Harnack Inequality for Some Classes of Markov Processes*

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Abstract

In this paper we establish a Harnack inequality for nonnegative harmonic functions of some classes of Markov processes with jumps.

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1 Introduction

Let $X = (X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ be a conservative strong Markov process on $\mathbb{R}^d$. A nonnegative Borel function h on $\mathbb{R}^d$ is said to be harmonic with respect to X in a domain (i.e., a connected open set) $D \subset \mathbb{R}^d$ if it is not identically infinite in D and if for any bounded open subset $B \subset \overline{B} \subset D$,

$$h(x) = \mathbb{E}_x[h(X(\tau_B))1_{\tau_B<\infty}], \quad \forall x \in B,$$

where $\tau_B = \inf\{t > 0 : X_t \notin B\}$ is the first exit time of B.

We say that the Harnack inequality holds for X if for any domain $D \subset \mathbb{R}^d$ and any compact subset K of D, there is a constant $C > 0$ depending only on D and K such that for any nonnegative function h harmonic with respect to X in D,

$$\sup_{x\in K} h(x) \le C \inf_{x\in K} h(x).$$

The Harnack inequality is a very important tool in studying harmonic functions. For instance, the Harnack inequality for diffusion processes is extremely important in the study of partial differential equations. It is well known that, when X is a Brownian motion (or a diffusion process satisfying certain conditions), the Harnack inequality holds. Until very recently almost all results concerning the Harnack inequality were restricted to Markov processes with continuous paths, i.e., to harmonic functions corresponding to local operators. The only exception was the rotationally invariant α-stable process, $\alpha \in (0, 2)$, in which case the Harnack inequality follows directly from the explicit form of the exit distribution from a ball (i.e., the corresponding Poisson kernel).

The first result on the Harnack inequality for processes with jumps (other than rotationally invariant stable processes) was obtained by Bass and Levin in [1]. They studied a jump process whose jump kernel is symmetric and comparable to the jump kernel of the rotationally invariant α-stable process and proved the Harnack inequality for bounded nonnegative harmonic functions of this process. Vondraček [15] adapted the arguments of [1] and proved that, when X is a (not necessarily rotationally invariant) strictly α-stable process, $\alpha \in (0, 2)$, the Harnack inequality holds. In a recent preprint [3], the Harnack inequality was proved by using a different method for symmetric α-stable processes under the assumptions that $\alpha \in (0, 1)$ and the Lévy measure is comparable to the Lévy measure of the rotationally invariant α-stable process. In [7] Kolokoltsov proved detailed estimates on the transition density of symmetric α-stable processes whose Lévy measures are comparable to the Lévy measure of the rotationally invariant α-stable process. The estimates of [7] could be used to prove the Harnack inequality for the symmetric stable processes studied in [7]. We would like to emphasize that the processes in [1], [3], [7] and [15] satisfy a scaling property, a fact often used in the arguments of these papers.

The goal of this paper is to extend Bass-Levin's method and prove the Harnack inequality for quite general classes of Markov processes. These classes include processes that need not have any scaling property and are not necessarily symmetric. In Section 2, we extract the essential ingredients of the Bass-Levin method by isolating three conditions that suffice to prove a Harnack inequality. Then we prove that a Markov process satisfying those conditions indeed satisfies the Harnack inequality. The rest of the paper is devoted to verifying that various classes of processes satisfy those conditions. In Section 3 we study Lévy processes and give sufficient conditions on the Lévy measure for a Harnack inequality to hold. In particular, in Examples 3.6 – 3.10 we show that various mixtures of stable processes, as well as relativistic stable processes, satisfy the Harnack inequality. In Section 4 we show that the conditions are satisfied for a class of symmetric Markov processes with no diffusion component. In Section 5 we deal with non-symmetric Markov processes with Lévy type generators, again with no diffusion component.

For any subset A of $\mathbb{R}^d$, we use τ_A and T_A to denote the exit and hitting times of A respectively.

In this paper, we use $C_c^2(\mathbb{R}^d)$ to denote the family of C^2 functions with compact support, and $C_b^2(\mathbb{R}^d)$ to denote the family of C^2 functions f such that f and its partials up to order 2 are bounded.

2 General Results

Suppose that $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ is a conservative Borel right process such that its sample paths have left limits (in the Euclidean topology of $\mathbb{R}^d$) in $(0, \infty)$. We assume that X admits a Lévy system (N, H), where $N(x, dy)$ is a kernel on $\mathbb{R}^d$ and H_t is a positive continuous additive functional of X with bounded 1-potential such that for any nonnegative Borel function f on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left(\sum_{s \le t} f(X_{s-}, X_s) \right) = \mathbb{E}_x \left(\int_0^t \int_{\mathbb{R}^d} f(X_s, y) N(X_s, dy) dH_s \right).$$

In this paper we assume that $N(x, dy)$ has a density $j(x, y)$ with respect to the Lebesgue measure and that $H_t = t$.

Now we state the conditions that are needed for the proof of the Harnack inequality:

(A1) There exists a constant $C_1 > 0$ such that for any $r \in (0, 1)$ and $x \in \mathbb{R}^d$,

$$\sup_{z \in B(x,r)} \mathbb{E}_z \tau_{B(x,r)} \le C_1 \inf_{z \in B(x,r/2)} \mathbb{E}_z \tau_{B(x,r)} < \infty.$$

(A2) There exists a constant $C_2 > 0$ such that for all $r \in (0,1)$, $x \in \mathbb{R}^d$ and $A \subset B(x,r)$ we have

$$\mathbb{P}_y(T_A < \tau_{B(x,3r)}) \geq C_2 \frac{|A|}{|B(x,r)|}, \quad \forall y \in B(x,2r).$$

(A3) There exist positive constants C_3 and C_4 such that for all $x \in \mathbb{R}^d$, $r \in (0,1)$, $z \in B(x,r/2)$ and every nonnegative bounded function H with support in $B(x,2r)^c$ we have

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \leq C_3(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(x,y) dy$$

and

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \geq C_4(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(x,y) dy.$$

Lemma 2.1 *If a bounded nonnegative function h on $\mathbb{R}^d$ is harmonic with respect to X in a domain $D \subset \mathbb{R}^d$, then for any $x \in D$,*

$$\mathbb{P}_x(h(X_t) \textit{ is right continuous in } [0, \tau_D)) = 1.$$

Proof. For any bounded open set $B \subset \overline{B} \subset D$, let $X^B = (X_t^B, \mathbb{P}_x, x \in B)$ be the process obtained by killing the process X upon exiting from B, then it follows from Corollary (12.24) of [14] that X^B is a right process. Using the fact that h is harmonic in D and the strong Markov property, one can see that for every $x \in B$,

$$\begin{aligned}
h(X(t \wedge \tau_B)) &= h(X(t))1_{t<\tau_B} + h(X(\tau_B))1_{t \geq \tau_B} \\
&= \mathbb{E}_{X(t)}[h(X(\tau_B))1_{\tau_B<\infty}]1_{t<\tau_B} + h(X(\tau_B))1_{t \geq \tau_B} \\
&= \mathbb{E}_x[(h(X(\tau_B))1_{\tau_B<\infty}) \circ \theta(t) \,|\, \mathcal{M}_t]1_{t<\tau_B} + h(X(\tau_B))1_{t \geq \tau_B} \\
&= \mathbb{E}_x[h(X(\tau_B))1_{t<\tau_B<\infty} \,|\, \mathcal{M}_t] + \mathbb{E}_x[h(X(\tau_B))1_{t \geq \tau_B} \,|\, \mathcal{M}_t] \\
&= \mathbb{E}_x[h(X(\tau_B))1_{\tau_B<\infty} \,|\, \mathcal{M}_t],
\end{aligned}$$

Thus $(h(X(t \wedge \tau_B)))$ is a $\mathbb{P}_x$-martingale for every $x \in B$. It follows that for any $x \in B$ and any $t > 0$,

$$\begin{aligned}
\mathbb{E}_x[h(X(\tau_B))1_{\tau_B<\infty}|\mathcal{M}_t] &\geq \mathbb{E}_x[h(X(\tau_B))1_{t<\tau_B<\infty}|\mathcal{M}_t] \\
&= \mathbb{E}_x[h(X(\tau_B))1_{\tau_B<\infty}|\mathcal{M}_t]1_{t<\tau_B} \\
&= h(X(t))1_{t<\tau_B} = h(X_t^B),
\end{aligned}$$

thus

$$h(x) \geq \mathbb{E}_x h(X_t^B).$$

Taking expectation we get

$$\mathbb{E}_x[h(X_t^B)] = \mathbb{E}_x[h(X(\tau_B))1_{t<\tau_B<\infty}].$$

Since for any $x \in B$, we have $\lim_{t\downarrow 0} \mathbb{P}_x(t < \tau_B) = 1$, it follows that

$$\lim_{t\downarrow 0} \mathbb{E}_x[h(X_t^B)] = \mathbb{E}_x[h(X(\tau_B))1_{\tau_B<\infty}] = h(x),$$

which implies that h is excessive with respect to X^B. Therefore for any $x \in B$,

$$\mathbb{P}_x(h(X_t) \text{ is right continuous in } [0, \tau_B)) = 1.$$

□

Theorem 2.2 *Suppose that the conditions (A1)–(A3) are satisfied. Then there exists $C_5 > 0$ such that, for any $r \in (0, 1/4)$, $x_0 \in \mathbb{R}^d$, and any function h which is nonnegative, bounded on $\mathbb{R}^d$, and harmonic with respect to X in $B(x_0, 16r)$, we have*

$$h(x) \le C_5 h(y), \quad \forall x, y \in B(x_0, r).$$

Proof. This proof is basically the proof given in [1]. Without loss of generality we may assume that h is strictly positive in $B(x_0, 16r)$. Using the harmonicity of h and the condition (A2), one can show that h is bounded from below on $B(x_0, r)$ by a positive number. To see this, let $\epsilon > 0$ be such that $F = \{x \in B(x_0, 3r) \setminus B(x_0, 2r) : h(x) > \epsilon\}$ has positive Lebesgue measure. Take a compact subset K of F so that it has positive Lebesgue measure. Then by condition (A2), for $x \in B(x_0, r)$, we have

$$h(x) = \mathbb{E}_x\left[h(X(T_K \wedge \tau_{B(x_0,3r)}))1_{\{T_K\wedge\tau_{B(x_0,3r)}<\infty\}}\right] > \epsilon\, C_2 \frac{|K|}{|B(x_0, 3r)|}.$$

By taking a constant multiple of h we may assume that $\inf_{B(x_0,r)} h = 1/2$. Choose $z_0 \in B(x_0, r)$ such that $h(z_0) \le 1$. We want to show that h is bounded above in $B(x_0, r)$ by a constant $C > 0$ independent of h and $r \in (0, 1/4)$. We will establish this by contradiction: If there exists a point $x \in B(x_0, r)$ with $h(x) = K$ where K is too large, we can obtain a sequence of points in $B(x_0, 2r)$ along which h is unbounded.

Using conditions (A1) and (A3), one can see that there exists $c_1 > 0$ such that if $x \in \mathbb{R}^d$, $s \in (0, 1)$ and H is nonnegative bounded function with support in $B(x, 2s)^c$, then for any $y, z \in B(x, s/2)$,

$$\mathbb{E}_z H(X(\tau_{B(x,s)})) \le c_1 \mathbb{E}_y H(X(\tau_{B(x,s)})). \tag{2.1}$$

By (A2), there exists $c_2 > 0$ such that if $A \subset B(x_0, 4r)$ then

$$\mathbb{P}_y(T_A < \tau_{B(x_0,16r)}) \ge c_2 \frac{|A|}{|B(x_0, 4r)|}, \quad \forall y \in B(x_0, 8r). \tag{2.2}$$

Again by (A2), there exists $c_3 > 0$ such that if $x \in \mathbb{R}^d$, $s \in (0,1)$ and $F \subset B(x, s/3)$ with $|F|/|B(x,s/3)| \geq 1/3$, then

$$\mathbb{P}_x(T_F < \tau_{B(x,s)}) \geq c_3. \tag{2.3}$$

Let

$$\eta = \frac{c_3}{3}, \qquad \zeta = (\frac{1}{3} \wedge \frac{1}{c_1})\eta. \tag{2.4}$$

Now suppose there exists $x \in B(x_0, r)$ with $h(x) = K$ for $K > \frac{2|B(x_0,4r)|}{c_2\zeta} \vee \frac{2(12)^d}{c_2\zeta}$. Let s be chosen so that

$$|B(x, \frac{s}{3})| = \frac{2|B(x_0,4r)|}{c_2\zeta K} < 1. \tag{2.5}$$

Note that this implies

$$s = 12(\frac{2}{c_2\zeta})^{1/d} r K^{-1/d} < r. \tag{2.6}$$

Let us write B_s for $B(x,s)$, τ_s for $\tau_{B(x,s)}$, and similarly for B_{2s} and τ_{2s}. Let A be a compact subset of

$$A' = \{u \in B(x, \frac{s}{3}) : h(u) \geq \zeta K\}.$$

It follows from Lemma 2.1 that $h(X_t)$ is right continuous in $[0, \tau_{B(x_0,16r)})$. Since $z_0 \in B(x_0, r)$ and $A' \subset B(x, \frac{s}{3}) \subset B(x_0, 2r)$, we can apply (2.2) to get

$$\begin{aligned} 1 &\geq h(z_0) \geq \mathbb{E}_{z_0}[h(X(T_A \wedge \tau_{B(x_0,16r)}))1_{\{T_A < \tau_{B(x_0,16r)}\}}] \\ &\geq \zeta K \mathbb{P}_{z_0}[T_A < \tau_{B(x_0,16r)}] \\ &\geq c_2 \zeta K \frac{|A|}{|B(x_0,4r)|}. \end{aligned}$$

Hence

$$\frac{|A|}{|B(x,\frac{s}{3})|} \leq \frac{|B(x_0,4r)|}{c_2\zeta K|B(x,\frac{s}{3})|} = \frac{1}{2}.$$

This implies that $|A'|/|B(x,s/3)| \leq 1/2$. Let F be a compact subset of $B(x,s/3) \setminus A'$ such that

$$\frac{|F|}{|B(x,\frac{s}{3})|} \geq \frac{1}{3}. \tag{2.7}$$

Let $H = h \cdot 1_{B_{2s}^c}$. We claim that

$$\mathbb{E}_x[h(X(\tau_s)); X(\tau_s) \notin B_{2s}] \leq \eta K.$$

If not, $\mathbb{E}_x H(X(\tau_s)) > \eta K$, and by (2.1), for all $y \in B(x, s/3)$, we have

$$\begin{aligned} h(y) &= \mathbb{E}_y h(X(\tau_s)) \geq \mathbb{E}_y[h(X(\tau_s)); X(\tau_s) \notin B_{2s}] \\ &\geq c_1^{-1}\mathbb{E}_x H(X(\tau_s)) \geq c_1^{-1}\eta K \geq \zeta K, \end{aligned}$$

contradicting (2.7) and the definition of A'.

Let $M = \sup_{B_{2s}} h$. We then have

$$\begin{aligned} K &= h(x) = \mathbb{E}_x h(X(\tau_s)) \\ &= \mathbb{E}_x[h(X(T_F)); T_F < \tau_s] + \mathbb{E}_x[h(X(\tau_s)); \tau_s < T_F; X(\tau_s) \in B_{2s}] \\ &\quad + \mathbb{E}_x[h(X(\tau_s)); \tau_s < T_F; X(\tau_s) \notin B_{2s}] \\ &\leq \zeta K \mathbb{P}_x(T_F < \tau_s) + M \mathbb{P}_x(\tau_s < T_F) + \eta K \\ &= \zeta K \mathbb{P}_x(T_F < \tau_s) + M(1 - \mathbb{P}_x(T_F < \tau_s)) + \eta K, \end{aligned}$$

or equivalently

$$\frac{M}{K} \geq \frac{1 - \eta - \zeta \mathbb{P}_x(T_F < \tau_s)}{1 - \mathbb{P}_x(T_F < \tau_s)}.$$

Using (2.3) and (2.4) we see that there exists $\beta > 0$ such that $M \geq K(1 + 2\beta)$. Therefore there exists $x' \in B(x, 2s)$ with $h(x') \geq K(1 + \beta)$.

Now suppose there exists $x_1 \in B(x_0, r)$ with $h(x_1) = K_1$. Define s_1 in terms of K_1 analogously to (2.5). Using the above argument (with x_1 replacing x and x_2 replacing x'), there exists $x_2 \in B(x_1, 2s_1)$ with $h(x_2) = K_2 \geq (1 + \beta)K_1$. We continue and obtain s_2 and then x_3, K_3, s_3, etc. Note that $x_{i+1} \in B(x_i, 2s_i)$ and $K_i \geq (1 + \beta)^{i-1} K_1$. In view of (2.6), $\sum_i |x_{i+1} - x_i| \leq c_4 r K_1^{-1/d}$. So if $K_1 > c_4^d$, then we have a sequence $x_1, x_2, \ldots$ contained in $B(x_0, 2r)$ with $h(x_i) \geq (1 + \beta)^{i-1} K_1 \to \infty$, a contradiction to h being bounded. Therefore we can not take K_1 larger than c_4^d, and thus $\sup_{y \in B(x_0, r)} h(y) \leq c_4^d$, which is what we set out to prove. $\Box$

By using standard chain argument, we can easily get the following consequence of the theorem above.

Corollary 2.3 *Suppose that (A1)–(A3) are satisfied. For any domain D of $\mathbb{R}^d$ and any compact subset K of D, there exists a constant $C_6 = C_6(D, K) > 0$ such that for any function h which is nonnegative bounded in $\mathbb{R}^d$ and harmonic with respect to X in D, we have*

$$h(x) \leq C_6 h(y), \quad x, y \in K.$$

In the next result, we remove the boundedness assumption on the harmonic functions in Corollary 2.3.

Theorem 2.4 *Suppose that (A1)–(A3) are satisfied. For any domain D of $\mathbb{R}^d$ and any compact subset K of D, there exists a constant $C_7 = C_7(D, K) > 0$ such that for any function h which is nonnegative in $\mathbb{R}^d$ and harmonic with respect to X in D, we have*

$$h(x) \leq C_7 h(y), \quad x, y \in K.$$

Proof. Choose a bounded domain U such that $K \subset U \subset \overline{U} \subset D$. If h is harmonic with respect to X in D, then

$$h(x) = \mathbb{E}_x[h(X(\tau_U))1_{\{\tau_U<\infty\}}], \quad x \in U.$$

For any $n \geq 1$, define

$$h_n(x) = \mathbb{E}_x[(h \wedge n)(X(\tau_U))1_{\{\tau_U<\infty\}}], \quad x \in \mathbb{R}^d.$$

Then h_n is a bounded nonnegative function on $\mathbb{R}^d$, harmonic with respect to X in U, and

$$\lim_{n\uparrow\infty} h_n(x) = h(x), \quad x \in \mathbb{R}^d.$$

It follows from Corollary 2.3 that there exists a constant $c = c(U, K) > 0$ such that

$$h_n(x) \leq c h_n(y), \quad x, y \in K, n \geq 1.$$

Letting $n \uparrow \infty$, we get that

$$h(x) \leq c h(y), \quad x, y \in K.$$

□

3 Lévy Processes

In this section we consider Lévy processes in $\mathbb{R}^d$ with no Gaussian component. Any Lévy process is a conservative Feller process, so it is a Borel right process with left limits. Our goal is to find conditions on the Lévy measure of the process that are sufficient for the Harnack inequality.

Let $X = (X_t, \mathbb{P}_x)$ be a Lévy process in $\mathbb{R}^d$ such that the characteristic function $\hat{\mu}$ of X_1 is given by

$$\hat{\mu}(z) = \exp\left[\int_{\mathbb{R}^d} (e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle 1_{(|x|\leq 1)})\, \nu(dx)\right], \quad z \in \mathbb{R}^d.$$

Here ν is the Lévy measure of X, i. e., a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\, \nu(dx) < \infty$. Moreover, throughout this section we assume that $\nu(\mathbb{R}^d) = \infty$, thus excluding the compound Poisson case. It is well known that $(\nu(x - dy), t)$ is a Lévy system for X. The infinitesimal generator $\mathbf{L}$ of the corresponding semigroup is given by

$$\mathbf{L}f(x) = \int_{\mathbb{R}^d} \left(f(x + y) - f(x) - y \cdot \nabla f(x)1_{(|y|\leq 1)}\right)\, \nu(dy) \tag{3.1}$$

for $f \in C_b^2(\mathbb{R}^d)$. Moreover, for every $f \in C_b^2(\mathbb{R}^d)$

$$f(X_t) - f(X_0) - \int_0^t \mathbf{L}f(X_s)\, ds$$

is a $\mathbb{P}_x$-martingale for every $x \in \mathbb{R}^d$.

3.1 Radially symmetric case

Assume that the Lévy measure ν has a radially symmetric non-increasing density j, i. e. , $\nu(dy) = j(|y|)\,dy$. An important consequence of this assumption is the fact that for every $r \in (0,1)$,

$$\int_{(r\le|y|\le1)} y\,\nu(dy) = 0\,,$$

implying that

$$\mathbf{L}f(x) = \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - y\cdot\nabla f(x)1_{(|y|<r)}\right)\,\nu(dy)\,, \tag{3.2}$$

for every $r \in (0,1)$, $f \in C_b^2(\mathbb{R}^d)$.

We will assume that j satisfies the following hypotheses: There exists $c>0$ such that

$$j(u) \le cj(2u)\,, \quad \text{for all } u\in(0,2)\,, \tag{3.3}$$

$$j(u) \le cj(u+1)\,, \quad \text{for all } u\ge 1\,. \tag{3.4}$$

Note that these hypotheses imply that $j(u) > 0$ for all $u>0$.

Let

$$\phi_1(r) = r^{-2}\int_0^r u^{d+1}j(u)du, \quad \phi_2(r) = \int_r^\infty u^{d-1}j(u)du\,, \tag{3.5}$$

and let $\phi(r) = \phi_1(r) + \phi_2(r)$.

Lemma 3.1 *There exists a constant $C_1>0$ such that for every $r\in(0,1)$ and every $t>0$,*

$$\mathbb{P}_x(\sup_{s\le t}|X_s - X_0| > r) \le C_1\phi(r)t\,.$$

Proof. It suffices to prove the lemma for $x=0$. Let $f\in C_b^2(\mathbb{R}^d)$, $0\le f\le 1$, $f(0)=0$, and $f(y)=1$ for all $|y|\ge1$. Let $c_1 = \sup_y \sum_{j,k}|(\partial^2/\partial y_j\partial y_k)f(y)|$. Then $|f(z+y)-f(z)-y\cdot\nabla f(z)| \le (1/2)\sum_{j,k}|(\partial^2/\partial y_j\partial y_k)f(y)||y|^2 \le c_1|y|^2$. For $r\in(0,1)$, let $f_r(y) = f(y/r)$. Then the following estimate is valid:

$$\begin{aligned}
|f_r(z+y) - f_r(z) - y\cdot\nabla f_r(z)1_{(|y|\le r)}| &\le \frac{c_1}{2}\frac{|y|^2}{r^2}1_{(|y|\le r)} + 1_{(|y|\ge r)} \\
&\le c_2(1_{(|y|\le r)}\frac{|y|^2}{r^2} + 1_{(|y|\ge r)})\,.
\end{aligned}$$

By using (3.2), we get the following estimate:

$$\begin{aligned}
|\mathbf{L}f_r(z)| &\le \int_{\mathbb{R}^d} |f_r(z+y) - f_r(z) - y\cdot\nabla f_r(z)1_{(|y|\le r)}|\,\nu(dy) \\
&\le c_2\int_{\mathbb{R}^d}(1_{(|y|\le r)}\frac{|y|^2}{r^2} + 1_{(|y|\ge r)})\,\nu(dy) \\
&\le c_3(\phi_1(r)+\phi_2(r)) = c_3\phi(r)\,,
\end{aligned}$$

where the constant c_3 depends on f and dimension d, but not on r. Further, by the martingale property,

$$\mathbb{E}_0 f_r(X(\tau_{B(0,r)} \wedge t)) - f_r(0) = \mathbb{E}_0 \int_0^{\tau_{B(0,r)}\wedge t} \mathbf{L} f_r(X_s)\, ds$$

implying the estimate

$$\mathbb{E}_0 f_r(X(\tau_{B(0,r)} \wedge t)) \le c_3 \phi(r) t \,.$$

If X exits $B(0,r)$ before time t, then $f_r(X(\tau_{B(0,r)} \wedge t)) = 1$, so the left hand side is larger than $\mathbb{P}_0(\tau_{B(0,r)} \le t)$. □

Lemma 3.2 *For every $r \in (0,1)$, and every $x \in \mathbb{R}^d$,*

$$\inf_{z\in B(x,r/2)} \mathbb{E}_z \tau_{B(x,r)} \ge \frac{1}{4C_1\phi(r/2)} \,.$$

Proof. Let $z \in B(x, r/2)$. Then

$$\mathbb{P}_z(\tau_{B(x,r)} \le t) \le \mathbb{P}_z(\tau_{B(z,r/2)} \le t) \le C_1 \phi(r/2) t \,.$$

Therefore,

$$\mathbb{E}_z \tau_{B(x,r)} \ge t \mathbb{P}_z(\tau_{B(x,r)} \ge t) \ge t(1 - C_1\phi(r/2)t) \,.$$

Choose $t = 1/(2C_1\phi(r/2))$ so that $1 - C_1\phi(r/2)t = 1/2$. Then

$$\mathbb{E}_z \tau_{B(x,r)} \ge \frac{1}{2C_1\phi(r/2)} \frac{1}{2} = \frac{1}{4C_1\phi(r/2)} \,.$$

□

Lemma 3.3 *There exists a constant $C_2 > 0$ such that for every $r \in (0,1)$ and every $x \in \mathbb{R}^d$,*

$$\sup_{z\in B(x,r)} \mathbb{E}_z \tau_{B(x,r)} \le \frac{C_2}{\phi_2(r)} \,.$$

Proof. Let $r \in (0,1)$, and let $x \in \mathbb{R}^d$. By a well-known formula connecting the Lévy measure and the harmonic measure (e. g. [6]), we have

$$\begin{aligned} 1 &\ge \mathbb{P}_z(|X(\tau_{B(x,r)}) - x| > r) \\ &= \int_{B(x,r)} \nu(B(x,r)^c - y) G_{B(x,r)}(z,y)\, dy \\ &= \int_{B(x,r)} G_{B(x,r)}(z,y) \int_{B(x,r)^c} j(|u-y|)\, du\, dy \,, \end{aligned}$$

where $G_{B(x,r)}$ denotes the Green function of the process X killed upon exiting $B(x,r)$. Now we estimate the inner integral. Let $y \in B(x,r)$, $u \in B(x,r)^c$. If $u \in B(x,2)$, then $|u-y| \le 2|u-x|$, while for $u \notin B(x,2)$ we use $|u-y| \le |u-x|+1$. Then

$$\begin{aligned}
\int_{B(x,r)^c} j(|u-y|)\, du &= \int_{B(x,r)^c\cap B(x,2)} j(|u-y|)\, du + \int_{B(x,r)^c\cap B(x,2)^c} j(|u-y|)\, du \\
&\ge \int_{B(x,r)^c\cap B(x,2)} j(2|u-x|)\, du + \int_{B(x,r)^c\cap B(x,2)^c} j(|u-x|+1)\, du \\
&\ge \int_{B(x,r)^c\cap B(x,2)} c^{-1} j(|u-x|)\, du + \int_{B(x,r)^c\cap B(x,2)^c} c^{-1} j(|u-x|)\, du \\
&= \int_{B(x,r)^c} c^{-1} j(|u-x|)\, du\,,
\end{aligned}$$

where in the next to last line we used hypotheses (3.3) and (3.4). It follows that

$$\begin{aligned}
1 &\ge \int_{B(x,r)} G_{B(x,r)}(z,y)\, dy \int_{B(x,r)^c} c^{-1} j(|u-x|)\, du \\
&= \mathbb{E}_z \tau_{B(x,r)} c^{-1} c_1 \int_r^\infty v^{d-1} j(v)\, dv \\
&= c_2 \mathbb{E}_z \tau_{B(x,r)} \phi_2(r)
\end{aligned}$$

which implies the lemma. □

Lemma 3.4 *There exists a constant $C_3 > 0$ such that for every $r \in (0,1)$, every $x \in \mathbb{R}^d$, and any $A \subset B(x,r)$*

$$\mathbb{P}_y(T_A < \tau_{B(x,3r)}) \ge C_3 |A| \frac{j(4r)}{\phi(r)}\,, \quad \textit{for all } y \in B(x,2r)\,.$$

Proof. Note first that since

$$\phi(r) \ge \int_r^{4r} u^{d-1} j(u) du \ge c_1 r^d j(4r)$$

and $|A| \le c_2 r^d$ for some constants c_1 and c_2, we get that

$$|A| \frac{j(4r)}{\phi(r)} \le \frac{c_2}{c_1}.$$

Therefore, by choosing $C_3 < c_1/(4c_2)$, we may assume without loss of generality that $\mathbb{P}_y(T_A < \tau_{B(x,3r)}) < 1/4$. Set $\tau = \tau_{B(x,3r)}$. By Lemma 3.1, $\mathbb{P}_y(\tau \le t) \le \mathbb{P}_y(\tau_{B(y,r)} \le t) \le C_1 \phi(r) t$.

Choose $t_0 = 1/(4C_1\phi(r))$, so that $\mathbb{P}_y(\tau \le t_0) \le 1/4$. Further, if $z \in B(x,3r)$ and $u \in A \subset B(x,r)$, then $|u-z| \le 4r$. Since j is decreasing, $j(|u-z|) \ge j(4r)$. Thus,

$$\begin{aligned}
\mathbb{P}_y(T_A < \tau) &\ge \mathbb{E}_y \sum_{s \le T_A \wedge \tau \wedge t_0} 1_{\{X_{s-} \neq X_s, X_s \in A\}} \\
&= \mathbb{E}_y \int_0^{T_A \wedge \tau \wedge t_0} \int_A j(|u - X_s|)\, du\, ds \\
&\ge \mathbb{E}_y \int_0^{T_A \wedge \tau \wedge t_0} \int_A j(4r)\, du\, ds \\
&= j(4r)|A| \mathbb{E}_y (T_A \wedge \tau \wedge t_0)\,,
\end{aligned}$$

where in the second line we used properties of the Lévy system. Next,

$$\begin{aligned}
\mathbb{E}_y(T_A \wedge \tau \wedge t_0) &\ge \mathbb{E}_y(t_0;\, T_A \ge \tau \ge t_0) \\
&= t_0 \mathbb{P}_y(T_A \ge \tau \ge t_0) \\
&\ge t_0[1 - \mathbb{P}_y(T_A < \tau) - \mathbb{P}_y(\tau < t_0)] \\
&\ge \frac{t_0}{2} = \frac{1}{8C_1\phi(r)}\,.
\end{aligned}$$

The last two displays give that

$$\mathbb{P}_y(T_A < \tau) \ge j(4r)|A| \frac{1}{8C_1\phi(r)} = \frac{1}{8C_1}|A|\frac{j(4r)}{\phi(r)}.$$

Hence the claim follows by choosing $C_3 = \frac{1}{8C_1} \wedge \frac{c_1}{4c_2}$. □

Lemma 3.5 *There exist positive constant C_4 and C_5, such that if $r \in (0,1)$, $x \in \mathbb{R}^d$, $z \in B(x,r)$, and H is a bounded nonnegative function with support in $B(x,2r)^c$, then*

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \le C_4(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(|y-x|)\, dy\,,$$

and

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \ge C_5(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(|y-x|)\, dy\,.$$

Proof. Let $y \in B(x,r)$ and $u \in B(x,2r)^c$. If $u \in B(x,2)$ we use the estimates

$$2^{-1}|u-x| \le |u-y| \le 2|u-x|, \tag{3.6}$$

while if $u \notin B(x,2)$ we use

$$|u-x| - 1 \le |u-y| \le |u-x| + 1. \tag{3.7}$$

Let $B \subset B(x,2r)^c$. Then using the Lévy system we get

$$\mathbb{E}_z 1_B(X(\tau_{B(x,r)})) = \mathbb{E}_z \int_0^{\tau_{B(x,r)}} \int_B j(|u-X_s|)\, du\, ds\,.$$

By use of (3.3), (3.4), (3.6), and (3.7), the inner integral is estimated as follows:

$$\begin{aligned}
\int_B j(|u-X_s|)\, du &= \int_{B\cap B(x,2)} j(|u-X_s|)\, du + \int_{B\cap B(x,2)^c} j(|u-X_s|)\, du \\
&\leq \int_{B\cap B(x,2)} j(2^{-1}|u-x|)\, du + \int_{B\cap B(x,2)^c} j(|u-x|-1)\, du \\
&\leq \int_{B\cap B(x,2)} cj(|u-x|)\, du + \int_{B\cap B(x,2)^c} cj(|u-x|)\, du \\
&= c\int_B j(|u-x|)\, du
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}_z 1_B(X(\tau_{B(x,r)})) &\leq \mathbb{E}_z \int_0^{\tau_{B(x,r)}} c\int_B j(|u-x|)\, du \\
&= c\,\mathbb{E}_z(\tau_{B(x,r)}) \int 1_B(u) j(|u-x|)\, du\,.
\end{aligned}$$

Using linearity we get the above inequality when 1_B is replaced by a simple function. Approximating H by simple functions and taking limits we have the first inequality in the statement of the lemma.

The second inequality is proved in the same way. □

The last lemma shows that hypothesis (A3) is satisfied in the current setting. Therefore, it remains to analyze hypotheses (A1) and (A2). By Lemmas 3.2 and 3.3, a sufficient condition for (A1) to hold is that there exists a constant $C_6 > 0$ such that for all $r \in (0,1)$,

$$\phi(r/2) \leq C_6 \phi_2(r)\,. \tag{3.8}$$

Since $\phi = \phi_1 + \phi_2$, this is equivalent to the condition that there exists a constant $C_7 > 0$ such that for all $r \in (0,1)$, $\phi_1(r/2) \leq C_7\phi_2(r)$.

From Lemma 3.4 it follows that a sufficient condition for (A2) to hold is that there exists a constant $C_8 > 0$ such that for all $r \in (0,1)$

$$\frac{j(4r)}{\phi(r)} \geq C_8 r^{-d} \tag{3.9}$$

We discuss now several examples satisfying all four hypotheses (A1)-(A3). Those examples are not covered by the paper [1].

Example 3.6 Assume that, in addition to (3.3) and (3.4), there exist $\alpha > 0$ and two positive constants C_9 and C_{10} such that

$$C_9 r^{-d-\alpha} \le j(r) \le C_{10} r^{-d-\alpha}, \quad \text{for all } r > 0 \text{ sufficiently small.} \tag{3.10}$$

We write $j(r) \sim r^{-d-\alpha}$ as $r \to 0$. The fact that ν is a Lévy measure implies that $\alpha < 2$. Therefore, in a neighborhood of zero in $\mathbb{R}^d$, the Lévy measure looks like $|x|^{-d-\alpha}$ which corresponds to the rotationally invariant strictly stable process. It is easy to check that the following is valid: $\phi_1(r) \sim r^{-\alpha}$, $\phi_2(r) \sim r^{-\alpha}$ and $j(r) \sim r^{-d-\alpha}$ as $r \to 0$. Therefore, both (3.8) and (3.9) hold true.

(a) Mixture of symmetric strictly stable processes. Let $m \in \mathbb{N}$, let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 2$, and let $a_1, a_2, \dots, a_m$ be positive real numbers. Define $j(r) = \sum_{i=1}^m a_i r^{-d-\alpha_i}$. Then j satisfies assumptions (3.3), (3.4) and (3.10) (with $\alpha = \alpha_m$). This case corresponds to the Lévy measure of the form $\nu(dx) = \sum_{i=1}^m a_i |x|^{-d-\alpha_i}\, dx$.

(b) Relativistic stable processes. Let m be a positive real number. A relativistic α-stable process is a Lévy processes whose infinitesimal generator can be written as $m - (-\Delta + m^{2/\alpha})^{\alpha/2}$, $0 < \alpha < 2$. We refer to [4] or [12] for details. The Lévy measure of this process has the form $\nu(dx) = j(|x|)\, dx$ where j is given by

$$j(r) = c(\alpha, d) r^{-d-\alpha} \psi(m^{1/\alpha} r)\,,$$

and

$$\psi(r) = c_1 \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}}\, ds\,.$$

Clearly, j is decreasing. Moreover, since

$$\psi(r) \sim e^{-r}(1 + r^{\frac{d+\alpha-1}{2}}), \quad r \to \infty\,,$$

and

$$\psi(r) = 1 + \frac{\psi''(0)}{2} r^2 + o(r^4), \quad \text{near } r = 0\,,$$

(see [4]), j satisfies (3.3), (3.4) and (3.10).

(c) Let $j(r) = r^{-d-\alpha}(1 - e^{-r})$ with $1 < \alpha < 3$. This function also satisfies all the required conditions.

Example 3.7 This example generalizes the mixture of symmetric stable processes and need not satisfy (3.10).

Let κ be a finite measure on $(0, 2)$ with $\text{supp}(\kappa) \subset [a, b]$, $0 < a < b < 2$. Define

$$j(r) = \int_{(0,2)} r^{-d-\alpha}\, \kappa(d\alpha) = r^{-d} \int_{(0,2)} r^{-\alpha}\, \kappa(d\alpha)\,.$$

Note that $j(r) \le 2^{d+2} j(2r)$ for all $r > 0$ which implies (3.3) and (3.4). A simple computation shows that

$$\begin{aligned} \phi_1(r) &= \int_{(0,2)} \frac{r^{-\alpha}}{2-\alpha}\,\kappa(d\alpha)\,, \\ \phi_2(r) &= \int_{(0,2)} \frac{r^{-\alpha}}{\alpha}\,\kappa(d\alpha)\,. \end{aligned}$$

Since the support of κ is contained in $[a, b]$, both $\phi_1(r)$ and $\phi_2(r)$ are comparable with $\int_{(0,2)} r^{-\alpha}\,\kappa(d\alpha)$. Therefore, we have both (3.8) and (3.9).

In case when κ is the Lebesgue measure on $[a, b]$, we have

$$j(r) = \frac{-r^{-a} + r^{-b}}{r^d \log r}$$

which does not satisfy (3.10).

3.2 Non-symmetric case

In this section we consider Lévy measures of the form $\nu(dy) = k(y) j(|y|)\,dy$ where j is a non-increasing function satisfying (3.3) and (3.4), and $k : \mathbb{R}^d \setminus \{0\} \to (0, \infty)$ is a function satisfying $\tilde{c}^{-1} \le k(y) \le \tilde{c}$, for all $y \ne 0$, for a positive constant $\tilde{c}$.

We will distinguish between two cases. In the first case we assume that for every $r \in (0, 1)$

$$\int_{(r \le |y| \le 1)} y\,\nu(dy) = \int_{(r \le |y| \le 1)} y k(y) j(|y|)\,dy = 0\,. \tag{3.11}$$

This assumption will hold true if $k(-y) = k(y)$ for each $y \in B(0, 1)$. A consequence of this assumption is that the infinitesimal generator $\mathbf{L}$ has the form (3.2) for every $r \in (0, 1)$. A careful reading of proofs of Lemmas 3.1-3.5 reveals that the proofs carry over to the current setting. The only difference is that constants will change and will depend on $\tilde{c}$ as well. For example, wherever we estimated the function $j(|y|)$, we now estimate $k(y)j(|y|)$ instead.

Example 3.8 Let j satisfy (3.10), and let k be bounded and bounded away from zero, such that (3.11) is valid. Then (A1)-(A3) are satisfied for $\nu(dx) = k(x)j(|x|)$. In particular, if $j(r) = r^{-d-\alpha}$ and k is such that the corresponding process is strictly α-stable (not necessarily symmetric), then (3.11) holds true (see, for example, [13]).

Similarly, let κ be a bounded measure on $(0, 2)$ supported in $[a, b]$, $0 < a < b < 2$, let $j(r) = \int_{(0,2)} r^{-d-\alpha}\,\kappa(d\alpha)$, and let k be bounded and bounded away from zero, such that (3.11) is valid. Then again, (A1)-(A3) are satisfied for $\nu(dx) = k(x)j(|x|)$.

The second case we consider is genuinely non-symmetric case when (3.11) does not hold. Then the infinitesimal generator $\mathbf{L}$ has the following form for each $r \in (0,1)$:

$$\mathbf{L}f(x) = \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - y \cdot \nabla f(x) 1_{(|y|<r)} \right) \nu(dy) - \int_{(r \le |y| \le 1)} y \cdot \nabla f(x) \, \nu(dy) \, . \quad (3.12)$$

Both the statement and the proof of Lemma 3.1 should be modified so that the additional term of the generator is taken into account. Let

$$\phi_3(r) = r^{-1} \int_r^1 u^d j(u) \, du \, ,$$

and let $\phi(r) = \phi_1(r) + \phi_2(r) + \phi_3(r)$. Let f and f_r be the functions from the proof of Lemma 3.1. Let $c_1 = \sup_y \sum_j |(\partial/\partial y_j) f(y)|$. Then $|y \cdot \nabla f_r(y)| \le c_1 r^{-1} |y|$. Hence, the estimate for $\mathbf{L}f_r(z)$ will read

$$\begin{aligned} |\mathbf{L}f_r(z)| &\le \int_{\mathbb{R}^d} |f_r(z+y) - f_r(z) - y \cdot \nabla f_r(z) 1_{(|y| \le r)}| \, \nu(dy) \\ &+ \int_{\mathbb{R}^d} |y \cdot \nabla f_r(z) 1_{(r \le |y| \le 1)}| \\ &\le c_1 \int_{\mathbb{R}^d} \left(1_{(|y| \le r)} \frac{|y|^2}{r^2} + 1_{(|y| \ge r)} + 1_{(r \le |y| \le 1)} \frac{|y|}{r} \right) \nu(dy) \\ &\le c_2 (\phi_1(r) + \phi_2(r) + \phi_3(r)) = c_2 \phi(r) \, , \end{aligned}$$

The rest of the proof remains the same, with the new function ϕ. Similarly, statements of Lemmas 3.2 and 3.4 now involve the modified function ϕ. The proofs of all lemmas carry over with new constants depending on $\tilde{c}$. In order that (A1) and (A2) hold true, it suffices to have conditions (3.8) and (3.9), but now with modified ϕ.

Example 3.9 This is analogous to Example 3.6. Assume, in addition to (3.3) and (3.4), that j satisfies (3.10). Let k be bounded between two positive numbers. Then

$$\phi_3(r) \sim \frac{r^{-\alpha} - r^{-1}}{\alpha - 1} \, .$$

If $\alpha \in (1,2)$ then $(r^{-\alpha} - r^{-1})/(\alpha - 1)$ is of the order $r^{-\alpha}$, same as the order of $\phi_1(r)$, $\phi_2(r)$ and $j(r)r^d$. Therefore, hypotheses (A1)-(A3) are valid in case $1 < \alpha < 2$. For $\alpha \in (0,1]$ this argument is no longer valid, and we cannot establish (A1) and (A2).

Example 3.10 This example generalizes Example 3.7. Let κ be a finite measure on $(0,2)$ with support $\mathrm{supp}(\kappa) \subset [a,b]$, $1 < a < b < 2$, and let j be defined as in Example 3.7. We compute

$$\phi_3(r) = \int_{(0,2)} \frac{r^{-\alpha} - r^{-1}}{\alpha - 1} \, \kappa(d\alpha) = \int_{(0,2)} \frac{r^{-\alpha}}{\alpha - 1} \, \kappa(d\alpha) - \int_{(0,2)} \frac{r^{-1}}{\alpha - 1} \, \kappa(d\alpha) \, .$$

Since $\mathrm{supp}(\kappa) \subset [a,b] \subset (1,2)$, it easily follows that $\phi_3(r) \sim \int_{(0,2)} r^{-\alpha} \, \kappa(d\alpha)$. Therefore, the analogue of (3.8) and (3.9) are true, hence (A1)-(A3) are also valid.

4 Symmetric Markov Processes with no diffusion component

In this section we are going to assume that $0 < \alpha_1 \le \alpha_2 < 2$ and that $k_1(x,y)$ and $k_2(x,y)$ are symmetric functions on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$\tilde{\kappa}_i \le k_i(x,y) \le \kappa_i, \quad i = 1,2, \;\; x,y \in \mathbb{R}^d,$$

for some positive constants $\tilde{\kappa}_i \le \kappa_i$, $i = 1,2$. The symmetric form $(\mathcal{E}, C_c^2(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x)-f(y))(g(x)-g(y)) \left(\frac{k_1(x,y)}{|x-y|^{d+\alpha_1}} + \frac{k_2(x,y)}{|x-y|^{d+\alpha_2}} \right) dxdy$$

is closable, so its minimal extension $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form. Therefore by the general theory of Dirichlet forms there is a symmetric Hunt process X on $\mathbb{R}^d$, associated with $(\mathcal{E}, \mathcal{F})$, starting from every point in $E := \mathbb{R}^d \setminus N$, where N is a set of zero capacity. Put

$$j(x,y) = \frac{k_1(x,y)}{|x-y|^{d+\alpha_1}} + \frac{k_2(x,y)}{|x-y|^{d+\alpha_2}},$$

then it is well known (see, for instance, [5]) that $(j(x,y)dy, dt)$ is a Lévy system for X.

Throughout this section we assume that, for $i = 1,2$, when $\alpha_i \ge 1$, the first partials of k_i are bounded on $\mathbb{R}^d \times \mathbb{R}^d$. We are going to use M_i to denote the bounds on the first partials of k_i.

For any $\alpha \in (0,2)$, any bounded function $k(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ and any $f \in C_c^2(\mathbb{R}^d)$, we define

$$\begin{aligned}
\mathbf{L}_{\alpha,k} f(x) &= \int_{\mathbb{R}^d} (f(x+y)-f(x)) \frac{k(x,x+y)}{|y|^{d+\alpha}} dy, \qquad \alpha \in (0,1), \\
\mathbf{L}_{\alpha,k} f(x) &= \int_{\mathbb{R}^d} (f(x+y)-f(x)-(y,\nabla f(x))1_{|y|<1}) \frac{k(x,x)}{|y|^{d+\alpha}} dy \\
&\quad + \int_{\mathbb{R}^d} (f(x+y)-f(x))(k(x,x+y)-k(x,x)) \frac{1}{|y|^{d+\alpha}} dy, \quad \alpha \in [1,2).
\end{aligned}$$

To make sure the last integral converges, we need to assume that, when $\alpha \in [1,2)$, the first partials of k are bounded on $\mathbb{R}^d \times \mathbb{R}^d$. Using symmetry, one can easily see that, when $\alpha \in [1,2)$,

$$\begin{aligned}
\mathbf{L}_{\alpha,k} f(x) &= \int_{\mathbb{R}^d} (f(x+y)-f(x)-(y,\nabla f(x))1_{|y|<r}) \frac{k(x,x)}{|y|^{d+\alpha}} dy \\
&\quad + \int_{\mathbb{R}^d} (f(x+y)-f(x))(k(x,x+y)-k(x,x)) \frac{1}{|y|^{d+\alpha}} dy
\end{aligned}$$

for any $r \in (0,1)$.

If $\mathbf{L}$ stands for the generator of X in the L^2 sense, then the following result is elementary. See, for instance, [9].

Lemma 4.1 *For any $f \in C_c^2(\mathbb{R}^d)$, we have*

$$\mathbf{L}f(x) = \mathbf{L}_{\alpha_1,k_1}f(x) + \mathbf{L}_{\alpha_2,k_2}f(x),$$

for almost every $x \in \mathbb{R}^d$.

This lemma tells us that $\mathbf{L}_{\alpha_1,k_1}f + \mathbf{L}_{\alpha_2,k_2}$ is a representative of $\mathbf{L}f$ for any $f \in C_c^2(\mathbb{R}^d)$. From now on we will always take this representative when dealing with $\mathbf{L}f$. It follows from the proof of Theorem 5.2.2 of [5] that for any $f \in C_c^2(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) - \int_0^t \mathbf{L}f(X_s)ds$$

is a $\mathbb{P}_x$-martingale for any $x \in E$.

Lemma 4.2 *For any $x \in E$ and any $r \in (0,1)$ we have*

$$\mathbb{P}_x(\sup_{s\le t}|X_s - X_0| > r) \le Cr^{-\alpha_2}t,$$

where C is a positive constant depending on $\alpha_1, \alpha_2, \kappa_1, \kappa_2, M_1, M_2$ and d.

Proof. Suppose that $x \in E$ is fixed. Let f be a C^2 function on $\mathbb{R}^d$ taking values in $[0,1]$ such that $f(y) = 0$ for $|y| \le 1/2$ and $f(y) = 1$ for $|y| \ge 1$. Let f_n be a sequence of C^2 functions such that $0 \le f_n \le 1$,

$$f_n(y) = \begin{cases} f(y), & |y| \le n+1 \\ 0, & |y| > n+2, \end{cases}$$

and that $|\sum_{j,k}(\partial^2/\partial x_j \partial x_k)f_n|$ is uniformly bounded. Then there exist positive constants c_1 and c_2 such that

$$|\nabla f_n(y)| \le c_1, \quad y \in \mathbb{R}^d,$$

and

$$|f_n(y+z) - f_n(y) - (z \cdot \nabla f_n(y))| \le c_2|z|^2, \quad y, z \in \mathbb{R}^d.$$

Put $f_r(y) = f((y-x)/r)$ and $f_{n,r}(y) = f_n((y-x)/r)$.

We claim that for any $r \in (0,1)$, any $y \in \mathbb{R}^d$ and any $n \ge 1$,

$$|\mathbf{L}f_{n,r}(y)| \le C_1 r^{-\alpha_2}$$

positive constant C_1 depending on $\alpha_1, \alpha_2, \kappa_1, \kappa_2, M_1, M_2$ and d. We prove this claim by dealing with $\mathbf{L}_{\alpha_1,k_i}$ $i = 1, 2$ separately.The proof of the two cases are identical, so we will give a proof of the case $i = 2$ only, and we will do this by dealing separately with the cases $\alpha_2 \in (0, 1)$ and $\alpha_2 \in [1, 2)$.

(i) $\alpha_2 \in (0, 1)$. In this case we have for any $r > 0$ and any $y \in \mathbb{R}^d$,

$$\begin{aligned}
|\mathbf{L}_{\alpha_2,k_2} f_{n,r}(y)| &= |\int_{\mathbb{R}^d} (f_{n,r}(y+z) - f_{n,r}(y)) \frac{k_2(y, y+z)}{|z-y|^{d+\alpha_2}} dz| \\
&\leq c_1\kappa_2 r^{-1} \int_{|z|<r} |z|^{-(d+\alpha_2-1)} dz + 2\kappa_2 \int_{|z|>r} |z|^{-(d+\alpha_2)} dz \\
&\leq c_3\kappa_2 r^{-1} \int_0^r s^{-\alpha_2} ds + c_4\kappa_2 \int_r^\infty s^{-\alpha_2-1} ds \\
&= \frac{c_3\kappa_2}{1-\alpha_2} r^{-\alpha_2} + \frac{c_4\kappa_2}{\alpha_2} r^{-\alpha_2}.
\end{aligned}$$

(ii) $\alpha_2 \in [1, 2)$. In this case we have for any $r \in (0, 1)$ and any $y \in \mathbb{R}^d$,

$$\begin{aligned}
|\mathbf{L}_{\alpha_2,k_2} f_{n,r}(y)| &\leq |\int_{\mathbb{R}^d} (f_{n,r}(y+z) - f_{n,r}(y) - (z, \nabla f_{n,r}(y)) 1_{|z|<r}) \frac{k_2(y,y)}{|z|^{d+\alpha_2}} dz| \\
&\quad + |\int_{\mathbb{R}^d} (f_{n,r}(y+z) - f_{n,r}(y))(k_2(y, y+z) - k_2(y,y)) \frac{1}{|z|^{d+\alpha_2}} dz| \\
&\leq c_2\kappa_2 r^{-2} \int_{|z|<r} |z|^{-(d+\alpha_2-2)} dz + 2\kappa_2 \int_{|z|\geq r} |z|^{-(d+\alpha_2)} dz \\
&\quad + c_1 M_2 \kappa_2 r^{-1} \int_{|z|<r} |z|^{-(d+\alpha_2-2)} dz + \kappa_2 \int_{|z|\geq r} |z|^{-(d+\alpha_2)} dz \\
&\leq \frac{c_5\kappa_2}{2-\alpha_2} r^{-\alpha_2} + \frac{c_6 M_2 \kappa_2}{2-\alpha_2} r^{-\alpha_2+1} + \frac{c_7\kappa_2}{\alpha_2} r^{-\alpha_2} \\
&\leq (\frac{c_8}{2-\alpha_2} + \frac{c_9}{\alpha_2}) \kappa_2 r^{-\alpha_2}.
\end{aligned}$$

Thus we finished the proof of the claim.

Therefore we have for any $r \in (0, 1)$ and any $n \geq 1$,

$$\mathbb{E}_x f_{n,r}(X(\tau_{B(x,r)} \wedge t)) = \mathbb{E}_x \int_0^{\tau_{B(x,r)}\wedge t} \mathbf{L} f_{n,r}(X_s) ds \leq C_1 r^{-\alpha_2} t.$$

Letting $n \uparrow \infty$, we get

$$\mathbb{E}_x f_r(X(\tau_{B(x,r)} \wedge t)) \leq C_1 r^{-\alpha_2} t.$$

If X exits $B(x, r)$ before time t, then $f_r(X(\tau_{B(x,r)} \wedge t)) = 1$, so the left hand side is greater than $\mathbb{P}_x(\tau_{B(x,r)} \leq t)$. □

For any $\lambda > 0$, we define

$$G_\lambda f(x) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(X_t) dt, \quad x \in E.$$

We use $C_0(\mathbb{R}^d)$ to denote the collection of continuous functions f such that $\lim_{|x|\to\infty} f(x) = 0$. From the lemma above we immediately get the following result:

Corollary 4.3 *If $f \in C_0(\mathbb{R}^d)$, then $\mathbb{E}_x f(X_t)$ tends to $f(x)$ uniformly in $x \in E$ as $t \downarrow 0$ and $\lambda G_\lambda f(x)$ tends to $f(x)$ uniformly in $x \in E$ as $\lambda \uparrow \infty$.*

Proof. Omitted. □

Remark 4.4 *Note that, for $i = 1, 2$, when $\alpha_i \in [1, 2)$ we have for any $r > 1$,*

$$\begin{aligned}
|\mathbf{L}_{\alpha_i,k_i} f_{n,r}(y)| &\le |\int_{\mathbb{R}^d} (f_{n,r}(y+z) - f_{n,r}(y) - (z, \nabla f_{n,r}(y)) 1_{|z|<r}) \frac{k_i(y,y)}{|z|^{d+\alpha_i}} dz| \\
&\quad + |\int_{\mathbb{R}^d} (f_{n,r}(y+z) - f_{n,r}(y))(k_i(y, y+z) - k_i(y,y)) \frac{1}{|z|^{d+\alpha_i}} dz| \\
&\le c_2 \kappa_i r^{-2} \int_{|z|<r} |z|^{-(d+\alpha_i-2)} dz + 2\kappa_i \int_{|z|\ge r} |z|^{-(d+\alpha_i)} dz \\
&\quad + c_1 M_i \kappa_i r^{-1} \int_{|z|<1} |z|^{-(d+\alpha_i-2)} dz + 2c_1 \kappa_i r^{-1} \int_{1\le|z|<r} |z|^{-(d+\alpha_i-1)} dz \\
&\quad + \kappa_i \int_{|z|\ge r} |z|^{-(d+\alpha_i)} dz \\
&\le c_3 r^{-1},
\end{aligned}$$

where c_1 and c_2 are the constants in the proof above, and c_3 is a positive constant depending on α_i, κ_i, M_i and d. Combining this with (i) in the proof above we get for any $r > 1$, any $y \in \mathbb{R}^d$ and any $n \ge 1$,

$$|\mathbf{L} f_{n,r}(y)| \le C_1 r^{-(\alpha_1 \wedge 1)}$$

for some positive constant C_1 depending on $\alpha_1, \alpha_2, \kappa_1, \kappa_2, M_1, M_2$ and d. Repeating the argument in the last paragraph in the proof above we can show that for any $x \in E$, any $r > 1$ and any $t > 0$ we have

$$\mathbb{P}_x(\sup_{s\le t} |X_s - X_0| > r) \le C_2 r^{-(\alpha_1 \wedge 1)} t,$$

for some positive constant C_2 depending on $\alpha_1, \alpha_2, \kappa_1, \kappa_2, M_1, M_2$ and d. From this one can easily see that for bounded function f with compact support, the function $x \mapsto \mathbb{E}_x[f(X_t)]$ tends to zero as $|x| \to \infty$.

Lemma 4.5 *Suppose $\varepsilon \in (0,1)$ is a constant. Then for every $x \in \mathbb{R}^d$ and $r \in (0,1)$ we have*

$$\inf_{z \in B(x,(1-\varepsilon)r)\cap E} \mathbb{E}_z \tau_{B(x,r)} \geq \frac{\varepsilon^{\alpha_2}}{4C} r^{\alpha_2},$$

where C is the constant in Lemma 4.2.

Proof. The proof is the same as that of Lemma 3.2. □

Lemma 4.6 *For any $x \in \mathbb{R}^d$ and any $r > 0$ we have*

$$\sup_{z \in B(x,r)\cap E} \mathbb{E}_z \tau_{B(x,r)} \leq \frac{\alpha_2 2^{d+\alpha_2}}{\tilde{\kappa}_2} r^{\alpha_2}.$$

Proof. It is elementary to show that, for any $y \in B(x,r)$,

$$\int_{B(x,r)^c} \frac{1}{|y-u|^{d+\alpha_2}} du \geq 2^{-(d+\alpha_2)} \int_{B(x,r)^c} \frac{1}{|x-u|^{d+\alpha_2}} du = \frac{1}{\alpha_2 2^{d+\alpha}} r^{-\alpha_2}.$$

Thus for any $z \in B(x,r) \cap E$,

$$\begin{aligned}
1 &\geq \mathbb{P}_z(|X(\tau_{B(x,r)}) - x| > r) \\
&= \int_{B(x,r)} G_{B(x,r)}(z,y) \int_{B(x,r)^c} j(y,u) du dy \\
&\geq \tilde{\kappa}_2 \int_{B(x,r)} G_{B(x,r)}(z,y) \int_{B(x,r)^c} \frac{1}{|y-u|^{d+\alpha_2}} du \\
&\geq \frac{\tilde{\kappa}_2}{\alpha_2 2^{d+\alpha_2}} r^{-\alpha_2} \int_{B(x,r)} G_{B(x,r)}(z,y) dy \\
&= \frac{\tilde{\kappa}_2}{\alpha_2 2^{d+\alpha_2}} r^{-\alpha_2} \mathbb{E}_z \tau_{B(x,r)},
\end{aligned}$$

where $G_{B(x,r)}$ is the Green function of the process X killed upon exiting $B(x,r)$. Therefore

$$\mathbb{E}_z \tau_{B(x,r)} \leq \frac{\alpha_2 2^{d+\alpha_2}}{\tilde{\kappa}_2} r^{\alpha_2}.$$

□

Lemmas 4.5 and 4.6 imply that X satisfies the analogue of condition (A1) on E. The following Lemma implies that X satisfies the analogue of (A2) on E.

Lemma 4.7 *For all $r \in (0,1)$ and $A \subset B(x,r)$ we have*

$$\mathbb{P}_y(T_A < \tau_{B(x,3r)}) \geq \frac{\tilde{\kappa}_2 |B(0,1)|}{4^{d+\alpha_2} 8C} \frac{|A|}{|B(x,r)|}, \quad \forall y \in B(x,2r) \cap E,$$

where C is the constant in Lemma 4.2.

Proof. The proof is similar to that of Lemma 3.4 and we omit the details. □

The following lemma says that X satisfies the analogue of (A3) on E.

Lemma 4.8 *There exist positive constants C_7 and C_8 such that if $x \in \mathbb{R}^d$, $r > 0$, $z \in B(x,r) \cap E$ and H is a bounded nonnegative function with support in $B(x,2r)^c$, then*

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \le C_7(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(x,y) dy$$

and

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \ge C_8(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(x,y) dy.$$

Proof. The proof is similar to that of Lemma 3.5. □

Lemma 4.6 implies in particular that for any bounded open set B in $\mathbb{R}^d$, τ_B is finite almost surely. Let D be a domain in $\mathbb{R}^d$. A function h defined on E is said to be harmonic in $D \cap E$ if it is not identically infinite in $D \cap E$ and if for any bounded open subset $B \subset \overline{B} \subset D$,

$$h(x) = \mathbb{E}_x[h(X(\tau_B))] \quad \forall x \in B \cap E.$$

It is clear that constant functions are harmonic on $D \cap E$. Since X satisfies the analogue of (A1)–(A3) on E, we can repeat the argument of Section 2 to show that X satisfies the Harnack inequality on E. But our goal is to establish that X satisfies the Harnack inequality on $\mathbb{R}^d$. To do that we need to guarantee that we can start our process from every point in $\mathbb{R}^d$.

The result below is the analogue of Theorem 4.1 in [1].

Theorem 4.9 *Let $r \in (0,1)$. If h is bounded in E and harmonic in $B(x_0,r) \cap E$ for some $x_0 \in \mathbb{R}^d$, then there exist positive constants C and β independent of x_0 and r such that*

$$|h(x) - h(y)| \le C\|h\|_\infty |x-y|^\beta, \quad \forall x, y \in B(x_0, r/2) \cap E.$$

Proof. By Lemma 4.7 there exists $c_1 > 0$ such that if $x \in E$, $s \in (0,1/2)$, and $A \subset B(x,s/3)$ with $|A|/|B(x,s/3)| \ge 1/3$, then

$$\mathbb{P}_x(T_A < \tau_{B(x,s)}) \ge c_1.$$

By Lemma 4.6 and Lemma 4.8 with $H = 1_{B(x,s')^c}$ with $s' \in (2s,1)$ we get that

$$\mathbb{P}_x(X(\tau_{B(x,s)}) \notin B(x,s')) \le c_2 \frac{s^{\alpha_2}}{(s')^{\alpha_2}}.$$

Let

$$\gamma = 1 - \frac{c_1}{4}, \quad \rho = \frac{1}{3} \wedge (\frac{\gamma}{2})^{1/\alpha_2} \wedge (\frac{c_1\gamma^2}{8c_2})^{1/\alpha_2}.$$

Adding a constant to h if necessary we may assume $0 \le h \le M$. Now we can repeat the argument in the proof of Theorem 4.1 in [1] to get that

$$\sup_{B(x,\rho^k r)\cap E} h - \inf_{B(x,\rho^k r)\cap E} h \le M\gamma^k$$

for all $k \ge 0$. If $x, y \in B(x_0, r/2) \cap E$, let k be the smallest integer such that $|x - y| < \rho^k$. Then $\log|x - y| \ge (k+1)\log\rho, y \in B(x, \rho^k)$, and

$$\begin{aligned} |h(y) - h(x)| &\le M\gamma^k = Me^{k\log\gamma} \le c_3 Me^{\log|x-y|(\log\gamma/\log\rho)} \\ &= c_3 M|x-y|^{\log\gamma/\log\rho}. \end{aligned}$$

□

Recall that for any $\lambda > 0$,

$$G_\lambda f(x) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(X_t)dt, \quad x \in E.$$

It is obvious that $\lambda\|G_\lambda f\|_\infty \le \|f\|_\infty$. Since the process X may be recurrent, $G_0 f$ is not defined in general. For any integer $n \ge 2$ and $\lambda \ge 0$, we define

$$G^{(n)}_\lambda f(x) = \mathbb{E}_x \int_0^{\tau_{B(0,n)}} e^{-\lambda t} f(X_t)dt, \quad x \in E.$$

Lemma 4.10 *There exist positive constants C and $\beta > 0$ such that for any $n \ge 2$, any bounded function f on $\mathbb{R}^d$, and any $x, y \in B(0, n-1) \cap E$ with $|x - y| < 1/4$,*

$$|G^{(n)}_0 f(x) - G^{(n)}_0 f(y)| \le C(\|G^{(n)}_0 f\|_\infty + \|f\|_\infty)|x - y|^\beta.$$

Proof. For any $r \in (0, 1)$, we have

$$\begin{aligned} G^{(n)}_0 f(x) &= \mathbb{E}_x \int_0^{\tau_{B(x,r)}} f(X_t)dt + \mathbb{E}_x G^{(n)}_0 f(X(\tau_{B(x,r)})), \\ G^{(n)}_0 f(y) &= \mathbb{E}_y \int_0^{\tau_{B(x,r)}} f(X_t)dt + \mathbb{E}_y G^{(n)}_0 f(X(\tau_{B(x,r)})). \end{aligned}$$

Using the theorem above and the fact that $\mathbb{E}_z G^{(n)}_0 f(X(\tau_{B(x,r)}))$ is harmonic in $B(x, r)$ we get

$$|G^{(n)}_0 f(x) - G^{(n)}_0 f(y)| \le 2\|f\|_\infty \sup_{z\in B(x,r)\cap E} \mathbb{E}_z \tau_{B(x,r)} + c_1\|G^{(n)}_0 f\|_\infty |x - y|^{\beta_1},$$

where β_1 is the constant β in the theorem above. Taking $r = |x - y|^{1/2}$ and using Lemma 4.6 we get our result. □

Theorem 4.11 *For any $\lambda > 0$, there exist positive constants C and $\beta > 0$ such that for any bounded function f on $\mathbb{R}^d$ and any $x, y \in E$ with $|x-y| < 1/4$,*

$$|G_\lambda f(x) - G_\lambda f(y)| \le C\|f\|_\infty |x-y|^\beta.$$

Proof. Without loss of generality we may assume that $f \ge 0$. Assume that $x, y \in B(0, n-1)$ for some n. Let $h_n = f - \lambda G_\lambda^{(n)} f$. Note that $\|h_n\|_\infty \le 2\|f\|_\infty$. By the resolvent equation we have $G_\lambda^{(n)} f = G_0^{(n)} h_n$, thus $\|G_0^{(n)} h_n\|_\infty \le \lambda^{-1}\|f\|_\infty$. Consequently, $\|G_0^{(n)} h_n\|_\infty + \|h_n\|_\infty \le (2+\lambda^{-1})\|f\|_\infty$. Now we can use the lemma above to conclude that

$$|G_\lambda^{(n)} f(x) - G_\lambda^{(n)} f(y)| \le C(2+\lambda^{-1})\|f\|_\infty |x-y|^\beta,$$

where C and β are the constants from the lemma above. Letting $n \uparrow \infty$ we get our result. □

From the theorem above we know that for any bounded f, $G_\lambda f$ is Hölder continuous in E. Thus we extend the resolvent $G_\lambda f(x)$ continuously to $\mathbb{R}^d$. From Corollary 4.3 and Remark 4.4 we can see that the extended resolvent on $\mathbb{R}^d$ is a Feller resolvent, that is, G_λ maps $C_0(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$ and for any $f \in C_0(\mathbb{R}^d)$, $\lambda G_\lambda f$ converges to f in the L^∞ norm on $\mathbb{R}^d$. Now we can use Hille-Yosida theorem to extend the semigroup of X continuously to $\mathbb{R}^d$. This semigroup is Feller and so we can start our process X from every point in $\mathbb{R}^d$. Now we can repeat the arguments in Lemmas 4.2–4.8 to check that X satisfies conditions (A1)–(A3) on $\mathbb{R}^d$.

Remark 4.12 *So far in this section we have proved that the Harnack inequality holds when the jumping measure is of the form*

$$\left(\frac{k_1(x,y)}{|x-y|^{d+\alpha_1}} + \frac{k_2(x,y)}{|x-y|^{d+\alpha_2}}\right) dxdy.$$

We can easily generalize this to the case when the jumping measure is of the form

$$\left(\sum_{i=1}^n \frac{k_i(x,y)}{|x-y|^{d+\alpha_i}}\right) dxdy,$$

where $n \ge 1$ is an integer, $0 < \alpha_1 \le \cdots \le \alpha_n < 2$, and $k_1, \dots, k_n$ satisfy the conditions posed on k_1 and k_2 at the beginning of this section.

Remark 4.13 *We can easily generalize the argument of this section to show that the Harnack inequality holds when the jumping measure is of the form*

$$(k_1(x,y) j_{\alpha_1}(|y-x|) + k_2(x,y) j_{\alpha_2}(|y-x|))dxdy,$$

where $j_{\alpha_1}(|x|)dx$ and $j_{\alpha_2}(|x|)dx$ are the Lévy measures of the relativistic α_1 and α_2-stable processes respectively.

Remark 4.14 *Assume that $j : (0,\infty) \to (0,\infty)$ is a non-increasing function such that $j(|x|)dx$ is the Lévy measure of a Lévy process and that conditions (3.3), (3.4), (3.8) and (3.9) are satisfied. If $k(x,y)$ is a symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ such that it is bounded by two positive numbers and that its partials are bounded, the symmetric form $(\mathcal{E}, C^2_c(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ defined by*

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(x)-f(y))(g(x)-g(y))k(x,y)j(|y-x|)dxdy$$

is closable, so its minimal extension $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. We can easily modify the argument of this section to show that the Harnack inequality holds for the symmetric Markov process associated with $(\mathcal{E},\mathcal{F})$.

5 Non-symmetric Markov Processes with no diffusion component

In this section we are going to show that for a wide class of non-symmetric Markov processes, conditions (A1)–(A3) are satisfied and therefore the Harnack inequality holds.

Before we describe the process we are going to deal with, let us first recall some basic facts about strictly stable processes. For any $\alpha \in (0,2)$, the characteristic functions of a strictly stable process on $\mathbb{R}^d$ is given by $\exp(-t\Psi(z))$ with the function Ψ specified below

$$\begin{aligned}
\Psi(z) &= -\int_S \lambda(d\xi)\int_0^\infty (e^{i(z,r\xi)}-1)r^{-(1+\alpha)}dr, && \alpha\in(0,1),\\
\Psi(z) &= -\int_S \lambda(d\xi)\int_0^\infty (e^{i(z,r\xi)}-1-i(z,r\xi)1_{(0,1)})r^{-2}dr - i(z,\gamma), && \alpha=1,\\
\Psi(z) &= -\int_S \lambda(d\xi)\int_0^\infty (e^{i(z,r\xi)}-1-i(z,r\xi)1_{(0,1)})r^{-(1+\alpha)}dr, && \alpha\in(1,2),
\end{aligned}$$

for some finite measure λ on the unit sphere $S=\{x\in\mathbb{R}^d : |x|=1\}$.

The Lévy measure ν of a strictly α-stable process is given by

$$\nu(B) = \int_S \lambda(d\xi)\int_0^\infty 1_B(r\xi)r^{-(1+\alpha)}dr$$

for every Borel set B in $\mathbb{R}^d$.

Suppose $0<\alpha_1<\alpha_2<2$. In this section we assume that $k_i(x,y)$, $i=1,2$, are functions on $\mathbb{R}^d\times\mathbb{R}^d$ bounded between two positive numbers and that $b_i(x)$, $i=1,2$, are bounded $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$. We are going to consider solutions to the martingale problem for the operator $\mathbf{L} = \mathbf{L}_{\alpha_2} + \mathbf{L}_{\alpha_1}$, where

$$\begin{aligned}
\mathbf{L}_{\alpha_1} f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x)) \frac{k_1(x,y)}{|y|^{d+\alpha_1}} dy, && \alpha_1 \in (0,1), \\
\mathbf{L}_{\alpha_1} f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x) - (y, \nabla f(x)) 1_{|y|<1}) \frac{k_1(x,y)}{|y|^{d+\alpha_1}} dy + (b_1(x), \nabla f(x)), && \alpha_1 \in [1,2), \\
\mathbf{L}_{\alpha_2} f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x)) \frac{k_2(x,y)}{|y|^{d+\alpha_2}} dy, && \alpha_2 \in (0,1), \\
\mathbf{L}_{\alpha_2} f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x) - (y, \nabla f(x)) 1_{|y|<1}) \frac{k_2(x,y)}{|y|^{d+\alpha_2}} dy + (b_2(x), \nabla f(x)), && \alpha_2 = 1, \\
\mathbf{L}_{\alpha_2} f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x) - (y, \nabla f(x)) 1_{|y|<1}) \frac{k_2(x,y)}{|y|^{d+\alpha_2}} dy, && \alpha_2 \in (1,2).
\end{aligned}$$

Our assumptions on k_2 and b are as follows:

(i) for each $x \in \mathbb{R}^d$, $\frac{k_2(x,y)}{|y|^{d+\alpha_2}}$ is the density (with respect to the Lebesgue measure) of the Lévy measure of a strictly α_2-stable process ;

(ii) the partial derivatives of $k_2(x,y)$ with respect to y up to order d are bounded continuous on $\mathbb{R}^d \times S$;

(iii) the function $b_2(x)$ is bounded on $\mathbb{R}^d$.

It follows from [8] and [10] that, under the assumptions above, the martingale problem for $\mathbf{L}$ is well-posed. That is, there is a conservative strong Markov process $X = (X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ on $(D([0,\infty), \mathbb{R}^d), \mathcal{B}(D([0,\infty), \mathbb{R}^d)))$ such that for any $f \in C_0^\infty(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) - \int_0^t \mathbf{L} f(X_s) ds$$

is a $\mathbb{P}_x$-martingale for each $x \in \mathbb{R}^d$. Here $D([0,\infty), \mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued cadlag functions on $[0,\infty)$, and $\mathcal{B}(D([0,\infty), \mathbb{R}^d))$ is the Borel σ-field on $D([0,\infty), \mathbb{R}^d)$.

Theorem 5.1 *Under the assumptions above, the Harnack inequality holds for X.*

Proof. It is routine to check that $(k_1(x,y)|y-x|^{-(d+\alpha_1)}dy + k_2(x,y)|y-x|^{-(d+\alpha_2)}dy, dt)$ is a Lévy system for X, see, for instance, the proof of Proposition 2.3 of [1]. Using the same argument as in the last section, we can check that X is Feller process satisfying the conditions (A1)–(A3) of Section 2, thus the Harnack inequality holds for X. We omit the details. □

Remark 5.2 *With the same argument, one can show that, for $0 < \alpha_1 < \alpha_2 < 2$, if Y and Z are independent and if they are respectively strictly α_1 and α_2-stable processes such that the densities of their Lévy measures with respect to the Lebesgue measure are bounded between two positive numbers, then the Harnack inequality holds for the process $X = Y + Z$. Here we do need to assume that the density of the Lévy measure of Z to be smooth. The smoothness assumptions on k_2 made before Theorem 5.1 are to guarantee that the martingale problem for $\mathbf{L}$ is well-posed.*

Remark 5.3 *Assume that $j : (0, \infty) \to (0, \infty)$ is a non-increasing function such that $j(|x|)dx$ is the Lévy measure of a Lévy process and that conditions (3.3), (3.4), (3.8) and (3.9) are satisfied. Suppose that condition (3.10) is satisfied for some $\alpha \in (0, 2)$. Let $k(x, y)$ and b be functions on $\mathbb{R}^d \times \mathbb{R}^d$ and $\mathbb{R}^d$ respectively satisfying the following conditions:*

(i) The function k is bounded between two positive numbers;

(ii) The partial derivatives with respect to x of k up to order 2 are bounded and continuous on $\mathbb{R}^d \times \mathbb{R}^d$;

(iii) b and its partial derivatives up to order 2 are bounded on $\mathbb{R}^d$.

For any $f \in C_b^2(\mathbb{R}^d)$, define

$$\begin{aligned}
\mathbf{L}f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x))k(x, y)j(|y|)dy, && \alpha \in (0, 1) \\
\mathbf{L}f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x) - (y \cdot \nabla f(x))1_{|y|<1})k(x, y)j(|y|)dy + (b(x), \nabla f(x)), && \alpha \in [1, 2).
\end{aligned}$$

It follows from [11] that the martingale problem for $\mathbf{L}$ is well posed. Let X be the conservative Markov process associated with L. Then by using the argument of this section we can show that the Harnack inequality is valid for X.

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