Accessibility, Martin boundary and minimal thinness for Feller processes in metric measure spaces

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Abstract

In this paper we study the Martin boundary at infinity for a large class of purely discontinuous Feller processes in metric measure spaces. We show that if ∞ is accessible from an open set D, then there is only one Martin boundary point of D associated with it, and this point is minimal. We also prove the analogous result for finite boundary points. As a consequence, we show that minimal thinness of a set is a local property.

AMS 2010 Mathematics Subject Classification: Primary 60J50, 31C40; Secondary 31C35, 60J45, 60J75.

Keywords and phrases: Martin boundary, Martin kernel, purely discontinuous Feller process, minimal thinness

1 Introduction and setup

The Martin kernel and Martin boundary of an open set with respect to a transient strong Markov process were introduced in [28] with the goal of representing non-negative harmonic functions (with respect to the underlying process) as an integral of the Martin kernel against a finite measure on the (minimal) Martin boundary. The identification of the Martin boundary for purely discontinuous Markov processes began in late nineties when it was shown in [4, 10] that for the isotropic α-stable process the Martin boundary of a bounded Lipschitz domain coincides with its Euclidean boundary. Soon after, the result was extended in [30] to the so-called κ-fat open sets. These results were subsequently extended in two directions: to more general processes and to general open sets.

In the first direction, the Martin boundary of bounded κ-fat open sets was studied in [19] for a class of subordinate Brownian motions and then in [24] for some symmetric Lévy processes. In both papers the Martin boundary was identified with the Euclidean boundary. In fact, the latter paper gives a local result: if an open set $D \subset \mathbb{R}^d$ is κ-fat at $z_0 \in \partial D$, then there is exactly one (minimal) Martin boundary point associated to z_0. A related result is

[*]This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. NRF-2015R1A4A1041675)

[†]Research supported in part by a grant from the Simons Foundation (208236)

[‡]Research supported in part by the Croatian Science Foundation under the project 3526

the identification of the Martin boundary at infinity of an unbounded open set with a single point provided the set is κ-fat at infinity, see [23]. In all of these papers an appropriate boundary Harnack principle for non-negative harmonic functions played a major role.

In the second direction, the boundary Harnack principle and the Martin kernel for arbitrary open sets were studied in [6] for isotropic α-stable processes. The authors of [6] introduced the concepts of accessible and inaccessible boundary points and proved a result that leads to the identification of the finite Martin boundary of an arbitrary bounded open with its Euclidean boundary. It was also proved in [6] that a finite Martin boundary point is minimal if and only if the corresponding Euclidean boundary point is accessible. By use of the Kelvin transform, they were able to identify the infinite part of the Martin boundary as well.

The main goal of this paper is to generalize results of [6, 23] to more general processes. Inspired by the paper [7] we will work with a class of purely discontinuous Feller processes in duality in a measure metric space $\mathfrak{X}$. The jumps of these processes are assumed to be quite regular as precisely described in Assumptions **C**, **C1** and **C2** below. Most of Lévy processes fall into our framework, see Section 4 for details. Our main results can be roughly stated as follows: let D be an open set in $\mathfrak{X}$. If $z_0 \in \partial D$ (the boundary of D in the original topology of $\mathfrak{X}$) is accessible, then there is exactly one Martin boundary point associated with z_0. In case $D \subset \mathfrak{X}$ is bounded and all its boundary points are accessible, the Martin boundary and the minimal Martin boundary of D are identified with ∂D. In case of unbounded open set such that infinity is accessible, we identify the Martin boundary at infinity with a single point.

Another goal of this paper is to show that minimal thinness is a local property. We will use our results on the Martin boundary to show that under certain geometric assumptions, if $E \subset D \subset \mathfrak{X}$ are open sets with a common boundary point z_0 which is accessible from both E and D, then $F \subset E$ is minimally thin at z_0 in E if and only if F is minimally thin at z_0 in D.

We now provide a precise description of the process and the assumptions it satisfies, introduce all necessary notation, state the results and explain the methods of proofs.

Let $(\mathfrak{X}, d, m)$ be a metric measure space with a countable base such that all bounded closed sets are compact and the measure m has full support. For $x \in \mathfrak{X}$ and $r > 0$, let $B(x, r)$ denote the ball centered at x with radius r. Let $R_0 \in (0, \infty]$ be the localization radius such that $\mathfrak{X} \setminus B(x, 2r) \neq \emptyset$ for all $x \in \mathfrak{X}$ and all $r < R_0$.

Let $X = (X_t, \mathcal{F}_t, \mathbb{P}_x)$ be a Hunt process on $\mathfrak{X}$. We will assume the following

Assumption A: X is a Hunt process admitting a strong dual process $\widehat{X}$ with respect to the measure m and $\widehat{X}$ is also a Hunt process. The transition semigroups (P_t) and $(\widehat{P}_t)$ of X and $\widehat{X}$ are both Feller and strongly Feller. Every semi-polar set of X is polar.

For the definition of Hunt processes see p. 45 of [3], and for the definition of a strong dual, see Definition VI.(1.2) on p.225 of [3]. For the definition of Feller processes see pp. 49–50 of [12], and for the definition of strong Feller processes see p. 129 of [12]. For the definitions of polar and semi-polar sets, see Definition II.(3.1) on p.79 of [3].

In the sequel, all objects related to the dual process $\widehat{X}$ will be denoted by a hat. Recall that a set is polar (semi-polar, respectively) for X if and only if it is polar (semi-polar, respectively) for $\widehat{X}$ (see VI. (1.19) in [3]). Under assumption **A** the process X admits a (possibly infinite) Green function $G(x,y)$ serving as a density of the occupation measure: $G(x,A) := \mathbb{E}_x \int_0^\infty \mathbf{1}_{(X_t \in A)} dt = \int_A G(x,y) m(dy)$. Moreover, $G(x,y) = \widehat{G}(y,x)$ for all $x, y \in \mathfrak{X}$, cf. VI.1 in [3] for details. Further, let D be an open subset of $\mathfrak{X}$ and $\tau_D = \inf\{t > 0 : X_t \notin D\}$ the exit time from D. The killed process X^D is defined by $X_t^D = X_t$ if $t < \tau_D$ and $X_t^D = \partial$ where ∂ is an extra point added to $\mathfrak{X}$. The killed process $\widehat{X}^D$ is defined analogously. By Hunt's switching identity (Theorem 1.16 in [3]), it holds that $\mathbb{E}_x[G(X_{\tau_D}, y)] = \widehat{\mathbb{E}}_y[\widehat{G}(\widehat{X}_{\widehat{\tau}_D}, x)]$ for all $x, y \in \mathfrak{X}$ which implies that X^D and $\widehat{X}^D$ are in duality, see p.43 in [13]. Again by VI.1 in [3], X^D admits a unique (possibly infinite) Green function (potential kernel) $G_D(x,y)$ such that for every non-negative Borel function f,

$$G_D f(x) := \mathbb{E}_x \int_0^{\tau_D} f(X_t) dt = \int_D G_D(x,y) f(y)\, m(dy)\,,$$

and $G_D(x,y) = \widehat{G}_D(y,x)$, $x, y \in D$, with $\widehat{G}_D(y,x)$ the Green function of $\widehat{X}^D$. It is assumed throughout the paper that $G_D(x,y) = 0$ for $(x,y) \in (D \times D)^c$. We also note that the killed process X^D is strongly Feller, see e.g. the first part of the proof of Theorem on pp. 68–69 in [11]. From now on, we will always assume that D is Greenian, that is, the Green function $G_D(x,y)$ is finite for all $x, y \in D$, $x \neq y$. Under this assumption, the killed process X^D is transient in the sense that there exists a non-negative Borel function f on D such that $0 < G_D f < \infty$ (and the same is true for $\widehat{X}$).

Recall that $z \in \partial D$ is said to be regular with respect to X if $\mathbb{P}_z(\tau_D = 0) = 1$ and irregular otherwise. We will denote the set of regular (respectively irregular) points of ∂D with respect to X by D^{reg} (respectively D^{irr}). $\widehat{D}^{\mathrm{reg}}$ (respectively $\widehat{D}^{\mathrm{irr}}$) stands for the sets of regular (respectively irregular) points of ∂D with respect to $\widehat{X}$ respectively. It is well known that D^{irr} and $\widehat{D}^{\mathrm{irr}}$ are semipolar, hence polar under **A**.

The process X, being a Hunt process, admits a Lévy system (J, H) where $J(x, dy)$ is a kernel on the state space $\mathfrak{X}$ (called the Lévy kernel of $\mathfrak{X}$), and $H = (H_t)_{t \geq 0}$ is a positive continuous additive functional of X. We assume that $H_t = t$ so that for every function $f : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ vanishing on the diagonal and every stopping time T,

$$\mathbb{E}_x \sum_{0 < s \leq T} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^T f(X_s, y) J(X_s, dy) ds\,.$$

By using τ_D in the displayed formula above and taking $f(x,y) = \mathbf{1}_D(x)\mathbf{1}_A(y)$ with $A \subset \overline{D}^c$, we get that

$$\mathbb{P}_x(X_{\tau_D} \in A, \tau_D < \zeta) = \mathbb{E}_x \int_0^{\tau_D} J(X_s, A) ds = \int_D G_D(x,y) J(y, A) m(dy)\,, \tag{1.1}$$

where ζ is the life time of X. Similar formulae hold for the dual process $\widehat{X}$ and $\widehat{J}(x, dy) m(dx) = J(y, dx) m(dy)$.

Assumption C: The Lévy kernels of X and $\widehat{X}$ are of the form $J(x,dy) = j(x,y)m(dy)$, $\widehat{J}(x,dy) = \widehat{j}(x,y)m(dy)$, where $j(x,y) = \widehat{j}(y,x) > 0$ for all $x, y \in \mathfrak{X}$, $x \neq y$.

The next two related assumptions control the decay of the density j.

Assumption C1(z_0, R): Let $z_0 \in \mathfrak{X}$ and $R \leq R_0$. For all $0 < r_1 < r_2 < R$, there exists a constant $c = c(z_0, r_2/r_1) > 0$ such that for all $x \in B(z_0, r_1)$ and all $y \in \mathfrak{X} \setminus B(z_0, r_2)$,

$$c^{-1} j(z_0, y) \leq j(x,y) \leq c j(z_0, y), \qquad c^{-1}\widehat{j}(z_0, y) \leq \widehat{j}(x,y) \leq c\widehat{j}(z_0, y).$$

In the next assumption we require that the localization radius $R_0 = \infty$.

Assumption C2(z_0, R): Let $z_0 \in \mathfrak{X}$ and $R > 0$. For all $R \leq r_1 < r_2$, there exists a constant $c = c(z_0, r_2/r_1) > 0$ such that for all $x \in B(z_0, r_1)$ and all $y \in \mathfrak{X} \setminus B(z_0, r_2)$,

$$c^{-1} j(z_0, y) \leq j(x,y) \leq c j(z_0, y), \qquad c^{-1}\widehat{j}(z_0, y) \leq \widehat{j}(x,y) \leq c\widehat{j}(z_0, y).$$

We *define* the Poisson kernel of an open set $D \in \mathfrak{X}$ by

$$P_D(x,z) = \int_D G_D(x,y) j(y,z) m(dy), \qquad x \in D, z \in D^c. \tag{1.2}$$

By (1.1), we see that $P_D(x,\cdot)$ is the density of the exit distribution of X from D restricted to $\overline{D}^c$:

$$\mathbb{P}_x(X_{\tau_D} \in A, \tau_D < \zeta) = \int_A P_D(x,z) m(dz), \qquad A \subset \overline{D}^c.$$

Recall that $f : \mathfrak{X} \to [0, \infty)$ is regular harmonic in D with respect to X if

$$f(x) = \mathbb{E}_x[f(X_{\tau_D}), \tau_D < \zeta], \quad \text{for all } x \in D,$$

and it is harmonic in D with respect to X if for every relatively compact open $U \subset \overline{U} \subset D$

$$f(x) = \mathbb{E}_x[f(X_{\tau_U}), \tau_U < \zeta], \quad \text{for all } x \in U.$$

Throughout the paper we will adopt the convention that $X_\zeta = \partial$ and $f(\partial) = 0$ for every function f. Thus we will drop $\tau_D < \zeta$ in expressions similar to the right-hand side in the penultimate display. A function $f : \mathfrak{X} \to [0, \infty)$ harmonic in D with respect to X^D if for every relatively compact open $U \subset \overline{U} \subset D$

$$f(x) = \mathbb{E}_x[f(X^D_{\tau_U})], \quad \text{for all } x \in U.$$

It follows from the Hunt switching formula that for every $y \in D$ and any open neighborhood U of y, $G_D(\cdot, y)$ is regular harmonic in $D \setminus \overline{U}$. In particular, $G_D(\cdot, y)$ is harmonic in $D \setminus \{y\}$.

The next pair of assumptions is about an approximate factorization of harmonic functions. This approximate factorization is a crucial tool in proving the oscillation reduction. The first one is an approximate factorization of harmonic functions at a finite boundary point.

Assumption F1(z_0, R): Let $z_0 \in \mathfrak{X}$ and $R \leq R_0$. For any $\frac{1}{2} < a < 1$, there exists $C(a) = C(z_0, R, a) \geq 1$ such that for every $r \in (0, R)$, every open set $D \subset B(z_0, r)$, every

non-negative function f on $\mathfrak{X}$ which is regular harmonic in D with respect to X and vanishes in $B(z_0,r)\cap(\overline{D}^c\cup D^{\rm reg})$, and all $x\in D\cap B(z_0,r/8)$,

$$C(a)^{-1}\mathbb{E}_x[\tau_D]\int_{\overline{B}(z_0,ar/2)^c} j(z_0,y)f(y)m(dy)$$
$$\le f(x)\le C(a)\mathbb{E}_x[\tau_D]\int_{\overline{B}(z_0,ar/2)^c} j(z_0,y)f(y)m(dy). \tag{1.3}$$

In the second assumption we require that the localization radius $R_0=\infty$.

Assumption F2(z_0,R): Let $z_0\in\mathfrak{X}$ and $R>0$. For any $1<a<2$, there exists $C(a)=C(z_0,R,a)\ge 1$ such that for every $r\ge R$, every unbounded open set $D\subset\overline{B}(z_0,r)^c$, every non-negative function f on $\mathfrak{X}$ which is regular harmonic in D with respect to X and vanishes on $\overline{B}(z_0,r)^c\cap(\overline{D}^c\cup D^{\rm reg})$, and all $x\in D\cap\overline{B}(z_0,8r)^c$,

$$C(a)^{-1}\,P_D(x,z_0)\int_{B(z_0,2ar)} f(z)m(dz)$$
$$\le f(x)\le C(a)\,P_D(x,z_0)\int_{B(z_0,2ar)} f(z)m(dz). \tag{1.4}$$

The approximate factorization of harmonic functions stated in **F1** and **F2** can be proved under somewhat stronger assumptions than the Assumptions **A**, **B**, **C** and **D** in [7]. This is done in the companion paper [26].

Let $D\subset\mathfrak{X}$ be an open set. A point $z\in\partial D$ is called accessible from D with respect to X if

$$P_D(x,z)=\int_D G_D(x,w)j(w,z)m(dw)=\infty \quad \text{for all } x\in D\,, \tag{1.5}$$

and inaccessible otherwise.

In case D is unbounded we say that ∞ is accessible from D with respect to X if

$$\mathbb{E}_x\tau_D=\int_D G_D(x,w)m(dw)=\infty \quad \text{for all } x\in D \tag{1.6}$$

and inaccessible otherwise. The concepts of accessible and inaccessible points were introduced in [7].

For $D\subset\mathfrak{X}$, let $\partial_M D$ denote the Martin boundary of D with respect to X^D in the sense of Kunita-Watanabe [28], see Section 3 for more details. A point $w\in\partial_M D$ is said to be minimal if the Martin kernel $M_D(\cdot,w)$ is a minimal harmonic function with respect to X^D. We will use $\partial_m D$ to denote the minimal Martin boundary of D with respect to X^D. A point $w\in\partial_M D$ is said to be a *finite Martin boundary point* if there exists a bounded (with respect to the metric d) sequence $(y_n)_{n\ge1}\subset D$ converging to w in the Martin topology. A point $w\in\partial_M D$ is said to be an *infinite Martin boundary point* if there exists an unbounded (with respect to the metric d) sequence $(y_n)_{n\ge1}\subset D$ converging to w in the Martin topology. We note that these two definitions do not rule out the possibility that a point $w\in\partial_M D$ is at

the same time finite and infinite Martin boundary point. We will show in Corollary 1.4(a) that under appropriate and natural assumptions this cannot happen. A point $w \in \partial_M D$ is said to be associated with $z_0 \in \partial D$ if there is a sequence $(y_n)_{n\geq 1} \subset D$ converging to w in the Martin topology and to z_0 in the topology of $\mathfrak{X}$. The set of Martin boundary points associated with z_0 is denoted by $\partial_M^{z_0} D$. A point $w \in \partial_M D$ is said to be associated with ∞ if w is an infinite Martin boundary point. The set of Martin boundary points associated with ∞ is denoted by $\partial_M^{\infty} D$. $\partial_M^f D$ and $\partial_m^f D$ will be used to denote the finite part of the Martin boundary and minimal boundary respectively. Note that $\partial_M^{\infty} D$ is the set of infinite Martin boundary points.

Now we can state the first main result of the paper. We will always assume that Assumptions **A** and **C** hold true.

Theorem 1.1 *Let $D \subset \mathfrak{X}$ be an open set. (a) Suppose that $z_0 \in \partial D$. Assume that there exists $R \leq R_0$ such that **C1**(z_0, R) holds, and that $\widehat{X}$ satisfies **F1**(z_0, R). If z_0 is accessible from D with respect to X, then there is only one Martin boundary point associated with z_0.*
(b) Suppose that, in addition to the assumptions in (a), for all $r \in (0, R]$,

$$\sup_{x\in D\cap B(z_0,r/2)} \sup_{y\in \mathfrak{X}\setminus B(z_0,r)} \max(G_D(x,y), \widehat{G}_D(x,y)) =: c(r) < \infty\,. \tag{1.7}$$

Then the Martin kernel $M_D(\cdot, z_0)$ is harmonic with respect to X^D.
*(c) Suppose, in addition, that X satisfies **F1**(z_0, R), that*

$$\lim_{D\ni x\to z} G_D(x,y) = 0 \quad \text{for every } z \in D^{\mathrm{reg}} \text{ and every } y \in D, \tag{1.8}$$

and that, if D is unbounded then for $r \in (0, R]$,

$$\lim_{x\to\infty} G_D(x,y) = 0 \quad \text{for all } y \in D \cap B(z_0, r). \tag{1.9}$$

Then the corresponding Martin boundary point is minimal.

Corollary 1.2 *Suppose that every point $z_0 \in \partial D$ is accessible from D with respect to X, and that the assumptions of Theorem 1.1(c) are satisfied for all $z_0 \in \partial D$ (with $c(r)$ in (1.7) independent of z_0).*
(a) The finite part of the Martin boundary $\partial_M D$ and the minimal Martin boundary $\partial_m D$ can be identified with ∂D.
(b) If D is bounded, then ∂D and $\partial_M D$ are homeomorphic.
(c) Let D be bounded. For any non-negative function u which is harmonic with respect to X^D, there exists a unique finite measure μ on ∂D such that

$$u(x) = \int_{\partial D} M_D(x,z)\mu(dz), \qquad x \in D.$$

Theorem 1.3 *(a) Suppose that $R_0 = \infty$, D is an unbounded open subset of $\mathfrak{X}$, and ∞ is accessible from D with respect to X. If there is a point $z_0 \in \mathfrak{X}$ and $R > 0$ such that $\boldsymbol{C2}(z_0, R)$ holds, and $\widehat{X}$ satisfies $\boldsymbol{F2}(z_0, R)$, then there is only one Martin boundary point associated with ∞.*
(b) Suppose that, in addition to the assumptions in (a), for all $r \geq R$

$$\sup_{x \in D \cap B(z_0, r/2)} \sup_{y \in \mathfrak{X} \setminus B(z_0, r)} \max(G_D(x, y), \widehat{G}_D(x, y)) =: c(r) < \infty \,. \tag{1.10}$$

Then the Martin kernel associated with ∞ is harmonic with respect to X^D.
(c) Suppose, in addition, that X satisfies $\boldsymbol{F2}(z_0, R)$, that (1.8) holds, and that

$$\lim_{x \to \infty} G_D(x, y) = 0 \quad \text{for all } y \in D. \tag{1.11}$$

Then the Martin boundary point associated with ∞ is minimal.

Corollary 1.4 *Let $R_0 = \infty$ and $D \subset \mathfrak{X}$ be unbounded. Suppose that every point $z_0 \in \partial D$ is accessible from D with respect to X, that ∞ is accessible from D with respect to X, that the assumptions of Theorem 1.1(c) are satisfied for all $z \in \partial D$ (with $c(r)$ in (1.7) independent of z) and that the assumptions of Theorem 1.3(c) are satisfied. Then*
(a) $\partial_M^f D \cap \partial_M^\infty D = \emptyset$.
(b) The Martin boundary $\partial_M D$ is homeomorphic with the one-point compactification of ∂D.
(c) For any non-negative function u which is harmonic with respect to X^D, there exists a unique finite measure μ on ∂D and $\mu_\infty \geq 0$ such that

$$u(x) = \int_{\partial D} M_D(x, z)\mu(dz) + M_D(x, \infty)\mu_\infty \,, \qquad x \in D,$$

where $M_D(\cdot, \infty)$ denotes the Martin kernel associated with ∞.

The preliminary version of the results of this paper (and the forthcoming paper [27]) was presented at the 11th Workshop on Markov Processes and Related Topics held in Shanghai Jiaotong University from June 27 to June 30 2015, and at the International Conference on Stochastic Analysis and Related Topics held in Wuhan University from August 3 to August 8 2015. In the recent preprint [15], Juszczyszyn and Kwaśnicki independently considered similar problems as those in Corollary 1.2 for bounded D. Our main motivation for the current paper was to investigate the Martin boundary at infinity. The investigation starts with the result stating that there is only one Martin boundary point associated with ∞ which should be understood as a local result about the Martin boundary in the sense that no other information about the remaining part of the boundary is required. This motivated our approach in studying the finite part of Martin boundary through the local approach – if $z_0 \in \partial D$ is accessible, then there is only one Martin boundary point associated to z_0. Again, no other information about the remaining part of boundary is used. We will first present

proofs for infinity. For readers' convenience, even though the structure of proofs is similar, we also provide the proofs for finite boundary points.

The case of inaccessible boundary points will be discussed in the forthcoming paper [27], the main reason being that the treatment of inaccessible points requires additional assumptions on $j(x,y)$ – see **E1** and **E2** in [27], and Theorem 3.1(ii) in [15].

Organization of the paper: In the next section we study the oscillation reduction at an accessible boundary point, first for the infinite point in Proposition 2.5, and then for a finite boundary point in Proposition 2.11. One of the main tools for this, borrowed from [6], is a decomposition of a regular harmonic function into two parts depending on where the process exits the open set. An estimate of one of the parts by the other is derived as a consequence of **F2**, respectively **F1**, cf. Lemma 2.1 and Lemma 2.7. The oscillation reduction result immediately leads to the existence of limits of ratios of non-negative harmonic functions which implies that the Martin kernel is the limit of ratios of Green functions. This is the key to associating a point on the topological boundary of D with a point on the Martin boundary. The third section is devoted to the study of the Martin kernel at infinity under the assumption that infinity is accessible from D and then of the Martin kernel at a finite accessible point of an open set D. We first prove that the Martin kernel is harmonic, and then that it is minimal, thus showing that a minimal Martin boundary point is associated with an accessible boundary point. In Section 4 we first briefly discuss examples satisfying our assumptions and then look at the case of a class of symmetric Lévy processes in detail. In the last section we look at minimal thinness at a minimal Martin boundary point of D. It is intuitively clear that minimal thinness of a set $F \subset D$ should be a local property depending only on the size of F near the boundary point. This suggests that if $F \subset E$, E open in D, and E and D have a common boundary point, then F should be minimally thin at that boundary point in E if and only if it is minimally thin in D. Clearly, the problem is that Martin spaces for E and D are different and one needs some sort of identification of the underlying boundary points. This is provided by Theorems 1.1 and 1.3. The second main ingredient in showing local character of minimal thinness is given in Proposition 5.4 where the Martin kernel with respect to E is given in terms of the Martin kernel with respect to D.

Notation: We will use the following conventions in this paper: $c, c_0, c_1, c_2, \dots$ stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. The labeling of the constants $c_0, c_1, c_2, \dots$ starts anew in the statement of each result. We will use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$. Notation $f \asymp g$ means that the quotient $f(t)/g(t)$ stays bounded between two positive numbers on their common domain of definition. For $x \in \mathfrak{X}$ and $r > 0$ we denote by $B(x,r)$ be the open ball centered at x with radius r and by $\overline{B}(x,r)$ the closure of $B(x,r)$.

2 Oscillation reduction under accessibility assumption

It follows easily from the strong Markov property that for all Greenian open sets U and D with $U \subset D$, $G_D(x,y) = G_U(x,y) + \mathbb{E}_x\left[G_D(X_{\tau_U}, y); \tau_U < \infty\right]$ for every $(x,y) \in \mathfrak{X} \times \mathfrak{X}$.

2.1 Infinity

In this subsection, we deal with the oscillation reduction at infinity. Throughout this subsection we will assume that $R_0 = \infty$ and that there exists a point $z_0 \in \mathfrak{X}$ such that $\mathbf{C2}(z_0, R)$ holds, and that $\widehat{X}$ satisfies $\mathbf{F2}(z_0, R)$ for some $R > 0$. We will fix z_0 and R and use the notation $B_r = B(z_0, r)$.

An immediate consequence of $\mathbf{F2}(z_0, R)$ for $\widehat{X}$ is the boundary Harnack principle at infinity in [26]: There exists $c > 1$ such that for any $r \geq R$, any open set $D \subset \overline{B}_r^c$ and any non-negative functions u and v on $\mathfrak{X}$ that are regular harmonic in D with respect to $\widehat{X}$ and vanish on $\overline{B}_r^c \cap (\overline{D}^c \cup \widehat{D}^{\rm reg})$, it holds that

$$c^{-1}\frac{u(y)}{v(y)} \leq \frac{u(x)}{v(x)} \leq c\frac{u(y)}{v(y)} \qquad \text{for all } x, y \in D \cap \overline{B}_{8r}^c. \tag{2.1}$$

Note that we can take $c = (C(3/2))^2$. By enlarging $C(3/2)$ in $\mathbf{F2}(z_0, R)$, without loss of generality we assume the above c is equal to $C(3/2)$.

For an open set D and $p > q > 0$, let $D^p = D \cap \overline{B}_p^c$ and $D^{p,q} = D^q \setminus D^p$. For $p > q > 1$, $r \geq R$ and non-negative function f on $\mathfrak{X}$ define

$$\begin{aligned} f^{pr,qr}(x) &= \mathbb{E}_x\left[f(\widehat{X}_{\widehat{\tau}_{D^{pr}}}) : \widehat{X}_{\widehat{\tau}_{D^{pr}}} \in D^{pr,qr}\right], \\ \widetilde{f}^{pr,qr}(x) &= \mathbb{E}_x\left[f(\widehat{X}_{\widehat{\tau}_{D^{pr}}}) : \widehat{X}_{\widehat{\tau}_{D^{pr}}} \in (D \setminus D^{qr}) \cup \overline{B}_r\right]. \end{aligned}$$

Lemma 2.1 *Suppose that $r \geq R$, $D \subset \overline{B}_r^c$ is an open set, f is a non-negative function on $\mathfrak{X}$ which is regular harmonic in D with respect to $\widehat{X}$, and vanishes on $\overline{B}_r^c \cap (\overline{D}^c \cup \widehat{D}^{\rm reg})$. There exists $C_1 = C_1(R) > 0$ (independent of D, f and $r \geq R$) such that for any $p/16 > q > 2$ and any $\epsilon > 0$, if*

$$\int_{\overline{B}_{qr}} f(y)m(dy) \leq \epsilon \int_{D^{pr/8,qr}} f(y)m(dy), \tag{2.2}$$

then for every $x \in D^{pr}$, $\widetilde{f}^{pr/8,qr}(x) \leq C_1\epsilon f^{pr/8,qr}(x)$.

Proof. Note that

$$\begin{aligned} \widetilde{f}^{pr/8,qr}(x) &= \mathbb{E}_x\left[f(\widehat{X}_{\widehat{\tau}_{D^{pr/8}}}) : \widehat{X}_{\widehat{\tau}_{D^{pr/8}}} \in \overline{B}_{qr}\right] \\ &= \int_{\overline{B}_{qr}} \int_{D^{pr/8}} \widehat{G}_{D^{pr/8}}(x,y)\widehat{j}(y,z)m(dy)f(z)m(dz). \end{aligned}$$

By $\mathbf{C2}(z_0,R)$, $\widehat{j}(y,z)\le c_1\widehat{j}(y,z_0)$, for all $(y,z)\in B^c_{2qr}\times\overline{B}_{qr}$, where the constant c_1 is independent of p and q. Thus

$$\begin{aligned}&\int_{\overline{B}_{qr}}\int_{D^{pr/8}}\widehat{G}_{D^{pr/8}}(x,y)\widehat{j}(y,z)m(dy)f(z)m(dz)\\ \le\ & c_1\int_{\overline{B}_{qr}}\int_{D^{pr/8}}\widehat{G}_{D^{pr/8}}(x,y)\widehat{j}(y,z_0)m(dy)f(z)m(dz)\\ =\ & c_1\widehat{P}_{D^{pr/8}}(x,z_0)\int_{\overline{B}_{qr}}f(z)m(dz).\end{aligned}$$

Now, using (2.2) and the fact that $f=f^{pr/8,qr}$ on $D^{pr/8,qr}$, we get that for every $x\in D^{pr}$,

$$\widetilde{f}^{pr/8,qr}(x)\le c_1\epsilon\widehat{P}_{D^{pr/8}}(x,z_0)\int_{D^{pr/8,qr}}f^{pr/8,qr}(y)m(dy),$$

which is less than or equal to

$$c_1\epsilon\widehat{P}_{D^{pr/8}}(x,z_0)\int_{B_{3pr/8}}f^{pr/8,qr}(y)m(dy).$$

Since $f^{pr/8,qr}$ is regular harmonic in $D^{pr/8}$ with respect to $\widehat{X}$, and vanishes on $\overline{B}^c_{pr/8}\cap(\overline{D}^c\cup\widehat{D}^{\rm reg})$, using $\mathbf{F2}(z_0,R)$ (with $a=3/2$), we conclude that for every $x\in D^{pr}$,

$$\widetilde{f}^{pr/8,qr}(x)\le c_1\epsilon\widehat{P}_{D^{pr/8}}(x,z_0)\int_{B_{3pr/8}}f^{pr/8,qr}(y)m(dy)\le c_1C(3/2)\epsilon f^{pr/8,qr}(x).$$

□

Again, by enlarging $C(3/2)$ in $\mathbf{F2}$, without loss of generality we assume $C_1=(C(3/2))^2$. From now on we let $C=C(3/2)$, so that $C_1=C^2$.

Let $r\ge R$ and $D\subset\overline{B}^c_r$ be an open set. Recall that for any $p>q>0$, $D^p=D\cap\overline{B}^c_p$ and $D^{p,q}=D^q\setminus D^p$. If f_1 and f_2 are non-negative functions on $\mathfrak{X}$, for any $p>1$, we let

$$m^{pr}:=\inf_{D^{pr}}\frac{f_1}{f_2},\qquad M^{pr}:=\sup_{D^{pr}}\frac{f_1}{f_2}.$$

Note that $f_i=f_i^{pr,qr}+\widetilde{f}_i^{pr,qr}$.

Lemma 2.2 *Let $r\ge R$, $D\subset\overline{B}^c_r$ an open set, and $p/16>q>2$. If f_1 and f_2 are non-negative functions on $\mathfrak{X}$ which are regular harmonic in D with respect to $\widehat{X}$, and vanish on $\overline{B}^c_r\cap(\overline{D}^c\cup\widehat{D}^{\rm reg})$, then*

$$(C+1)\left(\sup_{D^{pr}}\frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}}-\inf_{D^{pr}}\frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}}\right)\le(C-1)\left(M^{qr}-m^{qr}\right).\tag{2.3}$$

Proof. For any $x \in D^{pr/8}$, we define

$$\begin{aligned} g(x) &:= f_1^{pr/8,qr}(x) - m^{qr} f_2^{pr/8,qr}(x) \\ &= \mathbb{E}_x \left[(f_1 - m^{qr} f_2)(\widehat{X}_{\widehat{\tau}_{D^{pr/8}}}) : \widehat{X}_{\widehat{\tau}_{D^{pr/8}}} \in D^{pr/8,qr} \right], \end{aligned}$$

which is regular harmonic in $D^{pr/8}$ with respect to $\widehat{X}$, and vanishes on $\overline{B}^c_{pr/8} \cap (\overline{D}^c \cup \widehat{D}^{\rm reg})$. Next, it follows from (2.1) that for any $x_1, x_2 \in D^{pr}$ (we assume that $D^{pr} \neq \emptyset$), $g(x_1) f_2^{pr/8,qr}(x_2) \le C g(x_2) f_2^{pr/8,qr}(x_1)$. Therefore,

$$\begin{aligned} \sup_{D^{pr}} \frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}} - m^{qr} &= \sup_{D^{pr}} \frac{g}{f_2^{pr/8,qr}} \le C \inf_{D^{pr}} \frac{g}{f_2^{pr/8,qr}} \\ &= C \left(\inf_{D^{pr}} \frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}} - m^{qr} \right). \end{aligned} \tag{2.4}$$

We can similarly get that

$$M^{qr} - \inf_{D^{pr}} \frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}} \le C \left(M^{qr} - \sup_{D^{pr}} \frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}} \right). \tag{2.5}$$

Adding (2.4) and (2.5) and rearranging, we arrive at (2.3). □

For any positive function ϕ on a non-empty open set U, let

$$\mathrm{RO}_U \phi = \frac{\sup_U \phi}{\inf_U \phi}. \tag{2.6}$$

Lemma 2.3 *Let $r \ge R$, $D \subset \overline{B}^c_r$ an open set, $p/16 > q > 2$, and $\epsilon > 0$. Let f_1 and f_2 be non-negative functions on $\mathfrak{X}$ which are regular harmonic in D with respect to $\widehat{X}$ and vanish on $\overline{B}^c_r \cap (\overline{D}^c \cup \widehat{D}^{\rm reg})$. If*

$$\int_{\overline{B}_{qr}} f_i(y) m(dy) \le \epsilon \int_{D^{pr/8,qr}} f_i(y) m(dy), \quad i = 1, 2, \tag{2.7}$$

then

$$\mathrm{RO}_{D^{pr}} \frac{f_1}{f_2} \le (1 + C^2\epsilon)^2 + (1 + C^2\epsilon) \frac{C-1}{C+1} \left(\mathrm{RO}_{D^{qr}} \frac{f_1}{f_2} - 1 \right). \tag{2.8}$$

Proof. Applying Lemma 2.1 we get that

$$\begin{aligned} M^{pr} &= \sup_{D^{pr}} \frac{f_1}{f_2} = \sup_{D^{pr}} \frac{f_1^{pr/8,qr} + \widetilde{f}_1^{pr/8,qr}}{f_2^{pr/8,qr} + \widetilde{f}_2^{pr/8,qr}} \le \sup_{D^{pr}} \frac{(1 + C^2\epsilon) f_1^{pr/8,qr}}{f_2^{pr/8,qr}}, \\ m^{pr} &= \inf_{D^{pr}} \frac{f_1}{f_2} = \inf_{D^{pr}} \frac{f_1^{pr/8,qr} + \widetilde{f}_1^{pr/8,qr}}{f_2^{pr/8,qr} + \widetilde{f}_2^{pr/8,qr}} \ge \inf_{D^{pr}} \frac{f_1^{pr/8,qr}}{(1 + C^2\epsilon) f_2^{pr/8,qr}}. \end{aligned}$$

Inserting this in (2.3), we arrive at

$$\begin{aligned}
&(C+1)\left(\frac{M^{pr}}{1+C^2\epsilon}-(1+C^2\epsilon)m^{pr}\right)\\
&\le (C+1)\left(\sup_{D^{pr}}\frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}}-\inf_{D^{pr}}\frac{f_1^{pr/8,qr}}{f_2^{pr/8,qr}}\right)\\
&\le (C-1)(M^{qr}-m^{qr}).
\end{aligned}$$

Rearranging and using that $m^{pr}\ge m^{qr}$ we get

$$\frac{M^{pr}}{m^{pr}}\le (1+C^2\epsilon)^2+(1+C^2\epsilon)\frac{C-1}{C+1}\left(\frac{M^{qr}}{m^{qr}}-1\right),$$

which implies (2.8). □

In the rest of this subsection, we fix an open set D and a point $x_0\in D$.

Lemma 2.4 *Suppose that ∞ is accessible from D with respect to X. For any $q\ge 4$, $r\ge 2d(z_0,x_0)\vee R$ and $\epsilon>0$, there exists $p=p(\epsilon,q,D,x_0,r)>16q$ such that*

$$\int_{D^{pr,qr}}G_D(x_0,y)m(dy)>\epsilon\int_{B_{qr}}G_D(x_0,y)m(dy).$$

Proof. Since ∞ is accessible from D with respect to X, we have that

$$\mathbb{E}_{x_0}\tau_D=\int_D G_D(x_0,v)m(dv)=\infty.$$

The function $v\mapsto G_D(x_0,v)$ is regular harmonic in $D^r\supset D^{qr/3}$ with respect to $\widehat{X}$ and vanishes on $\overline{B}^{qr/3}\cap(\overline{D^{qr/3}}^c\cup\widehat{(D^{qr/3})}^{\mathrm{reg}})$. By using $\mathbf{F2}(z_0,R)$ for $\widehat{X}$ (with the open set $D^{qr/3}$, $a=3/2$ and radius $qr/3$)

$$\int_{B_{qr}}G_D(x_0,z)m(dz)\le c\inf_{v\in D^{8qr/3}}\frac{G_D(x_0,v)}{\widehat{P}_{D^{qr/3}}(v,z_0)}<\infty.$$

Thus

$$\infty=\int_{D^{qr}}G_D(x_0,z)m(dz)=\lim_{p\to\infty}\int_{D^{pr,qr}}G_D(x_0,z)m(dz)$$

and so we can choose $p=p(\epsilon,q,D,x_0,r)>16q$ large enough so that

$$\int_{D^{pr,qr}}G_D(x_0,z)m(dz)>\epsilon\int_{B_{qr}}G_D(x_0,z)m(dz).$$

□

Proposition 2.5 *Suppose that ∞ is accessible from D with respect to X. Let $r > 2d(z_0, x_0) \vee R$. For every $\eta > 0$, there exists $s = s(r, D, x_0, \eta) > 1$ such that for any two non-negative functions f_1, f_2 on $\mathfrak{X}$ which are regular harmonic in D^r with respect to $\widehat{X}$ and vanish on $\overline{B}_r^c \cap (\overline{D}^c \cup \widehat{(D^r)}^{\mathrm{reg}})$, we have*

$$\mathrm{RO}_{D^{sr}} \frac{f_1}{f_2} \le 1 + \eta\,. \tag{2.9}$$

Proof. Let $\eta > 0$ and define

$$\phi(t) := 1 + \frac{\eta}{2} + \frac{C}{C+1}(t-1), \quad t \ge 1. \tag{2.10}$$

Then $\phi(t) = t$ for $t = 1 + \eta(C+1)/2$, $\phi(t) < t$ if $t > 1 + \eta(C+1)/2$, and $\phi(t) > t$ if $t < 1 + \eta(C+1)/2$. Thus $\lim_{l\to\infty} \phi^l(C) = 1 + \eta(C+1)/2$, where ϕ^l is the l-fold composition of ϕ. Let $l \in \mathbb{N}$ be such that

$$\phi^l(C) < 1 + \eta(C+1). \tag{2.11}$$

Choose $\epsilon = \epsilon(\eta) > 0$ small enough so that

$$(C\epsilon + 1 + \epsilon)^2 (1+\epsilon)^2 < 1 + \eta \tag{2.12}$$

and

$$(1 + C^2\epsilon)^2 + (1 + C^2\epsilon)\frac{C-1}{C+1}(t-1) < 1 + \frac{\eta}{2} + \frac{C}{C+1}(t-1) = \phi(t) \tag{2.13}$$

for all $t \ge 1$. Let k be the smallest integer such that $k > C^2\epsilon^{-2}$ and denote $n = lk$. Let $q_0 = 8$ and choose $q_1 = p(\epsilon, q_0, r, D, x_0)$ as in Lemma 2.4. Inductively, using Lemma 2.4, we choose $q_{j+1} = p(\epsilon, q_j, r, D, x_0)$ for $j = 0, 1, \ldots, n-1$, and $s = q_n$. Then by Lemma 2.4, for $j = 0, 1, \ldots, n-1$, we have

$$\int_{D^{q_{j+1}r, q_j r}} G_D(x_0, y) m(dy) > \epsilon \int_{\overline{B}_{q_j r}} G_D(x_0, y) m(dy). \tag{2.14}$$

It follows from $\mathbf{F2}(z_0, R)$ (applied to the open set $D^{q_j r/3}$ with $a = 3/2$ and radius $q_j r/3$) that for every $j = 0, 1, \ldots, n-1$, $i = 1, 2$ and $x \in D^{q_{j+1}r, 8q_j r/3}$,

$$C\frac{f_i(x)}{\int_{B_{q_j r}} f_i(y) m(dy)} \ge \widehat{P}_{D^{q_j r/3}}(x, z_0) \ge C^{-1} \frac{G_D(x_0, x)}{\int_{B_{q_j r}} G_D(x_0, y) m(dy)}.$$

Hence, by integrating over $D^{q_{j+1}r, 8q_j r/3}$ we get

$$\frac{\int_{D^{q_{j+1}r, 8q_j r/3}} f_i(x) m(dx)}{\int_{B_{q_j r}} f_i(y) m(dy)} \ge C^{-1} \frac{\int_{D^{q_{j+1}r, 8q_j r/3}} G_D(x_0, x) m(dx)}{\int_{B_{q_j r}} G_D(x_0, y) m(dy)}, \quad i = 1, 2.$$

Together with (2.14) we get that

$$\int_{D^{q_{j+1}r,q_jr}} f_i(y)m(dy) \geq \int_{D^{q_{j+1}r,8q_jr/3}} f_i(y)m(dy) > C^{-2}\epsilon \int_{B_{q_jr}} f_i(y)m(dy)$$

for *both* $i = 1$ and $i = 2$, and all $j = 0, 1, \ldots n-1$. Let $0 \leq m < l$. By the definition of k,

$$\begin{aligned}
&\int_{D^{q_{(m+1)k^r}/8,q_{mk^r}}} f_i(y)m(dy) \geq \int_{D^{q_{(m+1)k-1^r,q_{mk^r}}}} f_i(y)m(dy) \\
&= \sum_{j=mk}^{(m+1)k-1} \int_{D^{q_{j+1}r,q_jr}} f_i(y)m(dy) \geq \ kC^{-2}\epsilon \int_{B_{q_{mk^r}}} f_i(y)m(dy) \\
&\geq \epsilon^{-1} \int_{B_{q_{mk^r}}} f_i(y)m(dy), \quad i = 1, 2.
\end{aligned}$$

By using Lemma 2.3 with $p = q_{(m+1)k}$ and $q = q_{mk}$ we conclude from (2.13) that for every integer m such that $0 \leq m < l$,

$$\begin{aligned}
\mathrm{RO}_{D^{q_{(m+1)k^r}}} \frac{f_1}{f_2} &\leq (1 + C^2\epsilon)^2 + (1 + C^2\epsilon)\frac{C-1}{C+1}\left(\mathrm{RO}_{D^{q_{mk^r}}} \frac{f_1}{f_2} - 1\right) \\
&< \phi\left(\mathrm{RO}_{D^{q_{mk^r}}} \frac{f_1}{f_2}\right).
\end{aligned}$$

By definition of the integer l, monotonicity of ϕ, and the fact that $\mathrm{RO}_{D^{r/2}}(f_1/f_2) \leq C$, it follows that

$$\mathrm{RO}_{D^{q_{lk}R}} \frac{f_1}{f_2} \leq \phi\left(\mathrm{RO}_{D^{q_{(l-1)k}R}}\right) \leq \cdots \leq \phi^l\left(\mathrm{RO}_{D^{q_0R}}\right) \leq 1 + \eta(C+1).$$

This means that $\mathrm{RO}_{D^{sr}} \frac{f_1}{f_2} \leq 1 + \eta(C+1)$. Since $\eta > 0$ is arbitrary, the proof is complete. □

Corollary 2.6 *Suppose that ∞ is accessible from D with respect to X. Let $r > R$ and let f_1 and f_2 be non-negative functions on $\mathfrak{X}$ which are regular harmonic in D^r with respect to $\widehat{X}$ and vanish on $\overline{B}_r^c \cap (\overline{D}^c \cup (\widehat{D^r})^{\mathrm{reg}})$. Then the limit*

$$\lim_{D \ni x \to \infty} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.

Proof. Since one can increase r so that $r > 2d(z_0, x_0) \vee R$ without loss of generality, the existence of the limit and its finiteness is a direct consequence of Proposition 2.5. □

2.2 Finite boundary point

In this subsection, we deal with the oscillation reduction at a boundary point z_0 of an open set D. Throughout the subsection, we assume that there exists $R \le R_0$ such that $\mathbf{C1}(z_0, R)$ holds, and that $\widehat{X}$ satisfies $\mathbf{F1}(z_0, R)$. We will see that the results and the estimates below have the same structure as those in the previous subsection, the difference being that $\widehat{P}_D(x, z_0)$ is replaced by $\mathbb{E}_x\widehat{\tau}_D$ and $\int_{B_{ar}} f(y)m(dy)$ is replaced by $\widehat{\Lambda}_{ar}(f)$ (see below for definition).

As in the previous subsection we begin by recording a simple consequences of $\mathbf{F1}(z_0, R)$ for $\widehat{X}$, the boundary Harnack principle in [26]: There exists $c > 1$ such that for any $r \in (0, R)$, any open set $D \subset B_r$ and any non-negative functions u and v on $\mathfrak{X}$ that are regular harmonic in D with respect to $\widehat{X}$ and vanish on $B_r \cap (\overline{D}^c \cup \widehat{D}^{\rm reg})$, it holds that

$$c^{-1}\frac{u(y)}{v(y)} \le \frac{u(x)}{v(x)} \le c\frac{u(y)}{v(y)} \qquad \text{for all } x, y \in D \cap B_{r/8}. \tag{2.15}$$

Note that we can take $c = (C(2/3))^2$. By enlarging $C(2/3)$ in $\mathbf{F1}(z_0, R)$, without loss of generality we assume the above c is equal to $C(2/3)$.

Let $D \subset \mathfrak{X}$ be an open set. For $0 < p < q$, let $D_p = D \cap B_p$ and $D_{p,q} = D_q \setminus D_p$. For a function f on $\mathfrak{X}$, and $0 < p < q$, let

$$\widehat{\Lambda}_p(f) := \int_{\overline{B}_p^c} \widehat{j}(z_0, y)f(y)m(dy), \qquad \widehat{\Lambda}_{p,q}(f) := \int_{D_{p,q}} \widehat{j}(z_0, y)f(y)m(dy). \tag{2.16}$$

Let $r \in (0, R]$. For $0 < p < q < 1$ and a non-negative function f on $\mathfrak{X}$ define

$$\begin{aligned} f_{pr,qr}(x) &= \mathbb{E}_x\left[f(\widehat{X}_{\widehat{\tau}_{D_{pr}}}) : \widehat{X}_{\widehat{\tau}_{D_{pr}}} \in D_{pr,qr}\right], \\ \widetilde{f}_{pr,qr}(x) &= \mathbb{E}_x\left[f(\widehat{X}_{\widehat{\tau}_{D_{pr}}}) : \widehat{X}_{\widehat{\tau}_{D_{pr}}} \in (D \setminus D_{qr}) \cup B_r^c\right]. \end{aligned}$$

Lemma 2.7 *Suppose that $r \le R$, D is an open subset of B_r and f is a non-negative function on $\mathfrak{X}$ that is regular harmonic in D with respect to $\widehat{X}$ and vanishes on $B_r \cap (\overline{D}^c \cup \widehat{D}^{\rm reg})$. There exists $C_2 > 0$ independent of f and $r \le R$ such that for any $0 < 16p < q < 1/2$ and any $\epsilon > 0$, if*

$$\widehat{\Lambda}_{qr}(f) \le \epsilon\widehat{\Lambda}_{8pr,qr}(f), \tag{2.17}$$

then for every $x \in D_{pr}$, $\widetilde{f}_{8pr,qr}(x) \le C_2\epsilon f_{8pr,qr}(x)$.

Proof. Note that

$$\begin{aligned} \widetilde{f}_{8pr,qr}(x) &= \mathbb{E}_x\left[f(\widehat{X}_{\widehat{\tau}_{D_{8pr}}}) : \widehat{X}_{\widehat{\tau}_{D_{pr}}} \in B_{qr}^c\right] \\ &= \int_{B_{qr}^c}\int_{D_{8pr}} \widehat{G}_{D_{8pr}}(x, y)\widehat{j}(y, z)m(dy)f(z)m(dz). \end{aligned}$$

By $\mathbf{C1}(z_0,R)$, $\widehat{j}(y,z)\le c_1\widehat{j}(z_0,z)$ for all $(y,z)\in \overline{B}_{qr/2}\times B_{qr}^c$, where the constant c is independent of p and q. Thus

$$\begin{aligned}&\int_{B_{qr}^c}\int_{D_{8pr}}\widehat{G}_{D_{8pr}}(x,y)\widehat{j}(y,z)m(dy)f(z)m(dz)\\ \le& c_1\int_{D_{8pr}}\widehat{G}_{D_{8pr}}(x,y)m(dy)\int_{B_{qr}^c}\widehat{j}(z_0,z)f(z)m(dz)\\ =& c_1\mathbb{E}_x\widehat{\tau}_{D_{8pr}}\int_{B_{qr}^c}\widehat{j}(z_0,z)f(z)m(dz)\end{aligned}$$

Now, using (2.17) and the fact that $f=f_{8pr,qr}$ on $D_{8pr,qr}$, we get that for every $x\in D_{pr}$,

$$\widetilde{f}_{8pr,qr}(x)\le c_1\epsilon(\mathbb{E}_x\widehat{\tau}_{D_{8pr}})\widehat{\Lambda}_{8pr,qr}(f_{8pr,qr}),$$

which is less than or equal to $c\epsilon(\mathbb{E}_x\widehat{\tau}_{D_{8pr}})\widehat{\Lambda}_{8pr/3}(f_{8pr,qr})$. Note that $f_{8pr,qr}$ is regular harmonic in D_{8pr} with respect to $\widehat{X}$ and vanishes on $B_{8pr}\cap(\overline{D}^c\cup\widehat{D}^{\rm reg})$. Thus applying $\mathbf{F1}(z_0,R)$ (with $a=2/3$) to $f_{8pr,qr}$ we have that for every $x\in D_{pr}$,

$$\widetilde{f}_{8pr,qr}(x)\le c_1\epsilon(\mathbb{E}_x\widehat{\tau}_{D_{8pr}})\widehat{\Lambda}_{8pr/3}(f_{8pr,qr})\le c_1C(2/3)\epsilon f_{8pr,qr}(x).$$

□

Again, by enlarging $C(2/3)$ in **F1**, without loss of generality we can assume $C_2=(C(2/3))^2$. From now on we let $C=C(2/3)$, so that $C_2=C^2$.

Let $r\in(0,R]$, $D\subset B_r=B(z_0,r)$ an open set and $z_0\in\partial D$. Recall that for $0<p<q$, $D_p=D\cap B_p$ and $D_{p,q}=D_q\setminus D_p$. If f_1 and f_2 are non-negative functions on $\mathfrak{X}$, for any $p\in(0,1)$, we let

$$m_{pr}:=\inf_{D_{pr}}\frac{f_1}{f_2},\qquad M_{pr}=\sup_{D_{pr}}\frac{f_1}{f_2}.$$

Note that $f_i=(f_i)_{pr,qr}+(\widetilde{f}_i)_{pr,qr}$.

Lemma 2.8 *Let $r\le R$, $D\subset B_r$ an open set, and $0<16p<q<1/2$. If f_1 and f_2 are non-negative functions on $\mathfrak{X}$ which are regular harmonic in D with respect to $\widehat{X}$ and vanish on $B_r\cap(\overline{D}^c\cup\widehat{D}^{\rm reg})$, then*

$$(C+1)\left(\sup_{D_{pr}}\frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}}-\inf_{D_{pr}}\frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}}\right)\le(C-1)\left(M_{qr}-m_{qr}\right).\tag{2.18}$$

Proof. For any $x\in D_{8pr}$, we define

$$\begin{aligned}g(x)&:=(f_1)_{8pr,qr}(x)-m_{qr}(f_2)_{8pr,qr}(x)\\ &=\mathbb{E}_x\left[(f_1-m_{qr}f_2)(\widehat{X}_{\widehat{\tau}_{D_{8pr}}}):\widehat{X}_{\widehat{\tau}_{D_{8pr}}}\in D_{8pr,qr}\right],\end{aligned}$$

which is regular harmonic in D_{8pr} with respect to $\widehat{X}$ and vanishes on $B_{8pr} \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$. Next, it follows from (2.15) that for any $x_1, x_2 \in D_{pr}$ (we assume that $D_{pr} \neq \emptyset$),

$$g(x_1)(f_2)_{8pr,qr}(x_2) \leq Cg(x_2)(f_2)_{8pr,qr}(x_1).$$

Therefore,

$$\begin{aligned} \sup_{D_{pr}} \frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}} - m_{qr} &= \sup_{D_{pr}} \frac{g}{(f_2)_{8pr,qr}} \leq C \inf_{D_{pr}} \frac{g}{(f_2)_{8pr,qr}} \\ &= C \left(\inf_{D_{pr}} \frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}} - m_{qr} \right). \end{aligned} \tag{2.19}$$

We can similarly get that

$$M_{qr} - \inf_{D_{pr}} \frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}} \leq C \left(M_{qr} - \sup_{D_{pr}} \frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}} \right). \tag{2.20}$$

Adding (2.19) and (2.20) and rearranging, we arrive at (2.18). □

Recall that $\mathrm{RO}_U\phi$ is defined in (2.6).

Lemma 2.9 *Let $r \leq R$, $D \subset B_r^c$ an open set, $16p < q < 1/2$, and $\epsilon > 0$. Let f_1 and f_2 be non-negative functions on $\mathfrak{X}$ which are regular harmonic in D with respect to $\widehat{X}$ and vanish on $B_r \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$. If $\widehat{\Lambda}_{qr}(f_i) \leq \epsilon \widehat{\Lambda}_{8pr,qr}(f_i)$, $i = 1, 2$, then*

$$\mathrm{RO}_{D_{pr}} \frac{f_1}{f_2} \leq (1 + C^2\epsilon)^2 + (1 + C^2\epsilon) \frac{C-1}{C+1} \left(\mathrm{RO}_{D_{qr}} \frac{f_1}{f_2} - 1 \right). \tag{2.21}$$

Proof. Applying Lemma 2.7, we get that

$$\begin{aligned} M_{pr} &= \sup_{D_{pr}} \frac{f_1}{f_2} = \sup_{D_{pr}} \frac{(f_1)_{8pr,qr} + (\widetilde{f_1})_{8pr,qr}}{(f_2)_{8pr,qr} + (\widetilde{f_2})_{8pr,qr}} \leq \sup_{D_{pr}} \frac{(1 + C^2\epsilon)(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}}, \\ m_{pr} &= \inf_{D_{pr}} \frac{f_1}{f_2} = \inf_{D_{pr}} \frac{(f_1)_{8pr,qr} + (\widetilde{f_1})_{8pr,qr}}{(f_2)_{8pr,qr} + (\widetilde{f_2})_{8pr,qr}} \geq \inf_{D_{pr}} \frac{(f_1)_{8pr,qr}}{(1 + C^2\epsilon)(f_2)_{8pr,qr}}. \end{aligned}$$

By inserting this in (2.18), we arrive at

$$\begin{aligned} &(C+1) \left(\frac{M_{pr}}{1 + C^2\epsilon} - (1 + C^2\epsilon) m_{pr} \right) \\ &\leq (C+1) \left(\sup_{D_{pr}} \frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}} - \inf_{D_{pr}} \frac{(f_1)_{8pr,qr}}{(f_2)_{8pr,qr}} \right) \\ &\leq (C-1)(M_{qr} - m_{qr}). \end{aligned}$$

Rearranging and using that $m_{pr} \geq m_{qr}$ we get

$$\frac{M_{pr}}{m_{pr}} \leq (1+C^2\epsilon)^2 + (1+C^2\epsilon)\frac{C-1}{C+1}\left(\frac{M_{qr}}{m_{qr}} - 1\right),$$

which implies (2.21). □

In the remainder of this subsection, we fix an open set D such that $z_0 \in \partial D$, and a point $x_0 \in D$.

Lemma 2.10 *Suppose that z_0 is accessible from D with respect to X. Assume that $r \leq R \wedge (\frac{1}{2}d(z_0, x_0))$, $q \leq 1/4$ and $\epsilon > 0$. Then there exists $p = p(\epsilon, q, D, x_0, r) < q/16$ such that $\widehat{\Lambda}_{pr,qr}(G_D(x_0,\cdot)) > \epsilon\widehat{\Lambda}_{qr}(G_D(x_0,\cdot))$.*

Proof. Since z_0 is accessible from D with respect to X, we have that

$$P_D(x_0, z_0) = \int_D G_D(x_0, v) j(v, z_0) m(dv) = \infty.$$

The function $v \mapsto G_D(x_0, v)$ is regular harmonic in $D_r \supset D_{3qr}$ with respect to $\widehat{X}$ and vanishes on $B_{3qr} \cap (\overline{D_{3qr}}^c \cup \widehat{(D_{3qr})}^{\mathrm{reg}})$. By using $\mathbf{F1}(z_0, R)$ for $\widehat{X}$ (with the open set D_{3qr}, $a = 2/3$ and radius $3qr$),

$$\int_{\overline{B}^c_{qr}} G_D(x_0, y) j(y, z_0) m(dv) = \widehat{\Lambda}_{qr}(G_D(x_0,\cdot)) \leq c \inf_{v \in D_{3qr/8}} \frac{G_D(x_0, v)}{\mathbb{E}_v \widehat{\tau}_{D_{3qr}}} < \infty.$$

Thus

$$\infty = \int_{D_{qr}} G_D(x_0, v) j(v, z_0) m(dv) = \lim_{p \to 0} \int_{D_{pr,qr}} G_D(x_0, v) j(v, z_0) m(dy)$$

and so we can choose $p = p(\epsilon, q, D, x_0, r) < q/16$ small so that

$$\int_{D_{pr,qr}} G_D(x_0, y) j(y, z_0) m(dy) > \epsilon \int_{\overline{B}^c_{qr}} G_D(x_0, y) j(y, z_0) m(dy).$$

□

Proposition 2.11 *Suppose that z_0 is accessible from D with respect to X. Assume that $r \leq R \wedge (\frac{1}{2}d(z_0, x_0))$. For every $\eta > 0$, there exists $s = s(r, D, x_0, \eta) \in (0, 1)$ such that for any two non-negative functions f_1, f_2 on $\mathfrak{X}$ which are regular harmonic in D_r with respect to $\widehat{X}$ and vanish on $B_r \cap (\overline{D}^c \cup \widehat{(D_r)}^{\mathrm{reg}})$, we have*

$$\mathrm{RO}_{D_{sr}} \frac{f_1}{f_2} \leq 1 + \eta. \tag{2.22}$$

Proof. Let $\eta > 0$ and define ϕ as in (2.10) and let ϕ^l be the l-fold composition of ϕ. Let $l \in \mathbb{N}$ be such that (2.11) holds. Choose $\epsilon > 0$ small enough so that (2.12) and (2.13) holds. Let k be the smallest integer such that $k > C^2\epsilon^{-2}$ and denote $n = lk$. Let $q_0 = 1/8$ and choose $q_1 = p(\epsilon, q_0, r, D, x_0)$ as in Lemma 2.10. Inductively, using Lemma 2.10, we choose $q_{j+1} = p(\epsilon, q_j, r, D, x_0)$ for $j = 0, 1, \ldots, n-1$, and $s = q_n$. Then it follows from Lemma 2.10 that for $j = 0, 1, \ldots, n-1$, we have

$$\widehat{\Lambda}_{q_{j+1}r, q_j r}(G_D(x_0, \cdot)) > \epsilon \widehat{\Lambda}_{q_j r}(G_D(x_0, \cdot)). \tag{2.23}$$

It follows from $\mathbf{F1}(z_0, R)$ (applied to the open set D_{3q_jr} with $a = 2/3$ and radius $3q_jr$) that for every $j = 0, 1, \ldots, n-1$,

$$C \frac{f_i(x)}{\widehat{\Lambda}_{q_j r}(f_i)} \geq \mathbb{E}_x \widehat{\tau}_{D_{3q_j r}} \geq C^{-1} \frac{G_D(x_0, x)}{\widehat{\Lambda}_{q_j r}(G_D(x_0, \cdot))}, \quad x \in D_{q_{j+1}r, 3q_j r/8}.$$

Hence, by integrating over $D_{q_{j+1}r, 3q_j r/8}$ we get

$$\frac{\widehat{\Lambda}_{q_{j+1}r, 3q_j r/8}(f_i)}{\widehat{\Lambda}_{q_j r}(f_i)} \geq C^{-1} \frac{\widehat{\Lambda}_{q_{j+1}r, 3q_j r/8}(G_D(x_0, \cdot))}{\widehat{\Lambda}_{q_j r}(G_D(x_0, \cdot))}, \quad i = 1, 2.$$

Together with (2.23) it follows that $\widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) \geq \widehat{\Lambda}_{q_{j+1}r, 3q_j r/8}(f_i) > C^{-2}\epsilon \widehat{\Lambda}_{q_j r}(f_i)$ for *both* $i = 1$ and $i = 2$, and all $j = 0, 1, \ldots n-1$. Let $0 \leq m < l$; then

$$\begin{aligned} \Lambda_{8q_{(m+1)k}r, q_{mk}r}(f_i) &\geq \widehat{\Lambda}_{q_{(m+1)k-1}r, q_{mk}r}(f_i) = \sum_{j=mk}^{(m+1)k-1} \widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) \\ &\geq kC^{-2}\epsilon \widehat{\Lambda}_{q_{mk}r}(f_i) \geq \epsilon^{-1} \widehat{\Lambda}_{q_{mk}r}(f_i), \quad i = 1, 2. \end{aligned}$$

By using Lemma 2.9 with $p = q_{(m+1)k}$ and $q = q_{mk}$ we conclude that for every integer m such that $0 \leq m < l$,

$$\begin{aligned} \mathrm{RO}_{D_{q_{(m+1)k^r}}} \frac{f_1}{f_2} &\leq (1 + C^2\epsilon)^2 + (1 + C^2\epsilon) \frac{C-1}{C+1} \left(\mathrm{RO}_{D_{q_{mk^r}}} \frac{f_1}{f_2} - 1 \right) \\ &< \phi \left(\mathrm{RO}_{D_{q_{mk^r}}} \frac{f_1}{f_2} \right). \end{aligned}$$

The remainder of the proof is the same as the corresponding part of the proof of Proposition 2.5. □

Corollary 2.12 *Suppose that z_0 is accessible from D with respect to X. Let $r \leq R$ and let f_1 and f_2 be non-negative functions on $\mathfrak{X}$ which are regular harmonic in D_r with respect to $\widehat{X}$ and vanish on $B_r \cap (\overline{D}^c \cup (\widehat{D_r})^{\mathrm{reg}})$. Then the limit*

$$\lim_{D \ni x \to z_0} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.

Proof. Since one can decrease r so that $r \le R \wedge (\frac{1}{2} d(z_0, x_0))$ without loss of generality, the existence of the limit is a direct consequence of Proposition 2.11. □

3 Martin boundary for accessible points

Recall that D is a Greenian open subset of $\mathfrak{X}$ and that X^D is the process X killed upon exiting D. In order to apply the theory of Martin boundary developed in [28], we have to check that their Hypothesis (B), see p.498 in [28], holds in our setting. Since X^D is strongly Feller, it follows by the dominated convergence theorem that the α-resolvent operator $G_D^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t^D f(x)\, dt$, $\alpha > 0$, is also strongly Feller. Here $(P_t^D)_{t\ge 0}$ denotes the semigroup of X^D. In particular, $G_D^\alpha f$ is continuous for every bounded non-negative measurable f on D. It follows that $G_D^\alpha f$ is lower semi-continuous for every non-negative f on D and every $\alpha > 0$. Since $G_D f = \uparrow \lim_{\alpha\to 0} G_D^\alpha f$, we see that $G_D f$ is also lower semi-continuous for every non-negative f. Hence, conditions (11) and (12) on p.126 of [12] are satisfied. It follows from Theorem 2 on p.268 of [12] that $G_D \mathbf{1}_K$ is bounded for every compact set $K \subset D$. Let $f : D \to [0, \infty)$ be bounded measurable and vanish outside of a compact set $K \subset D$. Then $0 \le f \le \|f\|_\infty \mathbf{1}_K$. Thus $G_D f \le G_D(\|f\|_\infty \mathbf{1}_K) \le \|f\|_\infty G_D \mathbf{1}_K$ is bounded. Since X^D is strongly Feller, it follows that $P_t^D G_D f$ is continuous. Further,

$$G_D f - P_t^D G_D f = \int_0^t P_s^D f ds \le \|f\|_\infty \int_0^t P_s^D 1 ds \le \|f\|_\infty t\,.$$

The right-hand side converges to 0 uniformly in $x \in D$. Hence

$$G_D f = \lim_{t\to 0} P_t^D G_D f$$

uniformly in D. Thus $G_D f$ is a uniform limit of continuous functions, hence continuous. Finally, if $f \in C_c(D)$ (continuous functions on D with compact support), it is clear that $\alpha G_D^\alpha f(x) = \mathbb{E}_x \int_0^\infty e^{-t} f(X_{t/\alpha})\, dt \to f(x)$ boundedly on compacts as $\alpha \to \infty$. Since the same conclusions are valid for $\widehat{X}$, we have checked that Hypothesis (B) from [28] holds true.

Fix $x_0 \in D$ and define

$$M_D(x, y) := \frac{G_D(x, y)}{G_D(x_0, y)}, \qquad x, y \in D,\ y \ne x_0.$$

By Theorem 3 in [28], D has a Martin boundary $\partial_M D$ with respect to X^D satisfying the following properties:

(M1) $D \cup \partial_M D$ is a compact metric space (with the metric denoted by d_M);

(M2) D is open and dense in $D \cup \partial_M D$, and its relative topology coincides with its original topology;

(M3) $M_D(x, \cdot\,)$ can be uniquely extended to $\partial_M D$ in such a way that

(a) $M_D(x,y)$ converges to $M_D(x,w)$ as $y \to w \in \partial_M D$ in the Martin topology;

(b) for each $w \in D \cup \partial_M D$, the function $x \mapsto M_D(x,w)$ is excessive with respect to X^D;

(c) the function $(x,w) \mapsto M_D(x,w)$ is jointly continuous on $D \times ((D \setminus \{x_0\}) \cup \partial_M D)$ in the Martin topology and

(d) $M_D(\cdot, w_1) \neq M_D(\cdot, w_2)$ if $w_1 \neq w_2$ and $w_1, w_2 \in \partial_M D$.

Recall that a positive harmonic function f for X^D is minimal if, whenever g is a positive harmonic function for X^D with $g \leq f$ on D, one must have $f = cg$ for some constant c. If $M_D(\cdot, z)$, $z \in \partial_M D$, is a minimal harmonic function, the point z is called a minimal Martin boundary point. The set of all minimal Martin boundary points is denoted by $\partial_m D$. Then the following Martin representation is valid, see Theorem 4 in [28]: For every non-negative function h harmonic with respect to X^D, there is a unique finite measure μ on $\partial_M D$ concentrated on $\partial_m D$ such that

$$h(x) = \int_{\partial_M D} M_D(x,z)\, \mu(dz) = \int_{\partial_m D} M_D(x,z)\, \mu(dz)\,, \quad x \in D\,. \tag{3.1}$$

Recall that a point $w \in \partial_M D$ is a finite Martin boundary point if there exists a bounded sequence $(y_n)_{n\geq 1} \subset D$ converging to w in the Martin topology. The finite part of the Martin boundary will be denoted by $\partial_M^f D$. Recall that a point w on the Martin boundary $\partial_M D$ of D is said to be associated with $z_0 \in \partial D$ if there is a sequence $(y_n)_{n\geq 1} \subset D$ converging to w in the Martin topology and to z_0 in the topology of $\mathfrak{X}$. The set of Martin boundary points associated with z_0 is denoted by $\partial_M^{z_0} D$.

The proof of part (b) of the following result is a direct extension of that of Lemma 4.18 in [23] and part (a) is even simpler. So we omit the proof.

Lemma 3.1 *(a) Let D be a bounded open set and suppose that u is a bounded non-negative harmonic function for X^D. If there exists a polar set $N \subset \partial D$ such that for any $z \in \partial D \setminus N$*

$$\lim_{D \ni x \to z} u(x) = 0\,, \tag{3.2}$$

then u is identically equal to zero.
(b) Let D be an unbounded open set and suppose that u is a bounded non-negative harmonic function for X^D. If there exists a polar set $N \subset \partial D$ such that for any $z \in \partial D \setminus N$ (3.2) holds true and additionally,

$$\lim_{D \ni x \to \infty} u(x) = 0\,,$$

then u is identically equal to zero.

3.1 Martin boundary at infinity

In this subsection we assume that $R_0 = \infty$, that there exists a point $z_0 \in \mathfrak{X}$ such that $\mathbf{C2}(z_0, R)$ holds, and that $\widehat{X}$ satisfies $\mathbf{F2}(z_0, R)$ for some $R > 0$. We will fix z_0 and R and use the notation $B_r = B(z_0, r)$. Let D be an unbounded open subset of $\mathfrak{X}$ such that ∞ is accessible from D with respect to X. We will deal with the Martin boundary of D at infinity. Recall that x_0 is a fixed point in D.

Lemma 3.2 *For every $x \in D$ the limit*

$$M_D(x, \infty) := \lim_{D \ni v \to \infty} \frac{G_D(x, v)}{G_D(x_0, v)} \tag{3.3}$$

exists and is finite.

Proof. Fix $x \in D$, and let $r \geq 2 \min\{d(z_0, x), d(z_0, x_0), R\}$. As before, let $D^r = D \cap \overline{B}_r^c$. The functions $G_D(x, \cdot)$ and $G_D(x_0, \cdot)$ are regular harmonic in D^r with respect to $\widehat{X}$ and vanish in $\overline{B}_r^c \cap (\overline{D}^c \cup \widehat{(D^r)}^{\mathrm{reg}})$, hence by Corollary 2.6 we deduce that the limit

$$M_D(x, \infty) := \lim_{D \ni v \to \infty} \frac{G_D(x, v)}{G_D(x_0, v)}$$

exists and is finite. □

Proof of Theorem 1.3(a): We first note that $\partial_M^\infty D$ is not empty. Indeed, let $(y_n)_{n \geq 1} \subset D$ converge to ∞ in the topology of $\mathfrak{X}$. Since $D \cup \partial_M D$ is a compact metric space with the Martin metric d_M, there exist a subsequence $(y_{n_k})_{k \geq 1}$ and $w \in D \cup \partial_M D$ such that $\lim_{k \to \infty} d_M(y_{n_k}, w) = 0$. Clearly, $w \notin D$ (since relative topologies on D are equivalent). Thus we have found an unbounded sequence $(y_{n_k})_{k \geq 1} \subset D$ which converges to $w \in \partial_M D$ in the Martin topology and to ∞ in the topology of $\mathfrak{X}$.

Let $w \in \partial_M^\infty D$ and let $M_D(\cdot, w)$ be the corresponding Martin kernel. If $(y_n)_{n \geq 1}$ is a sequence in D converging to w in the Martin topology and to ∞ in the topology of $\mathfrak{X}$, then, by (M3)(a), $M_D(x, y_n)$ converges to $M_D(x, w)$. On the other hand, since y_n converges ∞ in the topology of $\mathfrak{X}$, by Lemma 3.2, $\lim_{n \to \infty} M_D(x, y_n) = M_D(x, \infty)$. Hence, for each $w \in \partial_M^\infty D$ it holds that $M_D(\cdot, w) = M_D(\cdot, \infty)$. Since, by (M3)(d), for two different Martin boundary points $w^{(1)}$ and $w^{(2)}$ it always holds that $M_D(\cdot, w^{(1)}) \neq M_D(\cdot, w^{(2)})$, we conclude that $\partial_M^\infty D$ consists of exactly one point. □

Proof of Theorem 1.3(b): We claim that for every $r > 4 \max(d(z_0, x_0), R)$ and $U := D \cap B_r$ it holds that

$$M_D(x, \infty) = \mathbb{E}_x \left[M_D(X_{\tau_U}, \infty) \right], \qquad x \in U. \tag{3.4}$$

For any $z \in D^{2r}$, since $G_D(\cdot, z)$ is regular harmonic in U, we have

$$\frac{G_D(x, z)}{G_D(x_0, z)} = \mathbb{E}_x \left[\frac{G_D(X_{\tau_U}, z)}{G_D(x_0, z)} \right], \qquad x \in U.$$

Hence, in view of Lemma 3.2, in order to prove (3.4) it suffices to show that, for any fixed $x \in U$, there exists $s > 16r$ such that the family

$$\left\{ \frac{G_D(X_{\tau_U}, z)}{G_D(x_0, z)} : z \in D^{4s} \right\}$$

is uniformly integrable with respect to the distribution of X_{τ_U} under $\mathbb{P}_x$.

In the remainder of this proof, we fix an $x \in U$. Let $s > 8r$. Then for any Borel set $E \subset D^r$,

$$\begin{aligned} &\mathbb{E}_x \left[\frac{G_D(X_{\tau_U}, z)}{G_D(x_0, z)}, X_{\tau_U} \in E \right] \\ &\leq \mathbb{E}_x \left[\frac{G_D(X_{\tau_U}, z)}{G_D(x_0, z)}, X_{\tau_U} \in (D^r \setminus D^{s/3}) \cap E \right] + \mathbb{E}_x \left[\frac{G_D(X_{\tau_U}, z)}{G_D(x_0, z)}, X_{\tau_U} \in D^{s/3} \right] \\ &= : I + II \,. \end{aligned}$$

We first show that II is small for large s. Let $w \in U$ and $d(z_0, y) > 4r/3$. By $\mathbf{C2}(z_0, R)$ we have that $j(w, y) \leq c_1 j(z_0, y)$ with $c_1 = c_1(z_0, 4/3)$. It follows that

$$\begin{aligned} P_U(x, y) &= \int_U G_U(x, w) j(w, y) m(dw) \leq c_1 (\mathbb{E}_x \tau_U) j(z_0, y) \\ &\leq c_1 (\mathbb{E}_x \tau_{B_r}) j(z_0, y) = c_2 j(z_0, y), \end{aligned}$$

where $c_2 = c_2(z_0, x, r)$. Hence,

$$\begin{aligned} \mathbb{E}_x[G_D(X_{\tau_U}, z), X_{\tau_U} \in D^{s/3}] &= \int_{D^{s/3}} G_D(y, z) P_U(x, y) m(dy) \\ &\leq c_2 \int_{D^{s/3}} G_D(y, z) j(z_0, y) m(dy). \end{aligned} \tag{3.5}$$

Next, for $z \in D^{4s}$,

$$\begin{aligned} G_D(x_0, z) &\geq \int_{D^{s/3}} G_D(y, z) P_{D \setminus \overline{D}^{s/4}}(x_0, y) m(dy) \\ &= \int_{D^{s/3}} \int_{D \setminus \overline{D}^{s/4}} G_D(y, z) G_{D \setminus \overline{D}^{s/4}}(x_0, u) j(u, y)\, m(du)\, m(dy). \end{aligned}$$

Let $y \in D^{s/3}$ and $u \in D \setminus \overline{D}^{s/4}$. By $\mathbf{C2}(z_0, R)$ we have that $j(z_0, y) \leq c_4 j(u, y)$ with $c_3 = c_3(z_0)$. Continuing the above display, we get that

$$G_D(x_0, z) \geq c_3^{-1} \left(\int_{D^{s/3}} G_D(y, z) j(z_0, y) m(dy) \right) \left(\int_{D \setminus \overline{D}^{s/4}} G_{D \setminus \overline{D}^{s/4}}(x_0, u) m(du) \right). \tag{3.6}$$

By combining (3.5) and (3.6) we get that for all $z \in D^{4s}$,

$$II = \int_{D^{s/3}} \frac{G_D(y, z)}{G_D(x_0, z)} P_U(x, y) m(dy) \leq c_2 c_3 \left(\int_{D \setminus \overline{D}^{s/4}} G_{D \setminus \overline{D}^{s/4}}(x_0, u) m(du) \right)^{-1}.$$

Since

$$\lim_{s\to\infty}\int_{D\setminus\overline{D}^{s/4}} G_{D\setminus\overline{D}^{s/4}}(x_0,u)m(du)=\int_D G_D(x_0,u)m(du)=\mathbb{E}_{x_0}\tau_D=\infty,$$

we see that for any $\epsilon>0$ we can find $s>16r$ such that

$$\left(\int_{D\setminus\overline{D}^{s/4}} G_{D\setminus\overline{D}^{s/4}}(x_0,u)m(du)\right)^{-1}<\frac{\epsilon}{2c_2c_3}.$$

Thus $II<\epsilon/2$ for all $z\in D^{4s}$.

We now fix an $s>16r$ as above and estimate I for all $z\in D^{4s}$. If $y\in D^r\setminus D^{s/3}$, then both $G_D(y,\cdot)$ and $G_D(x_0,\cdot)$ are regular harmonic with respect to $\widehat{X}$ in $D^{s/2}$ and vanish on $B_{s/2}\cap(\overline{D}^c\cup\widehat{D}^{\rm reg})$. Choose $z_1\in D^s$. By the boundary Harnack principle (2.1), we have that

$$\frac{G_D(y,z)}{G_D(x_0,z)}\le c_4\frac{G_D(y,z_1)}{G_D(x_0,z_1)},\qquad z\in D^{4s}.$$

Since $z_1\in D^s$ it follows from (1.10) that $c_5:=\sup_{y\in D^r\setminus D^{s/3}}G_D(y,z_1)<\infty$. Hence,

$$\begin{aligned} I\le\ & c_4\mathbb{E}_x\left[\frac{G_D(X_{\tau_U},z_1)}{G_D(x_0,z_1)},X_{\tau_U}\in(D^r\setminus D^{s/3})\cap E\right]\\ \le\ & \frac{c_4c_5}{G_D(x_0,z_1)}\mathbb{P}_x\big(X_{\tau_U}\in(D^r\setminus D^{s/3})\cap E\big)\le c_6\mathbb{P}_x(X_{\tau_U}\in E)\,, \end{aligned}\tag{3.7}$$

where $c_6=c_4c_5/G_D(x_0,z_1)$. Thus, given $\epsilon>0$, for any set $E\subset D^r$ with $\int_E P_U(x,y)m(dy)<\epsilon/(2c_7)$, we have $I<\epsilon/2$ for all $z\in D^{4s}$.

Therefore we have proved the claimed uniform integrability for the s chosen above, and consequently (3.4).

Now let $U_1\subset D$ be any bounded open set such that $\overline{U}_1\subset D$. Then there is $r>4R$ such that $U_1\subset D\cap B_r=:U$. Then by (3.4) and the strong Markov property we have that

$$M_D(x,\infty)=\mathbb{E}_x\left[M_D(X_{\tau_{U_1}},\infty)\right],\qquad x\in U_1,\tag{3.8}$$

which finishes the proof. □

Because of Theorem 1.3(a), we will also use ∞ to denote the Martin boundary point $\partial_M^\infty D$ associated with ∞. Note that it follows from the proof of Theorem 1.3(a) that if $(y_n)_{n\ge1}$ converges to ∞ in the topology of $\mathfrak{X}$, then it also converges to ∞ in the Martin topology.

For any $\epsilon>0$, define

$$K_\epsilon^\infty:=\left\{w\in\partial_M^f D:d_M(w,\infty)\ge\epsilon\right\}.\tag{3.9}$$

By the definition of the finite part of the Martin boundary, for each $w\in K_\epsilon^\infty$ there exists a bounded sequence $(y_n^w)_{n\ge1}\subset D$ such that $\lim_{n\to\infty}d_M(y_n^w,w)=0$. Without loss of generality we may assume that $d_M(y_n^w,w)<\frac{\epsilon}{2}$ for all $n\ge1$.

Lemma 3.3 *There exists $c = c(\epsilon) > 0$ such that $d(y_n^w, z_0) \le c$ for all $w \in K_\epsilon^\infty$ and all $n \ge 1$.*

Proof. We first claim that for any sequence $(y_n)_{n\ge1}$ in D satisfying $d(y_n, z_0) \to \infty$, we have $\lim_{n\to\infty} d_M(y_n, \infty) = 0$, i.e., $(y_n)_{n\ge1}$ converges to ∞ in the Martin topology. Indeed, since $D\cup \partial_M D$ is a compact metric space, (y_n) has a convergent subsequence (y_{n_k}). Let $w = \lim_{k\to\infty} y_{n_k}$ (in the Martin topology). Then $\lim_{k\to\infty} M_D(\cdot, y_{n_k}) = M_D(\cdot, w)$. On the other hand, from Lemma 3.2 and Theorem 1.3(a) we see that $\lim_{k\to\infty} M_D(\cdot, y_{n_k}) = M_D(\cdot, \infty)$. Therefore, $M_D(\cdot, w) = M_D(\cdot, \infty)$, which implies that $w = \infty$ by (M3)(d). Since this argument also holds for any subsequence of $(y_n)_{n\ge1}$, we conclude that $y_n \to \infty$ in the Martin topology.

Now suppose the lemma is not true. Then $\{y_n^w : w \in K_\epsilon^\infty, n \in \mathbb{N}\}$ contains a sequence $(y_{n_k}^{w_k})_{k\ge1}$ such that $\lim_{k\to\infty} d(y_{n_k}^{w_k}, z_0) = \infty$. By the paragraph above, we have that $\lim_{k\to\infty} d_M(y_{n_k}^{w_k}, \infty) = 0$. On the other hand, $d_M(y_{n_k}^{w_k}, \infty) \ge d_M(w_k, \infty) - d_M(y_{n_k}^{w_k}, w_k) \ge \epsilon/2$. This contradiction proves the claim. □

Proof of Theorem 1.3(c): Let h be a positive harmonic function for X^D such that $h \le M_D(\cdot, \infty)$. By the Martin representation (3.1), there is a finite measure μ on $\partial_M D$ (concentrated on $\partial_m D$) such that

$$h(x) = \int_{\partial_M D} M_D(x, w)\, \mu(dw) = \int_{\partial_M D\setminus\{\infty\}} M_D(x, w)\, \mu(dw) + M_D(x, \infty)\mu(\{\infty\})\,.$$

In particular, $h(x_0) = \mu(\partial_M D) \le M_D(x_0, \infty) = 1$ (because of the normalization at x_0). Hence, μ is a sub-probability measure.

For $\epsilon > 0$, let K_ϵ^∞ be the closed subset of $\partial_M^f D$ defined in (3.9). Define

$$u(x) := \int_{K_\epsilon^\infty} M_D(x, w)\, \mu(dw). \tag{3.10}$$

Then u is a positive harmonic function with respect to X^D and bounded above by

$$u(x) \le h(x) - \mu(\{\infty\})M_D(x, \infty) \le \big(1 - \mu(\{\infty\})\big)M_D(x, \infty)\,. \tag{3.11}$$

We claim that $\lim_{x\to\infty} u(x) = 0$. Let $p = c(\epsilon) \vee R$, where $c(\epsilon)$ is the constant from Lemma 3.3. Hence, for $w \in K_\epsilon^\infty$ and $(y_n^w)_{n\ge1}$ a sequence such that $\lim_{n\to\infty} d_M(y_n^w, w) = 0$, it holds that $d(y_n^w, z_0) \le p$. Fix a point $x_1 \in D^{8p}$ and choose an arbitrary point $y_0 \in D_p$. Then for any $x \in D^{8p}$ and any $y \in D_p$ we have that

$$\frac{G_D(x, y)}{G_D(x_0, y)} = \frac{G_D(x, y)}{G_D(x_1, y)}\,\frac{G_D(x_1, y)}{G_D(x_0, y)} \le c_1 \frac{G_D(x, y_0)}{G_D(x_1, y_0)}\,\frac{G_D(x_1, y)}{G_D(x_0, y)},$$

where the inequality follows from the dual version of (2.1) since X satisfies $\mathbf{F2}(z_0, R)$. Therefore for each $w \in K_\epsilon^\infty$ we have

$$M_D(x, w) \;=\; \lim_{n\to\infty} \frac{G_D(x, y_n^w)}{G_D(x_0, y_n^w)} \le c_1 \frac{G_D(x, y_0)}{G_D(x_1, y_0)}\, \lim_{n\to\infty} \frac{G_D(x_1, y_n^w)}{G_D(x_0, y_n^w)}$$

$$= c_1 \frac{G_D(x,y_0)}{G_D(x_1,y_0)}\, M_D(x_1,w) \le c_1 \frac{G_D(x,y_0)}{G_D(x_1,y_0)} \sup_{w\in K_\epsilon} M_D(x_1,w)$$
$$= c_2 G_D(x,y_0)$$

by the continuity of the Martin kernel (M3)(c). This inequality together with the definition of u shows that $u(x) \le c_2 G_D(x,y_0)$. Now using (1.11), we can conclude that $\lim_{x\to\infty, x\in D} u(x) = 0$ uniformly for $w \in K_\epsilon^\infty$.

Choose $r > 16p$. For any $x \in D_{r/2}$ and $y, y_1 \in D^{8r}$, by (2.1) applied to $G_D(x,\cdot)$ and $G_D(x_0,\cdot)$, we have

$$\frac{G_D(x,y)}{G_D(x_0,y)} \le c_3 \frac{G_D(x,y_1)}{G_D(x_0,y_1)}.$$

Letting $D \ni y \to \infty$, by Lemma 3.2 we get

$$M_D(x,\infty) \le c_3 \frac{G_D(x,y_1)}{G_D(x_0,y_1)} = c_4 G_D(x,y_1)\,, \quad x \in D_{r/2}. \tag{3.12}$$

Recall that by (1.8) $\lim_{D\ni x\to z} G_D(x,y) = 0$ for every $z \in \partial D$ which is regular for D^c with respect to X. Since $r > 16p$ can be arbitrarily large, we see from (3.12) and (3.11) that $\lim_{D\ni x, x\to z} u(x) = 0$ for every $z \in \partial D$ which is regular for D^c with respect to X.

Fix $r > 16p$ and $y_1 \in D^{8r}$. It follows from (1.10) that for all $x \in D_{r/2}$,

$$G_D(x,y_1) \le c_5\,. \tag{3.13}$$

From (3.11)–(3.13) we conclude that u is bounded in $x \in D_{r/2}$. Similarly by (1.10), for every $x \in D^{8p}$ we have that $G_D(x,y_0) \le c_6$ (recall $y_0 \in D_p$). Since $M_D(x,w) \le c_2 G_D(x,y_0)$ for each $x \in D^{8p}$ and each $w \in K_\epsilon^\infty$, by using (3.10) we see that u is bounded on D^{8p}. Thus u is bounded on D.

Now it follows from Lemma 3.1(b) that $u \equiv 0$ in D. This means that $\nu = \mu_{|K_\epsilon^\infty} = 0$. Since $\epsilon > 0$ was arbitrary and $\partial_M D \setminus \{\infty\} = \cup_{\epsilon>0} K_\epsilon^\infty$, we see that $\mu_{|\partial_M D\setminus\{\infty\}} = 0$. Hence $h = \mu(\{\infty\}) M(\cdot,\infty)$ showing that $M(\cdot,\infty)$ is minimal. Therefore we have proved the theorem. □

3.2 Finite part of Martin boundary

In this subsection, we deal with the oscillation reduction at a boundary point z_0 of an open set D. We will fix D and z_0 in this subsection, and use the notation $B_r = B(z_0,r)$. In this subsection, we will always assume that there exists $R \le R_0$ such that $\mathbf{C1}(z_0,R)$ holds, and that $\widehat{X}$ satisfies $\mathbf{F1}(z_0,R)$. We also assume that z_0 is accessible from D with respect to X.

Lemma 3.4 *For every $x \in D$, the limit*

$$M_D(x,z_0) := \lim_{D\ni v\to z_0} \frac{G_D(x,v)}{G_D(x_0,v)}$$

exists and is finite.

Proof. Fix $x \in D$, and let $r \le \frac{1}{2}\min\{d(z_0,x), d(z_0,x_0), R\}$. As before, let $D_r = D \cap B_r$. The functions $G_D(x,\cdot)$ and $G_D(x_0,\cdot)$ are regular harmonic in D_r with respect to $\widehat{X}$ and vanish in $B_r \cap (\overline{D}^c \cup (\widehat{D_r})^{\mathrm{reg}})$, hence by Corollary 2.12 we deduce that the limit

$$M_D(x,z_0) := \lim_{D \ni v \to z_0} \frac{G_D(x,v)}{G_D(x_0,v)}$$

exists and is finite. □

Proof of Theorem 1.1(a): We first note that $\partial_M^{z_0} D$ is not empty. Indeed, let $(y_n)_{n\ge1} \subset D$ converge to z_0 in the topology of $\mathfrak{X}$. Since $D \cup \partial_M D$ is a compact metric space with the Martin metric d_M, there exist a subsequence $(y_{n_k})_{k\ge1}$ and $w \in D \cup \partial_M D$ such that $\lim_{k\to\infty} d_M(y_{n_k}, w) = 0$. Clearly, $w \notin D$ (since relative topologies on D are equivalent). Thus we have found a sequence $(y_{n_k})_{k\ge1} \subset D$ which converges to $w \in \partial_M D$ in the Martin topology and to z_0 in the topology of $\mathfrak{X}$.

Let $w \in \partial_M^{z_0} D$ and let $M_D(\cdot, w)$ be the corresponding Martin kernel. If $(y_n)_{n\ge1}$ is a sequence in D converging to w in the Martin topology and to z_0 in the topology of $\mathfrak{X}$, then, by (M3)(a), $M_D(x, y_n)$ converge to $M_D(x,w)$. On the other hand, $d(y_n, z_0) \to 0$, thus by Lemma 3.4, $\lim_{n\to\infty} M_D(x,y_n) = M_D(x,z_0)$. Hence, for each $w \in \partial_M^{z_0} D$ it holds that $M_D(\cdot,w) = M_D(\cdot,z_0)$. Since, by (M3)(d), for two different Martin boundary points $w^{(1)}$ and $w^{(2)}$ it always holds that $M_D(\cdot,w^{(1)}) \ne M_D(\cdot,w^{(2)})$, we conclude that $\partial_M^{z_0} D$ consists of exactly one point. □

Proof of Theorem 1.1(b): We claim that for every $r \le \frac{1}{4}\min\{d(z_0,x_0), R\}$ and $U := D \setminus \overline{B}_r$ it holds that

$$M_D(x,z_0)) = \mathbb{E}_x\left[M_D(X_{\tau_U}, z_0)\right], \qquad x \in U. \tag{3.14}$$

For any $z \in D_{r/2}$, since $G_D(\cdot,z)$ is regular harmonic in U, we have

$$\frac{G_D(x,z)}{G_D(x_0,z)} = \mathbb{E}_x\left[\frac{G_D(X_{\tau_U},z)}{G_D(x_0,z)}\right], \quad x \in U.$$

Hence, in view of Lemma 3.4, in order to prove (3.14) it suffices to show that, for any fixed $x \in U$, there exists $s < r/(16)$ such that the family

$$\left\{\frac{G_D(X_{\tau_U},z)}{G_D(x_0,z)} : z \in D_{s/4}\right\}$$

is uniformly integrable with respect to the distribution of X_{τ_U} under $\mathbb{P}_x$.

In the remainder of this proof we fix an $x \in U$. Let $0 < s < r/8$. Then for any Borel set $E \subset D_r$,

$$\begin{aligned}\mathbb{E}_x\left[\frac{G_D(X_{\tau_U},z)}{G_D(x_0,z)}, X_{\tau_U} \in E\right] \;\le\; & \mathbb{E}_x\left[\frac{G_D(X_{\tau_U},z)}{G_D(x_0,z)}, X_{\tau_U} \in (D_r \setminus D_{3s}) \cap E\right] \\ & + \mathbb{E}_x\left[\frac{G_D(X_{\tau_U},z)}{G_D(x_0,z)}, X_{\tau_U} \in D_{3s}\right]\end{aligned}$$

$$= \quad : I + II\,.$$

We first show that II is small for small s. We claim that $P_U(x,\cdot)$ is bounded on $D_{3r/4}$. Indeed, let $y \in D_{3r/4}$. If $w \in U$, then by $\mathbf{C1}(z_0,R)$, we have that $j(w,y) \le c_1 j(w,z_0)$ where $c_1 = c_1(z_0,4/3)$. Hence,

$$\begin{aligned} P_U(x,y) &= \int_U G_U(x,w)j(w,y)m(dw) \le c_1 \int_U G_U(x,w)j(w,z_0)m(dw) \\ &= c_1 P_U(x,z_0) =: c_2. \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{E}_x[G_D(X_{\tau_U},z)), X_{\tau_U} \in D_{3s}] &= \int_{D_{3s}} G_D(y,z)P_U(x,y)m(dy) \\ &\le c_2 \int_{D_{3s}} G_D(y,z)m(dy). \end{aligned} \tag{3.15}$$

Next, for $z \in D_{s/4}$,

$$\begin{aligned} G_D(x_0,z) &\ge \int_{D_{3s}} G_D(y,z)P_{D\setminus\overline{D}_{4s}}(x_0,y)m(dy) \\ &= \int_{D_{3s}}\int_{D\setminus\overline{D}_{4s}} G_D(y,z)G_{D\setminus\overline{D}_{4s}}(x_0,u)j(u,y)\,m(du)\,m(dy). \end{aligned}$$

Let $y \in D_{3s}$ and $u \in D\setminus\overline{D}_{4s}$. By $\mathbf{C1}(z_0,R)$, we have that $j(u,z_0) \le c_3 j(u,y)$ with $c_3 = c_3(z_0)$. Continuing the above display, we get that

$$G_D(x_0,z) \ge c_3^{-1}\left(\int_{D_{3s}} G_D(y,z)m(dy)\right)\left(\int_{D\setminus\overline{D}_{4s}} G_{D\setminus\overline{D}_{4s}}(x_0,u)j(u,z_0)m(du)\right). \tag{3.16}$$

Combining (3.15) and (3.16) we arrive at

$$\begin{aligned} II &= \int_{D_{3s}} \frac{G_D(y,z)}{G_D(x_0,z)} P_U(x,y)m(dy) \\ &\le c_2c_3\left(\int_{D\setminus\overline{D}_{4s}} G_{D\setminus\overline{D}_{4s}}(x_0,u)j(u,z_0)m(du)\right)^{-1}. \end{aligned}$$

Since z_0 is accessible from D,

$$\begin{aligned} &\lim_{s\to 0}\int_{D\setminus\overline{D}_{4s}} G_{D\setminus\overline{D}_{4s}}(x_0,u)j(u,z_0)m(du) \\ &= \int_D G_D(x_0,u)j(u,z_0)m(du) = P_D(x_0,z_0) = \infty. \end{aligned}$$

Therefore, for any $\epsilon > 0$ one can find $s > 0$ such that

$$\left(\int_{D\setminus\overline{D}_{4s}} G_{D\setminus\overline{D}_{4s}}(x_0,u)j(u,z_0)m(du)\right)^{-1} < \frac{\epsilon}{2c_2c_3}.$$

Thus $II < \epsilon/2$ for all $z \in D_{s/4}$.

We fix $s < r/16$ as above and estimate I for all $z \in D_{s/4}$. Choose $z_1 \in D_s$. If $y \in D_r \setminus D_{3s}$, then both $G_D(y,\cdot)$ and $G_D(x_0,\cdot)$ are regular harmonic in D_{2s} with respect to $\widehat{X}$ and vanish on $B_{2s} \cap (\overline{D}^c \cup \widehat{(D_{2s})}^{\rm reg})$. Hence, by the boundary Harnack principle (2.15), we have that

$$\frac{G_D(y,z)}{G_D(x_0,z)} \le c_4 \frac{G_D(y,z_1)}{G_D(x_0,z_1)}, \qquad z \in D_{s/4}.$$

Since $z_1 \in D_s$ it follows from (1.7) that $c_5 := \sup_{y \in D_r \setminus D_{3s}} G_D(y,z_1) < \infty$. Hence,

$$\begin{aligned} I &\le c_4 \mathbb{E}_x \left[\frac{G_D(X_{\tau_U}, z_1)}{G_D(x_0,z_1)}, X_{\tau_U} \in (D_r \setminus D_{3s}) \cap E \right] \\ &\le \frac{c_4 c_5}{G_D(x_0,z_1)} \mathbb{P}_x\big(X_{\tau_U} \in (D_r \setminus D_{3s}) \cap E\big) \le c_6 \mathbb{P}_x(X_{\tau_U} \in E), \end{aligned} \tag{3.17}$$

where $c_6 = c_4 c_5 / G_D(x_0,z_1)$. Thus, given $\epsilon > 0$, for any set $E \subset D_r$ with $\mathbb{P}_x(X_{\tau_U} \in E) < \epsilon/(2c_6)$ we have that $I < \epsilon/2$ for all $z \in D_{s/4}$.

Therefore we have proved the claimed uniform integrability for the s chosen above, and consequently (3.14).

Now let $U_1 \subset D$ be any open set such that z_0 is not in $\overline{U}_1$. Then there is $r > 0$ such that $U_1 \subset D \setminus \overline{B}_r =: U$. Then by (3.14) and the strong Markov property we get that

$$M_D(x,z_0) = \mathbb{E}_x \left[M_D(X_{\tau_{U_1}}, z_0) \right], \qquad x \in U_1, \tag{3.18}$$

which finishes the proof. □

Because of Theorem 1.1(a), we will also use z_0 to denote the Martin boundary point $\partial_M^{z_0} D$ associated with $z_0 \in \partial D$. Note that it follows from the proof of Theorem 1.1(a) that if $(y_n)_{n\ge1}$ converges to z_0 in the topology of $\mathfrak{X}$, then it also converges to z_0 in the Martin topology.

For any $\epsilon > 0$, define

$$K_\epsilon^{z_0} := \left\{ w \in \partial_M^f D : d_M(w,z_0) \ge \epsilon \right\}. \tag{3.19}$$

By the definition of the finite part of the Martin boundary, for each $w \in K_\epsilon^{z_0}$ there exists a bounded sequence $(y_n^w)_{n\ge1} \subset D$ such that $\lim_{n\to\infty} d_M(y_n^w, w) = 0$. Without loss of generality we may assume that $d_M(y_n^w, w) < \frac{\epsilon}{2}$ for all $n \ge 1$.

Lemma 3.5 *There exists $c = c(\epsilon) > 0$ such that $d(y_n^w, z_0) \ge c$ for all $w \in K_\epsilon^{z_0}$ and all $n \ge 1$.*

Proof. Suppose the lemma is not true. Then $\{y_n^w : w \in K_\epsilon^{z_0}, n \in \mathbb{N}\}$ contains a subsequence $(y_{n_k}^{w_k})_{k\ge1}$ such that $\lim_{k\to\infty} d(y_{n_k}^{w_k}, z_0) = 0$. We also have $\lim_{k\to\infty} d_M(y_{n_k}^{w_k}, z_0) = 0$. On the other hand,

$$d_M(y_{n_k}^{w_k}, z_0) \ge d_M(w_k, z_0) - d_M(y_{n_k}^{w_k}, w_k) \ge \epsilon/2.$$

This contradiction proves the claim. □

Proof of Theorem 1.1(c): Let h be a positive harmonic function for X^D such that $h \le M_D(\cdot, z_0)$. By the Martin representation (3.1), there is a finite measure μ on $\partial_M D$ (concentrated on $\partial_m D$) such that

$$h(x) = \int_{\partial_M D} M_D(x,w)\,\mu(dw) = \int_{\partial_M D\setminus\{z_0\}} M_D(x,w)\,\mu(dw) + M_D(x,z_0)\mu(\{z_0\})\,.$$

In particular, $\mu(\partial_M D) = h(x_0) \le M_D(x_0,z_0) = 1$ (because of the normalization at x_0). Hence, μ is a sub-probability measure.

For $\epsilon > 0$, let $K_\epsilon^{z_0}$ be the closed subset of $\partial_M D$ defined in (3.19). Define

$$u(x) := \int_{K_\epsilon^{z_0}} M_D(x,w)\,\mu(dw). \tag{3.20}$$

Then u is a positive harmonic function with respect to X^D satisfying

$$u(x) \le h(x) - \mu(\{z_0\})M_D(x,z_0) \le \big(1-\mu(\{z_0\})\big)M_D(x,z_0)\,. \tag{3.21}$$

Let $p = c(\epsilon)\wedge R$, where $c(\epsilon)$ is the constant from Lemma 3.5. Hence, for $w \in K_\epsilon^{z_0}$ and $(y_n^w)_{n\ge1}$ a sequence such that $\lim_{n\to\infty} d_M(y_n^w, w) = 0$, it holds that $d(y_n^w, z_0) \ge p$. Fix $x_1 \in D_{p/8}$ and choose arbitrary $y_0 \in D^p$. For any $x \in D_{p/8}$ and any $y \in D^p$ we have that

$$\frac{G_D(x,y)}{G_D(x_0,y)} = \frac{G_D(x,y)}{G_D(x_1,y)}\,\frac{G_D(x_1,y)}{G_D(x_0,y)} \le c_1\frac{G_D(x,y_0)}{G_D(x_1,y_0)}\,\frac{G_D(x_1,y)}{G_D(x_0,y)}\,.$$

Here the inequality follows from the dual version of (2.15) applied to functions $G_D(\cdot,y)$ and $G_D(\cdot,y_0)$ which are regular harmonic in D_p with respect to X and vanish in $B(z_0,c)\cap(\overline{D}^c \cup D^{\mathrm{reg}})$. Now fix $w \in K_\epsilon^{z_0}$ and apply the above inequality to y_n^w to get

$$\begin{aligned} M_D(x,w) &= \lim_{n\to\infty}\frac{G_D(x,y_n^w)}{G_D(x_0,y_n^w)} \le c_1\frac{G_D(x,y_0)}{G_D(x_1,y_0)}\lim_{n\to\infty}\frac{G_D(x_1,y_n^w)}{G_D(x_0,y_n^w)} \\ &= c_1\frac{G_D(x,y_0)}{G_D(x_1,y_0)}\,M_D(x_1,w) \le c_1\frac{G_D(x,y_0)}{G_D(x_1,y_0)}\sup_{w\in K_\epsilon^{z_0}} M_D(x_1,w) \\ &\le c_2\frac{G_D(x,y_0)}{G_D(x_1,y_0)} = c_3 G_D(x,y_0)\,. \end{aligned}$$

In the last inequality we used property (M3)(c) of the Martin kernel. Thus,

$$M_D(x,w) \le c_3 G_D(x,y_0)\,, \qquad x \in D_{p/8}, w \in K_\epsilon^{z_0}\,. \tag{3.22}$$

Choose $r < p/16$. For any $x \in D^{2r}$ and $y, y_1 \in D_{r/8}$, by (2.15) applied to $G_D(x,\cdot)$ and $G_D(x_0,\cdot)$, we have

$$\frac{G_D(x,y)}{G_D(x_0,y)} \le c_4\frac{G_D(x,y_1)}{G_D(x_0,y_1)}.$$

Letting $D \ni y \to z_0$, we get

$$M_D(x, z_0) \le c_4 \frac{G_D(x, y_1)}{G_D(x_0, y_1)} = c_5 G_D(x, y_1)\,, \quad x \in D^{2r}. \tag{3.23}$$

Recall that by (1.8) $\lim_{D \ni x \to z} G_D(x, y) = 0$ for every $z \in \partial D$ which is regular for D^c with respect to X. Since $r < p/16$ can be arbitrarily small, we see from (3.23) and (3.21) that $\lim_{D \ni x, x \to z} u(x) = 0$ for every $z \in \partial D$, $z \neq z_0$, which is regular for D^c with respect to X.

Assume D is bounded. Fix $r < p/16$. It follows from (1.7) that for all $x \in D^{2r}$,

$$G_D(x, y_1) \le c_7\,. \tag{3.24}$$

From (3.23) and (3.21) we conclude that u is bounded in $x \in D^{2r}$. Similarly by (1.7), for every $x \in D_{p/8}$ we have that $G_D(x, y_0) \le c_9$ (recall $y_0 \in D^p$). Hence by (3.22) and (3.20) we see that u is bounded on $D_{p/8}$. Thus u is bounded on D. Now it follows from Lemma 3.1(a) that $u \equiv 0$ in D.

If D is unbounded, we argue as follows. It follows from (3.22) and the assumption (1.9) that $\lim_{D \ni x \to \infty} M_D(x, z_0) = 0$. Hence by (3.21) $\lim_{D \ni x \to \infty} u(x) = 0$. Thus, there exists $\overline{r} \ge 2$ such that $u(x) \le 1$ for all $x \in D^{\overline{r}})$. Fix $r < p/16 \wedge 1$ and let $x \in D \cap (B(z_0, \overline{r}) \setminus B(z_0, 2r))$. By (3.23) and (1.7),

$$M_D(x, z_0) \le c_5 G_D(x, y_1) \le c_{11}\,.$$

It follows that u is bounded in $D \cap (B(z_0, \overline{r}) \setminus B(z_0, 2r))$. The proof that u is bounded on $D \cap B(z_0, p/16)$ is the same as in the case of a bounded D. Hence, u is bounded, and again we conclude from Lemma 3.1 (b) that $u \equiv 0$ in D.

We see from (3.20) that $\nu = \mu_{|K_\epsilon} = 0$. Since $\epsilon > 0$ was arbitrary and $\partial_M D \setminus \{z_0\} = \cup_{\epsilon > 0} K_\epsilon^{z_0}$, it follows that $\mu_{|\partial_M D \setminus \{z_0\}} = 0$. Therefore $h = \mu(\{z_0\}) M_D(\cdot, z_0)$ showing that $M_D(\cdot, z_0)$ is minimal. □

Proof of Corollary 1.2 (a) We first note that since D is bounded, all Martin boundary points are finite, hence $\partial_M^f D = \partial_M D$. Let $\Xi : \partial D \to \partial_M^f D$ so that $\Xi(z)$ is the unique Martin boundary point associated with $z \in \partial D$. Since every finite Martin boundary point is associated with some $z \in \partial D$, we see that Ξ is onto. We show now that Ξ is 1-1. If not, there are $z, z' \in \partial D$, $z \neq z'$, such that $\Xi(z) = \Xi(z') = w$. Then $M_D(\cdot, z) = M_D(\cdot, w) = M_D(\cdot, z')$. Choose $r > 0$ small enough and satisfying $r < d(z, z')/4$. By (3.23) and (3.24) we see that there exists a constant $c_1 = c_1(z)$ such that $M_D(x, z) \le c_1$ for all $x \in D \setminus B(z, 2r)$. Similarly, there exists $c_2 = c_2(z')$ such that $M_D(x, z') \le c_2$ for all $x \in D \setminus B(z', 2r)$. Since $B(z, 2r)$ and $B(z', 2r)$ are disjoint, we conclude that $M_D(\cdot, z) = M_D(\cdot, z')$ is bounded on D by $c_1 \vee c_2$. Again by (3.23), $\lim_{D \ni x \to \zeta} M_D(x, z) = 0$ for all regular $\zeta \in \partial D$. In case of unbounded D, we showed in the proof of Theorem 1.1(b) that $\lim_{x \to \infty} M_D(x, z) = 0$. Hence by Lemma 3.1 we conclude that $M_D(\cdot, z) \equiv 0$. This is a contradiction with $M_D(x_0, z) = 1$.

The statement about the minimal Martin boundary follows from Theorem 1.1(c).

(b) We will show that $\Xi : \partial D \to \partial_M^f D$ is actually a homeomorphism. Let $z_0 \in \partial D$ and $x \in D$. Choose $r < \frac{1}{2}\min\{R, \operatorname{dist}(x, z_0), \operatorname{dist}(x_0, z_0)\}$ so that $x \in D \setminus B(z_0, 2r)$. It follows from Lemma 3.4 that for any $s < 1$ and $y \in D_{sr}$,

$$\Big|\frac{G_D(x,y)}{G_D(x_0,y)} - M_D(x,z_0)\Big| \le M_D(x,z_0)\left(\mathrm{RO}_{D_{sr}}\frac{G_D(x,\cdot)}{G_D(x_0,\cdot)} - 1\right). \tag{3.25}$$

Let $s < 1$ and $z' \in \partial D \cap B(z_0, sr/2)$. It follows from Lemma 3.4 that there exists $M_D(x,z') = \lim_{D\ni y\to z'} M_D(x,y)$. Letting $y \to z'$ in (3.25) we get that

$$|M_D(x,z') - M_D(x,z_0)| \le M_D(x,z_0)\left(\mathrm{RO}_{D_{sr}}\frac{G_D(x,\cdot)}{G_D(x_0,\cdot)} - 1\right).$$

Together with Proposition 2.11 we get that if $(z_n)_{n\ge1}$ is a sequence of points in ∂D converging to $z_0 \in \partial D$, then $M_D(\cdot, z_0) = \lim_{n\to\infty} M_D(\cdot, z_n)$.

In order to show that Ξ is continuous we proceed as follows. Let $z_n \to z_0$ in ∂D. Since $\partial_M D$ is compact, $(\Xi(z_n))_{n\ge1}$ has a subsequence $(\Xi(z_{n_k}))_{k\ge1}$ converging in the Martin topology to some $w \in \partial_M D$. By property (M3), $M_D(\cdot, \Xi(z_{n_k})) \to M_D(\cdot, w)$. On the other hand, by the first part of the proof, $M_D(\cdot, \Xi(z_{n_k})) = M_D(\cdot, z_{n_k}) \to M_D(\cdot, z_0)$, implying that $w = \Xi(z_0)$. This shows in fact that $(\Xi(z_n))_{n\ge1}$ is convergent with the limit $\Xi(z_0)$. Using the fact that ∂D is compact, the proof of the continuity of the inverse is similar.
(c) The Martin representation for non-negative harmonic functions is now a consequence of the general result from [28], cf. (3.1). □

Proof of Corollary 1.4 (a) Assume that $w \in \partial_M^\infty D \cap \partial_M^f D$. Then there exist an unbounded sequence $(y_n)_{n\ge1} \subset D$ and a bounded sequence $(z_n)_{n\ge1} \subset D$ both converging to w in the Martin topology. Since there is a subsequence $(y_{n_k})_{k\ge1}$ such that $y_{n_k} \to \infty$, we have that $w = \infty$, i.e., $M_D(\cdot, w) = M_D(\cdot, \infty)$. Similarly, there is a subsequence $(z_{n_k})_{k\ge1}$ and $z \in \partial D$ such that $z_{n_k} \to z$, hence $M_D(\cdot, w) = M(\cdot, z)$. This implies that $M_D(\cdot, \infty) = M_D(\cdot, z)$. We are going to show now that this is impossible. The proof of this fact is similar to the proof of Corollary 1.2(a).

As in the proof of Theorem 1.1(c), choose r small enough so that $M_D(x,z) \le c_1$ for all $x \in D \setminus B(z, 2r)$, cf. (3.23) and (3.24). Let $z_0 \in \mathfrak{X}$ be the point in the statement of Theorem 1.3. As in the proof of Theorem 1.3(c), choose r' large enough satisfying $r' > 2(d(z,z_0) + 4r)$ so that $M_D(x,\infty) \le c_2$ for all $x \in D \cap B(z_0, r'/2)$, cf. (3.12) and (3.13). Since $(D \setminus B(z,2r)) \cup B(z_0, r'/2) = D$, we conclude that $M_D(\cdot,\infty) = M_D(\cdot, z)$ is bounded on D by $c_1 \vee c_2$. In the same way as in the proof of Corollary 1.2(a) we conclude that $M_D(\cdot, z) \equiv 0$ which is a contradiction.
(b) In the proof of Corollary 1.2(a) we defined the mapping $\Xi : \partial D \to \partial_M^f D$ and showed that it is 1-1 and onto. By inspecting the proof of Corollary 1.2(b), we can see that it carries over to the case when D is unbounded. Hence, Ξ is a homeomorphism from ∂D to $\partial_M^f D$. Let $\partial D \cup \{\partial_\infty\}$ be the one-point compactification of ∂D. Extend Ξ to this compactification by defining $\Xi(\partial_\infty) = \infty \in \partial_M^\infty D$. By part (a), Ξ is 1-1 and onto.

Let $x \in D$. Choose $r > 2\max\{R, \mathrm{dist}(x, z_0), \mathrm{dist}(x_0, z_0)\}$ so that $x \in D \cap B(z_0, r/2)$. It follows from Lemma 3.2 that for any $s > 1$

$$\left| \frac{G_D(x,y)}{G_D(x_0,y)} - M_D(x,\infty) \right| \le M_D(x,\infty) \left(\mathrm{RO}_{D^{sr}} \frac{G_D(x,\cdot)}{G_D(x_0,\cdot)} - 1 \right), \qquad y \in D^{sr}. \tag{3.26}$$

Let $s > 1$ and $z' \in \partial D \cap B(z_0, 2sr)^c$. It follows from Lemma 3.4 that there exists $M_D(x, z') = \lim_{D \ni y \to z'} M_D(x, y)$. Letting $y \to z'$ in (3.26) we get that

$$|M_D(x,z') - M_D(x,\infty)| \le M_D(x,\infty) \left(\mathrm{RO}_{D^{sr}} \frac{G_D(x,\cdot)}{G_D(x_0,\cdot)} - 1 \right).$$

Together with Proposition 2.5 we get that if $(z_n)_{n\ge1}$ is a sequence of points in ∂D converging to ∞, then $M_D(\cdot,\infty) = \lim_{n\to\infty} M_D(\cdot, z_n)$. 1.2(b). The rest of the proof of (b) and the proof of (c) is exactly the same as the proof of Corollary 1.2(b) and (c), respectively.

4 Examples

Several classes of Feller processes satisfying the assumptions of [26] were studied in that paper. These examples include some symmetric and isotropic Lévy processes in $\mathbb{R}^d$, strictly stable (not necessarily symmetric) processes in $\mathbb{R}^d$, processes obtained by subordinating a Feller diffusion on unbounded Ahlfors regular n-spaces, and space non-homogeneous processes on $\mathbb{R}^d$ whose Dirichlet form is comparable to the Dirichlet forms of certain subordinate Brownian motions. Since the conditions of the present paper are implied by the conditions of [26], we refer the readers to that paper for details. Here we will focus on certain symmetric and isotropic Lévy processes where we can say more regarding accessible boundary points, and a class of subordinate Brownian motions not covered by [15].

4.1 Symmetric and isotropic Lévy processes

Let $X = (X_t, \mathbb{P}_x)$ be a purely discontinuous symmetric Lévy process in $\mathbb{R}^d$ with Lévy exponent $\Psi(\xi)$ so that

$$\mathbb{E}_x \left[e^{i\xi\cdot(X_t - X_0)} \right] = e^{-t\Psi(\xi)}, \qquad t > 0, x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$

Thus the state space $\mathfrak{X} = \mathbb{R}^d$, the measure m is the d-dimensional Lebesgue measure and the localization radius $R_0 = \infty$. Assume that $r \mapsto j_0(r)$ is a strictly positive and nonincreasing function on $(0,\infty)$ satisfying

$$j_0(r) \le c j_0(r+1), \qquad r > 1\,, \tag{4.1}$$

for some $c > 1$ and that the Lévy measure of X has a density j such that

$$\gamma^{-1} j_0(|y|) \le j(y) \le \gamma j_0(|y|), \qquad y \in \mathbb{R}^d \tag{4.2}$$

for some $\gamma \geq 1$. Since $\int_0^\infty j_0(r)(1 \wedge r^2)r^{d-1}dr < \infty$ by (4.2), the function $x \to j_0(|x|)$ is the Lévy density of an isotropic unimodal Lévy process whose characteristic exponent is

$$\Psi_0(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))j_0(|y|)dy. \tag{4.3}$$

The Lévy exponent Ψ can be written as

$$\Psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))j(y)dy$$

and, clearly by (4.2), it satisfies

$$\gamma^{-1}\Psi_0(|\xi|) \leq \Psi(\xi) \leq \gamma\Psi_0(|\xi|), \quad \text{for all } \xi \in \mathbb{R}^d. \tag{4.4}$$

Under the above assumptions, the process X satisfies Assumptions **(A)** and **(C)** (with $j(y,z) = \widehat{j}(y,z) = j(z-y)$), It also satisfies the assumption **(B)**, **B1-a**$(0,R)$, **B1-b**$(0,R)$, **B1-c**$(0,R)$, **B2-a**$(0,R)$ of [26] (for some $R > 0$); see [26] for more details.

Assume further that Ψ_0 satisfies the following scaling condition at infinity:

H1: There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \left(\frac{t}{s}\right)^{2\delta_1} \leq \frac{\Psi_0(t)}{\Psi_0(s)} \leq a_2 \left(\frac{t}{s}\right)^{2\delta_2}, \quad t \geq s \geq 1. \tag{4.5}$$

Then by (15) and Corollary 22 in [5], for every $R > 0$, there exists $c = c(R) > 1$ such that

$$c^{-1}\frac{\Psi_0(r^{-1})}{r^d} \leq j(r) \leq c\frac{\Psi_0(r^{-1})}{r^d} \quad \text{for } r \in (0, R]. \tag{4.6}$$

Let $\Phi(r) = (\Psi_0(r^{-1}))^{-1}$. Using (4.1) and (4.6), one can easily see that there exists $R > 0$ such that Assumption **C1**$(0,R)$ is satisfied. It is shown in Example 5.1 in [26] that X also satisfies assumptions **C1**$(0,R)$ and **D1**$(0,R)$ of that paper (for some $R > 0$). Consequently, Theorem 4.1 of [26] is valid which is precisely the assumption **F1**$(0,R)$. Further, it follows from Lemma 2.7 in [24] that (1.7) is also satisfied. Using **F1**$(0,R)$ and the fact that open balls are Greenian, we can apply Proposition 6.5 in [26]. Thus for any Greenian open set D, $\lim_{x\to z} G_D(x,y) = 0$ for every regular point $z \in \partial D$, so (1.8) holds. In case of an unbounded D we assume that X is transient. Then $\lim_{x\to\infty} G_D(x,y) = 0$ by Lemma 2.10 in [24]. We conclude that Theorem 1.1 and Corollary 1.2 apply.

Instead of **H1**, assume that Ψ_0 satisfies the following scaling condition at zero:

H2: There exist constants $0 < \delta_3 \leq \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \left(\frac{t}{s}\right)^{2\delta_3} \leq \frac{\Psi_0(t)}{\Psi_0(s)} \leq a_4 \left(\frac{t}{s}\right)^{2\delta_4}, \quad s \leq t \leq 1. \tag{4.7}$$

It is shown in Example 5.1 of [26] that for every $R > 0$ there exists $c = c(R) > 1$ such that

$$c^{-1}\frac{\Psi_0(r^{-1})}{r^d} \leq j_0(r) \leq c\frac{\Psi_0(r^{-1})}{r^d} \quad \text{for } r \in [R, \infty). \tag{4.8}$$

Together with (4.2) this implies that there is $R > 0$ such that $\mathbf{C2}(0, R)$ is true.

Let $d \geq 3$. Then X is transient and let $G(x) = G(x, 0)$ be its Green function. Then by Lemma 2.10 of [24], (1.10) holds. Assume that there exists a non-increasing function $r \mapsto G_0(r)$ and a constant $c \geq 1$ such that

$$c^{-1} G_0(|x|) \leq G(x) \leq c G_0(|x|)\,, \qquad x \in \mathbb{R}^d\,. \tag{4.9}$$

It is shown in Example 5.1 of [26] that X also satisfies assumptions $\mathbf{B2\text{-}b}(0, R)$, $\mathbf{C2}(0, R)$ and $\mathbf{D2}(0, R)$ of that paper (for some $R > 0$). Consequently, Theorem 2.1 of [26] is valid which is precisely the assumption $\mathbf{F2}(0, R)$. If we assume that the Green function of X is continuous then using the upper bound $G(x) \leq c|x|^{-1}\Psi_0(|x|^{-1})^{-1}$ in (5.16) in [26] and the strong Markov property, the Green function of X^D is continuous for all open set D. Thus by Proposition 6.2 of [26], (1.8) holds. Further, it follows from (5.16) in [26] that, if $d \geq 3$, (1.11) is also satisfied. We conclude that Theorem 1.3 applies for $d \geq 3$ under the assumption that G is continuous and satisfies (4.9). In fact, it is also shown in Example 5.1 of [26] that, if X is a subordinate Brownian motion whose Laplace exponent ϕ is a complete Bernstein function and that $\xi \to \phi(|\xi|^2)$ satisfies Assumption **H2**, then (1.11) is satisfied for $d > 2\delta_4$. Since $G(x) = g(|x|)$ is continuous and $r \mapsto g(r)$ is decreasing, in this case Theorem 1.3 applies for $d > 2\delta_4$.

In the next proposition we give a criterion for the accessibility of infinity and a finite boundary point. Let $B_r = B(0, r)$ and for an open set D, $D_r = D \cap B_r$ and $D^r = D \cap \overline{B}_r^c$.

Proposition 4.1 *(a) Let $D \subset \mathbb{R}^d$ be a Greenian open set such that $0 \in \partial D$ and assume that* **H1** *holds. Then 0 is inaccessible from D with respect to X if and only if*

$$\int_{D_1} (\mathbb{E}_y \tau_{D_1}) j(y)\, dy < \infty\,. \tag{4.10}$$

(b) Let $D \subset \mathbb{R}^d$ be a Greenian open set and assume that **H2** *holds. Then ∞ is inaccessible from D with respect to X if and only if*

$$\int_{D^1} P_{D^1}(y, 0)\, dy < \infty\,. \tag{4.11}$$

Proof. (b) Recall that ∞ is inaccessible from D if there exists $x \in D$ such that $\mathbb{E}_x \tau_D < \infty$. Let $r = \max(2|x|, R, 1)$ where $R > 0$ is the constant from $\mathbf{C2}(0, R)$ and $\mathbf{F2}(0, R)$. We write

$$\begin{aligned} \mathbb{E}_x \tau_D &= \int_{B(x,4r)} G_D(x, y)\, dy + \int_{D_{8r} \setminus B(x,4r)} G_D(x, y)\, dy + \int_{D^{8r}} G_D(x, y)\, dy \\ &=: I + II + III. \end{aligned}$$

Since X^D is transient, $I = G_D \mathbf{1}_{B(x,4r)}$ is bounded, hence finite. By (1.10) we have that $G_D(x, y) \leq c(r)$ for $y \in D_{8r} \setminus B(x, 4r)$, hence $II \leq c(r)|D_{8r}| < \infty$. Since $G_D(x, \cdot)$ is regular

harmonic in D^r, it follows from $\mathbf{F2}(0,R)$ that $G_D(x,y) \asymp P_{D^r}(y,0)$ for all $y \in D^{8r}$. Thus $III \asymp \int_{D^{8r}} P_{D^r}(y,0)\,dy$. Hence, $\mathbb{E}_x\tau_D < \infty$ if and only if $\int_{D^{8r}} P_{D^r}(y,0)\,dy < \infty$. Next,

$$\begin{aligned}\int_{D^1} P_{D^1}(y,0)\,dy &= \int_{D\cap\{1<|y|\le 8r\}} P_{D^1}(y,0)\,dy + \int_{D^{8r}} P_{D^1}(y,0)\,dy \\ &\le \int_{\int_{D\cap\{1<|y|\le 8r\}}} P_{B^1}(y,0)\,dy + \int_{D^{8r}} P_{D^1}(y,0)\,dy \\ &= : IV + V\,.\end{aligned}$$

By Proposition 3.1 of [26], $P_{B^1}(y,0) \le c_1$ for all $y \in B^1$, hence $IV \le c_1|B_{8r}| < \infty$. Finally, by repeatedly applying Lemma 3.9 of [26] we deduce that

$$P_{D^r}(y,0) \le P_{D^1}(y,0) \le c_2 P_{D^r}(y,0)\,, \quad y \in D^{8r},$$

with a constant $c_2 > 0$ depending on r. Thus, V is comparable to $\int_{D^{8r}} P_{D^r}(y,0)\,dy$, proving that $\int_{D^1} P_{D^1}(y,0)\,dy < \infty$ if and only if $\int_{D^{8r}} P_{D^r}(y,0)\,dy < \infty$. This finishes the proof.
(a) This can be proved in the similar way, so we omit the proof. □

Remark 4.2 (a) Note that the criterion in Proposition 4.1 does not depend on $x \in D$. Hence, if $\mathbb{E}_x\tau_D < \infty$ for one $x \in D$, then $\mathbb{E}_x\tau_D < \infty$ for all $x \in D$. Similarly, if $P_D(x,0) < \infty$ for one $x \in D$, then $P_D(x,0) < \infty$ for all $x \in D$.
(b) By inspecting the proof of Proposition 4.1 one can see that it carries over to the case of the process satisfying the assumptions in [26]. In particular, $x \mapsto \mathbb{E}_x\tau_D$ (respectively $x \mapsto P_D(x,z_0)$) is either identically infinite or finite for all $x \in D$.

For any open set V, let $s_V(x) = \mathbb{E}_x\tau_V$ and let $\omega_V^x = \mathbb{P}_x(X_{\tau_V} \in \cdot)$ be the harmonic measure. By the strong Markov property, for $V \subset D$ we have

$$s_D(x) = s_V(x) + \int_{D\setminus V} s_D(y)\omega_V^x(dy)\,.$$

If $\partial V \cap D$ is Lipschitz, it follows from [31] that $\omega_V^x(\partial V) = 0$ and hence

$$s_D(x) = s_V(x) + \int_{D\setminus V} s_D(y)P_V(x,y)dy\,. \tag{4.12}$$

We now record the following lower bound on the expected exit time from a ball: There exists a constant $c > 0$ such that for every $r > 0$ and every $x \in \mathbb{R}^d$

$$\mathbb{E}_x\tau_{B(x,r)} \ge \frac{c}{\Psi_0(r^{-1})}\,. \tag{4.13}$$

This follows, for example, from the last display in the proof of Theorem 2.2 of [8] and the proof of Lemma 13.4.2 in [20].

Let $\kappa \in (0, 1/2]$. Recall that an open set D in $\mathbb{R}^d$ is said to be κ-fat at $z_0 \in \partial D$ if there exists $r_0 > 0$ such that for every $r \in (0, r_0]$ there exists $A_r \in D$ such that $B(A_r, \kappa r) \subset D \cap B(0, r)$. An open set D in $\mathbb{R}^d$ is said to be κ-fat at infinity if there exists $r_0 > 0$ such that for every $r \geq r_0$ there exists $A_r \in D$ such that $B(A_r, \kappa r) \subset D \cap \overline{B}(0, r)^c$ and $|A_r| < \kappa^{-1} r$, cf. Definition 1.3 in [23].

Proposition 4.3 *(a) Suppose that* **H1** *holds. If $D \subset \mathbb{R}^d$ is κ-fat at $z_0 \in \partial D$, then z_0 is accessible from D with respect to X.*
(b) Suppose that **H2** *holds. If $D \subset \mathbb{R}^d$ is κ-fat at infinity, then infinity is accessible from D with respect to X.*

Proof. We prove part (b). The proof of (a) is similar.

Let $A_0 = A_{r_0}$ be a point in D such that $B(A_0, \kappa r_0) \subset D \cap \overline{B}(0, r_0)^c$ and $|A_0| < \kappa^{-1} r_0$. We inductively define the sequence $r_n = 4\kappa^{-1} r_{n-1}$, $n \geq 1$, and a sequence of points $A_n = A_{r_n}$ such that $B(A_n, \kappa r_n) \subset D \cap \overline{B}(0, r_n)^c$ and $r_n < |A_n| < \kappa^{-1} r_n$. It is easy to see that the family of balls $(B(A_n, \kappa r_n)_{n \geq 0}$ is pairwise disjoint.

Let $U := \cup_{n=0}^{\infty} B(A_n, \kappa r_n)$. Then by (4.12) with $V = B(A_0, \kappa r_0)$,

$$\begin{aligned}
\mathbb{E}_{A_0} \tau_D &\geq \int_{D \setminus B(A_0, \kappa r_0)} s_D(y) P_{B(A_0, \kappa r_1)}(A_1 0, y)\, dy \\
&\geq \int_{D \setminus B(A_0, \kappa r_0)} s_U(y) P_{B(A_0, \kappa r_0)}(A_0, y)\, dy \\
&\geq \sum_{n=1}^{\infty} \int_{B(A_n, \kappa r_n/2)} s_{B(A_n, \kappa r_n)}(y) P_{B(A_0, \kappa r_1)}(A_0, y)\, dy \\
&\geq \sum_{n=1}^{\infty} \big(\inf_{y \in B(A_n, \kappa r_n/2)} s_{B(A_n, \kappa r_n)}(y) \big) \int_{B(A_n, \kappa r_n/2)} P_{B(A_0, \kappa r_1)}(A_0, y)\, dy.
\end{aligned}$$

By (4.13),

$$s_{B(A_n, \kappa r_n)}(y) \geq c_1 \Psi_0((\kappa r_n)^{-1})^{-1} \geq c_2 \Psi_0(r_n^{-1})^{-1}$$

for all $y \in B(A_n, \kappa r_n/2)$. Further, if $y \in B(A_n, \kappa r_n)$, then $r_n/2 \leq |y - A_0| \leq 3\kappa^{-1}$. Hence, by Lemma 3.3 of [20], (4.13), (4.2) and (4.8), we have that for $y \in B(A_n, \kappa r_n/2)$,

$$P_{B(A_0, \kappa r_0)}(A_0, y) \geq c_3 \frac{\Psi_0(|y - A_0|^{-1})}{|y - A_0|^d} \Psi_0((\kappa r_0)^{-1})^{-1} \geq c_4 \frac{\Psi_0(r_n^{-1})}{r_n^d} \Psi_0(r_1^{-1})^{-1}\,.$$

Therefore,

$$\mathbb{E}_{A_0} \tau_D \geq \sum_{n=2}^{\infty} c_5 \Psi_0(r_n^{-1})^{-1} \frac{\Psi_0(r_n^{-1})}{r_n^d} \Psi_0(r_1^{-1})^{-1} r_n^d = \infty\,.$$

By using Remark 4.2 we see that ∞ is accessible form D. □

In the next result we give a criterion for accessibility of infinity from a thorn-like domain.

Let $f : (2, \infty) \to (0, \infty)$ be a positive non-decreasing function such that $f(t) \le t$ for all $t > 0$ and define

$$D = D^f := \{(y_1, \widetilde{y}) \in \mathbb{R}^d : y_1 > 2, |\widetilde{y}| < f(y_1)\}\,.$$

Here $y = (y_1, \widetilde{y})$ with $\widetilde{y} = (y_2, \dots, y_d) \in \mathbb{R}^{d-1}$.

Proposition 4.4 *Suppose that* **H1** *and* **H2** *hold. Then infinity is accessible from D if and only if*

$$\int_4^\infty \frac{\Psi_0(t^{-1})}{\Psi_0(f(t)^{-1})} \frac{f(t)^{d-1}}{t^d}\, dt = \infty\,. \tag{4.14}$$

Proof. Assume that the integral in (4.14) is infinite. Fix $x \in D$ and denote $\delta_D(x)$ by r. Let $U := \{(y_1, \widetilde{y}) \in \mathbb{R}^d : y_1 > 4(1+|x|), |\widetilde{y}| < f(y_1)/2\}$. Since $|x-y| \ge y_1 - x_1 \ge 4(1+|x|) - x_1 > 2x_1 > 2f(x_1) > r$ for all $y \in U$, we have $U \subset D \setminus B(x, r)$.

Moreover, $B(y, f(y_1/2)/2) \subset D$ for all $y \in U$. In fact, for $z \in B(y, f(y_1/2)/2)$ with $y \in U$ we have $z_1 > y_1 - f(y_1/2)/2 > y_1/2$, which implies that $z_1 > 6$ and $f(y_1/2) \le f(z_1)$. Using the last inequality we see that $|\widetilde{z}| \le |\widetilde{y} - \widetilde{z}| + |\widetilde{y}| < f(y_1/2) \le f(z_1)$. Thus for $y \in U$,

$$s_D(y) \ge s_{B(y, f(y_1/2)/2)}(y) \ge \frac{c_1}{\Psi_0((f(y_1/2)/2)^{-1})}\,,$$

where the last inequality follows from (4.13).

Notice that for $y \in U$, $|y| \asymp y_1$. Thus, since $|z - y| \le |x| + |z - x| + |y| \le 6y_1$ for $z \in B(x, r)$, using $j(y_1) \asymp \frac{\Psi_0(y_1^{-1})}{y_1^d}$ we have

$$P_{B(x,r)}(x, y) \ge c_2 \mathbb{E}_x[\tau_{B(x,r)}] j(y_1) \asymp \frac{\Psi_0(y_1^{-1})}{y_1^d}, \qquad y \in U.$$

Therefore

$$\begin{aligned}
s_D(x) &\ge \int_U s_D(y) P_{B(x,r)}(x, y) dy \\
&\ge c_3 \int_{4(1+|x|)}^\infty \frac{\Psi_0(y_1^{-1})}{\Psi_0((f(y_1/2)/2)^{-1})} \frac{f(y_1/2)^{d-1}}{y_1^d} dy_1 \\
&= c_4 \int_{2(1+|x|)}^\infty \frac{\Psi_0(2^{-1}t^{-1})}{\Psi_0(2f(t)^{-1})} \frac{f(t)^{d-1}}{t^d} dt \\
&\ge c_5 \int_{2(1+|x|)}^\infty \frac{f(t)^{d-1}}{t^d} \frac{\Psi_0(t^{-1})}{\Psi_0(f(t)^{-1})} dt = \infty\,,
\end{aligned}$$

where the last inequality follows from Lemma 1 of [14]. Thus ∞ is accessible from D.

Assume that the integral in (4.14) is finite. For $r \ge 4$, let $D_r := D \cap B(0, r)$. Then, by Lemma 2.5 and (2.1) in [24], and considering the infinite cylinder, we get $\sup_{x_1 = r} s_{D_{4r}}(x) \le c_6 \Psi_0(f(4r)^{-1})^{-1}$. Thus, by (4.12), we have that for $x \in D$ with $x_1 = r$

$$s_D(x) = s_{D_{4r}}(x) + \int_{D \setminus D_{4r}} s_D(y) P_{D_{4r}}(x, y) dy \tag{4.15}$$

$$\leq c_6\Psi_0(f(4r)^{-1})^{-1} + \int_{D\setminus D_{4r}} s_D(y)P_{D_{4r}}(x,y)dy. \tag{4.16}$$

By the argument in the paragraph before Theorem 3.12 in [21], Lemma 5.4 in [22] is valid for all $r > 0$. Hence

$$P_{D_{4r}}(x,y) \leq c_7 s_{D_{4r}}(x)\left(\int_{D\setminus B(0,2r)} j(|z|)P_{D_{4r}}(z,y)dz + j(|y|)\right)$$

Thus using (4.15)

$$\begin{aligned}
&\int_{D\setminus D_{4r}} s_D(y)P_{D_{4r}}(x,y)dy \\
&\leq c_8 s_{D_{4r}}(x)\left(\int_{D\setminus D_{4r}} s_D(y)\int_{D\setminus B(0,2r)} j(|z|)P_{D_{4r}}(z,y)dzdy\right. \\
&\qquad \left. + \int_{D\setminus D_{4r}} s_D(y)j(|y|)dy\right) \\
&= c_8 s_{D_{4r}}(x)\left(\int_{D\setminus B(0,2r)} j(|z|)\Big(\int_{D\setminus D_{4r}} s_D(y)P_{D_{4r}}(z,y)dy\Big)dz\right. \\
&\qquad \left. + c_8\int_{D\setminus D_{4r}} s_D(y)j(|y|)dy\right) \\
&\leq c_8 s_{D_{4r}}(x)\left(\int_{D\setminus B(0,2r)} j(|z|)s_D(z)dz + \int_{D\setminus D_{4r}} s_D(y)j(|y|)dy\right) \\
&\leq 2c_8 s_{D_{4r}}(x)\int_{D\setminus B(0,2r)} j(|z|)s_D(z)dz \\
&\leq 2c_6c_8\Psi_0(f(4r)^{-1})^{-1}\int_{D\setminus B(0,2r)} j(|z|)s_D(z)dz.
\end{aligned}$$

Applying this to (4.16), we get

$$s_D(x) \leq c_9\Psi_0(f(4r)^{-1})^{-1}\left(1 + \int_{D\setminus B(0,2r)} j(|z|)s_D(z)dz\right). \tag{4.17}$$

Let $M(r) := \sup_{x_1=r} s_D(x)\Psi_0(f(4r)^{-1})$. From (4.17), for $r > 4$,

$$\begin{aligned}
M(r) &\leq c_{10}\left(1 + \int_{2r}^{\infty}\int_{|\widetilde{z}|<f(s)} |(s,\widetilde{z})|^{-d}\Psi_0(|(s,\widetilde{z})|^{-1})M(s)\Psi_0(f(4s)^{-1})^{-1}d\widetilde{z}ds\right) \\
&\leq c_{11}\left(1 + \int_{2r}^{\infty} f(s)^{d-1}s^{-d}\Psi_0(s^{-1})M(s)\Psi_0(f(4s)^{-1})^{-1}ds\right) \\
&\leq c_{11}\left(1 + \int_{r}^{\infty} f(s)^{d-1}s^{-d}\Psi_0(s^{-1})M(s)\Psi_0(f(4s)^{-1})^{-1}ds\right).
\end{aligned}$$

Let $m(r) = M(1/r)$; by a change of variable we have that for $r < 1/4$,

$$m(r) \leq c_{11}\left(1 + \int_{1/r}^{\infty} f(s)^{d-1}s^{-d}\Psi_0(s^{-1})M(s)\Psi_0(f(4s)^{-1})^{-1}ds\right)$$

$$= c_{11}\left(1 + \int_0^r f(v^{-1})^{d-1} v^d \Psi_0(v) m(v) \Psi_0(f(4v^{-1})^{-1})^{-1} v^{-2} dv\right).$$

By Gronwall's inequality, for $r < 1/4$,

$$m(r) \le c_{12} \exp\left(\int_0^r f(v^{-1})^{d-1} v^d \Psi_0(v) \Psi_0(f(4v^{-1})^{-1})^{-1} v^{-2} dv\right).$$

Therefore, under the assumption that the integral in (4.14) is finite, we have for all $x \in D$ with $x_1 = r > 4$,

$$\begin{aligned}
s_D(x) &\le \Psi_0(f(4r)^{-1}) M(r) = \Psi_0(f(4r)^{-1}) m(1/r) \\
&\le c_{12} \Psi_0(f(4r)^{-1}) \exp\left(\int_r^\infty f(s)^{d-1} s^{-d} \Psi_0(s^{-1}) \Psi_0(f(4s)^{-1})^{-1} ds\right) \\
&\le c_{12} \Psi_0(f(16)^{-1}) \exp\left(\int_4^\infty f(4s)^{d-1} s^{-d} \Psi_0(s^{-1}) \Psi_0(f(4s)^{-1})^{-1} ds\right) \\
&\le c_{12} \Psi_0(f(16)^{-1}) \exp\left(c_{13} \int_{16}^\infty f(t)^{d-1} t^{-d} \Psi_0(4t^{-1}) \Psi_0(f(t)^{-1})^{-1} ds\right) \\
&\le c_{12} \Psi_0(f(16)^{-1}) \exp\left(c_{14} \int_{16}^\infty f(t)^{d-1} t^{-d} \Psi_0(t^{-1}) \Psi_0(f(t)^{-1})^{-1} ds\right) < \infty.
\end{aligned}$$

Here the last inequality follows from Lemma 1 of [14]. Hence infinity is inaccessible. □

Suppose that $f(t) = t(\log t)^{-\beta}$, $\beta \ge 0$. Then

$$\begin{aligned}
I &= \int_4^\infty \frac{\Psi_0(t^{-1})}{\Psi_0(t^{-1}(\log t)^\beta)} \frac{t^{d-1}(\log t)^{-\beta(d-1)}}{t^d} dt \\
&= \int_4^\infty \frac{\Psi_0(t^{-1})}{\Psi_0(t^{-1}(\log t)^\beta)} (\log t)^{-\beta(d-1)} \frac{dt}{t} \\
&\ge c_1 \int_4^\infty (\log t)^{-\beta(2\delta_4 + d - 1)} \frac{dt}{t},
\end{aligned}$$

where the inequality follows from **H2**. When $\beta \le 1/(d - 1 + 2\delta_4)$, the integral above is divergent and hence infinity is accessible. Note that when $\beta > 0$, D is not κ-fat at infinity for any $\kappa \in (0, 1/2]$. Similarly,

$$I \le c_2 \int_4^\infty (\log t)^{-\beta(2\delta_3 + d - 1)} \frac{dt}{t}.$$

When $\beta > 1/(d - 1 + 2\delta_3)$, the integral above is convergent and hence the infinity is inaccessible.

A result analogous to Proposition 4.4 is valid for a finite boundary point. Let $f : (0, 1) \to (0, \infty)$ be a bounded increasing function such that $f(t) \le t$ and define

$$D_f := \{x = (x_1, \widetilde{x}) : 0 < x_1 < 1, |\widetilde{x}| < f(x_1)\}.$$

Proposition 4.5 *Assume that* **H1** *holds. Then the point* 0 *is accessible from* D *if and only if*

$$\int_0^1 \frac{\Psi_0(t^{-1})}{\Psi_0(f(t)^{-1})} \frac{f(t)^{d-1}}{t^d} dt = \infty. \tag{4.18}$$

4.2 Subordinate Brownian motions

Let $Y = (Y_t, \mathbb{P}_x)$ be a standard Brownian motion in $\mathbb{R}^d$, and $S = (S_t)$ an independent subordinator with the Laplace exponent ϕ, $\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}$. The subordinate Brownian motion $X = (X_t, \mathbb{P}_x)$ is defined as $X_t = Y(S_t)$. Assume that ϕ is a complete Bernstein function with infinite Lévy measure μ satisfying the following hypothesis

H: There exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta \in (0, 1]$ such that

$$\frac{\phi'(\lambda t)}{\phi'(\lambda)} \le \sigma\, t^{-\delta} \quad \text{for all} \quad t \ge 1 \quad \text{and} \quad \lambda \ge \lambda_0\,.$$

When $d \le 2$, assume that $d + 2\delta - 2 > 0$ and there are $\sigma' > 0$ and

$$\delta' \in \left(1 - \tfrac{d}{2}, (1 + \tfrac{d}{2}) \wedge (2\delta + \tfrac{d-2}{2})\right) \tag{4.19}$$

such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \ge \sigma'\, x^{-\delta'} \quad \text{for all } x \ge 1 \quad \text{and} \quad \lambda \ge \lambda_0\,; \tag{4.20}$$

Assumption **H** was introduced and used in [17] and [18]. It is easy to check that if ϕ is a complete Bernstein function satisfying a weak lower scaling condition at infinity

$$a_1 \lambda^{\delta_1} \phi(t) \le \phi(\lambda t) \le a_2 \lambda^{\delta_2} \phi(t)\,, \qquad \lambda \ge 1, t \ge 1\,, \tag{4.21}$$

with $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0, 1)$, then **H** is automatically satisfied. In that case the process X belongs to the class of isotropic unimodal Lévy process considered in the previous subsection. The reason for assuming hypothesis **H** here is to cover the case of geometric stable and iterated geometric stable subordinators. Suppose that $\alpha \in (0, 2)$ for $d \ge 2$ and that $\alpha \in (0, 2]$ for $d \ge 3$. A geometric $(\alpha/2)$-stable subordinator is a subordinator with Laplace exponent $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$. Let $\phi_1(\lambda) := \log(1 + \lambda^{\alpha/2})$, and for $n \ge 2$, $\phi_n(\lambda) := \phi_1(\phi_{n-1}(\lambda))$. A subordinator with Laplace exponent ϕ_n is called an iterated geometric subordinator. It is easy to check that the functions ϕ and ϕ_n satisfy **H**, but they do not satisfy (4.21).

The process X clearly satisfies assumption **A** and **C**, and by symmetry, every semipolar set is polar. Suppose that X is transient. Then it follows from Lemma 5.4 of [18] that for all $z_0 \in \mathbb{R}^d$, **C1**(z_0, R), **F1**(z_0, R), and (1.7) (with a uniform constant) hold true. Moreover, since all Green functions are continuous, by Proposition 6.2 of [26], $\lim_{x \to z} G_D(x, y) = 0$ for every regular boundary point z of ∂D. Therefore the conclusions of Corollary 1.2 hold true.

Suppose now that X is an (iterated) geometric α-stable process with $0 < \alpha < 2$. Then X satisfies condition **H2** from the previous subsection (see Example 5.1 of [26]) and by the same arguments we conclude that Theorem 1.3 is true.

5 Minimal thinness is a local property

The purpose of this section is to establish several results analogous to those in Section 9.5 of [1] and to conclude that minimal thinness is a local property.

The setting is the following: $(\mathfrak{X}, d, m)$ is a metric measure space with countable base as before. Since bounded closed sets are compact, the topology of $\mathfrak{X}$ is locally compact. Let $X = (X_t, \mathbb{P}_x)$ be a Hunt process in $\mathfrak{X}$ satisfying Assumption **A**. The cone of excessive functions with respect to X is denoted by $\mathcal{S}(X)$. We assume that $(\mathfrak{X}, \mathcal{S}(X))$ is a balayage space in the sense of [2]. Let $D \subset \mathfrak{X}$ be an open set, X^D the killed process and $\mathcal{S}(X^D)$ the cone of excessive function with respect to X^D. By Proposition V.1.1 and Proposition VI.3.20 of [2], $(D, \mathcal{S}(X^D))$ is also a balayage space in the sense of [2]. In particular, all functions in $\mathcal{S}(X^D)$ are lower semi-continuous (l.s.c.) Moreover, by definitions and results from p. 94 and Lemma III 1.2 of [2], bounded harmonic functions on D with respect to X^D are continuous. Since we will be interested only in X^D, all notions defined below are relative to X^D.

For any (numerical) function $f : D \to (-\infty, \infty]$ we define its lower semi-continuous (l.s.c.) regularization $\widehat{f}$ by

$$\widehat{f}(x) = f(x) \wedge \left(\liminf_{y \to x} f(y) \right) .$$

Then $\widehat{f}$ is the largest l.s.c. function dominated by f: $\widehat{f} \leq f$. We remark that in this section the hat $\widehat{\ }$ denotes the l.s.c regularization and *not* the notions related to the dual process. For a Borel set $A \subset D$, let $S_A = \inf\{t \geq 0 : X_t \in A\}$ be the debut of A, and $T_A = \inf\{t > 0 : X_t \in A\}$ the hitting time of A. For $u \in \mathcal{S}(X^D)$, the reduced function of u on A is defined as (see p.243 of [2]):

$$\begin{aligned} R_u^A &= \inf\{v \in \mathcal{S}(X^D) : v \geq u \text{ on } A\} \\ &= \inf\{v \in \mathcal{S}(X^D) : v \leq u, v = u \text{ on } A\} . \end{aligned}$$

Its l.s.c. regularization $\widehat{R}_u^A := \widehat{R_u^A}$ is called the balayage of u on A. Then $\widehat{R}_u^A \in \mathcal{S}(X^D)$. The probabilistic interpretations of the reduced function and the balayage are (cf. VI.3 of [2])

$$R_u^A(x) = \mathbb{E}_x[u(X_{S_A})] , \qquad \widehat{R}_u^A(x) = \mathbb{E}_x[u(X_{T_A})] .$$

We have the following properties of R_u^A and $\widehat{R}_u^A$: $R_u^A = u$ on A, $\widehat{R}_u^A \leq R_u^A \leq u$ (p.243 of [2]), $\widehat{R}_u^A = R_u^A$ on A^c (Proposition VI.2.3 of [2]), $\{\widehat{R}_u^A < R_u^A\}$ is semipolar (Proposition VI.5.11 of [2]), hence polar by **A**.

Let $u : D \to [0, \infty)$ be continuous and harmonic in D with respect to X^D, $E \subset D$ an open set, and $w : E \to [0, \infty)$ harmonic in E with respect to X^E such that $w \leq u - R_u^{D \setminus E}$. We set $w \equiv 0$ on $D \setminus E$.

Lemma 5.1 *For every bounded open set $U \subset \overline{U} \subset D$, it holds that*

$$w(x) = \mathbb{E}_x[w(X_{\tau_{U \cap E}})] , \quad x \in U \cap E .$$

Proof. We first show that there exists a polar set $N \subset \partial E \cap D$ such that for every $z \in (\partial E \cap D) \setminus N$,

$$\lim_{x \to z, x \in E} w(x) = 0 . \tag{5.1}$$

Note that $R_u^{D\setminus E}(x) = \widehat{R}_u^{D\setminus E}(x)$ for all $x \in E$. Hence by continuity of u and lower semi-continuity of $\widehat{R}_u^{D\setminus E}$,

$$\begin{aligned}\limsup_{x\to z, x\in E} w(x) &\le \limsup_{x\to z, x\in E} \big(u(x) - \widehat{R}_u^{D\setminus E}(x)\big) \\ &= u(z) - \liminf_{x\to z, x\in E} \widehat{R}_u^{D\setminus E}(x) \le u(z) - \widehat{R}_u^{D\setminus E}(z)\,.\end{aligned}$$

Let $N = \partial E \cap D \cap \{\widehat{R}_u^{D\setminus E} < R_u^{D\setminus E}\}$. Then N is polar and it follows from the last display that for all $z \in (\partial E \cap D) \setminus N$ we have

$$\limsup_{x\to z, x\in E} w(x) \le u(z) - R_u^{D\setminus E}(z) = 0\,.$$

For each $n \ge 1$ define $U_n := \{x \in U \cap E : d(x, E^c) > \frac{1}{n}\}$. Then U_n is bounded and open in E, $U_n \subset \overline{U}_n \subset U \cap E$, $\overline{U}_{n+1} \subset U_n$, and $U \cap E = \cup_{n=1}^\infty U_n$. By harmonicity of w, for any $x \in U \cap E$ and n large enough,

$$\begin{aligned} w(x) &= \mathbb{E}_x[w(X^E_{\tau_{U_n}})] = \mathbb{E}_x[w(X^E_{\tau_{U_n}}) : \tau_{U_m} = \tau_{U\cap E} \text{ for some } m \ge 1] \\ &\quad + \mathbb{E}_x[w(X^E_{\tau_{U_n}}) : \tau_{U_m} < \tau_{U\cap E} \text{ for all } m \ge 1]\,.\end{aligned}$$

Since w is dominated by u which is continuous on D, it is bounded on the relatively compact set U. Hence by the dominated convergence theorem and (5.1),

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{E}_x[w(X^E_{\tau_{U_n}}) : \tau_{U_m} < \tau_{U\cap E} \text{ for all } m \ge 1] \\ &= \mathbb{E}_x\Big[\lim_{n\to\infty} w(X^E_{\tau_{U_n}}) \mathbf{1}_{(X^E_{\tau_{U\cap E}^-} \in (\partial E\cap D)\setminus N)} : \tau_{U_m} < \tau_{U\cap E} \text{ for all } m \ge 1\Big] = 0\,.\end{aligned}$$

Further,

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{E}_x[w(X^E_{\tau_{U_n}}) : \tau_{U_m} = \tau_{U\cap E} \text{ for some } m \ge 1\} \\ &= \mathbb{E}_x\Big[\lim_{n\to\infty} w(X^E_{\tau_{U_n}}) : \tau_{U_m} = \tau_{U\cap E} \text{ for some } m \ge 1\}\Big] \\ &= \mathbb{E}_x\big[w(X^E_{\tau_{U\cap E}}) : \tau_{U_m} = \tau_{U\cap E} \text{ for some } m \ge 1\}\big] \\ &= \mathbb{E}_x[w(X^E_{\tau_{U\cap E}}) : \tau_U < \tau_E]\,.\end{aligned}$$

This proves the lemma. □

Lemma 5.2 *Let*

$$v(x) := \begin{cases} w(x) + R_u^{D\setminus E}(x)\,, & x \in E \\ u(x)\,, & x \in D \setminus E\,. \end{cases} \tag{5.2}$$

For every bounded open set $U \subset \overline{U} \subset D$ it holds that

$$\mathbb{E}_x[v(X^D_{\tau_U})] \le v(x)\,, \qquad x \in U\,. \tag{5.3}$$

Proof. We first note that $v \le (u - R_u^E) + R_u^E = u$ in E, and clearly $v = u$ in $D \setminus E$. Hence, if $x \in U \cap (D \setminus E)$, then $\mathbb{E}_x[v(X^D_{\tau_U})] \le \mathbb{E}_x[u(X^D_{\tau_U})] = u(x) = v(x)$.

Assume now that $x \in U \cap E$. Since $R_u^{D\setminus E} = u$ on $D \setminus E$, we have

$$\begin{aligned}
\mathbb{E}_x[v(X^D_{\tau_U})] &= \mathbb{E}_x[v(X^D_{\tau_U});\, X^D_{\tau_U} \in E] + \mathbb{E}_x[v(X^D_{\tau_U});\, X^D_{\tau_U} \in D \setminus E] \\
&= \mathbb{E}_x[w(X^D_{\tau_U});\, X^D_{\tau_U} \in E] + \mathbb{E}_x[R_u^{D\setminus E}(X^D_{\tau_U});\, X^D_{\tau_U} \in E] \\
&\quad + \mathbb{E}_x[R_u^{D\setminus E}(X^D_{\tau_U});\, X^D_{\tau_U} \in D \setminus E] \\
&= \mathbb{E}_x[w(X^D_{\tau_U});\, X^D_{\tau_U} \in E] + \mathbb{E}_x[R_u^{D\setminus E}(X^D_{\tau_U})] =: A + B\,.
\end{aligned}$$

Next, by using that $w = 0$ on $D \setminus E$, and the fact that $X_t^D = X_t^E$ for all $t < \tau_E$,

$$\begin{aligned}
A &= \mathbb{E}_x[w(X^D_{\tau_U});\, X^D_{\tau_U} \in E, \tau_U < \tau_E] + \mathbb{E}_x[w(X^D_{\tau_U});\, X^D_{\tau_U} \in E, \tau_E < \tau_U] \\
&= \mathbb{E}_x[w(X^E_{\tau_U})] + \mathbb{E}_x[w(X^D_{\tau_U});\, \tau_E < \tau_U] \\
&= \mathbb{E}_x[w(X^E_{\tau_{U\cap E}})] + \mathbb{E}_x[w(X^D_{\tau_U});\, \tau_E < \tau_U] \\
&= w(x) + \mathbb{E}_x[w(X^D_{\tau_U});\, \tau_E < \tau_U] =: w(x) + A_2\,.
\end{aligned}$$

In the last line we used Lemma 5.1. We split B into two parts:

$$B = \mathbb{E}_x[R_u^{D\setminus E}(X^D_{\tau_U});\, \tau_E < \tau_U] + \mathbb{E}_x[R_u^{D\setminus E}(X^D_{\tau_U});\, \tau_U \le \tau_E] := B_1 + B_2\,,$$

and combine B_1 with A_2:

$$\begin{aligned}
A_2 + B_1 &= \mathbb{E}_x[w(X^D_{\tau_U});\, \tau_E < \tau_U] + \mathbb{E}_x[R_u^{D\setminus E}(X^D_{\tau_U});\, \tau_E < \tau_U] \\
&= \mathbb{E}_x[(w + R_u^{D\setminus E})(X^D_{\tau_U});\, \tau_E < \tau_U] \\
&\le \mathbb{E}_x[u(X^D_{\tau_U});\, \tau_E < \tau_U] \\
&= \mathbb{E}_x\left[\mathbb{E}_{X^D_{\tau_E}}\left(u(X^D_{\tau_U})\right);\, \tau_E < \tau_U\right] \\
&= \mathbb{E}_x[u(X^D_{\tau_E});\, \tau_E < \tau_U].
\end{aligned}$$

In the penultimate line we used the strong Markov property at time τ_E, and in the last line harmonicity of u (note that $X^D_{\tau_E} \in U \setminus E$ on $\tau_E < \tau_U$).

Finally, for B_2 we use that $N := \{\widehat{R}_u^{D\setminus E} \ne R_u^{D\setminus E}\}$ is polar, hence $\mathbb{P}_x(X^D_{\tau_U} \in N) = 0$. Therefore, by using that $\widehat{R}_u^{D\setminus E}(y) = \mathbb{E}_y[u(X^D_{\tau_E})]$ in the second line, and the strong Markov property in the third,

$$\begin{aligned}
B_2 &= \mathbb{E}_x[\widehat{R}_u^{D\setminus E}(X^D_{\tau_U});\, \tau_U \le \tau_E] \\
&= \mathbb{E}_x\left[\mathbb{E}_{X^D_{\tau_U}}\left(u(X^D_{\tau_E})\right);\, \tau_U \le \tau_E\right] \\
&= \mathbb{E}_x[u(X^D_{\tau_E});\, \tau_U \le \tau_E]\,.
\end{aligned}$$

Putting everything together we get

$$\mathbb{E}_x[v(X^D_{\tau_U})] = w(x) + A_1 + B_1 + B_2$$

$$
\begin{aligned}
&\leq w(x) + \mathbb{E}_x[u(X^D_{\tau_E});\, \tau_E < \tau_U] + \mathbb{E}_x[u(X^D_{\tau_E});\, \tau_U \leq \tau_E] \\
&= w(x) + \mathbb{E}_x[u(X^D_{\tau_E})] = w(x) + R_u^{D\setminus E}(x) \\
&= w(x) + \widehat{R}_u^{D\setminus E}(x) = v(x)\,.
\end{aligned}
$$

In the last line we used that $\widehat{R}_u^{D\setminus E} = R_u^{D\setminus E}$ on E. □

Lemma 5.3 *Let v be defined by (5.2) and let $\widehat{v}(x) := \liminf_{y\to x} v(y)$ be its lower semi-continuous regularization. Then $\widehat{v}$ is excessive for X^D. Moreover, $\widehat{v} \leq u$ and there exists a polar set $N \subset \partial E \cap D$ such that $\widehat{v} = u$ on $(D \setminus E) \setminus N$.*

Proof. First note that $\widehat{v} \leq v$ on D. Let $U \subset \overline{U} \subset D$ be open. Define

$$
\widetilde{v}(x) := \mathbb{E}_x[\widehat{v}(X^D_{\tau_U})]\,, \quad x \in D\,.
$$

By the proof of Lemma III.1.2 in [2], $\widetilde{v}$ is lower semi-continuous in U. Moreover, by Lemma 5.2,

$$
\widetilde{v}(x) = \mathbb{E}_x[\widehat{v}(X^D_{\tau_U})] \leq \mathbb{E}_x[v(X^D_{\tau_U})] \leq v(x)\,, \qquad x \in U\,.
$$

Hence, by lower semi-continuity of $\widetilde{v}$ in U, for every $x \in U$,

$$
\widetilde{v}(x) \leq \liminf_{y\to x, y\in U} \widetilde{v}(y) \leq \liminf_{y\to x, y\in U} v(y) = \widehat{v}(x)\,.
$$

This proves that

$$
\mathbb{E}_x[\widehat{v}(X^D_{\tau_U})] \leq \widehat{v}(x) \qquad \text{for all } x \in U\,. \tag{5.4}
$$

Now, for any open $U \subset \overline{U} \subset D$, let $H_U(x, dy) = \mathbb{P}_x(X^D_{\tau_U} \in dy)$. Then the family $H_U(x, \cdot)$ (over all relatively compact open $U \subset D$) forms a family of harmonic kernels, cf. Chapter II of [2]. In the notation of [2], (5.4) means that $\widehat{v} \in {}^*\mathcal{H}^+(D)$. By Corollary III.2.1 of [2], the latter family is equal to $\mathcal{S}(X^D)$. Hence, $\widehat{v}$ is excessive with respect to X^D.

Clearly, $\widehat{v} \leq v \leq u$ on D. Recall that $v = u$ on $D \setminus E$. Let $z \in \partial E \cap D$. Then

$$
\begin{aligned}
\liminf_{x\to z, x\in E} v(x) &= \liminf_{x\to z, x\in E} (w(x) + R_u^{D\setminus E}(x)) \\
&\geq \liminf_{x\to z, x\in E} w(x) + \liminf_{x\to z, x\in E} R_u^{D\setminus E}(x) \\
&\geq \liminf_{x\to z, x\in E} w(x) + \widehat{R}_u^{D\setminus E}(z)
\end{aligned}
$$

(since $\widehat{R}_u^{D\setminus E}$ is the l.s.c. regularization of $R_u^{D\setminus E}$). By (5.1), $\liminf_{x\to z, x\in E} w(x) = 0$ for all $z \in (\partial E \cap D) \setminus N_1$ with N_1 being a polar set. Also, $\widehat{R}_u^{D\setminus E} = R_u^{D\setminus E}$ except on a polar set N_2. By setting $N = N_1 \cup N_2$, we see that for all $z \in (\partial E \cap D) \setminus N$,

$$
\liminf_{x\to z, x\in E} v(x) \geq R_u^{D\setminus E}(z) = u(z)\,.
$$

Clearly, for all $z \in \partial E \cap D$,

$$\liminf_{x \to z, x \in D \setminus E} v(x) = \liminf_{x \to z, x \in D \setminus E} u(x) \geq u(z)\,.$$

Together the last two displays give that for all $z \in (\partial E \cap D) \setminus N$,

$$\widehat{v}(z) = \liminf_{x \to z} v(x) \geq u(z)\,.$$

□

We note that for every $g : D \setminus E \to [0, \infty)$ the function $x \mapsto \mathbb{E}_x[g(X_{\tau_E})] = \widehat{R}_g^{D \setminus E}(x)$ is harmonic in E with respect to X^D. Since $R_g^{D \setminus E} = \widehat{R}_g^{D \setminus E}$ on E, it follows that $R_g^{D \setminus E}$ is harmonic in E with respect to X^D.

In what follows, $\partial_M D$ denotes the Martin boundary of D with respect to X^D, $\partial_m D$ the minimal Martin boundary, and $D_M = D \cup \partial_M D$ the Martin space (with the Martin topology). For $z \in \partial_m D$ let $M_D(\cdot, z)$ be the Martin kernel (based at $x_0 \in E$). Then $M_D(\cdot, z)$ is continuous and harmonic in D with respect to X^D. We recall that $E \subset D$ is *minimally thin* in D at $z \in \partial_m D$ with respect to X^D if $\widehat{R}^E_{M_D(\cdot,z)} \neq M_D(\cdot, z)$.

Proposition 5.4 *Let $E \subset D$ be an open set in D, $z \in \partial_m D$ such that z is in the closure of E in D_M. Assume that $D \setminus E$ is minimally thin at z in D with respect to X^D. Let*

$$h(x) := M_D(x, z) - R^{D \setminus E}_{M_D(\cdot,z)}(x)\,, \qquad x \in E\,.$$

Then h is a minimal harmonic function in E with respect to X^E.

Proof. We first prove that h is harmonic with respect to X^E. Let $U \subset \overline{U} \subset E$ be relatively compact open in E. Then

$$\begin{aligned}
\mathbb{E}_x\left[h(X^E_{\tau_U})\right] &= \mathbb{E}_x\left[M_D(X^E_{\tau_U}, z)\right] - \mathbb{E}_x\left[\mathbb{E}_{X^E_{\tau_U}}\left[M_D\left(X^D_{S_{D\setminus E}}, z\right)\right]\right] \\
&= \mathbb{E}_x\left[M_D(X^D_{\tau_U}, z)\right] - \mathbb{E}_x\left[M_D(X^D_{\tau_U}, z);\, \tau_U = \tau_E\right] \\
&\qquad -\mathbb{E}_x\left[\mathbb{E}_{X^D_{\tau_U}}\left[M_D\left(X^D_{S_{D\setminus E}}, z\right)\right];\, \tau_U < \tau_E\right] \\
&= \mathbb{E}_x\left[M_D(X^D_{\tau_U}, z)\right] - \mathbb{E}_x\left[M_D(X^D_{\tau_E}, z);\, \tau_U = \tau_E\right] \\
&\qquad -\mathbb{E}_x\left[M_D\left(X^D_{S_{D\setminus E}}, z\right);\, \tau_U < \tau_E\right] \\
&= \mathbb{E}_x\left[M_D(X^D_{\tau_U}, z)\right] - \mathbb{E}_x\left[M_D(X^D_{D\setminus E}, z);\, \tau_U = S_{D\setminus E}\right] \\
&\qquad -\mathbb{E}_x\left[M_D\left(X^D_{D_{D\setminus E}}, z\right);\, \tau_U < S_{D\setminus E}\right] \\
&= M_D(x, z) - \mathbb{E}_x\left[M_D\left(X^D_{S_{D\setminus E}}, z\right)\right] = h(x).
\end{aligned}$$

Now suppose that $w : E \to [0, \infty)$ is harmonic in E with respect to X^E and $w \leq h$. Define v analogously to (5.2) by

$$v(x) := \begin{cases} w(x) + R^{D\setminus E}_{M_D(\cdot,\cdot,z)}(x)\,, & x \in E \\ M_D(x, z)\,, & x \in D \setminus E\,, \end{cases}$$

and let $\widehat{v}$ be its l.s.c. regularization. By Lemma 5.3, $\widehat{v} \in \mathcal{S}(X^D)$, $\widehat{v} \le M_D(\cdot, z)$ on D, and $\widehat{v} = M_D(\cdot, z)$ on $(D \setminus E) \setminus N$, N polar. By the Riesz decomposition,

$$\widehat{v} = aM_D(\cdot, z) + G_D\mu\,,$$

where $0 \le a \le 1$ and μ is a measure charging no polar set (since $\widehat{v}$ is locally bounded, the same holds for $G_D\mu$, hence μ cannot charge polar sets). Note that $\widehat{v} = w + R^{D\setminus E}_{M_D(\cdot,z)}$ on E. The function $R^{D\setminus E}_{M_D(\cdot,z)}$ is harmonic in E with respect to X^D. By assumption, w is harmonic in E with respect to X^E, and hence harmonic in E with respect to X^D (we extend $w = 0$ on $D \setminus E$). Therefore, $\widehat{v}$ is harmonic in E with respect to X^D which implies the same for $G_D\mu = \widehat{v} - aM_D(\cdot, z)$.

Recall that $D \setminus E$ is thin at $y \in D$ if and only if $\widehat{R}^{D\setminus E}_{G_D(\cdot,y)} \ne G_D(\cdot, y)$ (this can be proved along the same lines as the corresponding proof for minimal thinness, cf. Proposition 6.2 of [24]). Let

$$A = \{y \in \partial E \cap D :\ \widehat{R}^{D\setminus E}_{G_D(\cdot,y)} \ne G_D(\cdot, y)\}\,.$$

By Proposition VI.5.12 of [3], A is polar, and hence $\mu(A) = 0$.

Now consider $R^{D\setminus E}_{G_D\mu}$. This function is harmonic in E with respect to X^D. Moreover, $R^{D\setminus E}_{G_D\mu} \le G_D\mu$ on D. Hence, $G_D\mu - R^{D\setminus E}_{G_D\mu} \ge 0$ and is harmonic in E with respect to X^D. Note that $G_D\mu - R^{D\setminus E}_{G_D\mu} = 0$ on $D \setminus E$. Hence, $G_D\mu - R^{D\setminus E}_{G_D\mu}$ is harmonic in E with respect to X^E. On the other hand, for $x \in E$,

$$\begin{aligned}
G_D\mu(x) - R^{D\setminus E}_{G_D\mu}(x) &= G_D\mu(x) - \widehat{R}^{D\setminus E}_{G_D\mu}(x)\\
&= \int_D G_D(x,y)\,\mu(dy) - \int_D \widehat{R}^{D\setminus E}_{G_D(\cdot,y)}(x)\,\mu(dy)\\
&= \int_D \Big[G_D(x,y) - \widehat{R}^{D\setminus E}_{G_D(\cdot,y)}(x)\Big]\,\mu(dy)\\
&= \int_E \Big[G_D(x,y) - \widehat{R}^{D\setminus E}_{G_D(\cdot,y)}(x)\Big]\,\mu(dy) + \int_A \Big[G_D(x,y) - \widehat{R}^{D\setminus E}_{G_D(\cdot,y)}(x)\Big]\,\mu(dy)\\
&= \int_E G_E(x,y)\,\mu(dy) = G_E\mu(x)\,.
\end{aligned}$$

In the last line we used that $\mu(A) = 0$ and the formula for the Green function of X^E: $G_E(x,y) = G_D(x,y) - \mathbb{E}_x[G_D(X_{\tau_E}, y)]$. This shows that $G_D\mu - R^{D\setminus E}_{G_D\mu}$ is at the same time harmonic in E (with respect to X^E) and the potential of the measure $\mu_{|E}$. Hence, it is identically zero in E, that is, $G_D\mu = R^{D\setminus E}_{G_D\mu}$ in E, hence in D.

Since v and $\widehat{v}$ differ at most on a polar set, and $v = M_D(\cdot, z)$ on $D \setminus E$, we see that $G_D\mu = (1-a)M_D(\cdot, z)$ outside a polar set. Therefore

$$(1-a)R^{D\setminus E}_{M_D(\cdot,z)} = R^{D\setminus E}_{G_D\mu} = G_D\mu\,.$$

Hence, on E we have

$$w = \widehat{v} - R^{D\setminus E}_{M_D(\cdot,z)} = aM_D(\cdot, z) + (1-a)R^{D\setminus E}_{M_D(\cdot,z)} - R^{D\setminus E}_{M_D(\cdot,z)}$$

$$= a\left(M_D(\cdot,z) - R^{D\setminus E}_{M_D(\cdot,z)}\right) = ah\,,$$

which completes the proof. □

Remark 5.5 The assumption that $D \setminus E$ is minimally thin at z in D with respect to X^D is used to conclude that $h \neq 0$. If $D \setminus E$ is not minimally thin at z in D with respect to X^D, then $R^{D\setminus E}_{M_D(\cdot,z)} = M_D(\cdot,z)$.

Proposition 5.6 *Let $E \subset D$ be an open set in D, $z \in \partial_m D$ such that z is in the closure of E in D_M. Assume that $D \setminus E$ is minimally thin at z in D with respect to X^D and let*

$$h(x) := M_D(x,z) - R^{D\setminus E}_{M_D(\cdot,z)}(x)\,, \qquad x \in E\,.$$

Let $\zeta = \zeta(z)$ be the Martin boundary point of E associated with the minimal harmonic function h. Assume that $(x_n)_{n\geq 1}$ is a sequence of points in E that converges to z in D_M and also

$$\liminf_{n\to\infty} \frac{G_E(x_0,x_n)}{G_D(x_0,x_n)} > 0\,. \tag{5.5}$$

Assume further that for every subsequence (x_{n_k}), $G_E(\cdot,x_{n_k})/G_E(x_0,x_{n_k})$ converges to a harmonic function with respect to X^E. Then $(x_n)_{n\geq 1}$ converges to ζ in E_M (Martin space of E).

Proof. Let $(x_n)_{n\geq 1}$ be a sequence in E converging to z in D_M and such that (5.5) holds. Assume that (x_n) does not converge to ζ in E_M. This implies that there exists a subsequence (x_{n_k}) with the property that $G_E(\cdot,x_{n_k})/G_E(x_0,x_{n_k})$ converges in E to a function $u : E \to [0,\infty)$ such that $u \neq h/h(x_0)$. By assumption, u is harmonic with respect to X^E. It follows from (5.5), that by choosing a further subsequence (if necessary) we can arrange that

$$\lim_{k\to\infty} \frac{G_E(x_0,x_{n_k})}{G_D(x_0,x_{n_k})} = a > 0\,.$$

Therefore, on E we have that

$$\lim_{k\to\infty} \frac{G_E(\cdot,x_{n_k})}{G_D(x_0,x_{n_k})} = au\,.$$

Since $G_E(\cdot,y) = G_D(\cdot,y) - R^{D\setminus E}_{G_D(\cdot,y)}$, and since

$$\frac{R^{D\setminus E}_{G_D(\cdot,y)}(x_0)}{G_D(x_0,y)} = R^{D\setminus E}_{M_D(\cdot,y)}(x_0)$$

(which easily follows from the probabilistic representation of the reduced function), we get by use of Fatou's lemma in the last line that

$$au(x) \;=\; \lim_{k\to\infty}\left(\frac{G_D(x,x_{n_k})}{G_D(x_0,x_{n_k})} - \frac{R^{D\setminus E}_{G_D(\cdot,x_{n_k})}(x)}{G_D(x_0,x_{n_k})}\right)$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\left(M_D(x,x_{n_k}) - R^{D\setminus E}_{M_D(\cdot,x_{n_k})}(x)\right)\\
&\le M_D(x,z) - R^{D\setminus E}_{M_D(\cdot,z)}(x) = h(x)\,.
\end{aligned}
$$

Since u is harmonic for X^E, it follows from Proposition 5.4 that au is proportional to h. Since $u(x_0)=1$, that would imply $u = h/h(x_0)$ which contradicts the assumption. □

If $F \subset E \subset D$, and $v \in \mathcal{S}(X^E)$, let ${}^E R^F_v$ denote the reduced function of v on F with respect to X^E.

Lemma 5.7 *Let $F \subset E \subset D$, $u \in \mathcal{S}(X^D)$, and define $v := u - R^{D\setminus E}_u$. Then $v \in \mathcal{S}(X^E)$ and*

$$
{}^E R^F_v = R^{D\setminus F}_u - R^{D\setminus E}_u\,. \tag{5.6}
$$

Proof. Since the excessiveness implies that $u(x) \ge \mathbb{E}_x[u(X^D_{S_{D\setminus E}})]$, v is non-negative. If $x \in E$, by the strong Markov property,

$$
\begin{aligned}
\mathbb{E}_x\left[v(X^U_t)\right] &= \mathbb{E}_x\left[u(X^E_t)\right] - \mathbb{E}_x\left[\mathbb{E}_{X^E_t}\left[u\left(X^D_{S_{D\setminus E}}\right)\right]\right]\\
&= \mathbb{E}_x\left[u(X^D_t)\right] - \mathbb{E}_x\left[u(X^D_t) : t \ge \tau_E\right]\\
&\qquad -\mathbb{E}_x\left[\mathbb{E}_{X^D_t}\left[u\left(X^D_{S_{D\setminus E}}\right)\right] : t < \tau_E\right]\\
&= \mathbb{E}_x\left[u(X^D_t)\right] - \mathbb{E}_x\left[u(X^D_t) : t \ge \tau_E\right] - \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right) : t < \tau_E\right]\\
&= \mathbb{E}_x\left[u(X^D_t)\right] - \mathbb{E}_x\left[u\left(X^D_{t\vee\tau_E}\right)\right]\,.
\end{aligned}
$$

By the excessiveness of u for X^D, $\mathbb{E}_x\left[u(X^D_t)\right] \le u(x)$ and

$$
\mathbb{E}_x\left[u\left(X^D_{t\vee\tau_E}\right)\right] \ge \mathbb{E}_x\left[u\left(X^D_{\tau_E}\right)\right]\,.
$$

Thus

$$
\mathbb{E}_x\left[v(X^E_t)\right] \le u(x) - \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right)\right]\,.
$$

Moreover

$$
\begin{aligned}
\lim_{t\downarrow 0}\mathbb{E}_x\left[v(X^E_t)\right] &= \lim_{t\downarrow 0}\mathbb{E}_x\left[u(X^D_t)\right] - \lim_{t\downarrow 0}\mathbb{E}_x\left[u\left(X^D_{t\vee\tau_E}\right)\right]\\
&= u(x) - \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right)\right]\,.
\end{aligned}
$$

Note that for $x \in E$,

$$
\begin{aligned}
&\mathbb{E}_x\left[u\left(X^D_{S_{D\setminus F}}\right)\right]\\
=&\mathbb{E}_x\left[u\left(X^D_{S_{E\setminus F}}\right); X_{S_{D\setminus F}} \in E\right] + \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus F}}\right); S_{D\setminus F} = S_{D\setminus E}\right]\\
=&\mathbb{E}_x\left[u\left(X^E_{S_{E\setminus F}}\right)\right] + \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right); S_{D\setminus F} = S_{D\setminus E}\right]\\
=&\mathbb{E}_x\left[u\left(X^E_{S_{E\setminus F}}\right)\right] + \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right)\right] - \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right); S_{D\setminus F} < S_{D\setminus E}\right]\,.
\end{aligned}
$$

By strong Markov property,

$$\mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right); S_{D\setminus F} < S_{D\setminus E}\right] = \mathbb{E}_x\left[\mathbb{E}_{X^E_{S_{E\setminus F}}}\left[u\left(X^D_{S_{D\setminus E}}\right)\right]\right].$$

Thus

$$\begin{aligned}&\mathbb{E}_x\left[u\left(X^D_{S_{D\setminus F}}\right)\right]\\ =&\mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right)\right] + \mathbb{E}_x\left[u\left(X^E_{S_{E\setminus F}}\right)\right] - \mathbb{E}_x\left[\mathbb{E}_{X^U_{S_{E\setminus F}}}\left[u\left(X^D_{S_{D\setminus E}}\right)\right]\right].\end{aligned}$$

Therefore

$$\mathbb{E}_x\left[u\left(X^D_{S_{D\setminus F}}\right)\right] = \mathbb{E}_x\left[u\left(X^D_{S_{D\setminus E}}\right)\right] + \mathbb{E}_x\left[v\left(X^E_{S_{E\setminus F}}\right)\right],$$

which is (5.6). □

Proposition 5.8 *Let $E \subset D$ be an open set in D, $z \in \partial_m D$ such that z is in the closure of E in D_M. Assume that $D \setminus E$ is minimally thin at z in D with respect to X^D and let*

$$h(x) := M_D(x,z) - R^{D\setminus E}_{M_D(\cdot,z)}(x)\,, \qquad x \in E\,.$$

Let $\zeta = \zeta(z)$ be the Martin boundary point of E associated with the minimal harmonic function h. Let $F \subset E$. Then F is minimally thin at ζ in E with respect to X^E if and only if F is minimally thin at z in D with respect to X^D.

Proof. The set F is minimally thin at ζ with respect to X^E if and only if ${}^E R^F_h \neq h$. By Lemma 5.7 (with F replaced by $E \setminus F$ and $u = M_D(\cdot, z)$),

$${}^E R^{E\setminus F}_h = R^{D\setminus(E\setminus F)}_{M_D(\cdot,z)} - R^{D\setminus E}_{M_D(\cdot,z)} = R^{(D\setminus E)\cup F}_{M_D(\cdot,z)} - R^{D\setminus E}_{M_D(\cdot,z)}\,.$$

Since $h = M_D(\cdot,z) - R^{D\setminus E}_{M_D(\cdot,z)}$, we see that ${}^E R^F_h \neq h$ if and only if $R^{(D\setminus E)\cup F}_{M_D(\cdot,z)} \neq M_D(\cdot,z)$. The last condition is equivalent to $(D \setminus E) \cup F$ being minimally thin at z in D with respect to X^D. Since $D \setminus E$ is not minimally thin at z, the latter is equivalent to F being minimally thin at z in D with respect to X^D. □

Remark 5.9 Proposition 5.8 does not depend on Proposition 5.6.

Let $D \subset \mathfrak{X}$ be an open unbounded set. Suppose E is an open subset of D such that for some $R > 0$ it holds that $D \cap \overline{B}(z_0,R)^c = E \cap \overline{B}(z_0,R)^c$. Assume that ∞ is accessible both from E and from D. Assume that the assumptions **A**, **C**, **C2**(z_0,R) and **F2**(z_0,R) for X and $\widehat{X}$ are satisfied. By Theorem 1.3 there is only one Martin boundary point of E associated with ∞, say ∞^E, and this point is minimal, $\infty^E \in \partial_m E$. In the same way, there is only one Martin boundary point of D associated with ∞, say ∞^D, and this point is also minimal, $\infty^D \in \partial_m D$. Hence, the concept of minimal thinness at ∞ of a set $F \subset E$ makes sense with respect to both X^E and X^D. In fact, we have the following result.

Theorem 5.10 *Suppose that* ***A***, ***C***, ***C2***(z_0, R) *and* ***F2***(z_0, R) *for* X *and* $\widehat{X}$ *hold true. Let* $D \subset \mathfrak{X}$ *be an unbounded open set, and let* E *be an open subset of* D *such that for some* $R > 0$ *it holds that* $D \cap \overline{B}(z_0, R)^c = E \cap \overline{B}(z_0, R)^c$. *Assume that* ∞ *is accessible from* E *and from* D. *Suppose that* $F \subset E$. *Then* F *is minimally thin at* ∞ *with respect to* X^E *if and only if it is minimally thin at* ∞ *with respect to* X^D.

Proof. Let $x_0 \in E$ and choose $r_0 > 2(d(x_0, E^c) \wedge R)$. For every $r \in (0, r_0)$, both $G_D(x_0, \cdot)$ and $G_E(x_0, \cdot)$ are regular harmonic in $D \cap \overline{B}(z_0, r)^c$ with respect to X and vanish in $\overline{B}(z_0, r)^c \setminus D$. Let $x_1 \in E \cap \overline{B}(z_0, 2r)^c = D \cap \overline{B}(z_0, 2r)^c$ be fixed. By the boundary Harnack principle,

$$\frac{G_E(x_0, x)}{G_D(x_0, x)} \geq c^{-1} \frac{G_E(x_0, x_1)}{G_D(x_0, x_1)}, \qquad \text{for all } x \in D \cap \overline{B}(z_0, 8r)^c .$$

This implies that

$$\liminf_{E \ni x \to \infty} \frac{G_E(x_0, x)}{G_D(x_0, x)} > 0 . \tag{5.7}$$

Let $M_D(\cdot, \infty) = M_D(\cdot, \infty^D)$, respectively $M_E(\cdot, \infty) = M_E(\cdot, \infty^E)$, be the Martin kernels at ∞ for D, respectively E. Define

$$h(x) = M_D(x, \infty) - R^{D \setminus E}_{M_D(\cdot, \infty)}(x).$$

By Proposition 5.4, h is a minimal harmonic function with respect to X^E. Let $\zeta \in \partial_m E$ be the minimal Martin boundary point of E corresponding to h. Let $(x_n)_{n \geq 1}$ be a sequence of points in E converging to ∞. By (5.7),

$$\liminf_{n \to \infty} \frac{G_E(x_0, x_n)}{G_D(x_0, x_n)} > 0 .$$

Also, it follows from Lemma 3.4 and Theorem 1.3(b) that, for every subsequence (x_{n_k}), $G_E(\cdot, x_{n_k})/G_E(x_0, x_{n_k})$ converges to the harmonic function $M_E(\cdot, \infty)$. It follows from Proposition 5.6 that $(x_n)_{n \geq 1}$ converges to ζ in the Martin topology of E_M. Thus, $\zeta \in \partial_m E$ is associated to ∞. By uniqueness, $\zeta = \infty^E$ and therefore $h = M_E(\cdot, \infty^E) = M_E(\cdot, \infty)$. The claim of the theorem now follows from Proposition 5.8. □

Remark 5.11 Suppose that ∞ is accessible from E. Since $G_E(x, w) \leq G_D(x, w)$, $x, w \in E$, implies that $\mathbb{E}_x \tau_E \leq \mathbb{E}_x \tau_D$ for $x \in E$, we see that $\mathbb{E}_x \tau_D = \infty$ for all $x \in E$. If the assumptions of [26] are satisfied, it follows from Remark 4.2 that ∞ is also accessible from D.

One can similarly prove the following theorem saying that minimal thinness is a local property at a finite boundary point.

Theorem 5.12 *Suppose that* ***A***, ***C***, ***C1***(z_0, R) *and* ***F1***(z_0, R) *for* X *and* $\widehat{X}$ *hold true. Let* $D \subset \mathfrak{X}$, $z_0 \in \partial D$, *and let* E *be an open subset of* D *such that for some* $R > 0$ *it holds that* $D \cap B(z_0, R) = E \cap B(z_0, R)$. *Assume that* z_0 *is accessible from* E *and from* D. *Suppose that* $F \subset E$. *Then* F *is minimally thin at* z_0 *with respect to* X^E *if and only if it is minimally thin at* z_0 *with respect to* X^D.

Acknowledgements: Part of the research for this paper was done during the visit of Renming Song and Zoran Vondraček to Seoul National University from May 24 to June 8 of 2015. They thank the Department of Mathematical Sciences of Seoul National University for the hospitality. We also thank the referee for very helpful comments on the first version of this paper.

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