

On harmonic functions for trace processes

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Abstract

Let X be a standard Markov process with state space E and let F be a closed subset of E . A nonnegative function f on F is extended probabilistically to a function h_f on the whole space E . We show that the extension h_f is harmonic with respect to X provided that f is harmonic with respect to Y , the trace process on F of the process X . A consequence is that if the Harnack inequality holds for X , it also holds for the trace process Y . Several examples illustrating the usefulness of the result are given.

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1 Introduction

This paper is motivated by the recent work [5] which proposes to study d -dimensional non-local operators by means of $(d + 1)$ -dimensional local operators. A probabilistic interpretation of this approach is to consider a d -dimensional subordinate Brownian motion Y as the trace of a $(d + 1)$ -dimensional diffusion X on the hyperplane $\mathbb{H} = \mathbb{R}^d \times \{0\}$. In this paper we consider the case of a standard Markov process X on a locally compact separable metric space E and the trace of

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X on a closed subset F of E . Our main goal is to show that, under reasonable conditions, the Harnack inequality holds for the trace process if it holds for X . The main tool consists of studying the relationship between harmonic functions with respect to the trace process and their extensions from F to the whole space E . To be more precise, let $\sigma_F = \inf\{t > 0 : X_t \in F\}$ be the hitting time to F . For a nonnegative Borel function f on F we define its extension $h_f : E \rightarrow [0, \infty]$ by $h_f(x) = \mathbb{E}_x[f(X_{\sigma_F})]$, $x \in E$. It is a well-known fact that this extension is harmonic with respect to X outside F . The first result we prove is Theorem 2.1 which states that (under certain reasonable conditions) if f is harmonic in $F \cap D$ with respect to the trace process, then the extension h_f is harmonic with respect to X in D . Here D is an open subset of E . As an immediate consequence of this result we record the following fact: If the Harnack inequality holds for X , then it also holds for the trace process. We give several examples in which the Harnack inequality is established for some processes which can be realized as traces of other processes. When all harmonic functions with respect to X are continuous, we also prove a converse of Theorem 2.1.

The rest of the paper is organized as follows. In Section 2, we discuss trace processes of standard Markov processes and the relation between harmonic functions with respect to the original process and harmonic functions with respect to its trace process. Section 3 gives a few examples of trace processes and applications of the results in Section 2. In Section 4, we prove the converse of the Theorem 2.1.

In this paper, we denote “:=” to mean “is defined to be”. For two functions f and g , the notation “ $f \asymp g$ ” means that there exist constants $c_2 > c_1 > 0$ such that $c_1 g \leq f \leq c_2 g$. For two real numbers a and b , $a \wedge b := \min\{a, b\}$. For a set K in a space E , we use K^c to denote its complement in E , that is, $K^c := E \setminus K$. We will use ∂ to denote a cemetery point and for every function f , we extend its definition to ∂ by setting $f(\partial) = 0$. For a topological space E , we use $\mathcal{B}(E)$ to denote the Borel σ -field on E , $\mathcal{B}^\mu(E)$ the μ -completion of $\mathcal{B}(E)$, and we define $\mathcal{B}^*(E) := \bigcap_{\mu \in \mathcal{P}(E)} \mathcal{B}^\mu(E)$, where $\mathcal{P}(E)$ is the collection of all probability measures on E . Each element of $\mathcal{B}^*(E)$ is called a universally measurable subset of E . We use the notation $\mathbb{P}_x(X_t \in \cdot) = \mathbb{P}(X_t \in \cdot | X_0 = x)$ and for $\mu \in \mathcal{P}(E)$, $\mathbb{P}_\mu(X_t \in \cdot) := \int_E \mathbb{P}_x(X_t \in \cdot) \mu(dx)$.

Throughout this paper, we use c, c_1, c_2, \dots to denote generic constants, whose exact values are not important and can change from one appearance to another.

2 Setup and Main Result

Assume that (E, ρ) is a locally compact separable metric space with the metric ρ and that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x, x \in E)$ is a standard process on E , that is, a normal right continuous Markov process which is quasi-left continuous on $(0, \zeta)$, where $\zeta := \inf\{t \geq 0 : X_t = \partial\}$ is the lifetime of the process. The shift operators θ_t , $t \geq 0$, satisfy $X_s \circ \theta_t = X_{s+t}$ identically for $s, t \geq 0$.

The semigroup $\{P_t\}_{t \geq 0}$ of X on the space of nonnegative Borel functions on E is defined by

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[f(X_t) : t < \zeta].$$

$\{P_t\}_{t \geq 0}$ can be extended to the space of nonnegative universally measurable functions on E .

Throughout this paper, ξ is an excessive measure of X with $\text{supp}[\xi] = E$; that is, ξ is a σ -finite Borel measure on E with full support such that $\xi P_t \leq \xi$ for all $t \geq 0$. Here ξP_t denotes the measure ν defined by $\int_E f(x) \nu(dx) = \int_E P_t f(x) \xi(dx)$ for any Borel function $f \geq 0$ on E . Since X is right continuous, we have $\lim_{t \rightarrow 0} \xi P_t = \xi$ setwise.

Throughout this paper, $A = (A_t : t \geq 0)$ is a positive continuous additive functional of X in the strict sense, i.e., in the sense of [3]. Then there exists a unique measure μ on E , which is called the Revuz measure of $A = (A_t : t \geq 0)$, such that

$$\int_E f(x) \mu(dx) = \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}_\xi \left[\int_0^t f(X_s) dA_s \right] \quad (2.1)$$

for all nonnegative Borel function f on E (see [7, Theorem A.3.5]). Let

$$R(\omega) := \inf\{t > 0 : A_t(\omega) > 0\},$$

then the support of A is defined by

$$F := \{x \in E : \mathbb{P}_x(R = 0) = 1\}. \quad (2.2)$$

Since $t + R(\theta_t \omega) \downarrow R(\omega)$ as $t \downarrow 0$, F is a nearly Borel, finely closed set. Let $\sigma_F := \inf\{t > 0 : X_t \in F\}$ be the first hitting time of F . Then $\mathbb{P}_x(\sigma_F = R) = 1$ for every $x \in E$ (see [29, Section 64]). Furthermore, each point of F is regular for F with respect to X , that is

$$\mathbb{P}_x(\sigma_F = 0) = 1, \quad \text{for every } x \in F.$$

We are concerned with the trace of X on the subset F of E . We will use $(\tau_t : t \geq 0)$ to denote the right continuous inverse of $(A_t : t \geq 0)$ defined by

$$\tau_t = \begin{cases} \inf\{s \geq 0 : A_s > t\} & \text{for } t < A_{\zeta-}, \\ \infty & \text{for } t \geq A_{\zeta-}. \end{cases}$$

Let $\widehat{\zeta} := A_{\zeta-}$. The time changed process $Y = (\Omega, \mathcal{F}, \mathcal{F}_{\tau_t}, Y_t, \mathbb{P}_x, x \in F)$ defined by

$$Y_t = \begin{cases} X_{\tau_t} & \text{for } 0 \leq t < \widehat{\zeta}, \\ \partial & \text{for } t \geq \widehat{\zeta} \end{cases}$$

is called the trace of X on F . It is known (cf. [7, Theorem A.3.11] and [29, (65.9)]) that Y is a right process on $(F, \mathcal{B}^*(F))$ with the life time $\widehat{\zeta}$. Here, a right process on F is a right continuous, strong Markov process with $Y_t(\omega) = \partial$ for $t \geq \widehat{\zeta}(\omega)$ and $P_\mu(\lim_{s \downarrow t} u(Y_s) = u(Y_t), \forall t \geq 0) = 1$ for any $\mu \in \mathcal{P}(F)$ and any α -excessive function u with respect to Y and $\alpha \geq 0$. A $[0, \infty]$ -valued function u on F is said to be α -excessive with respect to Y if u is $\mathcal{B}^*(F)$ -measurable and $e^{-\alpha t} \mathbb{E}_x[u(Y_t)] \uparrow u(x)$ as $t \downarrow 0$ for all $x \in F$. For recent development on trace processes, we refer the readers to [7, 8, 9, 16, 18].

We will make the following assumptions on X and F throughout this paper.

A1. Every semipolar set with respect to X is polar with respect to X .

A2. For every point $x \in F^c$ we have

$$\mathbb{P}_x(\sigma_F < \infty) > 0.$$

For the definitions of semipolar set and polar set and their basic properties, we refer the readers to [3] and [7].

Since $\mathbb{P}_x(\sigma_F = R) = 1$, the assumption **A2** is simply saying that our positive continuous additive functional A of X is not trivial.

We introduce the following notations: For $U \subset E$, let $\tau_U := \inf\{t > 0 : X_t \notin U\}$ be the exit time of X from U and $\hat{\tau}_U := \inf\{t > 0 : Y_t \notin U \cap F\}$ be the exit time of Y from $U \cap F$. The notions pertinent to the process Y will be denoted by $\hat{\cdot}$.

Suppose that D is an open subset of E . A nonnegative Borel function h on E is said to be harmonic in D with respect to X if for any $x \in D$ and any open subset $U \subset \bar{U} \subset D$ we have

$$h(y) = \mathbb{E}_y[h(X_{\tau_U}) : \tau_U < \zeta], \quad \text{for all } y \in U. \quad (2.3)$$

Harmonic functions with respect to Y are defined in a similar fashion using the relative topology.

For any function $f : F \rightarrow [0, \infty)$, we define a function $h_f : E \rightarrow [0, \infty]$ by

$$h_f(x) = \mathbb{E}_x[f(X_{\sigma_F})], \quad x \in E.$$

Note that $h_f(x) = f(x)$ on F since every point on F is regular for F with respect to X . The function h_f is called the extension of f . Using the strong Markov property, it is easy to see, and is well known, that h_f is harmonic in $E \setminus F$ with respect to X .

In the remainder of this section, we will prove the following theorem.

Theorem 2.1 *Suppose that f is a nonnegative function on F and D is an open subset of E with $D \cap F \neq \emptyset$. If the function f is harmonic in $F \cap D$ with respect to Y , then its extension h_f is harmonic in D with respect to X .*

Fix an arbitrary open subset $U \subset \bar{U} \subset D$ and let

$$T := \inf\{t > 0 : X_t \in U^c \cap F\}$$

be the first time the process X hits the set F outside of U . For simplicity, we let $\tau := \tau_U = \inf\{t > 0 : X_t \notin U\}$, but retain the notation $\hat{\tau}_U$ for the first exit time of Y from $U \cap F$. An easy, but fundamental observation, is that $Y_{\hat{\tau}_U} = X_T$ \mathbb{P}_x -a.s. for every $x \in U \cap F$. Therefore, if f is harmonic in $F \cap U$ with respect to Y , we have

$$f(x) = \mathbb{E}_x[f(Y_{\hat{\tau}_U})] = \mathbb{E}_x[f(X_T)], \quad \text{for every } x \in U \cap F. \quad (2.4)$$

Note that $\tau \leq T$, \mathbb{P}_x -a.s. for every $x \in U$.

One simple, but important observation is that, since F is finely closed, $X_{\sigma_F} \in F$ and $X_T \in U^c \cap F$, \mathbb{P}_x -a.s. for every $x \in E$. (See, for example, [7, (A.2.5)] and [29, (10,6)]).

Lemma 2.2 *Suppose that f is a nonnegative function on F .*

(a) *Let $g(x) = \mathbb{E}_x[f(X_T) : T < \zeta]$, $x \in E$. Then $g(x) = \mathbb{E}_x[g(X_\tau) : \tau < \zeta]$ for every $x \in U$.*

(b) *If f is harmonic in $D \cap F$ with respect to Y , then*

$$\mathbb{E}_x[f(X_T); \sigma_F < T] = \mathbb{E}_x[f(X_{\sigma_F}) : \sigma_F < T], \quad x \in U.$$

Proof. (a) First note that on $\{\tau < T\}$ we have $X_T \circ \theta_\tau = X_{\tau+T \circ \theta_\tau} = X_T$, \mathbb{P}_x -a.s. for every $x \in U$. Therefore, for $x \in U$,

$$\mathbb{E}_x[g(X_\tau) : \tau < T] = \mathbb{E}_x[\mathbb{E}_{X_\tau}[f(X_T)] : \tau < T] = \mathbb{E}_x[f(X_T \circ \theta_\tau) : \tau < T] = \mathbb{E}_x[f(X_T) : \tau < T].$$

Let N be the set of irregular points of $U^c \cap F$ with respect to X . N is semi-polar with respect to X by [3, Proposition II.3.3], hence polar with respect to X by the assumption **A1**. Thus

$$\mathbb{P}_x(X_T \in N) = 0, \quad x \in U. \quad (2.5)$$

On the other hand, if X_T is in the set of regular points of $U^c \cap F$ with respect to X , then on $\{\tau = T\}$,

$$g(X_\tau) = g(X_T) = \mathbb{E}_{X_T}[f(X_T)] = f(X_T). \quad (2.6)$$

Thus, by (2.5)-(2.6), for $x \in U$,

$$\begin{aligned} \mathbb{E}_x[g(X_\tau) : \tau < \zeta] &= \mathbb{E}_x[g(X_\tau) : \tau < T < \zeta] + \mathbb{E}_x[g(X_\tau) : \tau = T < \zeta] \\ &= \mathbb{E}_x[g(X_\tau) : \tau < T < \zeta] + \mathbb{E}_x[g(X_\tau) \mathbf{1}_{(U^c \cap F) \setminus N}(X_\tau) : \tau = T < \zeta] \\ &= \mathbb{E}_x[f(X_T) : \tau < T < \zeta] + \mathbb{E}_x[f(X_T) \mathbf{1}_{(U^c \cap F) \setminus N}(X_\tau) : \tau = T < \zeta] \\ &= \mathbb{E}_x[f(X_T) : \tau < T < \zeta] + \mathbb{E}_x[f(X_T) : \tau = T < \zeta] \\ &= \mathbb{E}_x[f(X_T) : T < \zeta] = g(x). \end{aligned}$$

(b) Note that on $\{\sigma_F < T\}$ it holds that $X_{\sigma_F} \in U$. Since f is harmonic in $D \cap F$ with respect to Y , we have by (2.4) that $\mathbb{E}_{X_{\sigma_F}}[f(X_T)] = f(X_{\sigma_F})$ for every $X_{\sigma_F} \in U \cap F$. Moreover, if X_T is a regular point of $U^c \cap F$ with respect to X , $\mathbb{E}_{X_{\sigma_F}}[f(X_T)] = f(X_{\sigma_F})$. Hence, for $x \in U$,

$$\begin{aligned} \mathbb{E}_x[f(X_T) : \sigma_F < T] &= \mathbb{E}_x[f(X_T \circ \theta_{\sigma_F}) : \sigma_F < T] \\ &= \mathbb{E}_x[\mathbb{E}_{X_{\sigma_F}}[f(X_T)] : \sigma_F < T] \\ &= \mathbb{E}_x[\mathbb{E}_{X_{\sigma_F}}[f(X_T)] \mathbf{1}_{U \cup ((U^c \cap F) \setminus N)}(X_T) : \sigma_F < T] \\ &= \mathbb{E}_x[f(X_{\sigma_F}) \mathbf{1}_{U \cup ((U^c \cap F) \setminus N)}(X_T) : \sigma_F < T] \\ &= \mathbb{E}_x[f(X_{\sigma_F}) : \sigma_F < T]. \end{aligned}$$

□

Proof of Theorem 2.1: Let $x \in U$. Then

$$\begin{aligned}\mathbb{E}_x[f(X_T)] &= \mathbb{E}_x[f(X_T) : \sigma_F < T] + \mathbb{E}_x[f(X_T) : \sigma_F = T] \\ &= \mathbb{E}_x[f(X_{\sigma_F}) : \sigma_F < T] + \mathbb{E}_x[f(X_{\sigma_F}) : \sigma_F = T] \\ &= \mathbb{E}_x[f(X_{\sigma_F})] = h_f(x),\end{aligned}$$

where the equality in the second line follows from Lemma 2.2 (b). By Lemma 2.2 (a) we obtain that $h_f(x) = \mathbb{E}_x[h_f(X_\tau) : \tau < \zeta]$. □

We say that the Harnack inequality holds for X if for any open subset D of E and any compact subset K of D , there exists a constant $C > 0$ depending only on D and K such that for nonnegative function h harmonic in D with respect to X ,

$$\sup_{x \in K} h(x) \leq C \inf_{x \in K} h(x).$$

The Harnack inequality for Y is defined in the same way using the relative topology.

We say that the scale invariant Harnack inequality holds for X if there exist $R > 0$ and $C > 0$ such that for any $x_0 \in E$, any $r \leq R$ and any nonnegative function h harmonic in $B(x_0, r) := \{x \in E : \rho(x, x_0) < r\}$ with respect to X ,

$$\sup_{x \in B(x_0, r/2)} h(x) \leq C \inf_{x \in B(x_0, r/2)} h(x).$$

As an immediate consequence of Theorem 2.1, we get

Theorem 2.3 *If the Harnack inequality holds for X , then the Harnack inequality also holds for Y . If the scale invariant Harnack inequality holds for X , then the scale invariant Harnack inequality also holds for Y in the following sense: there exist $R > 0$ and $C > 0$ such that for any $x_0 \in F$, any $r \leq R$ and any nonnegative function h harmonic in $B(x_0, r) \cap F$ with respect to Y ,*

$$\sup_{x \in B(x_0, r/2) \cap F} h(x) \leq C \inf_{x \in B(x_0, r/2) \cap F} h(x).$$

Remark 2.4 We did not use the quasi-left continuity of X in this section. Thus one can easily see that our main results (Theorem 2.1 and Theorem 2.3) are also true for right processes. Moreover, all results except the second statement of Theorem 2.3 are true for right processes on a Radon space, i.e., a space that is homeomorphic to a universally measurable subset of a compact metric space.

Remark 2.5 By the strong Markov property, under the following assumption **A1'** instead of the assumption **A1**, Theorem 2.1 and Theorem 2.3 are also true. We omit the details.

A1' : For every open sets $O_1 \subset O_2$ with $O_2^c \cap F \neq \emptyset$, there exists an open subset U with $\overline{O_1} \subset U \subset \overline{U} \subset O_2$ such that every point $x \in U^c \cap F$ is regular point of $U^c \cap F$ with respect to X , that is

$$\mathbb{P}_x(\sigma_{U^c \cap F} = 0) = 1, \quad \text{for every } x \in U^c \cap F,$$

where $\sigma_{U^c \cap F} = \inf\{t > 0 : X_t \in U^c \cap F\}$ is the hitting time of $U^c \cap F$ for X .

3 Examples

The first example we give concerns subordinate Brownian motions.

Example 3.1 Suppose that $X_t^{(1)}$ is a Brownian motion on \mathbb{R}^d and $X_t^{(2)}$ is an independent diffusion on \mathbb{R} . Let m be the speed measure of $X^{(2)}$. It is well known that m is an excessive reference measure for $X^{(2)}$, and $X^{(2)}$ is a symmetric process with respect to m . Define a measure ξ on \mathbb{R}^{d+1} by $\xi(dx) = dx^{(1)} \times m(dx^{(2)})$, $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{d+1}$, where $dx^{(1)}$ stands for the d -dimensional Lebesgue measure. Then ξ is an excessive reference measure for the process $X_t = (X_t^{(1)}, X_t^{(2)})$ on \mathbb{R}^{d+1} and X is symmetric with respect to ξ . Suppose that 0 is regular for itself with respect to $X^{(2)}$, that is, starting from 0, $X^{(2)}$ returns to 0 immediately with probability 1. Let $L = (L_t : t \geq 0)$ be the local time of $X^{(2)}$ at 0. Then for any $t > 0$ and $x \in \mathbb{R}$ we have

$$\mathbb{E}_x[L_t] = \int_0^t p^{(2)}(s, x, 0) ds,$$

where $p^{(2)}$ stands for the transition density of $X^{(2)}$ with respect to m . Using this, one can easily check that for any nonnegative Borel function f on \mathbb{R}^{d+1} we have

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathbb{R}^{d+1}} \xi(dx) \mathbb{E}_x \left[\int_0^t f(X_s) dL_s \right] = \int_{\mathbb{R}^d} f(x^{(1)}, 0) dx^{(1)}, \quad \text{for all } x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{d+1},$$

that is to say, as a positive continuous additive functional of X , the Revuz measure of L is the d -dimensional Lebesgue measure on the hyperplane $\mathbb{H} := \mathbb{R}^d \times \{0\}$. Since the support of L is clearly equal to \mathbb{H} , we take $F = \mathbb{H}$. Let $(\tau_t : t \geq 0)$ be the right-continuous inverse of $(L_t : t \geq 0)$. It is well known that $(\tau_t : t \geq 0)$ is a subordinator. The process Y defined by $Y_t := X(\tau_t)$ is the trace of X on \mathbb{H} and it is a subordinate Brownian motion.

Since X is symmetric, semi-polar sets are polar by [17, Theorem 4.1.2]. Thus F and X satisfy the assumptions **A1** and **A2**.

For $\alpha \in (0, 2)$, let Z_t be a Bessel process of dimension $2 - \alpha$, that is a diffusion process on $[0, \infty)$ with infinitesimal generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \frac{1 - \alpha}{2x} \frac{d}{dx}.$$

Similar to [28, Exercise XII.2.16], by changing the sign of each excursion of Z with probability $1/2$, we obtain a diffusion process $X^{(2)}$ on \mathbb{R} whose generator on $\mathbb{R} \setminus \{0\}$ is given by

$$\frac{1}{2} \frac{d^2}{dx^2} + \frac{1 - \alpha}{2|x|} \frac{d}{dx}.$$

In this case the trace of $X = (X^{(1)}, X^{(2)})$ on \mathbb{H} is a symmetric α -stable process. Using results from [14] one can check (see [5]) that the scale invariant Harnack inequality holds for X . Thus it follows from Theorem 2.3 that the scale invariant Harnack inequality holds for the symmetric α -stable process Y .

Example 3.2 Suppose that $d \geq 2$ and X is a symmetric diffusion in \mathbb{R}^d whose infinitesimal generator is $\sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j)$ with $(a_{ij}(x))$ being measurable and uniformly elliptic. It is well known that the transition density $p(t, x, y)$ satisfies the following estimates:

$$c_1 t^{-d/2} e^{-c_2|x-y|^2/t} \leq p(t, x, y) \leq c_3 t^{-d/2} e^{-c_4|x-y|^2/t} \quad (3.1)$$

for some positive constants $c_i, i = 1, 2, 3, 4$, (see [15].)

Let μ be a Radon measure on \mathbb{R}^d satisfying

$$\mu(B(x, r)) \leq cr^\beta, \quad \text{for all } x \in \mathbb{R}^d, \quad 0 < r < 1,$$

for some $c > 0$ and $\beta > d - 2$. Then, using (3.1), one can follow the proof of [24, Proposition 2.3] line by line to conclude that μ is smooth in the strict sense (see [17, p.195] for the definition). Therefore it follows from [17, Theorem 5.1.7] that there exists a unique positive continuous additive functional $A = (A_t : t \geq 0)$ of X in the strict sense with Revuz measure μ . Let F be the support of A as in (2.2), which is also the quasi support of μ (see [17, p.168] for the definition). Then by [17, Theorem 6.2.1], the trace process Y is a Hunt process on F .

Since the scale invariant Harnack inequality holds for X by [15], in the case when F satisfies **A2**, we know by Theorem 2.3 that the scale invariant Harnack inequality also holds for Y . In particular, if F is a closed β -set (i.e., $\mu(B(x, r)) \asymp r^\beta$ for all $x \in F$ and $0 < r \leq 1$) and μ is the restriction to F of the β -dimensional Hausdorff measure for some $\beta > d - 2$, then F is nonpolar and **A2** is satisfied.

Example 3.3 Suppose that $d \geq 3$. We assume that D is a bounded domain whose boundary ∂D has zero Lebesgue measure and there exists a bounded linear extension operator $T : W^{1,2}(D) \rightarrow W^{1,2}(\mathbb{R}^d)$ such that $Tf = f$ a.e. on D for $f \in W^{1,2}(D)$. Here $W^{1,2}(D)$ is Sobolev space on D . In particular, bounded uniform domains satisfy the above condition. (For the definition of uniform domains, see [2, Definition 1.1].)

Let X be a symmetric reflecting diffusion in \bar{D} whose infinitesimal generator is $\sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j)$ with $(a_{ij}(x))$ being measurable and uniformly elliptic. See [6] and [2] for the definition and properties of X . It is well-known that the transition density $p(t, x, y)$ satisfies the following estimates:

$$c_1 t^{-d/2} e^{-c_2|x-y|^2/t} \leq p(t, x, y) \leq c_3 t^{-d/2} e^{-c_4|x-y|^2/t}, \quad \text{for all } (t, x, y) \in (0, 1] \times \bar{D} \times \bar{D} \quad (3.2)$$

for some positive constants $c_i, i = 1, 2, 3, 4$. (See [2, p.3] and [2, (3.6)].) From the above inequality and the semigroup property, we have

$$p(t, x, y) \leq c_5, \quad \text{for every } (t, x, y) \in (1, \infty) \times \bar{D} \times \bar{D}$$

for some $c_5 > 0$. Using the two displays above we easily show that

$$G_1(x, y) := \int_0^\infty e^{-t} p(t, x, y) dt \leq c_6 \frac{1}{|x - y|^{d-2}}, \quad x, y \in \overline{D} \quad (3.3)$$

for some $c_6 > 0$. Let μ be a Radon measure on \overline{D} satisfying

$$\mu(B(x, r)) \leq cr^\beta, \quad \text{for all } x \in \overline{D}, \quad 0 < r < 1,$$

for some $c > 0$ and $\beta > d - 2$. Then, using (3.3), one can follow the proof of [24, Proposition 2.3] line by line and conclude that μ is smooth in the strict sense. Therefore, as in Example 3.3, there exists a unique positive continuous additive functional $A = (A_t : t \geq 0)$ of X in the strict sense with Revuz measure μ and the trace process Y is a Hunt process on F , the support of A as in (2.2).

Since the scale invariant Harnack inequality for X follows easily from (3.2) (see [15]), in the case when F satisfies **A2**, we know by Theorem 2.3 that the scale invariant Harnack inequality also holds for Y . In particular, if F is a closed β -set contained in \overline{D} and μ is restriction to F of the β -dimensional Hausdorff measure for some $\beta > d - 2$, then F is nonpolar and **A2** is satisfied.

Example 3.4 Let E be a closed n -set in \mathbb{R}^d with $d \geq 2$ and $0 < n \leq d$. That is, there is a positive Borel measure ν on E such that $\nu(B(x, r)) \asymp r^n$ for all $x \in E$ and $0 < r \leq 1$.

Fix an n -measure ν on E and $0 < \alpha < 2$. Define

$$\begin{aligned} \mathcal{F} &= \left\{ u \in L^2(E, \nu) : \int_{E \times E} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} \nu(dx) \nu(dy) < \infty \right\} \\ \mathcal{E}(u, v) &= \frac{1}{2} \int_{E \times E} \frac{c(x, y)(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}} \nu(dx) \nu(dy) \end{aligned}$$

for $u, v \in \mathcal{F}$, where $c(x, y)$ is a symmetric function on $E \times E$ that is bounded between two positive constants. It is easy to check that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E, \nu)$ and therefore there is an associated ν -symmetric Hunt process X on E starting from every point in E except for an exceptional set that has zero capacity. The process X is called a stable-like process on E .

We further assume that there exists $c_1 > 0$ such that $\nu(B(x, r)) \leq c_1 r^n$ for every $x \in E$ and $r > 0$. Then, it is shown in [11] that, in fact X is a Feller process on E and it has a Hölder continuous transition density function $p(t, x, y)$. Furthermore,

$$p(t, x, y) \asymp \left(t^{-n/\alpha} \wedge \frac{t}{|x - y|^{n+\alpha}} \right), \quad \text{for all } (t, x, y) \in (0, 1] \times E \times E. \quad (3.4)$$

Using this and the semigroup property we can easily show that there exists $c_2 > 0$ such that

$$p(t, x, y) \leq c_2, \quad \text{for every } t \geq 1 \text{ and } x, y \in E \text{ with } |x - y| < 1.$$

Using the two displays above we can show that there exists $c_3 > 0$ such that

$$G_1(x, y) := \int_0^\infty e^{-t} p(t, x, y) dt \leq c_3 \frac{1}{|x - y|^{n-\alpha}}, \quad \text{for every } x, y \in E \text{ with } |x - y| < 1. \quad (3.5)$$

Let μ be a Radon measure on E satisfying

$$\mu(B(x, r)) \leq cr^\beta, \quad \text{for all } x \in E, \quad 0 < r < 1$$

for some $c > 0$ and $\beta > n - \alpha$. Using (3.5), one can follow the proof of [24, Proposition 2.3] line by line and conclude that μ is smooth in the strict sense. Therefore, as in Example 3.3, there exists a unique positive continuous additive functional $A = (A_t : t \geq 0)$ of X in the strict sense with Revuz measure μ and the trace process Y is a Hunt process on F , the support of A as in (2.2). Since the scale invariant Harnack inequality for X holds (see [11]), in the case when F satisfies **A2**, we know by Theorem 2.3 that the scale invariant Harnack inequality also holds for Y .

When E is the Euclidean closure of an open d -set in \mathbb{R}^d and ξ is the Lebesgue measure on \mathbb{R}^d , the corresponding process X is the reflected α -stable process on E studied in [4]. In this case, if F is a closed β -set contained in E and μ is the restriction to F of the β -dimensional Hausdorff measure for some $\beta > d - 2$, then F is nonpolar and **A2** is satisfied.

We can give a lot more examples of symmetric Markov processes and their traces where the (scale invariant) Harnack inequality holds for some trace processes. For instance, we can give explicit examples of trace processes of the subordinate Brownian motions studied in [26] and [23] satisfying the scale invariant Harnack inequality, and we can also give explicit examples of trace processes of the jump processes of mixed type studied in [12] satisfying the scale invariant Harnack inequality. Now we give an example of a non-symmetric Markov process X and its traces.

Example 3.5 Let $d \geq 3$. We say that a signed Radon measure ν on \mathbb{R}^d belongs to the Kato class $\mathbf{K}_{d,i}$ if $\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} |x-y|^{-d+i} |\nu|(dy) = 0$, for $i = 1, 2$. We assume that $\mu = (\mu^1, \dots, \mu^d)$ is fixed with each μ^i being a signed measure on \mathbb{R}^d belonging to $\mathbf{K}_{d,1}$. We also assume that the operator L is either L_1 or L_2 where

$$L_1 := \frac{1}{2} \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j) \quad \text{and} \quad L_2 := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \partial_i \partial_j$$

with $\mathbf{A} := (a_{ij}(x))$ being C^1 and uniformly elliptic but not necessarily symmetric. Informally speaking, a diffusion process in \mathbb{R}^d with drift μ is a diffusion process in \mathbb{R}^d with generator $L + \mu \cdot \nabla$. For the precise definition of the diffusion X with drift μ and its property, we refer the readers to [1, 20, 21, 22]. In [20] (also see Section 6 in [21]), it was shown that X has a density $q(t, x, y)$ which is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and that there exist positive constants c_i , $i = 1, \dots, 6$, such that

$$c_1 e^{-c_2 t} t^{-\frac{d}{2}} e^{-\frac{c_3 |x-y|^2}{2t}} \leq q(t, x, y) \leq c_4 e^{c_5 t} t^{-\frac{d}{2}} e^{-\frac{c_6 |x-y|^2}{2t}}. \quad (3.6)$$

Thus the process X satisfies the conditions (R) , (T_1) , (T_2) , (U_1) and (U_2) in Chapter 5 of [13]. It follows from [13, Theorem 5.4] and the Corollary to [13, Theorem 5.2] that X satisfies Hunt's Hypothesis (B) and the equilibrium principle (E) . By repeating the argument in the proof of [25,

Theorem 2.5.1] we know that X satisfies the maximum principle (M) in Chapter 5 of [13]. Thus by [13, Theorem 5.3] it follows that every semipolar set of X is polar for X , that is, **A1** is valid.

Let D be an arbitrary bounded domain and X^D be the subprocess of X killed upon leaving D with the transition density function $q^D(t, x, y)$ with respect to the Lebesgue measure. Define

$$h_D(x) := \int_D G_D(y, x) dy \quad \text{and} \quad \xi_D(dx) := h_D(x) dx,$$

where $G_D(x, y) := \int_0^\infty q^D(t, x, y) dt$ is the Green function of X^D . Then ξ_D is an excessive measure with respect to X^D [22, Proposition 2.2] and $q^D(t, x, y)/h_D(y)$ is the transition density function of X^D with respect to the reference measure ξ_D .

It is proved in [22] that for any measure $\nu \in \mathbf{K}_{d,2}$, the measure $h_D\nu$ is a smooth measure of X^D with respect to ξ . Thus for any measure ν in $\mathbf{K}_{d,2}$, we can construct the trace process Y of X on the support of ν . For example, if U is a Lipschitz domain with $\bar{U} \subset D$ and σ is the surface measure of ∂U , then it is easy to check that $\sigma \in \mathbf{K}_{d,2}$. Thus we can talk about the trace Y of X^D on ∂U . Since the scale invariant Harnack inequality is valid for X^D (see [20]), the scale invariant Harnack inequality is also valid for Y .

4 Converse of Theorem 2.1

In this section, we continue to assume that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x, x \in E)$ is a standard process on a locally compact separable metric space (E, ρ) .

Besides **A1** and **A2**, we further assume that X satisfies the following

A3. Every harmonic function in D with respect to X is continuous in D .

Recall that, for any function $f : F \rightarrow [0, \infty)$, the extension h_f of f is $h_f(x) := \mathbb{E}_x[f(X_{\sigma_F})]$, for $x \in E$.

Theorem 4.1 *Suppose that f is a nonnegative function on F . If the extension h_f of f is harmonic with respect to X in an open subset \tilde{D} of E and*

$$\mathbb{P}_x \left(\inf\{t > 0 : X_t \in \tilde{D}^c \cap F\} < \infty \right) = 1 \quad \text{for every } x \in F, \quad (4.1)$$

then f is harmonic with respect to Y in $D := \tilde{D} \cap F$.

In the remainder of this section, we will prove Theorem 4.1. We fix an open set \tilde{D} in E and put $D = \tilde{D} \cap F$. We assume that f is nonnegative function on F such that its extension h_f is harmonic with respect to X in \tilde{D} .

We fix a bounded open set B in F such that the closure of B in F is contained in D , and let \tilde{B} be any bounded open set strictly contained in \tilde{D} such that $\tilde{B} \cap F = B$. Let

$$S_0 := \sigma_F = \inf\{t > 0 : X_t \in F\}$$

be the first time the process X hits F ,

$$T := T_{\tilde{B}} = \inf\{t > 0 : X_t \in \tilde{B}^c \cap F\} \quad (4.2)$$

be the first time the process X hits the set F outside of \tilde{B} and

$$\tau_1 := \tau_{\tilde{B}} = \inf\{t > 0 : X_t \in \tilde{B}^c\}$$

the exit time of X from \tilde{B} . Note that, by (4.1)

$$\mathbb{P}_x(T < \infty) = 1 \text{ for every } x \in F. \quad (4.3)$$

We use the notation $\hat{\tau}_B$ for the first exit time of Y from $B = B \cap F$. Let us inductively introduce two families of stopping times. For $n \geq 1$ let

$$S_n := \begin{cases} \tau_n + S_0 \circ \theta_{\tau_n}, & \tau_n < T, \\ T, & \tau_n = T, \end{cases} \quad (4.4)$$

$$\tau_{n+1} := \begin{cases} S_n + \tau_1 \circ \theta_{S_n}, & S_n < T, \\ T, & S_n = T. \end{cases} \quad (4.5)$$

Note that for $n \geq 1$ we have

$$S_{n+1}(\omega) = S_n(\omega) + S_1 \circ \theta_{S_n}(\omega), \quad \text{for } S_n(\omega) < \infty. \quad (4.6)$$

Lemma 4.2 *Let $x \in B$. Then*

$$\begin{aligned} \mathbb{E}_x [\mathbb{E}_{X_{\tau_1}}[f(X_{S_0}) : S_0 = T]] &= \mathbb{E}_x [f(X_{S_1}) : S_1 = T], \\ \mathbb{E}_x [\mathbb{E}_{X_{\tau_1}}[f(X_{S_0}) : S_0 < T]] &= \mathbb{E}_x [f(X_{S_1}) : S_1 < T]. \end{aligned}$$

Proof. By the strong Markov property it follows that

$$\begin{aligned} \mathbb{E}_x [\mathbb{E}_{X_{\tau_1}}[f(X_{S_0})1_{\{S_0=T\}}]] &= \mathbb{E}_x [f(X_{S_0} \circ \theta_{\tau_1}) 1_{\{S_0=T\}} \circ \theta_{\tau_1}] \\ &= \mathbb{E}_x [f(X_T \circ \theta_{\tau_1}) 1_{\{S_0 \circ \theta_{\tau_1} = T \circ \theta_{\tau_1}\}}] \\ &= \mathbb{E}_x [f(X_{\tau_1+T \circ \theta_{\tau_1}}) 1_{\{S_0 \circ \theta_{\tau_1} = T - \tau_1\}}] \\ &= \mathbb{E}_x [f(X_T) 1_{\{\tau_1 + S_0 \circ \theta_{\tau_1} = T\}}] \\ &= \mathbb{E}_x [f(X_T) 1_{\{S_1=T\}}] = \mathbb{E}_x [f(X_{S_1}) 1_{\{S_1=T\}}]. \end{aligned}$$

The proof of the second equality is similar and uses that on $\{S_1 < T\}$ it holds that $\tau_1 < T$, and hence $X_{S_0} \circ \theta_{\tau_1} = X_{S_1}$. \square

Lemma 4.3 *Assume that h_f is harmonic with respect to X in \tilde{D} . Then for every $n \geq 1$ and every $x \in B$,*

$$f(x) = \mathbb{E}_x[f(X_{S_n})].$$

Proof. For $n = 1$ and $x \in B$ the result follows from the following computation:

$$\begin{aligned} f(x) &= h_f(x) = \mathbb{E}_x[h_f(X_{\tau_1})] = \mathbb{E}_x[\mathbb{E}_{X_{\tau_1}}[f(X_{S_0})]] \\ &= \mathbb{E}_x[\mathbb{E}_{X_{\tau_1}}[f(X_{S_0}) : S_0 = T]] + \mathbb{E}_x[\mathbb{E}_{X_{\tau_1}}[f(X_{S_0}) : S_0 < T]] \\ &= \mathbb{E}_x[f(X_{S_1}) : S_1 = T] + \mathbb{E}_x[f(X_{S_1}) : S_1 < T] = \mathbb{E}_x[f(X_{S_1})], \end{aligned}$$

where the last line follows from the previous lemma. The proof for $n \geq 2$ follows by induction. By use of (4.6), we have

$$\begin{aligned} \mathbb{E}_x[f(X_{S_{n+1}})] &= \mathbb{E}_x[f(X_{S_n+S_1 \circ \theta_{S_n}})] = \mathbb{E}_x[f(X_{S_1 \circ \theta_{S_n}})] \\ &= \mathbb{E}_x[\mathbb{E}_{X_{S_n}}[f(X_{S_1})]] = \mathbb{E}_x[f(X_{S_n})] = f(x). \end{aligned}$$

□

Proof of Theorem 4.1: Define $S = \lim_{n \rightarrow \infty} S_n \leq T$ and $\rho = \lim_{n \rightarrow \infty} \tau_n$. Note that $\{\widehat{\tau}_B < \widehat{\zeta}\} = \{T < \zeta\}$. If $S_n < T < \zeta$ for every $n \geq 1$, then it holds that $S_1 < S_2 < \dots < T < \zeta$. Then also $\tau_1 < \tau_2 < \dots < T < \zeta$. By the quasi-left continuity of X , $X_S = \lim_{n \rightarrow \infty} X_{S_n} = \lim_{n \rightarrow \infty} X_{\tau_n} = X_\rho$. Since $X_{S_n} \in B$ for all $n \geq 1$, it follows that $X_S \in \text{Cl}(B)$ where $\text{Cl}(B)$ is the closure of B in F . Similarly, $X_{\tau_n} \in \widetilde{B}^c$ for all $n \geq 1$, hence $X_\rho \in \widetilde{B}^c$. Therefore, $X_S = X_\rho \in \text{Cl}(B) \cap \widetilde{B}^c \subset \text{Cl}(B) \cap B^c = \partial B$ where ∂B denotes the boundary of B in F . In particular, it follows that $S = T$, and $X_T = Y_{\widehat{\tau}_B} \in \partial B$. Since h_f is harmonic with respect to X in \widetilde{D} , f is continuous in D by **A3**. Thus by the Lebesgue dominated convergence theorem, we have

$$\lim_{k \rightarrow \infty} \mathbb{E}_x[f(X_{S_k}) : \cap_{n=0}^{\infty} \{S_n < T < \zeta\}] = \mathbb{E}_x[f(Y_{\widehat{\tau}_B}) : \cap_{n=0}^{\infty} \{S_n < T < \zeta\}]. \quad (4.7)$$

Secondly, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathbb{E}_x[f(X_{S_k}) : \cup_{n=0}^{\infty} \{S_n = T < \zeta\}] &= \mathbb{E}_x[f(X_T) : \cup_{n=0}^{\infty} \{S_n = T < \zeta\}] \\ &= \mathbb{E}_x[f(Y_{\widehat{\tau}_B}) : \cup_{n=0}^{\infty} \{S_n = T < \zeta\}]. \end{aligned} \quad (4.8)$$

Finally, by (4.3), Lemma 4.3, (4.7) and (4.8), it follows that

$$\begin{aligned} f(x) &= \lim_{k \rightarrow \infty} \mathbb{E}_x[f(X_{S_k})] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_x[f(X_{S_k}) : \cap_{n=0}^{\infty} \{S_n < T < \zeta\}] + \lim_{k \rightarrow \infty} \mathbb{E}_x[f(X_{S_k}) : \cup_{n=0}^{\infty} \{S_n = T < \zeta\}] \\ &= \mathbb{E}_x[f(Y_{\widehat{\tau}_B}); \widehat{\tau}_B < \widehat{\zeta}]. \end{aligned}$$

Since B is an arbitrary bounded open set strictly contained in D , this proves that f is harmonic with respect to Y in D . □

In fact, it is easy to see that Theorem 4.1 is also true when X is a right process on a Radon space.

Remark 4.4 Many examples of the trace processes are pure jump processes. For a large class of pure jump processes, the following is true:

A4 : For all open sets D_1 and D_2 in F with $\text{Cl}(D_1) \subset D_2$, there exists an open set B in F with $\text{Cl}(D_1) \subset B \subset \text{Cl}(B) \subset D_2$ such that

$$\mathbb{P}_x(Y_{\widehat{\tau}_B} \in \partial B) = 0 \quad \text{for every } x \in B. \quad (4.9)$$

(See [10, 30, 31].) Here and below, $\text{Cl}(B)$ denotes the closure of B in F and ∂B denotes the boundary of B in F . For example, by [30, Corollary 4.3], the trace process we considered in Example 3.1 satisfies **A4** since (4.9) is true for every smooth open set in \mathbb{R}^d . Recall that T , S_n and τ_n are the stopping times in (4.2), (4.4)-(4.5) respectively. Then it is easy to see that if Y satisfies (4.1) and (4.9), then

$$\mathbb{P}_x(\cup_{n=0}^{\infty} \{S_n = T\}) = 1 \quad \text{for every } x \in B. \quad (4.10)$$

In fact, suppose, on the contrary, that on a set of positive \mathbb{P}_x probability it holds that $S_1 < S_2 < \dots < T$. Then also $\tau_1 < \tau_2 < \dots < T$. Define $S = \lim_{n \rightarrow \infty} S_n \leq T$ and $\rho = \lim_{n \rightarrow \infty} \tau_n$. Note that $\{\widehat{\tau}_B < \widehat{\zeta}\} = \{T < \zeta\}$. If $S_n < T < \zeta$ for every $n \geq 1$, then it holds that $S_1 < S_2 < \dots < T < \zeta$. Then also $\tau_1 < \tau_2 < \dots < T < \zeta$. By the quasi-left continuity of X , $X_S = \lim_{n \rightarrow \infty} X_{S_n} = \lim_{n \rightarrow \infty} X_{\tau_n} = X_\rho$. Since $X_{S_n} \in B$ for all $n \geq 1$, it follows that $X_S \in \text{Cl}(B)$. Similarly, $X_{\tau_n} \in \widetilde{B}^c$ for all $n \geq 1$, hence $X_\rho \in \widetilde{B}^c \subset B^c$. Therefore, $X_S = X_\rho \in \partial B$. In particular, it follows that $S = T$, and $X_T = Y_{\tau_B} \in \partial B$. Hence, $\mathbb{P}_x(Y_{\widehat{\tau}_B} \in \partial B) > 0$, which contradicts (4.9).

Now suppose f is a nonnegative real-valued function on F with the harmonic extension h_f in an open subset \widetilde{D} of E . Instead of **A3**, we assume that f is locally bounded on $D := \widetilde{D} \cap F$ and (4.10) is true. Then we get

$$\lim_{n \rightarrow \infty} \mathbb{E}_x[f(X_{S_n}) : S_n < T] \leq c \lim_{n \rightarrow \infty} \mathbb{P}_x(S_n < T) = 0. \quad (4.11)$$

Moreover, by (4.10), $(\{S_n = T\} : n \geq 0)$ is a sequence of events which increases to a \mathbb{P}_x -a.s. event. Therefore, by the monotone convergence theorem we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_x[f(X_{S_n}) : S_n = T] &= \lim_{n \rightarrow \infty} \mathbb{E}_x[f(X_T) : S_n = T] \\ &= \mathbb{E}_x[f(X_T) : \cup_{n=0}^{\infty} \{S_n = T\}] = \mathbb{E}_x[f(X_T)]. \end{aligned} \quad (4.12)$$

Thus, by use of Lemma 4.3 and (4.11)-(4.12), it follows that

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} \mathbb{E}_x[f(X_{S_n})] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_x[f(X_{S_n}) : S_n < T] + \lim_{n \rightarrow \infty} \mathbb{E}_x[f(X_{S_n}) : S_n = T] \\ &= \mathbb{E}_x[f(X_T)] = \mathbb{E}_x[f(Y_{\widehat{\tau}_B})]. \end{aligned}$$

Therefore, by the strong Markov property, we can conclude that Theorem 4.1 is true for locally bounded f without the assumption **A3** if Y satisfies **A4**.

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