of Z. The process $\overline{Z}_{L_t^{-1}}$ is also a subordinator and is called the ladder height process of X. (For basic properties of the ladder time and ladder height processes, we refer the readers to [2, Chapter 6].) Let χ denote the Laplace exponent of the ladder height process of Z, and let V be its potential measure. By a slight abuse of notation we also use V to denote the function $V(t) = V((0,t))$, $t > 0$.

From now on we assume that the process X is a subordinate Brownian motion satisfying **(H)**. Since X is transient, it has a Green function $G(x,y)$ given by

$$G(x,y) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}} u(t)\, dt\,,$$

where u is the potential density of the subordinator S. If we define

$$G(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} u(t)\, dt \quad r > 0,$$

then $G(\cdot)$ is a non-increasing function on $(0,\infty)$ and $G(x,y)=G(|x-y|)$ for all $x,y\in\mathbb{R}^d$.

When X is a subordinate Brownian motion satisfying **(H)**, the Green function $G(x,y)$ of X satisfies the following sharp estimates

$$G(x,y)\asymp \frac{1}{|x-y|^d\phi(|x-y|^{-2})}, \qquad |x-y|\to 0. \tag{2.3}$$

The Laplace exponent χ of the ladder height process of Z is a complete Bernstein function, the function V is a smooth function and satisfies

$$V(t)\asymp \phi(t^{-2})^{-1/2}, \qquad t\to 0\,. \tag{2.4}$$

For these two results see [19, 21, 22] in case $\alpha\in(0,2)$, and [23] in case $\alpha=2$. In fact, when $\alpha=2$, we have more precisely,

$$G(x,y)\sim \frac{\Gamma(d/2-1)}{4a\pi^{d/2}|x-y|^{d-2}}\,, \qquad |x-y|\to 0\,, \tag{2.5}$$

$$V(t)\sim a\,t\,, \qquad t\to 0\,. \tag{2.6}$$

We record two consequences of estimates (2.3) and (2.4).

Proposition 2.1 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Let $R>0$.*
(i) There exists a constant $C_1(R)=C_1(d,\phi,R)>1$ such that for all $x,y\in H$ satisfying $|x-y|<R$ it holds that

$$C_1(R)^{-1}|x-y|^{-d}\le \frac{G(|x-y|)}{V(|x-y|)^2}\le C_1(R)|x-y|^{-d}\,. \tag{2.7}$$

(ii) There exists a constant $C_2(R)=C_2(d,\phi,R)>1$ such that

$$C_2(R)^{-1}\le V(t)^{-2}\int_{B(0,t)}G(0,x)\,dx\le C_2(R)\,, \quad 0<t\le R\,. \tag{2.8}$$

Proof: Fix $R>0$. We have by (2.3) and (2.4) that for $|x-y|<R$

$$G(|x-y|)\asymp \frac{1}{|x-y|^d\phi(|x-y|)^{-2})} \quad \text{and} \quad V(|x-y|)\asymp\phi(|x-y|^2)^{-1/2}.$$

Now (2.7) follows immediately with the constant depending only on R,d,ϕ.

For part (ii) note that for $0<t\le R$,

$$\int_{B(0,t)}G(0,x)\,dx=c_1\int_0^t r^{d-1}G(r)\,dr\asymp\int_0^t r^{d-1}\frac{1}{r^{d-\alpha}\ell(r^{-2})}\,dr\asymp\frac{t^\alpha}{\ell(t^{-2})}\,,$$

where in case $\alpha\in(0,2)$ the last asymptotic equality follows by a property of slowly varying function ℓ (see [4, Theorem 1.5.11] - Karamata's theorem), while in the case $\alpha=2$ the last asymptotic equality is trivial. Combining this with (2.4) we obtain (2.8). $\square$

A nonnegative function h on $\mathbb{R}^d$ is harmonic in an open set $D \subset \mathbb{R}^d$ with respect to X if for every open set B such that $B \subset \overline{B} \subset D$ and every $x \in B$ it holds that $h(x) = \mathbb{E}_x[h(X_{\tau_B})]$. Here $\tau_B = \inf\{t > 0 : X_t \notin B\}$ is the first exit time of X from B. A nonnegative function h on $\mathbb{R}^d$ is regular harmonic in an open set $D \subset \mathbb{R}^d$ with respect to X if $h(x) = \mathbb{E}_x[h(X_{\tau_D})]$ for all $x \in D$. The following Harnack principle is a consequence of [21, Theorem 4.7] when $\alpha \in (0,2)$, and [27, Theorem 4.5] when $\alpha = 2$.

Theorem 2.2 (Harnack inequality) *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $R > 0$, there exists $c = c(R, \phi, d) > 0$ such that for every $r \in (0, R)$, every $x \in \mathbb{R}^d$, and every nonnegative function h on $\mathbb{R}^d$ which is harmonic in $B(x,r)$ with respect to X we have*

$$\sup_{y \in B(x,r/2)} h(y) \le c \inf_{y \in B(x,r/2)} h(y).$$

Recall that $H = \{x = (\widetilde{x}, x_d) : \widetilde{x} \in \mathbb{R}^{d-1}, x_d > 0\}$ is the half-space in $\mathbb{R}^d$ and that $\partial H = \{x = (\widetilde{x}, 0) : \widetilde{x} \in \mathbb{R}^{d-1}\}$ denotes its boundary. Let $z \in \partial H$. We will say that a function $h : \mathbb{R}^d \to \mathbb{R}$ vanishes continuously on $H^c \cap B(z,r)$ if $h = 0$ on $H^c \cap B(z,r)$ and h is continuous at every point of $\partial H \cap B(z,r)$. By [11, Lemma 4.2] and its proof (which works for all subordinate Brownian motions satisfying **(H)**), we see that, if h is a nonnegative function in $\mathbb{R}^d$ that is harmonic in $H \cap B(z,r)$ with respect to X and vanishes continuously on $H^c \cap B(z,r)$, then h is regular harmonic in $H \cap B(z,r)$ with respect to X. Thus, using the Harnack inequality and a Harnack chain argument, the following form of the boundary Harnack principle is a consequence of [22, Theorem 1.3] and [21, Theorem 4.22] in case $\alpha \in (0,2)$, and [23, Theorem 1.2] in case $\alpha = 2$. Note that the distance of the point $x \in H$ to the boundary ∂H will be denoted by $\delta_H(x)$. Clearly, $\delta_H(x) = x_d$ for $x \in H$.

Theorem 2.3 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Then, for every $R > 0$ there exists a constant $C_3(R) = C_3(d, \phi, R) > 0$ such that for $r \in (0, 2R]$, $z \in \partial H$ and any nonnegative function h in $\mathbb{R}^d$ that is harmonic in $H \cap B(z,r)$ with respect to X and vanishes continuously on $H^c \cap B(z,r)$, we have*

$$\frac{h(x)}{V(\delta_H(x))} \le C_3(R) \frac{h(y)}{V(\delta_H(y))} \qquad \text{for every } x, y \in H \cap B(z, r/2). \tag{2.9}$$

Recall that $\tau_H = \inf\{t > 0 : X_t \notin H\}$. The process $X^H = (X^H_t)_{t \ge 0}$ obtained by killing X upon exiting H is defined by

$$X^H_t := \begin{cases} X_t, & t < \tau_H, \\ \partial, & t \ge \tau_H, \end{cases}$$

where ∂ is the cemetery point. The killed process X^H is a symmetric Hunt process. Any function h on H is automatically extended to ∂ by setting $h(\partial) = 0$. A nonnegative function h is harmonic with respect to X^H if for every open set B such that $B \subset \overline{B} \subset H$ and every $x \in B$ it holds that $h(x) = \mathbb{E}_x[h(X^H_{\tau_B})]$. A nonnegative function s on H is said to be excessive with respect to X^H

if $s(x) \geq \mathbb{E}_x[s(X_t^H)]$ for all $t > 0$ and $x \in H$, and $\lim_{t\downarrow 0} \mathbb{E}_x[s(X_t^H)] = s(x)$ for all $x \in H$. Since a subordinate Brownian motion is always a strong Feller process, any function which is excessive with respect to X^H is lower semi-continuous.

For every $D \subset H$, X admits a Green function $G^D : D \times D \to (0, \infty]$ defined by

$$G^D(x, y) = G(x, y) - \mathbb{E}_x[G(X_{\tau_D}, y)]\,, \qquad x, y \in D\,.$$

The Green function G^D is symmetric and continuous (in extended sense). We extend the domain of the Green function G^D to $\mathbb{R}^d \times \mathbb{R}^d$ by setting G^D to be zero outside of $D \times D$.

We will frequently use the well-known fact that $G^D(\cdot, y)$ is harmonic in $D \setminus \{y\}$, and regular harmonic in $D \setminus \overline{B(y, \varepsilon)}$ for every $\varepsilon > 0$. Moreover, by the strong Markov property, for all open sets $D_1 \subset D_2 \subset H$,

$$G^{D_1}(x, y) = G^{D_2}(x, y) - \mathbb{E}_x[G^{D_2}(X_{\tau_{D_1}}, y)] \leq G^{D_2}(x, y), \quad x, y \in D_1. \tag{2.10}$$

The following sharp estimates of G^H for points close to the boundary and to each other will be crucial in the sequel.

Theorem 2.4 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Then, for every $R > 0$ there exists a constant $C_4(R) = C_4(d, \phi, R) > 0$ such that for all $x, y \in H$ satisfying $|x - y| < R$ and $\delta_H(x) \wedge \delta_H(y) < R$ it holds that*

$$\begin{aligned} C_4(R)^{-1}\left(1 \wedge \frac{V(\delta_H(x))}{V(|x-y|)}\right)\left(1 \wedge \frac{V(\delta_H(y))}{V(|x-y|)}\right) G(x, y) &\leq G^H(x, y) \\ \leq\ C_4(R)\left(1 \wedge \frac{V(\delta_H(x))}{V(|x-y|)}\right)\left(1 \wedge \frac{V(\delta_H(y))}{V(|x-y|)}\right) G(x, y)\,. \end{aligned} \tag{2.11}$$

Proof: Fix $R > 0$. We first note that by the argument in [10, Lemma 5.1] (using (2.4) and (2.6)), (2.11) is equivalent to

$$c^{-1}\left(1 \wedge \frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}\right) G(x, y) \leq G^H(x, y) \leq c\left(1 \wedge \frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}\right) G(x, y) \tag{2.12}$$

for some constant $c = c(d, \phi, R) > 1$. By translation invariance, we can assume $x, y \in B(0, \sqrt{5}R/2) \cap H$.

Recall that a domain in $\mathbb{R}^d$ is a connected open set in $\mathbb{R}^d$. A domain D in $\mathbb{R}^d$ is said to be a $C^{1,1}$ domain if there are $R_0 > 0$ and $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0, \nabla\varphi(0) = (0, \ldots, 0)$, $|\nabla\varphi(x) - \nabla\varphi(w)| \leq \Lambda_0|x - w|$, and an orthonormal coordinate system $CS_z : y = (y_1, \ldots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ with origin at z such that $B(z, R_0) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R_0) \text{ in } CS_z : y_d > \varphi(\widetilde{y})\}$. The pair (R_0, Λ_0) is called the $C^{1,1}$ characteristics of the $C^{1,1}$ domain D. Choose a *bounded* $C^{1,1}$ domain D such that $B(0, 3R) \cap H \subset D \subset H$ and such that its $C^{1,1}$ characteristics (R_0, Λ_0) depends only on d and R.

The estimates (2.12) with H replaced by D and for all $x, y \in D$ were proved in [22, Theorem 1.1] in case $\alpha \in (0, 2)$, and in [23, Theorem 1.4] in case $\alpha = 2$. Note that c depends on d, ϕ, R only.

It follows from (2.10), [22, Theorem 1.1] and [23, Theorem 1.4] that

$$G^H(x,y) \geq G^D(x,y) \geq c_1 \left(1 \wedge \frac{V(\delta_D(x))V(\delta_D(y))}{V(|x-y|)^2}\right) G(x,y) = c_2 \left(1 \wedge \frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}\right) G(x,y),$$

thus it suffices to show the second inequality in (2.12).

Let $x_R := (\tilde{0}, R/2)$. By Theorem 2.3 applied to $G^H(\cdot, w)$,

$$\begin{aligned}\int_{H\setminus D} G^H(y,w)\mathbb{P}_x(X_{\tau_D} \in dw) &\leq C_3(R)\frac{V(\delta_H(y))}{V(R/2)} \int_{H\setminus D} G^H(x_R,w)\mathbb{P}_x(X_{\tau_D} \in dw)\\ &\leq c_3 V(\delta_H(y)) \int_{H\setminus D} G(R)\mathbb{P}_x(X_{\tau_D} \in dw) \leq c_4 V(\delta_H(y))\mathbb{P}_x(X_{\tau_D} \in H \setminus D).\end{aligned}$$

Using the boundary Harnack principle for $\mathbb{P}_\cdot(X_{\tau_D} \in H \setminus D)$, we also have

$$\mathbb{P}_x(X_{\tau_D} \in H \setminus D) \leq c_5 \mathbb{P}_{x_R}(X_{\tau_D} \in H \setminus D)\frac{V(\delta_H(x))}{V(R/2)} \leq c_6 V(\delta_H(x)).$$

Since

$$V(\delta_H(x))V(\delta_H(y)) \leq c_7 \left(1 \wedge \frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}\right) \leq c_8 \left(1 \wedge \frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}\right) G(x,y),$$

we obtain

$$\int_{H\setminus D} G^H(y,w)\mathbb{P}_x(X_{\tau_D} \in dw) \leq c_9 \left(1 \wedge \frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}\right) G(x,y). \tag{2.13}$$

Since, by (2.10) and the symmetry of G^H,

$$G^H(x,y) = G^D(x,y) + \mathbb{E}_x[G^H(X_{\tau_D}, y)] = G^D(x,y) + \mathbb{E}_x[G^H(y, X_{\tau_D})],$$

we obtain the second inequality in (2.12) using (2.13), [22, Theorem 1.1] and [23, Theorem 1.4]. □

Remark 2.5 In case $\phi(\lambda) = a\lambda + b^\beta \lambda^{\beta/2}$ with $a, b > 0$ and $\beta \in (0, 2)$, the estimates (2.11) are valid for all $x, y \in H$. This was proved in [10, Theorem 1.7] by using the heat kernel estimates obtained in [10] and a lengthy computation.

3 Martin kernel and minimal thinness

In this section we always assume that X is a subordinate Brownian motion satisfying (**H**). Recall that X^H is the process obtained from X by killing it upon exiting H.

In the remainder of this paper we will use x_0 to denote the point $(\tilde{0}, 1) \in H$ and set

$$M^H(x,y) := \frac{G^H(x,y)}{G^H(x_0,y)}, \qquad x, y \in H, y \neq x_0 .$$

As the process X^H satisfies Hypothesis (B) in [24], H has a Martin boundary $\partial^M H$ with respect to X satisfying the following properties:

(M1) $H \cup \partial^M H$ is compact metric space;

(M2) H is open and dense in $H \cup \partial^M H$, and its relative topology coincides with its original topology;

(M3) $M^H(x, \cdot)$ can be uniquely extended to $\partial^M H$ in such a way that, $M^H(x, y)$ converges to $M^H(x, w)$ as $y \to w \in \partial^M H$, the function $x \to M^H(x, w)$ is excessive with respect to X^H, the function $(x, w) \to M^H(x, w)$ is jointly continuous on $H \times \partial^M H$ and $M^H(\cdot, w_1) \neq M^H(\cdot, w_2)$ if $w_1 \neq w_2$;

For any $w \in \partial^M H$, the function $x \to M^H(x, w)$ is called the Martin kernel of X^H corresponding to w. A point $w \in \partial^M H$ is called a finite Martin boundary point if there is a bounded sequence $\{w_n\} \subset H$ converging to w in the Martin topology.

Recall that a positive harmonic function h with respect to X^H is minimal if every positive harmonic function g with respect to X^H such that $g \leq h$ is proportional to h. The minimal Martin boundary of X^H is defined as

$$\partial^m H = \{z \in \partial^M H : M^H(\cdot, z) \text{ is minimal harmonic with respect to } X^H \}.$$

A point $z \in \partial^m H$ is called a finite minimal Martin boundary point if there is a bounded sequence $\{w_n\} \subset H$ converging to z in the Martin topology.

We will show that the finite part of the minimal Martin boundary of H with respect to X coincides with ∂H. The claim above can be proved by using Theorems 2.3–2.4 and following the methodology from [6], [19, Section 5] and [12, Section 6]. Since the Martin boundary of H with respect to X^H contains also non-finite boundary points, slight modifications of the argument are called for. In order to make the paper more readable, we provide below the details for most of the proofs.

The Lévy measure of the process X has a density J, called the Lévy density, given by

$$J(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \eta(t) dt, \qquad x \in \mathbb{R}^d.$$

Recall that η denotes the Lévy density of the subordinator S. Thus $J(x) = j(|x|)$ with

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \eta(t) dt, \qquad r > 0.$$

Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$. When hypothesis **(H)** is satisfied, j enjoys the following properties:

(a) For any $K > 0$, there exists $c = c(K) > 0$ such that

$$j(r) \leq c\, j(2r), \qquad \forall r \in (0, K); \tag{3.1}$$

(b) There exists $c > 0$ such that

$$j(r) \le c\, j(r+1), \qquad \forall r > 1. \tag{3.2}$$

(See [22, (2.17)–(2.18)] in case $\alpha \in (0,2)$ and [23, (2.5)–(2.6)] in case $\alpha = 2$.) We further have (see [21, Theorem 3.4]) that in case $\alpha \in (0,2)$,

$$j(r) \asymp \frac{\phi(r^{-2})}{r^d}\,, \quad r \to 0\,. \tag{3.3}$$

For an open set $U \subset \mathbb{R}^d$, let

$$K^U(x,z) := \int_U G^U(x,y) J(y-z) dy, \quad (x,z) \in U \times \overline{U}^c.$$

Then for any nonnegative measurable function f on $\mathbb{R}^d$,

$$\mathbb{E}_x \left[f(X_{\tau_U});\, X_{\tau_U -} \neq X_{\tau_U} \right] = \int_{\overline{U}^c} K^U(x,z) f(z) dz.$$

Note that when subordinate Brownian motion X satisfies **(H)** there exist $c_1, c_2, r_1 > 0$ such that for every $r \in (0, r_1]$ and $x_1 \in \mathbb{R}^d$,

$$\begin{aligned} K^{B(x_1,r)}(x,y) \;&\le\; c_1\, j(|y-x_1|-r) \left(\phi(r^{-2})\phi((r-|x-x_1|)^{-2})\right)^{-1/2} && (3.4)\\ &\le\; c_1\, j(|y-x_1|-r)\phi(r^{-2})^{-1} && (3.5) \end{aligned}$$

for all $(x,y) \in B(x_1,r) \times \overline{B(x_1,r)}^c$ and

$$K^{B(x_1,r)}(x_1,y) \;\ge\; c_2\, j(|y-x_1|)\phi(r^{-2})^{-1}, \qquad \text{for all } y \in \overline{B(x_1,r)}^c. \tag{3.6}$$

In case $\alpha \in (0,2)$, (3.4) and (3.6) were shown in [21, Proposition 4.10], while (3.5) follows from (3.4) and the fact that ϕ is increasing. In case $\alpha = 2$, the proof is analogous to the proof of [21, Proposition 4.10], except that one uses [20, Proposition 3.1] instead of [21, Proposition 4.9], and [27, Proposition 3.5] instead of [21, Lemma 4.2].

We use the notation that $A_r(Q) := (\tilde{Q}, r/2)$ for $Q \in \partial H$. Note that $H \cap B(Q,r)$ contains the ball $B(A_r(Q), r/2)$.

Using (**H**) and [4, Theorems 1.5.3 and 1.5.11], there exists a positive constant $R_* := R_*(d,\phi) < 1 \wedge r_1$ such that

$$\int_0^r \frac{1}{s\phi(s^{-1})} ds \;\le\; 4\, \frac{1}{\phi(r^{-1})}, \qquad \forall\, 0 < s < r \le 4R_*, \tag{3.7}$$

and for every $r \le 2R_*$,

$$\frac{1}{2} \;\le\; \min\left(\frac{\ell(8^{-2}r^{-2})}{\ell(r^{-2})}, \frac{\ell(16r^{-2})}{\ell(r^{-2})}\right) \;\le\; \max\left(\frac{\ell(8^{-2}r^{-2})}{\ell(r^{-2})}, \frac{\ell(16r^{-2})}{l(r^{-2})}\right) \;\le\; 2. \tag{3.8}$$

We will fix the constant R_* in remainder of this section.

The next lemma is proved by the same argument as that of [6, Lemma 5] and [19, Lemma 5.2] in case $\alpha \in (0,2)$.

Lemma 3.1 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *There exist positive constants $c = c(d, \phi)$ and $\gamma = \gamma(d, \phi) \in (0, \alpha)$ such that for all $Q \in \partial H$, $r \in (0, R_*]$, and nonnegative function h in $\mathbb{R}^d$ which is harmonic with respect to X in $H \cap B(Q, r)$ and vanishes continuously on $H^c \cap B(Q, r)$ we have*

$$\phi(r^{-2})\, h(A_r(Q)) \le c4^{-\gamma k}\phi(4^{2k}r^{-2})\, h(A_{4^{-k}r}(Q)) , \qquad k = 0, 1, \dots.$$

In case $\alpha = 2$ we may take $\gamma = 1$.

Proof: Without loss of generality, we may assume $Q = 0$. By Theorem 2.3, there exists a constant $C_3(R_*) = C_3(d, \phi, R_*) > 0$ such that for $r \in (0, 2R_*]$,

$$h(A_r(0)) \,\le\, C_3(R_*)\, \frac{V(\delta_H(A_r(0)))}{V(\delta_H(A_{4^{-k}r}(0)))} h(A_{4^{-k}r}(0)) \,=\, C_3(R_*)\, \frac{V(r/2)}{V(4^{-k}r/2)} h(A_{4^{-k}r}(0)).$$

This and (2.6) complete the proof of the lemma in case $\alpha = 2$.

Assume now that $\alpha \in (0, 2)$. Fix $r \le R_*$ and let $\eta_k := 4^{-k}r$, $A_k := A_{\eta_k}(0)$ and $B_k := B(A_k, \eta_{k+1})$, $k = 0, 1, \dots$. Note that the B_k's are disjoint. So by the harmonicity of h, we have

$$h(A_k) \,\ge\, \sum_{l=0}^{k-1} \mathbb{E}_{A_k}\left[h(X_{\tau_{B_k}}) :\, X_{\tau_{B_k}} \in B_l\right] \,=\, \sum_{l=0}^{k-1} \int_{B_l} K^{B_k}(A_k, z) h(z) dz.$$

Theorem 2.2 implies that

$$\int_{B_l} K^{B_k}(A_k, z) h(z) dz \,\ge\, c_0\, h(A_l) \int_{B_l} K^{B_k}(A_k, z) dz$$

for some constant $c_0 = c_0(d, \phi) > 0$. Since $\mathrm{dist}(A_k, B_l) \le 2\eta_l$, by (3.6) and the monotonicity of j we have

$$K^{B_k}(A_k, z) \ge\, c_1\, j(|2(A_k - z)|)\, \phi(\eta_{k+1}^{-2})^{-1} \,\ge\, c_1\, j(4\eta_l)\, \phi(\eta_{k+1}^{-2})^{-1} , \qquad z \in B_l. \tag{3.9}$$

Further, by (3.9), (3.1) and (3.3) we have

$$K^{B_k}(A_k, z) \ge c_1 \frac{j(4\eta_l)}{\phi(\eta_{k+1}^{-2})} \ge c_2 \frac{j(\eta_{l+1})}{\phi(\eta_{k+1}^{-2})} \ge c_3 \frac{\phi(\eta_{l+1}^{-2})}{\eta_{l+1}^{d}\phi(\eta_{k+1}^{-2})} , \quad z \in B_l$$

for some constat $c_3 = c_3(d, \phi) > 0$. Thus we have

$$\int_{B_l} K^{B_k}(A_k, z) dz \ge\, c_4 \frac{\phi(\eta_{l+1}^{-2})}{\phi(\eta_{k+1}^{-2})}, \qquad z \in B_l$$

for some constant $c_4 = c_4(d, \phi) > 0$. Therefore,

$$h(A_k)\phi(\eta_{k+1}^{-2}) \ge c_5 \sum_{l=0}^{k-1} h(A_l)\phi(\eta_{k+l}^{-2})$$

for some constant $c_5 = c_5(d, \phi) > 0$. Let $a_k := h(A_k)\phi(\eta_{k+1}^{-2})$ so that $a_k \geq c_5 \sum_{l=0}^{k-1} a_l$. By induction, one can easily check that $a_k \geq c_6(1 + c_5/2)^k a_0$ for some constant $c_6 = c_6(d, \alpha) > 0$. Thus, with $\gamma = \ln(1 + \frac{c_5}{2})(\ln 4)^{-1} > 0$ we get

$$\phi(r^{-2})h(A_r(Q)) \leq c_6 \, 4^{-\gamma k} \, \phi(4^{2k}r^{-2})h(A_{4^{-k}r}(Q)).$$

Note that we can choose $c_5 > 0$ small enough so that $\gamma < \alpha$. This completes the proof in case $\alpha \in (0, 2)$. $\Box$

By modifying the proof of [19, Lemma 5.3], one can obtain the following

Lemma 3.2 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Suppose $Q \in \partial H$ and $r \in (0, R_*]$. If $w \in H \setminus B(Q, r)$, then*

$$G^H(A_r(Q), w) \geq c \, \phi(r^{-2})^{-1} \int_{B(Q,r)^c} J(z - Q) G^H(z, w) dz$$

for some constant $c = c(d, \phi) > 0$.

Proof: This proof is similar that of [19, Lemma 5.3], we give the details here for completeness. Without loss of generality, we may assume $Q = 0$. Fix $w \in H \setminus B(0, r)$ and let $A := A_r(0)$ and $h(\cdot) := G^H(\cdot, w)$. Since h is regular harmonic in $H \cap B(0, 3r/4)$ with respect to X, we have

$$\begin{aligned} h(A) &\geq \mathbb{E}_A \left[h\left(X_{\tau_{H\cap B(0,3r/4)}} \right); X_{\tau_{H\cap B(0,3r/4)}} \in B(0, r)^c \right] = \int_{B(0,r)^c} K^{H\cap B(0,3r/4)}(A, z) h(z) dz \\ &= \int_{B(0,r)^c} \int_{H\cap B(0,3r/4)} G^{H\cap B(0,3r/4)}(A, y) \, J(y - z) dy h(z) dz. \end{aligned}$$

Since $B(A, r/4) \subset H \cap B(0, 3r/4)$, by the monotonicity of the Green functions,

$$G^{H\cap B(0,3r/4)}(A, y) \geq G^{B(A,r/4)}(A, y), \quad y \in B(A, r/4).$$

Thus

$$h(A) \geq \int_{B(0,r)^c} \int_{B(A,r/4)} G^{B(A,r/4)}(A, y) J(y - z) dy h(z) dz = \int_{B(0,r)^c} K^{B(A,r/4)}(A, z) h(z) dz,$$

which is greater than or equal to $c_2 \phi((r/4)^{-2})^{-1} \int_{B(0,r)^c} J(z - A) h(z) \, dz$ for some positive constant $c_2 = c_2(d, \phi)$ by (3.6). Now the conclusion of the lemma follows immediately from (3.8) and (3.1)–(3.2). $\Box$

Using the above lemma, (3.1), (3.2) and (3.5), the proof of next lemma is almost the same as that of [19, Lemma 5.4].

Lemma 3.3 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *There exist positive constants $c_1 = c_1(d,\phi)$ and $c_2 = c_2(d,\phi) < 1$ such that for any $Q \in \partial H$, $r \in (0, R_*]$ and $w \in H \setminus B(Q,4r)$, we have*

$$\mathbb{E}_x\left[G^H(X_{\tau_{H\cap B_k}}, w) : \, X_{\tau_{H\cap B_k}} \in B(Q,r)^c\right] \le c_1\, c_2^k\, G^H(x,w), \quad x \in H \cap B_k,$$

where $B_k := B(Q, 4^{-k}r)$, $k = 0, 1, \dots$.

Proof: Without loss of generality, we may assume $Q = 0$. Fix $r \le R_*$ and $w \in H \setminus B(0,4r)$. Let $\eta_k := 4^{-k}r$, $B_k := B(0,\eta_k)$ and

$$u_k(x) := \mathbb{E}_x\left[G^H(X_{\tau_{H\cap B_k}}, w); X_{\tau_{H\cap B_k}} \in B(0,r)^c\right], \quad x \in H \cap B_k.$$

Note that for $x \in H \cap B_{k+1}$

$$u_{k+1}(x) = \mathbb{E}_x\left[G^H(X_{\tau_{H\cap B_k}}, w); \, \tau_{H\cap B_{k+1}} = \tau_{H\cap B_k}, \; X_{\tau_{H\cap B_k}} \in B(0,r)^c\right] \le u_k(x). \tag{3.10}$$

Let $A_k := A_{\eta_k}(0)$. Since $G^H(\,\cdot\,, w)$ is zero on H^c, we have

$$\begin{aligned} u_k(A_k) &= \mathbb{E}_{A_k}\left[G^H(X_{\tau_{H\cap B_k}}, w); \, X_{\tau_{H\cap B_k}} \in B(0,r)^c\right] \\ &\le \mathbb{E}_{A_k}\left[G^H(X_{\tau_{B_k}}, w); \, X_{\tau_{B_k}} \in B(0,r)^c\right] = \int_{B(0,r)^c} K^{B_k}(A_k, z) G^H(z,w) dz. \end{aligned}$$

By (3.5) we have that there exists a constant $c_1 = c_1(d,\phi) > 0$ such that

$$u_k(A_k) \le c_1\, \phi(\eta_k^{-2})^{-1} \int_{B(0,r)^c} j(|z| - \eta_k) G^H(z,w)\, dz \quad k = 1, 2, \dots. \tag{3.11}$$

Note that $|z|/2 \le |z| - \eta_k \le |z|$ for $z \in B(0,4R_*) \setminus B(0,r)$ and $|z| - R_* \le |z| - \eta_k \le |z|$ for $z \in B(0,4R_*)^c$. Thus using (3.1)–(3.2), we see that Lemma 3.2 and (3.11) imply that

$$u_k(A_k) \le c_2 \frac{\phi(r^{-2})}{\phi((4^{-k}r)^{-2})} G^H(A_0, w), \quad k = 1, 2, \dots.$$

Now applying Lemma 3.1 we get

$$u_k(A_k) \le c_3 4^{-\gamma k} G^H(A_k, w), \quad k = 1, 2, \dots.$$

Finally, Theorem 2.3 gives that for $k = 1, 2, \dots$

$$\frac{u_k(x)}{G^H(x,w)} \le \frac{u_{k-1}(x)}{G^H(x,w)} \le c_4 \frac{u_{k-1}(A_{k-1})}{G^H(A_{k-1},w)} \le c_5\, 4^{-\gamma k}.$$

□

Now the next theorem follows from Theorem 2.3 and Lemma 3.3 in the same way as [6, Lemma 16] follows from [6, Lemmas 13–14]. We omit the details.

Theorem 3.4 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For each $x \in H$ and each $z \in \partial H$ there exists the limit*

$$M^H(x,z) := \lim_{y\to z} M^H(x,y)\,.$$

Moreover, the mapping $(x,z) \mapsto M^H(x,z)$ is continuous on $H \times \partial H$.

Remark 3.5 *Note that Theorem 3.4 shows that the finite part of the Martin boundary of H can be identified with a subset of ∂H.*

We recall here that the Martin kernel for the killed α-stable process in H is explicitly known (see [7]) and is given by

$$\begin{aligned} M^H(x,z) &= \frac{\delta_H(x)^{\alpha/2}}{|x-z|^d}\,(1+|z|^2)^{d/2}\,, \quad x\in H,\, z\in\partial H\,,\\ M^H(x,\infty) &= \delta_H(x)^{\alpha/2}\,, \quad x\in H\,. \end{aligned}$$

The same form of the Martin kernel is valid for the killed Brownian motion in H with α replaced by 2. We cannot hope to obtain explicit formulae for the Martin kernel for the process X^H, but the following sharp two-sided estimates for points close to the origin and to each other will suffice for our purpose.

Theorem 3.6 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $R>0$, there exists a constant $C_5(R)=C_5(d,\phi,R)>0$ such that for all $x\in H$ and all $z\in\partial H$ satisfying $|z|<R$ and $|x-z|<R/2$ it holds that*

$$C_5(R)^{-1}\frac{V(\delta_H(x))}{|x-z|^d}\,(1+|z|^2)^{d/2} \le M^H(x,z) \le C_5(R)\frac{V(\delta_H(x))}{|x-z|^d}\,(1+|z|^2)^{d/2}\,.$$

Proof: Let $x\in H$ and $z\in\partial H$ such that $|z|<R$ and $|x-z|<R/2$. Let $y\in H$ be such that $|y-z|<R/2$ and $\delta_H(y)\le\delta_H(x)\wedge|x-y|$. Then $|x-y|<R$, $\delta_H(x)\wedge\delta_H(y)<R$ and $\delta_H(x)\vee\delta_H(y)<2|x-y|$. Hence, by Theorem 2.4 and (2.12)

$$\frac{G^H(x,y)}{G^H(x_0,y)} \asymp \frac{\frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}\,G(x,y)}{\frac{V(\delta_H(x_0))V(\delta_H(y))}{V(|x_0-y|)^2}\,G(x_0,y)} = \frac{V(\delta_H(x))}{V(1)}\,\frac{G(x,y)}{V(|x-y|)^2}\,\frac{V(|x_0-y|)^2}{G(x_0,y)}.$$

Let $y\to z$; by Theorem 3.4 the left-hand side converges to $M^H(x,z)$, while the right-hand side converges to

$$\frac{V(\delta_H(x))}{V(1)}\,\frac{G(x,z)}{V(|x-z|)^2}\,\frac{V(|x_0-z|)^2}{G(x_0,z)}\,.$$

By use of Proposition 2.1(i) and the fact that $|x_0-z|=(1+|z|^2)^{1/2}$ it follows that

$$M^H(x,z)\asymp\frac{V(\delta_H(x))}{|x-z|^d}\,(1+|z|^2)^{d/2}\,,$$

which proves the theorem. $\square$

Theorem 3.6 in particular implies that $M^H(\cdot, z_1)$ differs from $M^H(\cdot, z_2)$ if z_1 and z_2 are two different points on ∂H. Together with Remark 3.5, this shows that the finite part of the Martin boundary of H can be identified with ∂H.

Now using Theorem 2.4, (3.4) and Lemma 3.1, we can prove the next two lemmas by the arguments in the proofs of [18, Lemmas 4.6–4.7] and [19, Lemmas 5.6–5.7].

Lemma 3.7 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $z \in \partial H$ and $B \subset \overline{B} \subset H$, $M^H(X_{\tau_B}, z)$ is $\mathbb{P}_x$-integrable.*

Proof: Take a sequence $\{z_m\}_{m\ge 1} \subset H \setminus \overline{B}$ converging to z. Since $M^H(\cdot, z_m)$ is regular harmonic with respect to X in B, by Fatou's lemma and Theorem 3.4,

$$\mathbb{E}_x\left[M^H\left(X_{\tau_B}, z\right)\right] \,=\, \mathbb{E}_x\left[\lim_{m\to\infty} M^H\left(X_{\tau_B}, z_m\right)\right] \,\le\, \liminf_{m\to\infty} M^H(x, z_m) \,=\, M^H(x,z) \,<\, \infty.$$

$\square$

Lemma 3.8 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $z \in \partial H$ and $x \in H$,*

$$M^H(x,z) \,=\, \mathbb{E}_x\left[M^H\left(X^H_{\tau_{B(x,r)}}, z\right)\right], \qquad \textit{for every } 0<r\le R_* \wedge \frac{1}{3}\delta_H(x).$$

Proof: Without loss of generality we assume that $z=0$, and fix $x \in H$ and $r \le R_* \wedge \frac{1}{3}\delta_H(x)$. Let $\eta_m := 4^{-m} r$ and $z_m := A_{\eta_m}(0)$, $m = 0, 1, \dots$. By the harmonicity of $M^H(\cdot, z_m)$, to prove the lemma, it suffices to show that $\{M^H(X_{\tau_{B(x,r)}}, z_m) : m \ge m_0\}$ is $\mathbb{P}_x$-uniformly integrable for some large m_0.

Using Theorem 2.3, there exist constants $m_0 \ge 0$ and $c_1 > 0$ such that for every $w \in H \backslash B(z, \eta_m)$,

$$M^H(w, z_m) \,=\, \frac{G^H(w, z_m)}{G^H(x_0, z_m)} \,\le\, c_1 \lim_{y\to z} \frac{G^H(w,y)}{G^H(x_0,y)} \,=\, c_1\, M^H(w,z), \quad m\ge m_0. \tag{3.12}$$

Using (3.12) and Lemma 3.7, for any $\varepsilon > 0$, there is an $N_0 > 1$ such that

$$\begin{aligned}
&\mathbb{E}_x\left[M^H\left(X_{\tau_{B(x,r)}}, z_m\right); M^H\left(X_{\tau_{B(x,r)}}, z_m\right) > N_0 \;\text{ and }\; X_{\tau_{B(x,r)}} \in H\setminus B(z,\eta_m)\right] \\
&\quad\le\; c_1\, \mathbb{E}_x\left[M^H\left(X_{\tau_{B(x,r)}}, z\right); c_1 M^H\left(X_{\tau_{B(x,r)}}, z\right) > N_0\right] \,<\, c_1 \frac{\varepsilon}{4c_1} \,=\, \frac{\varepsilon}{4}.
\end{aligned} \tag{3.13}$$

On the other hand, by (3.5), we have for $m \ge m_0$,

$$\begin{aligned}
&\mathbb{E}_x\left[M^H\left(X^H_{\tau_{B(x,r)}}, z_m\right); X_{\tau_{B(x,r)}} \in H\cap B(z,\eta_m)\right] \,=\, \int_{H\cap B(z,\eta_m)} M^H(w,z_m) K^{B(x,r)}(x,w)dw \\
&\le c_2 \int_{H\cap B(z,\eta_m)} M^H(w,z_m) j(|w-x|-r)\phi(r^{-2})^{-1}\, dw
\end{aligned}$$

for some $c_2 = c_2(d, \phi) > 0$. Since $|w-x| \geq |x-z| - |z-w| \geq \delta_H(x) - \eta_m \geq 3r - r = 2r$, using the monotonicity of j in the above equation, we see that

$$\begin{aligned}
&\mathbb{E}_x \left[M^H \left(X^H_{\tau_{B(x,r)}}, z_m \right) ; X_{\tau_{B(x,r)}} \in H \cap B(z, \eta_m) \right] \\
&\leq c_3 j(r) \phi(r^{-2})^{-1} \int_{B(z,\eta_m)} M^H(w, z_m) dw \leq c_4 \, G^H(x_0, z_m)^{-1} \int_{B(z,\eta_m)} G^H(w, z_m) dw \quad (3.14)
\end{aligned}$$

for some $c_3 = c_3(\phi) > 0$ and $c_4 = c_4(\phi, r) > 0$. By Lemma 3.1 there exists there exist $c_5 = c_5(\phi, m_0, r)$ and $\gamma \in (0, \alpha)$ such that

$$G^H(x_0, z_m)^{-1} \leq c_5 4^{-\gamma(m-m_0)} \frac{\phi(4^{2m} r^{-2})}{\phi(4^{2m_0} r^{-2})} \, G^H(x_0, z_{m_0})^{-1} \, . \tag{3.15}$$

Further, by (2.3) we have that there are constants $c_i = c_i(\phi, m_0, r) > 0$, $i = 6, 7, 8, 9$, such that

$$\begin{aligned}
\int_{B(0,\eta_m)} G^H(w, z_m) \, dw &\leq c_6 \int_{B(z_m, 2\eta_m)} \frac{dw}{|w - z_m|^d \phi(|w - z_m|^{-2})} \\
&\leq c_7 \int_0^{2\eta_m} \frac{ds}{s\phi(s^{-1})} \leq c_8 \phi((2\eta_m)^{-2})^{-1} \leq c_9 \phi(4^{2m} r^{-2})^{-1} \, , \quad (3.16)
\end{aligned}$$

where the third inequality follows from (3.7) and the fourth from (3.8). Putting together (3.15) and (3.16) we arrive at

$$G^H(x_0, z_m)^{-1} \int_{B(z,\eta_m)} G^H(w, z_m) \leq c_{10} 4^{-\gamma(m-m_0)} \frac{G^H(x_0, z_{m_0})}{\phi(4^{2m_0} r^{-2})} \tag{3.17}$$

for a constant $c_{10} = c_{10}(\phi, m_0, r) > 0$. Using (3.13), (3.14) and (3.17), we can take $m_1 = m_1(\varepsilon, \phi, m_0, r) \geq m_0$ large enough so that for $m \geq m_1$,

$$\begin{aligned}
&\mathbb{E}_x \left[M^H \left(X_{\tau_{B(x,r)}}, z_m \right) ; M^H \left(X_{\tau_{B(x,r)}}, z_m \right) > N_0 \right] \\
&\leq \mathbb{E}_x \left[M^H \left(X_{\tau_{B(x,r)}}, z_m \right) ; X_{\tau_{B(x,r)}} \in H \cap B(z, \eta_m) \right] \\
&\quad + \mathbb{E}_x \left[M^H \left(X_{\tau_{B(x,r)}}, z_m \right) ; M^H \left(X_{\tau_{B(x,r)}}, z_m \right) > N_0 \text{ and } X_{\tau_{B(x,r)}} \in H \setminus B(z, \eta_m) \right] \\
&\leq c_4 c_{10} 4^{-\gamma(m-m_0)} \frac{G^H(x_0, z_{m_0})}{\phi(4^{2m_0} r^{-2})} + \frac{\epsilon}{4}
\end{aligned}$$

is less than ϵ. As each $M^H(X_{\tau_{B(x,r)}}, z_m)$ is $\mathbb{P}_x$-integrable, we conclude that $\{M^H(X_{\tau_{B(x,r)}}, z_m) : m \geq m_0\}$ is uniformly integrable under $\mathbb{P}_x$. $\square$

The proof of next result is taken from [12, Theorem 6.10].

Theorem 3.9 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $z \in \partial H$, the function $x \mapsto M^H(\cdot, z)$ is harmonic in H with respect to X^H.*

Proof: Fix $z \in \partial H$ and let $h(x) := M^H(x,z)$. Consider an open set $D_1 \subset \overline{D_1} \subset H$, $x \in D_1$, and put $r(x) = R_* \wedge \frac{1}{3}\delta_H(x)$ and $B(x) = B(x, r(x))$. Define a sequence of stopping times $\{T_m, m \geq 1\}$ as follows: $T_1 := \inf\{t > 0 : X_t \notin B(X_0)\}$, and for $m \geq 2$,

$$T_m := \begin{cases} T_{m-1} + \tau_{B(X_{T_{m-1}})} \circ \theta_{T_{m-1}} & \text{if } X_{T_{m-1}} \in D_1 \\ \tau_{D_1} & \text{otherwise.} \end{cases}$$

Note that $X^H_{\tau_{D_1}} \in \partial D_1$ on $\cap_{n=1}^{\infty}\{T_n < \tau_{D_1}\}$. Thus, since $\lim_{m\to\infty} T_m = \tau_{D_1}$ $\mathbb{P}_x$-a.s. and h is continuous in D, using the quasi-left-continuity of X^H, we have $\lim_{m\to\infty} h(X^H_{T_m}) = h(X^H_{\tau_{D_1}})$ on $\cap_{n=1}^{\infty}\{T_n < \tau_{D_1}\}$. Now, by the dominated convergence theorem and Lemma 3.8,

$$\begin{aligned} h(x) &= \lim_{m\to\infty} \mathbb{E}_x\left[h(X^H_{T_m}); \cup_{n=0}^{\infty}\{T_n = \tau_{D_1}\}\right] + \lim_{m\to\infty} \mathbb{E}_x\left[h(X^H_{T_m}); \cap_{n=0}^{\infty}\{T_n < \tau_{D_1}\}\right] \\ &= \mathbb{E}_x\left[h(X^H_{\tau_{D_1}}); \cup_{n=0}^{\infty}\{T_n = \tau_{D_1}\}\right] + \mathbb{E}_x\left[h(X^H_{\tau_{D_1}}); \cap_{n=0}^{\infty}\{T_n < \tau_{D_1}\}\right] = \mathbb{E}_x\left[h(X^H_{\tau_{D_1}})\right]. \end{aligned}$$

□

The following two lemmas will serve as a counterpart of Theorem 3.6 for estimating the Martin kernel $M^H(x,w)$ when x and w are far away.

Lemma 3.10 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $R > 0$ and $\varepsilon \in (0,1)$, there exists a constant $c(R,\varepsilon) = c(d,\phi,R,\epsilon) > 0$ such that for all $z \in \partial H$ with $|z| < R$, all $x \in H \cap B(z,\varepsilon)$, and all $w \in \partial^M H \setminus (\partial H \cap B(z, 2\varepsilon))$ it holds that*

$$c^{-1} V(\delta_H(x)) \leq M^H(x,w) \leq c V(\delta_H(x)).$$

Proof: Fix $z \in \partial H$ with $|z| < R$ and $w \in \partial^M H \setminus (\partial H \cap B(z, 2\varepsilon))$. Let $y_0 := (\tilde{z}, \varepsilon/2)$. Consider a sequence $\{w_n\} \subset H \cap B(z, \varepsilon/2)$ such that $w_n \to w$. Then, by Theorem 2.3,

$$C_3(\varepsilon)^{-1} \frac{V(\delta_H(x))}{V(\varepsilon/2)} \leq \frac{G^H(x,w_n)}{G^H(y_0,w_n)} \leq C_3(\varepsilon)\frac{V(\delta_H(x))}{V(\varepsilon/2)}, \quad \text{for every } n \geq 1 \text{ and } x \in H \cap B(z,\varepsilon).$$

Now, using Theorem 2.2, we see that

$$c^{-1}\, C_3(\varepsilon)^{-1} \frac{V(\delta_H(x))}{V(\varepsilon/2)} \leq \frac{G^H(x,w_n)}{G^H(x_0,w_n)} \leq c\, C_3(\varepsilon)\frac{V(\delta_H(x))}{V(\varepsilon/2)}, \quad \text{for every } n \geq 1 \text{ and } x \in H \cap B(z,\varepsilon)$$

where $c = c(R,\phi,d) > 0$. Thus, letting $n \to \infty$, we conclude that

$$c^{-1}\, C_3(R)^{-1} \frac{V(\delta_H(x))}{V(1/2)} \leq M^H(x,w) \leq c\, C_3(R)\frac{V(\delta_H(x))}{V(1/2)} \quad \text{for every } x \in H \cap B(z,\varepsilon),$$

which finishes the proof. □

Lemma 3.11 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $R>0$, there exists a constant $c=c(d,\phi,R)>0$ such that for all $z\in\partial H$ with $|z|<R$ and all $x\in H$ such that $|x-z|>R/2$, it holds that $M^H(x,z)\ \le\ c\,G(|x-z|+1)$.*

Proof: Fix $z\in\partial H$ with $|z|<R$, $x\in H$ such that $|x-z|>R/2$, and let $z_0:=(\tilde{z},(R\wedge 1)/4)$. Consider a sequence $(z_n)_{n\ge 1}\subset H\cap B(z,(R\wedge 1)/4)$ such that $z_n\to z$. By Theorem 2.3 applied to function $G^H(x,\cdot)$ and $G^H(x_0,\cdot)$ we have that

$$M^H(x,z)=\lim_{n\to\infty}\frac{G^H(x,z_n)}{G^H(x_0,z_n)}\le C_3((R\wedge 1)/4))\frac{G^H(x,z_0)}{G^H(x_0,z_0)}\,.$$

It follows from Theorem 2.4, the fact that $|x_0-z_0|<R+1$ and the monotonicity of G and V that

$$G^H(x_0,z_0)\ge C_4(R+1)^{-1}G(R+1)\left(1\wedge\frac{V(1)}{V(R+1)}\right)\left(1\wedge\frac{V((R\wedge 1)/4))}{V(R+1)}\right)\ge c_1\,,$$

where $c_1=c_1(d,\phi,R)>0$. Hence, by using that $G^H(x,z_0)\le G(x,z_0)=G(|x-z_0|)\le G(|x-z|+1)$, we prove the lemma. □

Remark 3.12 By combining Theorem 3.6 and Lemma 3.11 we see that for every $z\in\partial H$ and every $w\in\partial^M H\setminus\{z\}$, it holds that $\lim_{x\to w}M^H(x,z)=0$. Indeed, if $w\in\partial H$, then there exists $R>1$ large enough such that $|z|<R$ and $|z-w|<R/3$ and the claim follows from Theorem 3.6. On the other hand, if $w\in\partial^M H\setminus\partial H$, then we use Lemma 3.11 and the fact that $\lim_{|x|\to\infty}G(|x-z|+1)=0$.

With the explicit estimates from Theorem 3.6 and Lemmas 3.10 and 3.11, a slight modification of the argument in [13, Theorem 3.7] gives the following.

Theorem 3.13 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *For every $z\in\partial H$, the function $M^H(\cdot,z)$ is minimal harmonic in H with respect to X. Therefore, the finite part of the minimal Martin boundary of H of X^H coincides with ∂H.*

Proof: Fix $z\in\partial H$ and suppose that h is nonnegative function harmonic with respect to X^H satisfying $h\le M^H(\cdot,z)$. By [24], there is a measure μ on $\partial^M H$ such that

$$h(x)=\int_{\partial^M H}M^H(x,w)\,\mu(dw)\,.$$

We want to prove that μ is a multiple of the point mass at z. Let $\delta\in(0,1)$ and denote by ν the restriction of μ to $\partial^M H\setminus(\partial H\cap B(z,\delta))$. If we show that $\nu=0$, we are done (since $\delta>0$ is arbitrary). Let

$$u(x):=\int_{\partial^M H}M^H(x,w)\,\nu(dw)\,. \tag{3.18}$$

Then u is a nonnegative harmonic functions with respect to X^H such that $u(x)\le h(x)\le M^H(x,z)$ for all $x\in H$. Let $\epsilon\in(0,1)$. It follows from Theorem 3.6 and Lemma 3.11 (as explained in

Remark 3.12) that for every $w \in \partial^M H \setminus (\partial H \cap B(z,\epsilon))$ we have that $\lim_{x\to w} M^H(x,z) = 0$. Hence, $\lim_{x\to w} u(x) = 0$. Suppose, additionally, that $4\epsilon < \delta$. It follows from Proposition 3.10 that

$$\lim_{x\to \partial H\cap B(z,2\epsilon)} M^H(x,w) = 0 \qquad \text{uniformly in } w \in \partial^M H \setminus (\partial H \cap B(z,\delta)) .$$

Applying the dominated convergence theorem in (3.18), we conclude that $\lim_{x\to w} u(x) = 0$ for every $w \in \partial H \cap B(z, 2\epsilon)$ as well. This shows that u is a nonnegative harmonic function with respect to X^H that vanishes continuously at $\partial^M H$. Therefore it is identically equal to zero proving that its representation measure $\nu = 0$. □

For simplicity we use the notation $\partial^m_\infty H := \partial^m H \setminus \partial H$. It is known (e.g., [24]) that every function s which is excessive with respect to X^H can be uniquely represented in the form

$$s(x) = G^H\mu(x) + M^H\nu(x) := \int_H G^H(x,y)\,\mu(dy) + \int_{\partial^m H} M^H(x,z)\,\nu(dz)\,, \quad x \in H\,, \tag{3.19}$$

where μ is a measure on H and ν a finite measure on $\partial^m H$.

For $z \in \partial H$, let $X^{H,z} = (X^H_t, \mathbb{P}^z_x)$ be the $M^H(\cdot,z)$-process. The existence of such a process is discussed in [16, 24]. Let ζ denote the lifetime of $X^{H,z}$. It is known (see [24]) that $\lim_{t\uparrow\zeta-} X^{H,z}_t = z$ a.s. $\mathbb{P}^z_x$.

For $A \subset H$, let $T_A := \inf\{t > 0 : X^H_t \in A\}$. A Borel set $A \subset H$ is said to be *minimally thin* in H with respect to X at $z \in \partial H$ if there exists $x \in H$ such that $\mathbb{P}^z_x(T_A < \zeta) \neq 1$, i.e., if with positive probability the conditioned process $X^{H,z}$ starting from x does not hit A. It is proved in [16, Satz 2.6] that A is minimally thin in H with respect to X at z if and only if

$$P_A M^H(\cdot,z) \neq M^H(\cdot,z)\,.$$

Here P_A denotes the hitting operator to A for X^H: $P_A f(x) = \mathbb{E}_x[f(X^H_{T_A})]$. In potential-theoretic language, $P_A M^H(\cdot,z) = \widehat{R}^A_{M^H(\cdot,z)}$ – the balayage of $M^H(\cdot,z)$ onto A. By following the proof of [1, Theorem 9.2.6] and using [16, Lemma 2.7] instead of [1, Lemma 9.2.2(c)] one can show that A is minimally thin in H with respect to X at $z \in \partial H$ if and only if there exists a function s excessive with respect to X^H such that

$$\liminf_{x\to z, x\in A} \frac{s(x)}{M^H(x,z)} > \nu(\{z\})\,, \tag{3.20}$$

where ν is the representing measure of the harmonic part $M^H\nu$ of s (as in (3.19)). By subtracting $\nu(\{z\})$ from the measure ν, we have

Proposition 3.14 *Suppose $A \subset H$ and $z \in \partial H$. The following are equivalent:*
(a) A is minimally thin in H with respect to X at z;
(b) There exists an excessive function $s(x) = G^H\mu(x) + M^H\nu(x)$ such that (3.20) *holds;*
(c) There exists an excessive function $s'(x) = G^H\mu(x) + M^H\nu'(x)$ with $\nu'(\{z\}) = 0$ such that

$$\liminf_{x\to z, x\in A} \frac{s'(x)}{M^H(x,z)} > 0\,.$$

4 Main result

The key to the proof of our generalization of Theorems 1.1–1.2 is the following lemma and proposition, which are generalizations of [30, Theorem 2].

Lemma 4.1 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Let ν be a finite measure on $\partial^m H$ with $\nu(\{0\})=0$. Then*

$$\int_{\{x\in H: M^H\nu(x)>M^H(x,0)/2\}\cap B(0,1)} |x|^{-d}\,dx<\infty\,.$$

Proof: Note that since $\nu(\{0\})=0$, $\nu(\{z\in\partial H:|z|<\epsilon\})$ can be made arbitrarily small by choosing ϵ small enough. Let $\epsilon<1/4$ be a positive constant such that

$$\nu(\{z\in\partial H:\,|z|<\epsilon\})<\frac{1}{10}\,C_5(1)^{-2}\left(\frac{\sqrt{3}}{8}\right)^d 2^{-d/2}. \tag{4.1}$$

Set $A:=\{x\in H: M^H\nu(x)>M^H(x,0)/2\}\cap B(0,1)$ and let

$$\nu^{(1)}=\nu|_{\{z\in\partial H:\,|z|<\epsilon\}}\,,\quad \nu^{(2)}=\nu|_{\{z\in\partial H:\,\epsilon\le|z|<1\}}\,,\quad \nu^{(3)}=\nu|_{\{z\in\partial H:\,|z|\ge1\}\cup\partial^m_\infty H}$$

so that $\nu=\nu^{(1)}+\nu^{(2)}+\nu^{(3)}$.

Let $y_0=(\tilde 0,1/8)$ and $x\in H$ be a point with $|x|<\kappa$ where $\kappa<1/4$ is to be chosen later. For $z\in\{z\in\partial H:\,|z|\ge1\}\cup\partial^m_\infty H$ the function $M^H(\cdot,z)$ is harmonic and vanishes continuously on $H^c\cap B(0,1/2)$ by Theorem 3.6 and Remark 3.12. Hence by Theorem 2.3 it holds that

$$\frac{M^H(x,z)}{M^H(y_0,z)}\le C_3(1/4)\frac{V(\delta_H(x))}{V(\delta_H(y_0))}\,.$$

Therefore, by Theorem 3.6, for $x\in H$ with $|x|<\kappa$,

$$\begin{aligned}
M^H\nu^{(3)}(x) &\le C_3(1/4)\frac{V(\delta_H(x))}{V(1/8)}\int_{\{z\in\partial H:\,|z|\ge1\}\cup\partial^m_\infty H} M^H(y_0,z)\,\nu(dz)\\
&\le C_3(1/4)\frac{\kappa^d V(\delta_H(x))}{V(1/8)}|x|^{-d}\int_{\{z\in\partial H:\,|z|\ge1\}\cup\partial^m_\infty H} M^H(y_0,z)\,\nu(dz)\\
&\le C_3(1/4)\frac{\kappa^d C_5(1)}{V(1/8)}M^H(x,0)\int_{\{z\in\partial H:\,|z|\ge1\}\cup\partial^m_\infty H} M^H(y_0,z)\,\nu(dz).
\end{aligned}$$

Now choose $\kappa<1/4$ small enough so that

$$C_3(1/4)\frac{\kappa^d C_5(1)}{V(1/8)}\int_{\{z\in\partial H:\,|z|\ge1\}\cup\partial^m_\infty H} M^H(y_0,z)\,\nu(dz)<1/10.$$

Then for $x\in H$ with $|x|<\kappa$,

$$M^H\nu^{(3)}(x)<\frac{1}{10}M^H(x,0)\,. \tag{4.2}$$

Let $0<\rho<1$, $x\in H$ with $|x|<\rho\epsilon$ and $z\in\partial H$ with $\epsilon\le|z|<1$. Then we have $2>|x-z|\ge|z|-|x|>(1-\rho)\epsilon$. Thus, by Theorem 3.6,

$$\begin{aligned}
M^H\nu^{(2)}(x) &= \int_{\{z\in\partial H:\epsilon\le|z|<1\}} M^H(x,z)\,\nu(dz)\\
&\le C_5(4)\,V(\delta_H(x))\int_{\{z\in\partial H:\epsilon\le|z|<1\}}|x-z|^{-d}(1+|z|^2)^{d/2}\,\nu(dz)\\
&\le C_5(4)\,2^{d/2}V(\delta_H(x))(1-\rho)^{-d}\epsilon^{-d}\nu(\{z\in\partial H:\epsilon\le|z|<1\})\\
&\le C_5(4)\,2^{d/2}\nu(\partial^m H)V(\delta_H(x))(1-\rho)^{-d}\rho^d|x|^{-d}\\
&\le C_5(4)^2\,2^{d/2}\nu(\partial^m H)\left(\frac{\rho}{1-\rho}\right)^d M^H(x,0)\,.
\end{aligned}$$

Now choose ρ small enough so that $C_5(4)^2\,2^{d/2}\nu(\partial^m H)\rho^d(1-\rho)^{-d}<1/10$. Then for $x\in H$ with $|x|<\rho\epsilon$,

$$M^H\nu^{(2)}(x)<\frac{1}{10}M^H(x,0)\,. \tag{4.3}$$

Consider now $M^H\nu^{(1)}(x)$ for $x\in H$ with $|x|<\rho\varepsilon\wedge\kappa$. If $\delta_H(x)>|x|/2$, then $|x|\le 2\delta_H(x)\le 2|x-z|$ for each $z\in\partial H$. Thus by Theorem 3.6,

$$\begin{aligned}
M^H(x,z) &\le C_5(1)\,V(\delta_H(x))|x-z|^{-d}(1+|z|^2)^{d/2}\\
&\le C_5(1)\,V(\delta_H(x))(|x|/2)^{-d}2^{d/2}\;\le\; C_5(1)^2\,2^{3d/2}M^H(x,0)\,,
\end{aligned}$$

for all $z\in\partial H$ with $|z|<\epsilon$. Hence, for $x\in H$ with $\delta_H(x)>|x|/2$ we have

$$\begin{aligned}
M^H\nu^{(1)}(x) &= \int_{\{z\in\partial H:|z|<\epsilon\}} M^H(x,z)\,\nu^{(1)}(dz)\\
&\le C_5(1)^2 2^{3d/2}M^H(x,0)\nu(\{z\in\partial H:|z|<\epsilon\})\;<\;\frac{1}{10}M^H(x,0)
\end{aligned}\tag{4.4}$$

by the choice of ϵ in (4.1).

It remains to study $M^H\nu^{(1)}$ on the set $E:=\{x\in H:\,\delta_H(x)<|x|/2\}\cap B(0,\rho\varepsilon\wedge\kappa)$. Note that for $x\in E$ we have $|x|<\frac{2}{\sqrt3}|\tilde x|$. For $j\ge 0$, set

$$R_j:=\{z\in\partial H:\,2^{-j-1}<|z|\le 2^{-j}\}\quad\text{ and }\quad \widetilde R_j:=\{x\in E:\,\widetilde x\in R_j\}.$$

Put $\nu_j=\nu^{(1)}|_{R_j}$. Assume that $x\in\widetilde R_j$ and $z\in R_k$ for $|j-k|>1$. If $k\ge 2+j$, $|x-z|\ge|\tilde x|-|z|\ge 2^{-j-1}-2^{-k}\ge\frac14 2^{-j}\ge\frac14|\tilde x|\ge\frac{\sqrt3}{8}|x|$. If $j\ge k+2$, $|x-z|\ge|z|-\frac{2}{\sqrt3}|\tilde x|\ge 2^{-k-1}-\frac{2}{\sqrt3}2^{-j}\ge 2^{-j+1}-\frac{2}{\sqrt3}2^{-j}\ge(2-\frac{2}{\sqrt3})|\tilde x|\ge\frac{\sqrt3}{2}(2-\frac{2}{\sqrt3})|x|=(\sqrt3-1)|x|$. Thus, if $|j-k|>1$, we have $|x-z|>\frac{\sqrt3}{8}|x|$ for every $x\in\widetilde R_j$ and $z\in R_k$, and so by Theorem 3.6, for $x\in\widetilde R_j$,

$$\begin{aligned}
M^H\nu_k(x) &= \int_{R_k} M^H(x,z)\,\nu^{(1)}(dz)\;\le\;C_5(1)\int_{R_k}V(\delta_H(x))|x-z|^{-d}(1+|z|^2)^{d/2}\,\nu^{(1)}(dz)\\
&\le C_5(1)\,(\frac{8}{\sqrt3})^d 2^{d/2}V(\delta_H(x))|x|^{-d}\nu^{(1)}(R_k)\;\le\;C_5(1)^2\,(\frac{8}{\sqrt3})^d 2^{d/2}\nu^{(1)}(R_k)M^H(x,0)\,,
\end{aligned}$$

implying

$$\sum_{|k-j|>1} M^H\nu_k(x) \le C_5(1)^2\,(\frac{8}{\sqrt{3}})^d 2^{d/2}\nu(\{|z|<\epsilon\})M^H(x,0) \le \frac{1}{10}M^H(x,0) \tag{4.5}$$

by the choice of ϵ in (4.1).

Let

$$M_j := \{x \in \widetilde{R}_j : M^H(\nu_{j-1}+\nu_j+\nu_{j+1})(x) > \frac{2}{5}M^H(x,0)\}.$$

Note that by Theorem 3.6,

$$\begin{aligned} M_j &\subset \{x \in \widetilde{R}_j : \int |x-z|^{-d}(1+|z|^2)^{d/2}(\nu_{j-1}(dz)+\nu_j(dz)+\nu_{j+1}(dz)) > \frac{2}{5C_5(1)^2}|x|^{-d}\} \\ &\subset \{x \in \widetilde{R}_j : \int |x-z|^{-d}2^{d/2}(\nu_{j-1}(dz)+\nu_j(dz)+\nu_{j+1}(dz)) > \frac{2}{5C_5(1)^2}|x|^{-d}\} \\ &= \{x \in \widetilde{R}_j : |x|^{-d} * (\nu_{j-1}+\nu_j+\nu_{j+1}) > \frac{2^{1-d/2}}{5C_5(1)^2}|x|^{-d}\}\,. \end{aligned}$$

For $x \in \widetilde{R}_j$ we have that $2^{-j-1} < |x| < \frac{2}{\sqrt{3}}2^{-j}$. Therefore,

$$M_j \subset \{|x|^{-d} * (\nu_{j-1}+\nu_j+\nu_{j+1}) > c_1^{-1}2^{jd}\}$$

where $c_1 := 2^{3d/2-1}3^{-d/2}5C_5(1)^2$. It follows from [29, Lemma 1 and Theorem] that

$$|M_j| \le c_1 2^{-jd}(\nu^{(1)}(R_{j-1})+\nu^{(1)}(R_j)+\nu^{(1)}(R_{j+1})).$$

Therefore by using again that $2^{-j-1} < |x| < \frac{2}{\sqrt{3}}2^{-j}$, we get that

$$\int_{M_j} |x|^{-d}\,dx \le c_2 2^{jd}|M_j| \le c_3(\nu^{(1)}(R_{j-1})+\nu^{(1)}(R_j)+\nu^{(1)}(R_{j+1}))$$

implying

$$\sum_j \int_{M_j} |x|^{-d}\,dx \le c_3\sum_j(\nu^{(1)}(R_{j-1})+\nu^{(1)}(R_j)+\nu^{(1)}(R_{j+1})) \le 3c_3\nu(\partial^m H)\,. \tag{4.6}$$

It follows from (4.2)–(4.6) that

$$\begin{aligned} &\int_{\{x\in H, |x|<\rho\varepsilon\wedge\kappa : M^H\nu(x)>M^H(x,0)/2\}} |x|^{-d}\,dx \\ \le\ &\int_{\{x\in H, |x|<\rho\varepsilon\wedge\kappa : M^H\nu^{(1)}(x)>3M^H(x,0)/10\}} |x|^{-d}\,dx \\ =\ &\int_{\{x\in E : M^H\nu^{(1)}(x)>3M^H(x,0)/10\}} |x|^{-d}\,dx \\ =\ &\sum_j \int_{\{x\in\tilde{R}_j : \sum_{|j-k|>1} M^H\nu_k(x)+\sum_{|j-k|\le 1} M^H\nu_k(x)>3M^H(x,0)/10\}} |x|^{-d}\,dx \\ \le\ &\sum_j \int_{M_j} |x|^{-d}\,dx \le 3c_3\nu(\partial^m H) < \infty. \end{aligned}$$

Therefore, using the trivial fact that $\int_{\{\rho\varepsilon\wedge\kappa<|x|<1\}}|x|^{-d}\,dx<\infty$, we conclude that

$$\int_A |x|^{-d}\,dx \;\le\; \sum_j \int_{M_j} |x|^{-d}\,dx + \int_{\{\rho\varepsilon\wedge\kappa<|x|<1\}} |x|^{-d}\,dx \;<\; \infty\,.$$

$\square$

Proposition 4.2 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Let $s=G^H\mu+M^H\nu$ be an excessive function with respect to X^H represented as* (3.19) *with $\nu(\{0\})=0$. Let $A:=\{x\in H:\, s(x)>M^H(x,0)\}$. Then*

$$\int_{A\cap B(0,1)} |x|^{-d}\,dx<\infty\,.$$

Proof: First note that $A\subset\{G^H\mu>M^H(\cdot,0)/2\}\cup\{M^H\nu>M^H(\cdot,0)/2\}$. By Lemma 4.1, we only need to consider $A_1:=\{G^H\mu>M^H(0,\cdot)/2\}\cap B(0,1)$.

Let $y_0=(\tilde{0},1/2)$ and $\kappa\ge 3$. By the boundary Harnack principle (Theorem 2.3) applied to $G^H(\cdot,y)$ we obtain that,

$$\int_{H\cap B(0,\kappa)^c} G^H(x,y)\,\mu(dy)\;\le\; C_3(1)\frac{V(\delta_H(x))}{V(\delta_H(y_0))}\int_{H\cap B(0,\kappa)^c} G^H(y_0,y)\mu(dy)$$

for every $\kappa\ge 3$ and $x\in B(0,1)\cap H$. Now choose κ large enough so that

$$\frac{C_3(1)}{V(\delta_H(y_0))}\int_{H\cap B(0,\kappa)^c} G^H(y_0,y)\mu(dy)<\frac{1}{4C_5(1)}, \tag{4.7}$$

where $C_5(1)$ is the constant from Theorem 3.6. Hence, for $x\in B(0,1)\cap H$, we have

$$\int_{H\cap B(0,\kappa)^c} G^H(x,y)\,\mu(dy)\;\le\;\frac{1}{4C_5(1)}V(\delta_H(x))\;\le\;\frac{1}{4C_5(1)}\frac{V(\delta_H(x))}{|x|^d}\;\le\;\frac14 M^H(x,0)\,. \tag{4.8}$$

Let $\mathcal{Q}=\{Q=(\prod_{i=1}^{d-1}(k_i2^{-j},(k_i+1)2^{-j}])\times(2^{-j},2^{-j+1}]:\,k_i\in\mathbb{Z},j\in\mathbb{N}\}$. Then $\mathcal{Q}$ is a cover of $H\cap\{x_d\le 1\}$. Clearly the cubes Q are disjoint and $d^{-1/2}\mathrm{diam}(Q)\le\delta_H(Q)\le\mathrm{diam}(Q)$. For $Q=(\prod_{i=1}^{d-1}(k_i2^{-j},(k_i+1)2^{-j}])\times(2^{-j},2^{-j+1}]\in\mathcal{Q}$, let $Q^*:=(\prod_{i=1}^{d-1}((k_i-1)2^{-j},(k_i+2)2^{-j}])\times(2^{-j-1},2^{-j+2}]$. Then $\mathcal{Q}^*=\{Q^*:\,Q\in\mathcal{Q}\}$ is a cover of $H\cap\{x_d\le 2\}$, $Q\subset Q^*\subset H$ for each $Q\in\mathcal{Q}$, and $2\delta_H(Q^*)=\delta_H(Q)$. Moreover, there exists a positive integer $K=K(d)$ such that each cube in $\mathcal{Q}^*$ intersects at most K other cubes from $\mathcal{Q}^*$. Let $\mathcal{Q}_1$ be the collection of cubes $Q\in\mathcal{Q}$ such that $H\cap B(0,1)\cap Q$ is non-empty. For $x\in H\cap B(0,1)$, let $Q(x)$ be the cube Q in $\mathcal{Q}_1$ containing x and denote by $Q^*(x)$ the corresponding Q^*.

Let $G^H\mu=s_1+s_2+s_3$ where

$$\begin{aligned}
s_1(x) &:= \int_{Q^*(x)} G^H(x,y)\,\mu(dy)\,,\\
s_2(x) &:= \int_{(H\setminus Q^*(x))\cap B(0,\kappa)} G^H(x,y)\,\mu(dy)\,,\\
s_3(x) &:= \int_{(H\setminus Q^*(x))\cap B(0,\kappa)^c} G^H(x,y)\,\mu(dy)\,,
\end{aligned}$$

where $\kappa \ge 3$ is the fixed constant in (4.7).

Define $\gamma(r) := \int_{B(0,r)} G(0,z)\,dz$, $r>0$. Since $G^H(z,y) \le G(z,y)$ we have for any $Q \in \mathcal{Q}_1$,

$$\begin{aligned}
\int_Q s_1(w)\,dw &= \int_Q \int_{Q^*} G^H(w,y)\,\mu(dy)\,dw = \int_{Q^*} \mu(dy) \int_Q G^H(w,y)\,dw \\
&\le \int_{Q^*} \mu(dy) \int_Q G(w,y)\,dw \le \int_{Q^*} \mu(dy) \int_{B(y,3\mathrm{diam}(Q))} G(w,y)\,dw \\
&\le \mu(Q^*)\gamma(3\mathrm{diam}(Q)) \le \mu(Q^*)\gamma(3\sqrt{d}\,\delta_H(Q))\,.
\end{aligned}$$

Since $|w| \le (1+\sqrt{d})\mathrm{dist}(0,Q)$ for $w\in Q$, we have by Theorem 3.6,

$$M^H(w,0) \ \ge\ C_5(1)^{-1}V(\delta_H(w))|w|^{-d} \ \ge\ C_5(1)^{-1}(1+\sqrt{d})^{-d}V(\delta_H(Q))\mathrm{dist}(0,Q)^{-d} \quad \text{for } w\in Q.$$

We need to estimate the Lebesgue measure of

$$B := \{w\in Q:\ s_1(w) > C_5(1)^{-1}(1+\sqrt{d})^{-d}V(\delta_H(Q))\,\mathrm{dist}(0,Q)^{-d}\}.$$

Since

$$\int_Q s_1(w)\,dw \ \ge\ C_5(1)^{-1}(1+\sqrt{d})^{-d}V(\delta_H(Q))\,\mathrm{dist}(0,Q)^{-d}|B|,$$

we have that

$$\begin{aligned}
|B| &\le C_5(1)(1+\sqrt{d})^{d}V(\delta_H(Q))^{-1}\,\mathrm{dist}(0,Q)^{d}\int_Q s_1(w)\,dw \\
&\le C_5(1)(1+\sqrt{d})^{d}V(\delta_H(Q))^{-1}\,\mathrm{dist}(0,Q)^{d}\mu(Q^*)\gamma(3\sqrt{d}\,\delta_H(Q)) \\
&\le C_5(1)(1+\sqrt{d})^{d}C_4(3\sqrt{d})V(\delta_H(Q))^{-1}V(3\sqrt{d}\delta_H(Q))^2\,\mathrm{dist}(0,Q)^{d}\,\mu(Q^*),
\end{aligned}$$

where in the third line we used Proposition 2.1(ii). Then

$$\begin{aligned}
&\int_{\{s_1>M^H(\cdot,0)/8\}\cap B(0,1)} |w|^{-d}\,dw \\
&= \sum_{Q\in\mathcal{Q}_1} \int_{\{s_1>M^H(\cdot,0)/8\}\cap B(0,1)\cap Q} |w|^{-d}\,dw \\
&\le \sum_{Q\in\mathcal{Q}_1} \int_{\{s_1>M^H(\cdot,0)/8\}\cap B(0,1)\cap Q} \mathrm{dist}(0,Q)^{-d}\,dw \\
&\le \sum_{Q\in\mathcal{Q}_1} \int_{\{s_1>C_5(1)^{-1}(1+\sqrt{d})^{-d}V(\delta_H(Q))\mathrm{dist}(0,Q)^{-d}/8\}\cap B(0,1)\cap Q} \mathrm{dist}(0,Q)^{-d}\,dw \\
&\le C_5(1)C_2(3\sqrt{d})(1+\sqrt{d})^{d}\sum_{Q\in\mathcal{Q}_1} V(\delta_H(Q))^{-1}V(3\sqrt{d}\delta_H(Q))^2\,\mu(Q^*) \\
&= C_5(1)C_2(3\sqrt{d})(1+\sqrt{d})^{d}\sum_{Q\in\mathcal{Q}_1} V(2\delta_H(Q^*))^{-1}V(6\sqrt{d}\delta_H(Q^*))^2\,\mu(Q^*) \\
&\le cC_5(1)C_2(2\sqrt{d})(1+\sqrt{d})^{d}\sum_{Q\in\mathcal{Q}_1} V(\delta_H(Q^*))\,\mu(Q^*)\,,
\end{aligned}$$

where the last line follows from (2.4) and (2.6). Note that $Q^* \subset B(0,4)$ for every $Q \in \mathcal{Q}_1$ and clearly $\delta_H(Q^*) \le \delta_H(y)$ for all $y \in Q^*$. Thus by first using these observations and the fact that each $Q^* \in \mathcal{Q}^*$ intersects at most K other cubes from $\mathcal{Q}^*$, and then using Theorem 2.3, we conclude that the last sum is dominated by

$$\begin{aligned}(K+1)\int_{H\cap B(0,4)} V(\delta_H(y))\,\mu(dy) &\le (K+1)\frac{C_3(4)V(2)}{G^H(w_1,y_1)}\int_{H\cap B(0,4)} G^H(w_1,y)\,\mu(dy)\\ &\le (K+1)\frac{C_3(4)V(2)}{G^H(w_1,y_1)}G^H\mu(w_1) < \infty\,,\end{aligned}$$

where $w_1 = (\widetilde{0},10)$ and $y_1 = (\widetilde{0},2)$. Therefore,

$$\int_{\{s_1 > M^H(\cdot,0)/8\}\cap B(0,1)} |w|^{-d}\,dw < \infty\,. \tag{4.9}$$

Consider now $x \in \{s_2 > M^H(\cdot,0)/4\}$. For $y \in H\setminus Q^*(x)$ it holds that $|x-y| \ge 4^{-1}\delta_H(x)$, and hence

$$s_2(x) \le \int_{\{\delta_H(x)\le 4|x-y|\}\cap B(0,\kappa)} G^H(x,y)\,\mu(dy)\,.$$

Let $\psi : H \to \partial H\setminus\{0\}$ be a Borel function such that $|y-\psi(y)| \le 2\delta_H(y)$ and define a measure μ' on $\partial H\setminus\{0\}$ by

$$\int_{\partial H\setminus\{0\}} f\,d\mu' = c_1\int_{H\cap B(0,\kappa)} (f\circ\psi)(y)V(\delta_H(y))\,\mu(dy)\,,$$

where $c_1 := C_4(1+\kappa)C_1(1+\kappa)(11)^d C_5(11(1+\kappa))$. Then by Theorem 2.3,

$$\mu'(\partial H\setminus\{0\}) = c_1\int_{H\cap B(0,\kappa)} V(\delta_H(y))\,\mu(dy) \le c_1\frac{C_3(\kappa)V(1)}{G^H(v_1,v_2)}G^H\mu(v_1) < \infty$$

where $v_1 = (\widetilde{0},3\kappa)$ and $v_2 = (\widetilde{0},1)$. Hence, μ' is positive and bounded.

We claim that

$$s_2(x) \le M^H\mu'(x) \quad \text{if } x \in \{s_2 > M^H(\cdot,0)/8\}\cap B(0,1). \tag{4.10}$$

Suppose that $y \in B(0,\kappa)\cap H$ and $x \in \{s_2 > M^H(\cdot,0)/8\}\cap B(0,1)$ with $\delta_H(x) \le 4|x-y|$. Then

$$\begin{aligned}&|x-\psi(y)| \le |x-y| + |y-\psi(y)| \le |x-y| + 2\delta_H(y)\\ &\le |x-y| + 2\delta_H(x) + 2|x-y| \le 11|x-y| \le 11(1+\kappa).\end{aligned}$$

Thus by Theorem 3.6,

$$\begin{aligned}M^H(x,\psi(y)) &\ge C_5(11(1+\kappa))^{-1}V(\delta_H(x))(1+|\psi(y)|^2)^{d/2}|x-\psi(y)|^{-d}\\ &\ge \frac{1}{(11)^d C_5(11(1+\kappa))}V(\delta_H(x))|x-y|^{-d}\,.\end{aligned}$$

Note that by Theorem 2.4 and Proposition 2.1(i),

$$G^H(x,y) \le C_4(1+\kappa)\frac{V(\delta_H(x))V(\delta_H(y))}{V(|x-y|)^2}G(x,y) \le C_4(1+\kappa)C_1(1+\kappa)V(\delta_H(x))V(\delta_H(y))|x-y|^{-d}.$$

Hence we get

$$G^H(x,y) \le c_1 V(\delta_H(y))M^H(x,\psi(y)).$$

Therefore, for $x \in \{s_2 > M^H(\cdot,0)/8\} \cap B(0,1)$,

$$\begin{aligned} s_2(x) &\le \int_{\{\delta_H(x)\le 2|x-y|\}\cap B(0,\kappa)} G^H(x,y)\,\mu(dy) \\ &\le c_1 \int_{\{\delta_H(x)\le 2|x-y|\}\cap B(0,\kappa)} V(\delta_H(y))M^H(x,\psi(y))\,\mu(dy) \\ &\le c_1 \int_{H\cap B(0,\kappa)} V(\delta_H(y))M^H(x,\psi(y))\,\mu(dy) \\ &= \int M^H(x,z)\,\mu'(dz) = M^H\mu'(x). \end{aligned}$$

Hence, we have proved (4.10), and so, by Lemma 4.1,

$$\int_{\{s_2>M^H(\cdot,0)/8\}\cap B(0,1)} |x|^{-d}\,dx \le \int_{\{M^H\mu'>M^H(\cdot,0)/8\}\cap B(0,1)} |x|^{-d}\,dx < \infty. \tag{4.11}$$

Together with (4.8), (4.9) and (4.11) we obtain that

$$\int_{A_1} |x|^{-d}\,dx \le \int_{\{s_1(x)+s_2(x)>M^H(\cdot,0)/4\}\cap B(0,1)} |x|^{-d}dx \le \sum_{i=1}^{2}\int_{\{s_i>M^H(\cdot,0)/8\}\cap B(0,1)} |x|^{-d}dx < \infty.$$

$\square$

Theorem 4.3 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Let A be a Borel subset of H and assume that* (1.1) *holds. Then A is not minimally thin in H with respect to X at $z=0$.*

Proof: Assume that A is minimally thin in H with respect to X at 0. Then, using Proposition 3.14 and multiplying s by a constant if necessary, there exists an excessive function $s = G^H\mu + M^H\nu$ with $\nu(\{0\}) = 0$ such that

$$\liminf_{x\to 0, x\in A}\frac{s(x)}{M^H(x,0)} = 2.$$

Let $B := \{x \in H : s(x) > M^H(x,0)\}$. Using the lower semi-continuity of s, we get that B is an open set. Thus there exists $\epsilon > 0$ such that $A \cap B(0,\epsilon) \subset B$. By Proposition 4.2,

$$\int_{A\cap B(0,\epsilon)} |x|^{-d}\,dx \le \int_B |x|^{-d}\,dx < \infty.$$

Since $\int_{\{\epsilon<|x|<1\}} |x|^{-d}\,dx < \infty$, we have proved the theorem. $\square$

Theorem 4.4 *Suppose that X is a subordinate Brownian motion satisfying* **(H)**. *Let $f:\mathbb{R}^{d-1}\to[0,\infty)$ be a Lipschitz function with Lipschitz constant $a>0$. The set $A:=\{x=(\widetilde{x},x_d)\in H: 0<x_d\le f(\widetilde{x})\}$ is minimally thin in H with respect to X at $z=0$ if and only if (1.2) holds.*

Proof: One direction follows by the same argument as the one in [17, Lemma 1]. In fact, assume that the integral in (1.2) diverges. For $x\in A=\{x=(\widetilde{x},x_d)\in H: 0<x_d\le f(\widetilde{x})\}$ we have

$$|x| \,=\, (|\widetilde{x}|^2+x_d^2)^{1/2} \,\le\, (|\widetilde{x}|^2+f(\widetilde{x})^2)^{1/2} \,\le\, (|\widetilde{x}|^2+a^2|\widetilde{x}|^2)^{1/2} \,=\, |\widetilde{x}|(1+a^2)^{1/2}\,.$$

Hence,

$$\int_{A\cap B(0,1)}\frac{1}{|x|^d}\,dx \ge \int_{\{|\widetilde{x}|<1\}}\int_0^{f(\widetilde{x})}\frac{1}{|\widetilde{x}|^d(1+a^2)^{1/2}}\,dx = (1+a^2)^{-1/2}\int_{\{|\widetilde{x}|<1\}} f(\widetilde{x})|\widetilde{x}|^{-d}\,d\widetilde{x}=\infty\,.$$

By Theorem 1.1, A is not minimally thin in H with respect to X at 0.

The proof of the other direction is modeled after the proof of [1, Theorem 9.7.1]. Suppose that (1.2) holds true. Then

$$V(\delta_H(x))\int_{\{|\widetilde{z}|<1\}}|x-\widetilde{z}|^{-d}|\widetilde{z}|^{-d}f(\widetilde{z})\,d\widetilde{z} = V(\delta_H(x))\int_{\partial H\cap B(0,1)}|x-z|^{-d}|z|^{-d}f(z)\,d\widetilde{\lambda}(z)<\infty$$

where $\widetilde{\lambda}$ denotes the $d-1$-dimensional Lebesgue measure. Define $s:H\to[0,\infty)$ by

$$s(x)=\int_{\partial H\cap B(0,1)}M^H(x,z)(1+|z|^2)^{-d/2}|z|^{-d}f(z)\,d\widetilde{\lambda}(z)\,. \tag{4.12}$$

Then by Theorem 3.6,

$$s(x)\,\le\, C_5(4)\,V(\delta_H(x))\int_{\partial H\cap B(0,1)}|x-z|^{-d}|z|^{-d}f(z)\,d\widetilde{\lambda}(z) \quad \text{for every } x\in B(0,1)\cap H.$$

Therefore, s is well-defined (finite), and is harmonic with the representing measure ν on ∂H having the density $1_{B(0,1)}(z)(1+|z|^2)^{-d/2}|z|^{-d}f(z)$. Notice that $\nu(\{0\})=0$. Further, $f(0)=0$ (otherwise the integral in (1.2) would diverge). Let $x\in A\cap B(0,1/2)$. Then $x_d\le f(\widetilde{x},0)\le a|\widetilde{x}|$, and so $x_d/(2a)\le|\widetilde{x}|/2\le 1/4$. Further, for $z\in B((\widetilde{x},0),\frac{x_d}{2a})\cap\partial H$ we have

$$\begin{aligned}
|z| \,&\le\, |z-\widetilde{x}|+|\widetilde{x}| \,\le\, \frac{x_d}{2a}+|\widetilde{x}| \,\le\, \frac{|\widetilde{x}|}{2}+|\widetilde{x}| \,=\, \frac{3}{2}|\widetilde{x}|<1\,,\\
|x-z| \,&=\, (|\widetilde{x}-z|^2+x_d^2)^{1/2} \,\le\, \left(\left(\frac{x_d}{2a}\right)^2+x_d^2\right)^{1/2} \,=\, x_d(1+(2a)^{-2})^{1/2}\,,\\
f(z) \,&>\, f(\widetilde{x},0)-\frac{x_d}{2} \,\ge\, x_d-\frac{x_d}{2} \,=\, \frac{x_d}{2}\,.
\end{aligned}$$

Therefore, with ω_{d-1} denoting the volume of the $(d-1)$-dimensional unit ball and using Theorem 3.6, we get

$$\begin{aligned}
s(x) &\geq C_5(4)^{-1}\, V(\delta_H(x)) \int_{B((\widetilde{x},0),\frac{x_d}{2a})\cap\partial H} |x-z|^{-d}|z|^{-d} f(z)\, d\lambda'(z)\\
&\geq C_5(4)^{-1}\, V(\delta_H(x)) \frac{1}{x_d^d(1+(2a)^{-2})^{d/2}} \frac{1}{\left(\frac{3}{2}|\widetilde{x}|\right)^d} \frac{x_d}{2}\omega_{d-1}\left(\frac{x_d}{2a}\right)^{d-1}\\
&\geq c_1 C_5(4)^{-1}\omega_{d-1}\, V(\delta_H(x)) x_d^{-d}|\widetilde{x}|^{-d} x_d x_d^{d-1}\\
&\geq c_1 C_5(4)^{-1}\omega_{d-1}\, \frac{V(\delta_H(x))}{|x|^d} \geq c_1 C_5(4)^{-2}\omega_{d-1} M^H(x,0)\,,
\end{aligned}$$

where c_1 depends on d and the Lipschitz constant a. This proves that for every $x \in A\cap B(0,1/2)$ it holds that

$$\frac{s(x)}{M^H(x,0)} \geq c_1 C_5(4)^{-2}\omega_{d-1} > \nu(\{0\})\,.$$

Hence, by Proposition 3.14, A is minimally thin in H with respect to X at 0. □

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