

Global uniform boundary Harnack principle with explicit decay rate and its application

Panki Kim* Renming Song[†] and Zoran Vondraček[‡]

Abstract

In this paper, we consider a large class of subordinate Brownian motions X via subordinators with Laplace exponents which are complete Bernstein functions satisfying some mild scaling conditions at zero and at infinity. We first discuss how such conditions govern the behavior of the subordinator and the corresponding subordinate Brownian motion for both large and small time and space. Then we establish a global uniform boundary Harnack principle in (unbounded) open sets for the subordinate Brownian motion. When the open set satisfies the interior and exterior ball conditions with radius $R > 0$, we get a global uniform boundary Harnack principle with explicit decay rate. Our boundary Harnack principle is global in the sense that it holds for all $R > 0$ and the comparison constant does not depend on R , and it is uniform in the sense that it holds for all balls with radii $r \leq R$ and the comparison constant depends neither on D nor on r . As an application, we give sharp two-sided estimates for the transition densities and Green functions of such subordinate Brownian motions in the half-space.

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1 Introduction

The study of potential theory of discontinuous Lévy processes in \mathbb{R}^d revolves around several fundamental questions such as sharp heat kernel and Green function estimates, exit time estimates and Poisson kernel estimates, Harnack and boundary Harnack principles for non-negative harmonic functions. One can roughly divide these studies in two categories: those on a bounded set and those on an unbounded set. For the former, it is the local behavior of the process that matters, while for the latter both local and global behaviors are important. The processes investigated in these studies are usually described in two ways: either the process is given explicitly through its characteristic exponent (such as the case of a symmetric stable process, a relativistically stable process, sum of two independent stable processes, etc.), or some conditions on the characteristic exponent are given. In the situation when one is interested in the potential theory on bounded sets,

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conditions imposed on the characteristic exponent govern the small time – small space (i.e., local) behavior of the process. Let us be more precise and describe in some detail one such condition and some of the results in the literature.

Let $S = (S_t)_{t \geq 0}$ be a subordinator (that is, an increasing Lévy process satisfying $S_0 = 0$) with Laplace exponent ϕ , and let $W = (W_t)_{t \geq 0}$ be a Brownian motion in \mathbb{R}^d , $d \geq 1$, independent of S with

$$\mathbb{E}_x \left[e^{i\xi(W_t - W_0)} \right] = e^{-t|\xi|^2}, \quad \xi \in \mathbb{R}^d, t > 0.$$

The process $X = (X_t)_{t \geq 0}$ defined by $X_t := W(S_t)$ is called a subordinate Brownian motion. It is a rotationally invariant Lévy process in \mathbb{R}^d with characteristic exponent $\phi(|\xi|^2)$. The function ϕ is a Bernstein function. Let us introduce the following upper and lower scaling conditions:

(H1): There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \left(\frac{R}{r} \right)^{\delta_1} \leq \frac{\phi(R)}{\phi(r)} \leq a_2 \left(\frac{R}{r} \right)^{\delta_2}, \quad 1 \leq r \leq R. \quad (1.1)$$

It follows from the definitions in [2, pp. 65 and 68] and [2, Proposition 2.2.1] that (1.1) is equivalent to saying that ϕ is in the class OR of O -regularly varying functions at ∞ with Matuszewka indices contained in $(0, 1)$. The advantage of the formulation above is that we can provide more direct proofs for some of the results below. (1.1) is a condition on the asymptotic behavior of ϕ at infinity and it governs the behavior of the subordinator S for small time and small space, which, in turn, implies the small time – small space behavior of the corresponding subordinate Brownian motion X . Very recently it has been shown in [20] (see also [16]) that if **(H1)** holds and ϕ is a complete Bernstein function, then the uniform boundary Harnack principle is true and various exit time and Poisson kernel estimates hold. Further, sharp two-sided Green function estimates for bounded $C^{1,1}$ open sets are given in [16]. The statements of these results usually take the following form: For some $R > 0$, there exists a constant $c = c(R) > 0$ (also depending on the process X) such that some quantities involving $r \in (0, R)$ can be estimated by expressions involving the constant c . The point is that although the constant c is uniform for small $r \in (0, R)$, it does depend on R , meaning that the result is local. It would be of interest to obtain a global and uniform version of such results, namely with the constant depending neither on R nor on the open set itself. This would facilitate the study of potential theory on unbounded sets. In order to accomplish this goal, it is clear that the assumption **(H1)** (or some similar condition) will not suffice, and that one needs additional assumptions that govern the behavior of the process for large time and large space.

In some recent papers (see [9, 10, 13]) potential-theoretic properties of stable and relativistically stable processes are studied in unbounded sets such as the half-space, half-space-like $C^{1,1}$ open sets and exterior $C^{1,1}$ open sets. Note that these processes are given explicitly by its characteristic exponent. In the current paper we would like to impose a condition similar to **(H1)** that governs the large time – large space behavior of the process and obtain global uniform potential-theoretic results. Thus, in addition to **(H1)**, we will also assume

(H2): There exist constants $0 < \delta_3 \leq \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \left(\frac{R}{r} \right)^{\delta_3} \leq \frac{\phi(R)}{\phi(r)} \leq a_4 \left(\frac{R}{r} \right)^{\delta_4}, \quad r \leq R \leq 1. \quad (1.2)$$

Similarly, (1.2) is equivalent to saying that ϕ is in the class of O -regularly varying functions at 0 with Matuszewka indices contained in $(0, 1)$. (1.2) is a condition about the asymptotic behavior of ϕ at zero and it governs the behavior of the subordinator S and the corresponding subordinate Brownian motion X for large time and large space. Also note that under **(H2)**, X is transient if $d \geq 2$.

Throughout the paper we will assume that ϕ is a complete Bernstein function satisfying **(H1)** and/or **(H2)**, and $X_t = W(S_t)$ will be the corresponding subordinate Brownian motion. First we study consequences of scaling conditions on the subordinator S , its Lévy density and potential density. This is done in Section 2 of the paper. In Section 3 we proceed to properties of the subordinate Brownian motion X . The first main result is about estimates of the Lévy density and the Green function of X for the whole space given in Theorem 3.4. These estimates allow us to repeat arguments from [17, 20] and obtain global uniform estimates of the exit times and the Poisson kernel, as well as global uniform Harnack and boundary Harnack principles. The latter will play a crucial role in this paper.

In Section 4 we prove the main result of the paper – the global uniform boundary Harnack principle with explicit decay rate in open sets satisfying both interior and exterior ball conditions (see Theorem 4.7(b)). The key technical contribution is Proposition 4.6 which has appeared in similar forms in several recent papers. The main novelty of the current version is that the estimate gets better as the radius grows larger. The quite technical part of the proof of this proposition is given in two auxiliary lemmas.

Theorem 4.7 is used in Section 5 to obtain sharp two-sided heat kernel and Green function estimates for the process X killed upon exiting the half-space $\mathbb{H} = \{x = (x_1, \dots, x_{d-1}, x_d) \in \mathbb{R}^d : x_d > 0\}$. To the best of our knowledge, this is the first time the heat kernel estimates are obtained in an unbounded set for a process which is not given by an explicit characteristic exponent.

The results of this paper, especially those of Sections 3 and 4, are used in the subsequent paper [22] to prove the boundary Harnack principle at infinity. This was the main motivation for the investigations in the current paper.

Using the tables at the end of [26], one can construct a lot of explicit examples of complete Bernstein functions satisfying both **(H1)** and **(H2)**. Here are a few of them:

- (1) $\phi(\lambda) = \lambda^\alpha + \lambda^\beta$, $0 < \alpha < \beta < 1$;
- (2) $\phi(\lambda) = (\lambda + \lambda^\alpha)^\beta$, $\alpha, \beta \in (0, 1)$;
- (3) $\phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^\beta$, $\alpha \in (0, 1)$, $\beta \in (0, 1 - \alpha)$;
- (4) $\phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^{-\beta}$, $\alpha \in (0, 1)$, $\beta \in (0, \alpha)$;

$$(5) \phi(\lambda) = (\log(\cosh(\sqrt{\lambda})))^\alpha, \alpha \in (0, 1);$$

$$(6) \phi(\lambda) = (\log(\sinh(\sqrt{\lambda})) - \log \sqrt{\lambda})^\alpha, \alpha \in (0, 1).$$

We remark here that relativistic stable processes do not satisfy **(H2)**, so the present paper does not cover this interesting case. We plan to address this important case in the near future.

Throughout this paper, $d \geq 1$ and the constants C_1, a_i and $\delta_i, i = 1, \dots, 4$, will be fixed. We use c_1, c_2, \dots to denote generic constants, whose exact values are not important and can change from one appearance to another. The labeling of the constants c_1, c_2, \dots starts anew in the statement of each result. The dependence of the constant c on the dimension d will not be mentioned explicitly. We will use “:=” to denote a definition, which is read as “is defined to be”. We will use dx to denote the Lebesgue measure in \mathbb{R}^d . For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure. We denote the Euclidean distance between x and y in \mathbb{R}^d by $|x - y|$ and denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For any two positive functions f and g , we use the notation $f(r) \asymp g(r)$, $r \rightarrow a$ to denote that $\frac{f(r)}{g(r)}$ stays between two positive constants as $r \rightarrow a$. $f \asymp g$ simply means that there is a positive constant $c \geq 1$ so that $c^{-1}g \leq f \leq cg$ on their common domain of definition. For any open $D \subset \mathbb{R}^d$ and $x \in D$, $\delta_D(x)$ stands for the Euclidean distance between x and D^c .

2 Scaling conditions and consequences

Recall that a function $\phi : (0, \infty) \rightarrow (0, \infty)$ is a Bernstein function if it has the representation

$$\phi(\lambda) = a + b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda t}) \mu(dt),$$

where $a, b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0, \infty)} (1 \wedge t) \mu(dt) < \infty$. A function $\phi : (0, \infty) \rightarrow (0, \infty)$ is a Bernstein function if and only if it is the Laplace exponent of a (killed) subordinator $S = (S_t)_{t \geq 0}$: $\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}$ for all $t \geq 0$ and $\lambda > 0$.

It is well-known that, if ϕ is a Bernstein function, then

$$\phi(\lambda t) \leq \lambda \phi(t) \quad \text{for all } \lambda \geq 1, t > 0, \tag{2.1}$$

implying

$$\frac{\phi(v)}{v} \leq \frac{\phi(u)}{u}, \quad 0 < u \leq v. \tag{2.2}$$

Note that (2.2) implies

$$\lambda \phi'(\lambda) \leq \phi(\lambda) \quad \text{for all } \lambda > 0. \tag{2.3}$$

We remark that, since ϕ is increasing, (2.1) is equivalent to that ϕ is an O -regularly varying function, see [2, Section 2.0.2].

Clearly (2.1) implies the following observation.

Lemma 2.1 *If ϕ is a Bernstein function, then for all $\lambda, t > 0$, $1 \wedge \lambda \leq \phi(\lambda t)/\phi(t) \leq 1 \vee \lambda$.*

Note that with this lemma, we can replace expressions of the type $\phi(\lambda t)$, when ϕ is a Bernstein function, with $\lambda > 0$ fixed and $t > 0$ arbitrary, by $\phi(t)$ up to a multiplicative constant depending on λ . We will often do this without explicitly mentioning it.

In the remainder of this paper, we will always assume that ϕ is a complete Bernstein function, that is, the Lévy measure μ of ϕ has a completely monotone density. We will denote this density by $\mu(t)$. For properties of complete Bernstein function, we refer our reader to [26].

We will assume that ϕ satisfies either **(H1)**, or **(H2)**, or both. Note that it follows from the right-hand side inequality in (1.1) that ϕ has no drift, i.e., $b = 0$. It also follows from the left-hand side inequality in (1.2) that ϕ has no killing term, i.e., $a = 0$. Since for most of this paper we assume both **(H1)** and **(H2)**, it is harmless to immediately assume that $a = b = 0$ (regardless whether the scaling conditions hold). So, from now on, $a = b = 0$.

Throughout this paper, we use $S = (S_t)_{t \geq 0}$ to denote a subordinator with Laplace exponent ϕ . Since ϕ is a complete Bernstein function, the potential measure U of S has a complete monotone density $u(t)$ (see [26, Theorem 10.3] or [17, Corollary 13.2.3]), called the potential density of S .

Without loss of generality we further assume that $\phi(1) = 1$. Then by taking $r = 1$ and $R = \lambda$ in **(H1)**, and $R = 1$ and $r = \lambda$ in **(H2)**, we get that

$$a_1 \lambda^{\delta_1} \leq \phi(\lambda) \leq a_2 \lambda^{\delta_2}, \quad \lambda \geq 1, \quad (2.4)$$

and

$$a_4^{-1} \lambda^{\delta_4} \leq \phi(\lambda) \leq a_3^{-1} \lambda^{\delta_3}, \quad \lambda \leq 1. \quad (2.5)$$

If $0 < r < 1 < R$, using (2.4) and (2.5), we have that under **(H1)**–**(H2)**,

$$\frac{\phi(R)}{\phi(r)} \leq a_2 a_4 \frac{R^{\delta_2}}{r^{\delta_4}} \leq a_2 a_4 \left(\frac{R}{r}\right)^{\delta_2 \vee \delta_4} \quad \text{and} \quad \frac{\phi(R)}{\phi(r)} \geq a_1 a_3 \frac{R^{\delta_1}}{r^{\delta_3}} \geq a_1 a_3 \left(\frac{R}{r}\right)^{\delta_1 \wedge \delta_3}.$$

Combining these with **(H1)** and **(H2)** we get

$$a_5 \left(\frac{R}{r}\right)^{\delta_1 \wedge \delta_3} \leq \frac{\phi(R)}{\phi(r)} \leq a_6 \left(\frac{R}{r}\right)^{\delta_2 \vee \delta_4}, \quad 0 < r < R < \infty. \quad (2.6)$$

For $a > 0$, we define $\phi^a(\lambda) = \phi(\lambda a^{-2})/\phi(a^{-2})$. Then ϕ^a is again a complete Bernstein function satisfying $\phi^a(1) = 1$. We will use $\mu^a(dt)$ and $\mu^a(t)$ to denote the Lévy measure and Lévy density of ϕ^a respectively, $S^a = (S_t^a)_{t \geq 0}$ to denote a subordinator with Laplace exponent ϕ^a , and $u^a(t)$ to denote the potential density of S^a . Since

$$\phi^a(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu^a(t) dt, \quad \int_0^\infty e^{-\lambda t} u^a(t) dt = \frac{1}{\phi^a(\lambda)}, \quad \lambda > 0,$$

it is straightforward to see that

$$\mu^a(t) = \frac{a^2}{\phi(a^{-2})} \mu(a^2 t), \quad t > 0, \quad (2.7)$$

and

$$u^a(t) = a^2 \phi(a^{-2})u(a^2t), \quad t > 0. \quad (2.8)$$

Now applying this to ϕ^a , we get that under **(H1)**–**(H2)**,

$$a_5 \left(\frac{R}{r}\right)^{\delta_1 \wedge \delta_3} \leq \frac{\phi^a(R)}{\phi^a(r)} \leq a_6 \left(\frac{R}{r}\right)^{\delta_2 \vee \delta_4}, \quad a > 0, \quad 0 < r < R < \infty. \quad (2.9)$$

The results in the next lemma will be used many times later in the paper.

Lemma 2.2 *Assume **(H1)** and **(H2)**. There exists $c = c(a_1, a_2, a_3, a_4, \delta_1, \delta_2, \delta_3, \delta_4) > 0$ such that*

$$\int_0^{\lambda^{-1}} \phi(r^{-2})^{1/2} dr \leq c\lambda^{-1}\phi(\lambda^2)^{1/2}, \quad \text{for all } \lambda > 0, \quad (2.10)$$

$$\lambda^2 \int_0^{\lambda^{-1}} r\phi(r^{-2}) dr + \int_{\lambda^{-1}}^{\infty} r^{-1}\phi(r^{-2}) dr \leq c\phi(\lambda^2), \quad \text{for all } \lambda > 0, \quad (2.11)$$

$$\int_0^{\lambda^{-1}} r^{-1}\phi(r^{-2})^{-1} dr \leq c\phi(\lambda^2)^{-1}, \quad \text{for all } \lambda > 0. \quad (2.12)$$

Proof. This result is essentially Karamata's theorem for O -regularly varying functions with constants controlled and its proof is hidden in the proofs in [2, Section 2.6]. Taking into account (2.6), direct proofs of (2.10)–(2.12) are the same as those of [20, Lemma 4.1]. We omit the proof here. \square

The following result plays a crucial role in this paper.

Proposition 2.3 *Suppose that w is a completely monotone function given by*

$$w(t) = \int_0^{\infty} e^{-st} f(s) ds,$$

where f is a nonnegative decreasing function.

(a) *It holds that*

$$f(s) \leq (1 - e^{-1})^{-1} s^{-1} w(s^{-1}), \quad s > 0. \quad (2.13)$$

(b) *If there exist $\delta \in (0, 1)$ and $a, s_0 > 0$ such that*

$$w(\lambda t) \leq a\lambda^{-\delta} w(t), \quad \lambda \geq 1, t \geq 1/s_0, \quad (2.14)$$

then there exists $c_1 = c_1(w, a, s_0, \delta) > 0$ such that

$$f(s) \geq c_1 s^{-1} w(s^{-1}), \quad s \leq s_0.$$

(c) If there exist $\delta \in (0, 1)$ and $a, s_0 > 0$ such that

$$w(\lambda t) \geq a\lambda^{-\delta}w(t), \quad \text{for all } \lambda \leq 1 \text{ and } t \leq 1/s_0, \quad (2.15)$$

then there exists $c_2 = c_2(w, a, s_0, \delta) > 0$ such that

$$f(s) \geq c_2s^{-1}w(s^{-1}), \quad s \geq s_0.$$

Proof. This result follows from Karamata's Tauberian theorem and monotone density theorem (together with their counterparts at 0) for O -regularly varying functions, see [2, Theorem 2.10.2 and Proposition 2.10.3]. Here we give a direct proof.

Direct proofs of (a) and (b) are given in [28] (see also [17, Proposition 13.2.5]).

(c) Let $\rho := \left(\int_0^{s_0} e^{-s} f(s) ds\right) \left(\int_{s_0}^{\infty} e^{-s} f(s) ds\right)^{-1}$. Note that $\rho = \rho(f, s_0) = \rho(w, s_0)$. For any $t \leq 1$, we have

$$\begin{aligned} \int_0^{s_0} e^{-ts} f(s) ds &= \int_0^{s_0} e^{(1-t)s} e^{-s} f(s) ds \leq e^{(1-t)s_0} \int_0^{s_0} e^{-s} f(s) ds \\ &= \rho e^{(1-t)s_0} \int_{s_0}^{\infty} e^{-s} f(s) ds \leq \rho \int_{s_0}^{\infty} e^{-ts} f(s) ds. \end{aligned}$$

Thus for any $t \leq 1$

$$w(t) \leq (\rho + 1) \int_{s_0}^{\infty} e^{-st} f(s) ds = \frac{\rho + 1}{t} \int_{s_0 t}^{\infty} e^{-s} f\left(\frac{s}{t}\right) ds.$$

Let $t \leq 1$ be arbitrary. For any $r \in (0, 1]$, we have

$$\begin{aligned} tw(t) &\leq (\rho + 1) \int_0^r \mathbf{1}_{\{s_0 t < s\}} e^{-s} f\left(\frac{s}{t}\right) ds + (\rho + 1) \int_r^{\infty} e^{-s} \mathbf{1}_{\{s_0 t < s\}} f\left(\frac{s}{t}\right) ds \\ &\leq (\rho + 1) \int_0^r \mathbf{1}_{\{s_0 t < s\}} e^{-s} f\left(\frac{s}{t}\right) ds + (\rho + 1) f\left(\frac{r}{t}\right) e^{-r} \\ &\leq (\rho + 1) (1 - e^{-1})^{-1} t \int_0^r \mathbf{1}_{\{s_0 t < s\}} e^{-s} \frac{1}{s} w\left(\frac{t}{s}\right) ds + (\rho + 1) f\left(\frac{r}{t}\right) e^{-r}, \quad (2.16) \end{aligned}$$

where in the last line we used (2.13).

Now we assume (2.15) and apply it to $w(\frac{t}{s})$ in (2.16). Note that $s \leq r \leq 1$, and since $s_0 t < s$ we also have that $t \leq s/s_0$. Thus $w(\frac{t}{s}) \leq a^{-1} s^{\delta} w(t)$, implying that

$$tw(t) \leq (\rho + 1) a^{-1} (1 - e^{-1})^{-1} tw(t) \int_0^r \mathbf{1}_{\{s_0 t < s\}} e^{-s} s^{\delta-1} ds + (\rho + 1) f\left(\frac{r}{t}\right) e^{-r}.$$

Choose $r = r(a, s_0, \delta) \in (0, 1]$ small enough so that

$$(\rho + 1) a^{-1} (1 - e^{-1})^{-1} \int_0^r e^{-s} s^{\delta-1} ds \leq \frac{1}{2}.$$

For this choice of r , we have $f(\frac{r}{t}) \geq c_1 tw(t)$, $t \leq 1$, for some $c_1 = c_1(a, w, a, s_0) > 0$. Thus

$$f(s) \geq c_1 \frac{r}{s} w\left(\frac{r}{s}\right) \geq c_2 s^{-1} w(s^{-1}), \quad s \geq r,$$

where $c_2 = c_1 r$. In order to extend the inequality to $s \geq s_0$ it suffices to use the continuity of w . \square

Corollary 2.4 (a) *The potential density u of S satisfies*

$$u(t) \leq (1 - e^{-1})^{-1} t^{-1} \phi(t^{-1})^{-1}, \quad t > 0. \quad (2.17)$$

(b) *If **(H1)** holds, then there exists $c_1 = c_1(\phi) > 0$ such that*

$$u(t) \geq c_1 t^{-1} \phi(t^{-1})^{-1}, \quad 0 < t \leq 1. \quad (2.18)$$

(c) *If **(H2)** holds, then there exists $c_2 = c_2(\phi) > 0$ such that*

$$u(t) \geq c_2 t^{-1} \phi(t^{-1})^{-1}, \quad 1 \leq t < \infty. \quad (2.19)$$

Proof. (a) The claim follows from Proposition 2.3(a) with $w(t) := \int_0^\infty e^{-st} u(s) ds = \frac{1}{\phi(t)}$.

(b) By the left-hand side of (1.1), $w(t) = \phi(t)^{-1}$ satisfies (2.14) with $\delta = \delta_1$, $a = a_1^{-1}$ and $s_0 = 1$. The claim follows from Proposition 2.3(b) with $c_1 = c_1(\phi, a_1, \delta_1)$.

(c) By the left-hand side of (1.2), $w(t) = \phi(t)^{-1}$ satisfies (2.15) with $\delta = \delta_3$, $a = a_3^{-1}$ and $s_0 = 1$. The claim follows from Proposition 2.3(c) with $c_2 = c_2(\phi, a_3, \delta_3)$. \square

Since ϕ is a complete Bernstein function, its conjugate function $\phi^*(\lambda) := \frac{\lambda}{\phi(\lambda)}$ is also complete Bernstein. It is immediate to see that, under **(H2)** for ϕ , the function ϕ^* satisfies

$$a_4^{-1} \left(\frac{R}{r} \right)^{1-\delta_4} \leq \frac{\phi(R)}{\phi(r)} \leq a_3^{-1} \left(\frac{R}{r} \right)^{1-\delta_3}, \quad r \leq R \leq 1.$$

Since the potential density u^* of ϕ^* is equal to the tail $\mu(t, \infty)$ of the Lévy measure μ (see [4, Corollary 5.5]), we conclude from Corollary 2.4 that

$$\mu(t, \infty) \leq (1 - e^{-1})^{-1} t^{-1} \phi^*(t^{-1})^{-1}, \quad t > 0, \quad (2.20)$$

$$\mu(t, \infty) \geq c t^{-1} \phi^*(t^{-1})^{-1}, \quad 1 \leq t < \infty, \quad \text{if **(H2)** holds.} \quad (2.21)$$

Proposition 2.5 (a) *The Lévy density μ of S satisfies*

$$\mu(t) \leq (1 - 2e^{-1})^{-1} t^{-1} \phi(t^{-1}), \quad t > 0. \quad (2.22)$$

(b) *If **(H1)** holds, then there exists $c_1 = c_1(\phi) > 0$ such that*

$$\mu(t) \geq c_1 t^{-1} \phi(t^{-1}), \quad 0 < t \leq 1. \quad (2.23)$$

(c) *If **(H2)** holds, then there exists $c_2 = c_2(\phi) > 0$ such that*

$$\mu(t) \geq c_2 t^{-1} \phi(t^{-1}), \quad 1 \leq t < \infty. \quad (2.24)$$

Proof. (a) This is proved in [15, Lemma A.1, Proposition 3.3].

(b) This is proved in [17, Theorem 13.2.10].

(c) The proof is similar to the proof of (b). It follows from (2.20) and (2.21) that there exists a constant $c_1 > 0$ such that $c_1^{-1}\phi(s^{-1}) \leq u^*(s) \leq c_1\phi(s^{-1})$ for $1 \leq s < \infty$. Fix $\lambda := (2c_1^2a_3^{-1})^{1/\delta_3} \vee 1 \geq 1$. Then by the left-hand side of **(H2)**, we have that for $s \geq \lambda$,

$$u^*(s) \leq c_1\phi(s^{-1}) = c_1\phi(\lambda^{-1}(\lambda^{-1}s)^{-1}) \leq c_1a_3^{-1}\lambda^{-\delta_3}\phi((\lambda^{-1}s)^{-1}) \leq c_1^2a_3^{-1}\lambda^{-\delta_3}u^*(\lambda^{-1}s) \leq \frac{1}{2}u^*(\lambda^{-1}s)$$

by our choice of λ . Further,

$$(1 - \lambda^{-1})s\mu(\lambda^{-1}s) \geq \int_{\lambda^{-1}s}^s \mu(t) dt = u^*(\lambda^{-1}s) - u^*(s) \geq u^*(\lambda^{-1}s) - \frac{1}{2}u^*(\lambda^{-1}s) = \frac{1}{2}u^*(\lambda^{-1}s).$$

This implies that for all $t \geq 1$

$$\mu(t) \geq \frac{1}{2(1 - \lambda^{-1})\lambda} t^{-1}u^*(t) = c_2t^{-1}u^*(t) \geq c_3t^{-1}\phi(t^{-1})$$

for some constants $c_2, c_3 > 0$. □

We conclude this section with some conditions on ϕ which imply **(H1)** and **(H2)**.

(H₀): There exist $\beta \in (0, 2)$ and a function $\ell : (0, \infty) \rightarrow (0, \infty)$ which is measurable, bounded on compact subsets of $(0, \infty)$ and slowly varying at 0 such that

$$\phi(\lambda) \asymp \lambda^{\beta/2}\ell(\lambda), \quad \lambda \rightarrow 0 + . \tag{2.25}$$

(H_∞): There exist $\alpha \in (0, 2)$ and a function $\tilde{\ell} : (0, \infty) \rightarrow (0, \infty)$ which is measurable, bounded on compact subsets of $(0, \infty)$ and slowly varying at infinity such that

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\tilde{\ell}(\lambda), \quad \lambda \rightarrow \infty . \tag{2.26}$$

Using Potter's theorem (cf. [2, Theorem 1.5.6]), it is proved in [17] that **(H_∞)** implies the right-hand side inequality of **(H1)**. One can similarly prove that **(H_∞)** also implies the left-hand side inequality of **(H1)** and that **(H₀)** implies **(H2)**.

3 Applications to subordinate Brownian motions

Recall that $S = (S_t)_{t \geq 0}$ is a subordinator with Laplace exponent ϕ . Let $W = (W_t, \mathbb{P}_x)_{t \geq 0}$ be a d -dimensional Brownian motion independent of S and with transition density

$$q(t, x, y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \mathbb{R}^d, \quad t > 0.$$

The process $X = (X_t)_{t \geq 0}$ defined by $X_t := W(S_t)$ is called a subordinate Brownian motion. It is a rotationally invariant Lévy process with characteristic exponent $\phi(|\xi|^2)$, $\xi \in \mathbb{R}^d$, and transition density given by

$$p(t, x, y) = \int_0^\infty q(s, x, y) \mathbb{P}(S_t \in ds).$$

By spatial homogeneity, the Lévy measure of X has a density $J(x) = j(|x|)$, where $j : (0, \infty) \rightarrow (0, \infty)$ is given by

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt. \quad (3.1)$$

Note that j is continuous and decreasing. We define $J(x, y) := J(y - x)$.

By the Chung-Fuchs criterion the process X is transient if and only if

$$\int_0^1 \frac{\lambda^{d/2-1}}{\phi(\lambda)} d\lambda < \infty. \quad (3.2)$$

Note that if $d \geq 3$, then X is always transient. If **(H2)** holds and $d > 2\delta_4$, then X is transient. In particular, if **(H2)** holds and $d \geq 2$, then X is transient. When X is transient, the mean occupation time measure of X admits a density $G(x, y) = g(|x - y|)$ which is called the Green function of X , and is given by the formula

$$g(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} u(t) dt. \quad (3.3)$$

Here u is the potential density of the subordinator S . Note that by the transience assumption, the integral converges. Moreover, g is continuous and decreasing.

We first record the upper bounds of $j(r)$ and $g(r)$.

Lemma 3.1

- (a) It holds that $j(r) \leq c_1 r^{-d} \phi(r^{-2})$ for all $r > 0$.
- (b) If $d \geq 3$ then $g(r) \leq c_2 r^{-d} \phi(r^{-2})^{-1}$ for all $r > 0$.

Proof. (a) We write

$$j(r) = \int_0^{r^2} (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt + \int_{r^2}^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt := J_1 + J_2.$$

To estimate J_2 from above we first use (2.22) and then the monotonicity of ϕ to obtain

$$\begin{aligned} J_2 &\leq c_1 \int_{r^2}^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} t^{-1} \phi(t^{-1}) dt \\ &\leq c_1 \phi(r^{-2}) \int_0^\infty t^{-d/2-1} e^{-r^2/(4t)} dt = c_2 \phi(r^{-2}) r^{-d}. \end{aligned}$$

By (2.2) we have $\phi(t^{-1})/t^{-1} \leq \phi(r^{-2})/r^{-2}$ for $t \leq r^2$ (i.e., $r^{-2} \leq t^{-1}$), thus by (2.22)

$$\begin{aligned} J_1 &\leq c_3 \int_0^{r^2} t^{-d/2} e^{-r^2/(4t)} t^{-1} \phi(t^{-1}) dt \leq c_3 r^2 \phi(r^{-2}) \int_0^{r^2} t^{-d/2-2} e^{-r^2/(4t)} dt \\ &\leq c_3 r^2 \phi(r^{-2}) \int_0^\infty t^{-d/2-2} e^{-r^2/(4t)} dt = c_4 r^{-d} \phi(r^{-2}). \end{aligned}$$

(b) We write

$$g(r) = \int_0^{r^2} (4\pi t)^{-d/2} e^{-r^2/(4t)} u(t) dt + \int_{r^2}^{\infty} (4\pi t)^{-d/2} e^{-r^2/(4t)} u(t) dt := L_1 + L_2. \quad (3.4)$$

By using (2.17) in the first inequality and the monotonicity of ϕ in the second inequality, we get

$$\begin{aligned} L_1 &\leq c_5 \int_0^{r^2} (4\pi t)^{-d/2} e^{-r^2/(4t)} t^{-1} \phi(t^{-1})^{-1} dt \leq c_6 \phi(r^{-2})^{-1} \int_0^{r^2} t^{-d/2-1} e^{-r^2/(4t)} dt \\ &\leq c_6 \phi(r^{-2})^{-1} \int_0^{\infty} t^{-d/2-1} e^{-r^2/(4t)} dt = c_7 \phi(r^{-2})^{-1} r^{-d}. \end{aligned} \quad (3.5)$$

Since $d \geq 3$, using that u is decreasing in the second inequality and (2.17) in the third, we get

$$L_2 \leq c_8 \int_{r^2}^{\infty} t^{-d/2} u(t) dt \leq c_8 u(r^2) \int_{r^2}^{\infty} t^{-d/2} dt \leq c_9 \phi(r^{-2})^{-1} r^{-d}.$$

□

Our next goal is to establish the asymptotic behaviors of $j(r)$ and $g(r)$ for small and/or large r under **(H1)** or **(H2)**, or both.

Lemma 3.2 *Assume **(H1)**.*

(a) *It holds that*

$$j(r) \asymp r^{-d} \phi(r^{-2}), \quad r \rightarrow 0. \quad (3.6)$$

(b) *If $d > 2\delta_2$ and X is transient, then*

$$g(r) \asymp r^{-d} \phi(r^{-2})^{-1}, \quad r \rightarrow 0. \quad (3.7)$$

Proof. (a) is proved in [17, Theorem 13.3.2], so we only prove (b). First note that the assumption $d > 2\delta_2$ is always satisfied when $d \geq 2$.

By Lemma 3.1 we only need to prove the upper bound in (3.7) for $d \leq 2$. To do that we write $g(r) = L_1 + L_2$ as in (3.4). First note that, by the same argument as for (3.5), we have $L_1 \leq c_1 \phi(r^{-2})^{-1} r^{-d}$.

Let $d \leq 2$ and $r \leq 1$. We split L_2 into two parts:

$$L_2 \leq c_2 \int_{r^2}^1 t^{-d/2} u(t) dt + c_2 \int_1^{\infty} t^{-d/2} e^{-r^2/(4t)} u(t) dt =: L_{21} + L_{22}.$$

For L_{21} we use (2.17) and the change of variables $t = r^2 s$ to get

$$L_{21} = c_3 r^{-d} \int_1^{r^{-2}} s^{-d/2-1} \phi(r^{-2} s^{-1})^{-1} ds.$$

Since $0 < r \leq 1$ and $r^2 \leq s^{-1} \leq 1$, it follows from (1.1) that $\phi(r^{-2}s^{-1})^{-1} \leq a_2 s^{\delta_2} \phi(r^{-2})^{-1}$. Hence

$$L_{21} \leq c_4 r^{-d} \phi(r^{-2})^{-1} \int_1^\infty s^{-d/2-1+\delta_2} ds = c_5 r^{-d} \phi(r^{-2})^{-1},$$

since the integral converges under the assumption $d > 2\delta_2$. Note that using **(H1)** and the assumption that $2\delta_2 < d$, we have $r^{-d} \phi(r^{-2})^{-1} \geq c_6 r^{2\delta_2-d} \geq c_6 > 0$. Since L_{22} is bounded for $r \leq 1$ by [15, Lemma 4.4], we have proved the upper bound.

To prove the converse inequality for all $d \geq 1$, we use (2.18) in the second inequality and (2.2) in the third to get that for $r \leq 1$,

$$\begin{aligned} g(r) &\geq (4\pi)^{-d/2} \int_0^{1/r^2} (tr^2)^{-d/2} e^{-r^2/(4tr^2)} u(r^2t) r^2 dt \\ &\geq c_7 r^{2-d} \int_0^1 t^{-d/2} e^{-1/(4t)} r^{-2} t^{-1} \phi(r^{-2}t^{-1})^{-1} dt \geq c_8 r^{-d} \phi(r^{-2})^{-1}. \end{aligned}$$

□

Lemma 3.3 *Assume (H2).*

(a) *It holds that*

$$j(r) \asymp r^{-d} \phi(r^{-2}), \quad r \rightarrow \infty. \quad (3.8)$$

(b) *If $d > 2\delta_4$, then X is transient and*

$$g(r) \asymp r^{-d} \phi(r^{-2})^{-1}, \quad r \rightarrow \infty. \quad (3.9)$$

Proof. (a) By Lemma 3.1 we only need to prove the lower bound in (3.8). For the lower bound we have

$$\begin{aligned} j(r) &\geq (4\pi)^{-d/2} \int_0^1 (r^2t)^{-d/2} e^{-1/(4t)} \mu(r^2t) r^2 dt \\ &\geq c_1 r^{-d+2} \mu(r^2) \int_0^1 t^{-d/2} e^{-1/(4t)} dt \geq c_2 r^{-d} \phi(r^{-2}), \end{aligned}$$

where in the last inequality we used (2.24).

(b) By (2.5), $a_4^{-1} \lambda^{\delta_4} \leq \phi(\lambda)$ for all $\lambda \leq 1$, so using the assumption $d > 2\delta_4$, X is transient by (3.2). Let $r \geq 1$. By the change of variables $s = r^2/t$ we get that

$$g(r) = c_3 r^{-d+2} \int_0^\infty s^{d/2-2} e^{-s/4} u(r^2s^{-1}) ds.$$

By (2.17), we have $u(r^2s^{-1}) \leq c_4 r^{-2} s \phi(r^{-2}s)^{-1}$. Hence

$$g(r) \leq c_5 r^{-d} \int_0^1 s^{d/2-1} \phi(r^{-2}s)^{-1} ds + c_5 r^{-d} \int_1^\infty s^{d/2-1} e^{-s/4} \phi(r^{-2}s)^{-1} ds =: L_1 + L_2.$$

To estimate L_1 from above, we note that, by **(H2)**, we have $\phi(r^{-2}s) \geq a_4^{-1}s^{\delta_4}\phi(r^{-2})$, $0 < s \leq 1$. Hence

$$L_1 \leq c_6 r^{-d} \phi(r^{-2})^{-1} \int_0^1 s^{d/2-1-\delta_4} e^{-s/4} ds = c_7 r^{-d} \phi(r^{-2})^{-1}$$

since the integral converges under the assumption $d/2 > \delta_4$. In order to estimate L_2 , we use that $r^{-2} \leq r^{-2}s$ for $s \geq 1$, hence by the monotonicity of ϕ , $\phi(r^{-2}s) \geq \phi(r^{-2})$. Therefore,

$$L_2 \leq c_8 r^{-d} \phi(r^{-2})^{-1} \int_1^\infty s^{d/2-1} e^{-s/4} ds = c_9 r^{-d} \phi(r^{-2})^{-1}.$$

For the lower bound we have

$$\begin{aligned} g(r) &\geq (4\pi)^{-d/2} \int_0^1 (r^2 t)^{-d/2} e^{-1/(4t)} u(r^2 t) r^2 dt \\ &\geq c_{10} r^{-d+2} u(r^2) \int_0^1 t^{-d/2} e^{-1/(4t)} dt \geq c_{11} r^{-d} \phi(r^{-2})^{-1}, \end{aligned}$$

where in the last inequality we used the left inequality in (2.19). \square

We now have the asymptotic behaviors of the Green function and Lévy density of X as an immediate consequence of Lemmas 3.2–3.3.

Theorem 3.4 *Assume both **(H1)** and **(H2)**.*

(a) *It holds that*

$$J(x) \asymp |x|^{-d} \phi(|x|^{-2}), \quad \text{for all } x \neq 0. \quad (3.10)$$

(b) *If $d > 2(\delta_2 \vee \delta_4)$ then the process X is transient and it holds*

$$G(x) \asymp |x|^{-d} \phi(|x|^{-2})^{-1}, \quad \text{for all } x \neq 0. \quad (3.11)$$

We record a simple consequence of Theorem 3.4.

Corollary 3.5 *Assume **(H1)** and **(H2)**. There exists $c > 0$ such that $J(x) \leq cJ(2x)$ for all $x \neq 0$ and, if $d > 2(\delta_2 \vee \delta_4)$ then $G(x) \leq cG(2x)$ for all $x \neq 0$.*

Proof. By Theorem 3.4 there exists $c_1 > 0$ such that

$$\frac{J(x)}{J(2x)} \leq c_1 \frac{|x|^{-d} \phi(|x|^{-2})}{|2x|^{-d} \phi(|2x|^{-2})} = 2^d c_1 \frac{\phi(|x|^{-2})}{\phi(4^{-1}|x|^{-2})} \leq c_2, \quad x \neq 0,$$

where the last inequality follows from Lemma 2.1. The statement about G is proved in the same way. \square

We also record the following property of j : There exists $c > 0$ such that

$$j(r) \leq cj(r+1), \quad \text{for all } r \geq 1. \quad (3.12)$$

This is a consequence of the similar property of $\mu(t)$ and is proved in [17, Proposition 13.3.5]. By Corollary 3.5 we also have

$$j(r) \leq cj(2r), \quad r > 0. \quad (3.13)$$

Let $a > 0$. Recall that ϕ^a was defined by $\phi^a(\lambda) = \phi(\lambda a^{-2})/\phi(a^{-2})$. Let $S^a = (S^a)_{t \geq 0}$ be a subordinator with Laplace exponent ϕ^a independent of the Brownian motion W . Let $X^a = (X_t^a)_{t \geq 0}$ be defined by $X_t^a := W_{S_t^a}$. Then X^a is a rotationally invariant Lévy process with characteristic exponent

$$\phi^a(|\xi|^2) = \frac{\phi(a^{-2}|\xi|^2)}{\phi(a^{-2})}, \quad \xi \in \mathbb{R}^d.$$

This shows that X^a is identical in law to the process $\{a^{-1}X_{t/\phi(a^{-2})}\}_{t \geq 0}$.

The Lévy measure of X^a has a density $J^a(x) = j^a(|x|)$, where j^a is given by

$$\begin{aligned} j^a(r) &= \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu^a(t) dt = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \frac{a^2}{\phi(a^{-2})} \mu(a^2 t) dt \\ &= a^d \phi(a^{-2})^{-1} \int_0^\infty (4\pi s)^{-d/2} e^{-a^2 r^2/(4s)} \mu(s) ds = a^d \phi(a^{-2})^{-1} j(ar). \end{aligned} \quad (3.14)$$

In the second line we used (2.7) and in the third the change of variables $s = a^2 t$. Together with Theorem 3.4(a), (3.14) gives the following corollary.

Corollary 3.6 *Assume (H1) and (H2). There exists $c > 1$ such that for all $a > 0$ and all $x \neq 0$,*

$$c^{-1}|x|^{-d}\phi^a(|x|^{-2}) \leq J^a(x) \leq c|x|^{-d}\phi^a(|x|^{-2}). \quad (3.15)$$

Define

$$\Phi(r) := \frac{1}{\phi(r^{-2})}, \quad r > 0.$$

Then Φ is a strictly increasing function satisfying $\Phi(1) = 1$. In terms of Φ , we can rewrite (3.15) as

$$c^{-1} \frac{1}{|x|^d \Phi(x)} \leq J(x) \leq c \frac{1}{|x|^d \Phi(x)}. \quad (3.16)$$

Further, (2.9) reads as

$$a_5 \left(\frac{R}{r}\right)^{2(\delta_1 \wedge \delta_3)} \leq \frac{\Phi(R)}{\Phi(r)} \leq a_6 \left(\frac{R}{r}\right)^{2(\delta_2 \vee \delta_4)}, \quad 0 < r < R < \infty. \quad (3.17)$$

This implies that

$$\int_0^r \frac{s}{\Phi(s)} ds \leq \frac{a_6}{2(1 - \delta_2 \vee \delta_4)} \frac{r^2}{\Phi(r)}, \quad \text{for all } r > 0. \quad (3.18)$$

The last three displays show that the process X satisfies conditions (1.4), (1.13) and (1.14) from [12]. Therefore, by [12, Theorem 4.12], X satisfies the parabolic Harnack inequality, hence also the Harnack inequality. Thus the following global Harnack inequality is true. We recall that a function

$u : \mathbb{R}^d \rightarrow [0, \infty)$ is *harmonic* with respect to the process X in an open set D if for every relatively compact open set $B \subset D$ it holds that

$$u(x) = \mathbb{E}_x[u(X_{\tau_B})] \quad \text{for all } x \in B,$$

where $\tau_B = \inf\{t > 0 : X_t \notin B\}$ is the exit time of X from B .

Theorem 3.7 *Assume (H1) and (H2). There exists $c = c(\phi) > 0$ such that, for any $r > 0$, $x_0 \in \mathbb{R}^d$, and any function u which is nonnegative on \mathbb{R}^d and harmonic with respect to X in $B(x_0, r)$, we have*

$$u(x) \leq cu(y), \quad \text{for all } x, y \in B(x_0, r/2).$$

This theorem can be also deduced by using the approach in [17].

We now give some other consequences of (2.9) and Corollary 3.6.

Let $B = (B_t, \mathbb{P}_x)_{t \geq 0}$ be a one-dimensional Brownian motion independent of S^a and let $Z^a = (Z_t^a)_{t \geq 0}$ be the one-dimensional subordinate Brownian motion defined by $Z_t := B(S_t^a)$. Let χ^a be the Laplace exponent of the ladder height process of Z^a , v^a be the potential density of the ladder height process of Z^a , and $V^a(t) = \int_0^t v^a(s) ds$ the corresponding renewal function. It follows from [14, Corollary 9.7] that

$$\chi^a(\lambda) = \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\log(\phi^a(\lambda^2 \theta^2))}{1 + \theta^2} d\theta\right), \quad \text{for all } a, \lambda > 0.$$

Using this and the fact that $\phi^a(\lambda) = \phi(\lambda a^{-2})/\phi(a^{-2})$ we see that $\chi^a(\lambda) = \phi(a^{-2})^{-1/2} \chi(\lambda/a)$. This and the identity $\int_0^\infty e^{-\lambda t} v^a(t) dt = \frac{1}{\chi^a(\lambda)}$ imply that for all $a > 0$ and $r > 0$, $v^a(t) = a\sqrt{\phi(a^{-2})}v(at)$ so that

$$V^a(r) = \int_0^r a\sqrt{\phi(a^{-2})}v(as) ds = \sqrt{\phi(a^{-2})}V(ar), \quad \text{for all } a, r > 0. \quad (3.19)$$

Furthermore, by combining [18, Proposition 2.6] and [1, Proposition III.1], we get

$$V^a(r) \asymp \frac{1}{\sqrt{\phi^a(r^{-2})}} = \frac{\sqrt{\phi(a^{-2})}}{\sqrt{\phi(r^{-2}a^{-2})}}, \quad \text{for all } a, r > 0. \quad (3.20)$$

Lemma 3.8 *Assume (H1) and (H2).*

(a) *There exists $c_1 = c_1(\phi) > 0$ such that for any $r > 0$ and $x_0 \in \mathbb{R}^d$,*

$$\mathbb{E}_x[\tau_{B(x_0, r)}] \leq c_1 (\phi(r^{-2})\phi((r - |x - x_0|)^{-2}))^{-1/2} \leq c\phi(r^{-2})^{-1}, \quad x \in B(x_0, r).$$

(b) *There exists $c_2 = c_2(\phi) > 0$ such that for every $r > 0$ and every $x_0 \in \mathbb{R}^d$,*

$$\inf_{z \in B(x_0, r/2)} \mathbb{E}_z[\tau_{B(x_0, r)}] \geq c_2\phi(r^{-2})^{-1}.$$

Proof. (a) Using our (3.20) instead of [20, Proposition 3.2], the proof of (a) is exactly the same as that of [20, Lemma 4.4].

(b) Using (2.11), we can repeat the proofs of [17, Lemmas 13.4.1–13.4.2] to see that the conclusions of [17, Lemmas 13.4.1–13.4.2] are valid for all $r > 0$. The conclusion of [17, Lemma 13.4.2] for all $r > 0$ is the desired conclusion in (b). \square

The function $J(x, y)$ is the Lévy intensity of X . It determines a Lévy system for X , which describes the jumps of the process X : For any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of X) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left[\sum_{s \leq T} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s, y) J(X_s, y) dy \right) ds \right]. \quad (3.21)$$

For every open subset $D \subset \mathbb{R}^d$, we denote by X^D the subprocess of X killed upon exiting D . A subset D of \mathbb{R}^d is said to be Greenian (for X) if X^D is transient. For an open Greenian set $D \subset \mathbb{R}^d$, let $G_D(x, y)$ denote the Green function of the killed process X^D , and let $K_D(x, z)$ be the Poisson kernel of D with respect to X . Then, by (3.21),

$$K_D(x, z) = \int_{D^c} G_D(x, y) J(y, z) dy. \quad (3.22)$$

Proposition 3.9 *Assume (H1) and (H2). There exist $c_1 = c_1(\phi) > 0$ and $c_2 = c_2(\phi) > 0$ such that for every $r > 0$ and $x_0 \in \mathbb{R}^d$,*

$$K_{B(x_0, r)}(x, y) \leq c_1 j(|y - x_0| - r) (\phi(r^{-2}) \phi((r - |x - x_0|)^{-2}))^{-1/2} \quad (3.23)$$

$$\leq c_1 j(|y - x_0| - r) \phi(r^{-2})^{-1} \quad (3.24)$$

for all $(x, y) \in B(x_0, r) \times \overline{B(x_0, r)}^c$ and

$$K_{B(x_0, r)}(x_0, y) \geq c_2 j(|y - x_0|) \phi(r^{-2})^{-1}, \quad \text{for all } y \in \overline{B(x_0, r)}^c. \quad (3.25)$$

Proof. With Lemma 3.8 in hand, the proof of this proposition is exactly the same as that of [17, Lemma 13.4.10]. \square

Let $C_b^2(\mathbb{R}^d)$ be the collection of C^2 functions in \mathbb{R}^d which, along with their partial derivatives of order up to 2, are bounded. Recall that the infinitesimal generator \mathcal{L} of the process X is given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x + y) - f(x) - y \cdot \nabla f(x) \mathbf{1}_{\{|y| \leq \varepsilon\}}) J(y) dy \quad (3.26)$$

for every $\varepsilon > 0$ and $f \in C_b^2(\mathbb{R}^d)$.

Lemma 3.10 *There exists $c = c(\phi) > 0$ such that for all $0 < r \leq R < \infty$ and $f \in C_b^2(\mathbb{R}^d)$ with $0 \leq f \leq 1$,*

$$\sup_{x \in \mathbb{R}^d} |\mathcal{L}f_r(x)| \leq c\phi(r^{-2}) \left(1 + \sup_y \sum_{j,k} |(\partial^2 / \partial y_j \partial y_k) f(y)| \right) + 2 \int_{|z| > R} J(z) dz$$

where $f_r(y) := f(y/r)$.

Proof. With Lemma 2.2 in hand, the proof of this lemma is exactly the same as that of [20, Lemma 4.2]. \square

Similarly, by following the proof of [20, Lemma 4.10] and using Lemma 3.10 instead of [20, Lemma 4.2], we obtain the next result.

Lemma 3.11 *For every $a \in (0, 1)$, there exists $c = c(\phi, a) > 0$ such that for any $r > 0$ and any open set D with $D \subset B(0, r)$ we have*

$$\mathbb{P}_x(X_{\tau_D} \in B(0, r)^c) \leq c\phi(r^{-2}) \mathbb{E}_x \tau_D, \quad x \in D \cap B(0, ar).$$

With the preparation above, we can use Corollary 3.6, Theorem 3.7, Lemma 3.8, Proposition 3.9 and Lemma 3.11 and repeat the argument of [20, Section 5] to get the following global uniform boundary Harnack principle. We omit the details here since the proof would be a repetition of the argument in [20, Section 5]. Recall that a function $f : \mathbb{R}^d \rightarrow [0, \infty)$ is said to be regular harmonic in an open set U with respect to X if for each $x \in U$, $f(x) = \mathbb{E}_x[f(X(\tau_U))]$.

Theorem 3.12 *Assume (H1) and (H2). There exists $c = c(\phi, d) > 0$ such that for every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r > 0$ and any nonnegative functions u, v in \mathbb{R}^d which are regular harmonic in $D \cap B(z_0, r)$ with respect to X and vanish a.e. in $D^c \cap B(z_0, r)$, we have*

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)}, \quad \text{for all } x, y \in D \cap B(z_0, r/2).$$

Remark 3.13 Very recently, the boundary Harnack principle for (discontinuous) Markov processes (not necessarily Lévy processes) on metric measure state spaces is discussed in [6]. In particular in case of a Lévy processes in \mathbb{R}^d , the boundary Harnack principle in [6] can be stated as follows (see [6, Theorem 3.5 and Example 5.5]):

Let $x_0 \in \mathbb{R}^d$, $0 < r < R$, and let $U \subset B(x_0, R)$ be open. Suppose that Y is purely discontinuous Lévy process satisfying [6, (2.10) and (5.2)]. There exists $c_{(1.1)} = c_{(1.1)}(x_0, r, R)$ such that if f, g are nonnegative functions on \mathbb{R}^d which are regular harmonic in U with respect to Y and vanish in $B(x_0, R) \setminus D$,

$$f(x)g(y) \leq c_{(1.1)} f(y)g(x), \quad x, y \in B(x_0, r). \quad (3.27)$$

Condition [6, (5.2)] holds for X by our (3.10). If $d > 2(\delta_2 \vee \delta_4)$, [6, (2.10)] holds for X by our (3.11). Comparing with our Theorem 3.12, the comparison constant $c_{(1.1)}$ in (3.27) depends on x_0, r and R

in general. It requires more accurate estimates to obtain the scale-invariant version of the boundary Harnack principle, that is, $c_{(1.1)}$ is independent of x_0 and depends on r and R only through r/R . In fact, in [6, Example 5.5], it is claimed, without proof, that one can prove the scale-invariant versions of the boundary Harnack inequalities in [16, 20] by checking all dependencies of $c_{(1.1)}$ in [6, (3.10) and (3.11)]. However, to accomplish this, one needs to estimate the Green function in order to check Assumption D in [6]. Especially when X is recurrent, to check Assumption D in [6] one may need upper bounds on the α -potential kernel with $\alpha > 0$ (see [6, Proposition 2.3 and the end of the second paragraph of Example 5.5]), which is not discussed in that paper.

4 Boundary Harnack principle with explicit decay rate

Let D be an open set in \mathbb{R}^d . For $x \in \mathbb{R}^d$, let $\delta_{\partial D}(x)$ denote the Euclidean distance between x and ∂D . Recall that for any $x \in D$, $\delta_D(x)$ is the Euclidean distance between x and D^c .

In this section we will assume that D satisfies the following types of ball conditions with radius R :

- (i) *uniform interior ball condition*: for every $x \in D$ with $\delta_D(x) < R$ there exists $z_x \in \partial D$ so that

$$|x - z_x| = \delta_{\partial D}(x) \quad \text{and} \quad B(x_0, R) \subset D, \quad x_0 := z_x + R \frac{x - z_x}{|x - z_x|};$$

- (ii) *uniform exterior ball condition*: D is equal to the interior of \bar{D} and for every $y \in \mathbb{R}^d \setminus \bar{D}$ with $\delta_{\partial D}(y) < R$ there exists $z_y \in \partial D$ so that

$$|y - z_y| = \delta_{\partial D}(y) \quad \text{and} \quad B(y_0, R) \subset \mathbb{R}^d \setminus D, \quad y_0 := z_y + R \frac{y - z_y}{|y - z_y|}.$$

The goal of this section is to obtain a global uniform boundary Harnack principle with explicit decay rate in open sets in \mathbb{R}^d satisfying the interior and exterior ball conditions with radius $R > 0$. This boundary Harnack principle is global in the sense that it holds for all $R > 0$ and the comparison constant does not depend on R , and it is uniform in the sense that it holds for all balls with radii $r \leq R$ and the comparison constant depends neither on D nor on r . Throughout the section we assume that **(H1)** and **(H2)** hold.

Let $Z = (Z_t)_{t \geq 0}$ be the one-dimensional subordinate Brownian motion defined by $Z_t := W^d(S_t)$. Recall that the potential measure of the ladder height process of Z is denoted by V and its density by v . We also use V to denote the renewal function of the ladder height process of Z . We use the notation $\mathbb{H} := \{x = (x_1, \dots, x_{d-1}, x_d) := (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ for the half-space.

Define $w(x) := V((x_d)^+)$. Note that $Z_t := W^d(S_t)$ has a transition density. Thus, using [27, Theorem 2], the proof of the next result is the same as that of [18, Theorem 4.1]. So we omit the proof.

Theorem 4.1 *The function w is harmonic in \mathbb{H} with respect to X and, for any $r > 0$, regular harmonic in $\mathbb{R}^{d-1} \times (0, r)$ with respect to X .*

Proposition 4.2 *There exists $c > 0$ such that for all $r > 0$, we have*

$$\sup_{x \in \mathbb{R}^d: 0 < x_d \leq 8r} \int_{B(x,r)^c \cap \mathbb{H}} w(y)j(|x-y|) dy \leq c\sqrt{\phi(r^{-2})}.$$

Proof. Without loss of generality, we assume $\tilde{x} = 0$. By the substitution $y = x + z$ we see that

$$\int_{B(x,r)^c \cap \mathbb{H}} w(y)j(|x-y|) dy = \int_{B(0,r)^c \cap \{z_d > -x_d\}} w(z+x)j(z) dz = \int_{B(0,r)^c \cap \{z_d > -x_d\}} V(z_d+x_d)j(z) dz.$$

The last integral is an increasing function of x_d implying that the supremum is attained for $x_d = 8r$. To conclude, take $x = (\tilde{0}, 8r)$. Then by Theorem 4.1 and (3.25),

$$V(8r) = w(x) = \int_{B(x,r)^c \cap \mathbb{H}} w(y)K_{B(x,r)}(x,y) dy \geq c_2\phi(r^{-2})^{-1} \int_{B(x,r)^c \cap \mathbb{H}} w(y)j(|x-y|) dy.$$

Hence,

$$\int_{B(x,r)^c \cap \mathbb{H}} w(y)j(|x-y|) dy \leq c_3V(8r)\phi(r^{-2}) \leq c_4\phi(r^{-2})^{1/2}.$$

□

For a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and $x \in \mathbb{R}^d$ we define

$$\mathcal{A}f(x) := \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |x-y| > \varepsilon\}} (f(y) - f(x))j(|y-x|) dy,$$

and use $\mathfrak{D}_x(\mathcal{A})$ to denote the family of all functions f such that $\mathcal{A}f(x)$ exists and is finite. It is well known that $C_c^2(\mathbb{R}^d) \subset \mathfrak{D}_x(\mathcal{A})$ for every $x \in \mathbb{R}^d$ and that, by the rotational symmetry of X , \mathcal{A} restricted to $C_c^2(\mathbb{R}^d)$ coincides with the infinitesimal generator \mathcal{L} of X which is given in (3.26).

Using [17, Corollary 13.3.8], Theorem 4.1, (3.12) and (3.13), the proof of the next result is the same as these of [18, Proposition 4.3] and [21, Theorem 3.4], so we omit the proof.

Theorem 4.3 *For any $x \in \mathbb{H}$, $w \in \mathfrak{D}_x(\mathcal{A})$ and $\mathcal{A}w(x) = 0$.*

Before we prove our main technical lemma, we first do some preparations.

Lemma 4.4 *If $f, g : (0, \infty) \rightarrow (0, \infty)$ are non-increasing, then for any $M > 0$ and any $x : [0, M] \rightarrow \mathbb{R}$ we have*

$$\int_0^M \int_0^M f(s)g(r + |s - x(r)|) dr ds \leq 2 \int_0^{3M/2} F(u)g(u) du,$$

where $F(u) = \int_0^u f(s) ds$.

Proof. Without loss of generality we may assume that g is right continuous. Then the inverse $g^{-1}(\lambda) := \sup\{x : g(x) \geq \lambda\}$ has the property that $g(x) \geq \lambda$ if and only if $x \leq g^{-1}(\lambda)$. Let $h(s) := g(r + |s - x(r)|)$, $s \in [0, M]$. Then

$$|\{s \in [0, M] : h(s) > \lambda\}| = |\{s \in [0, M] : g(r + |s - x(r)|) > \lambda\}|$$

$$\begin{aligned}
&= |\{s \in [0, M] : r + |s - x(r)| \leq g^{-1}(\lambda)\}| \\
&= |\{s \in [0, M] : |s - x(r)| \leq (g^{-1}(\lambda) - r)^+\}| \\
&\leq 2(g^{-1}(\lambda) - r)^+ .
\end{aligned}$$

Hence, the rearrangement $\{s \in [0, M] : h(s) > \lambda\}^*$ is contained in $[0, 2(g^{-1}(\lambda) - r)^+]$. Further note that $s \leq 2(g^{-1}(\lambda) - r)^+$ is equivalent to $r + \frac{s}{2} \leq g^{-1}(\lambda)$, which in turn is equivalent to $g(r + s/2) \geq \lambda$. Therefore the non-increasing rearrangement of h satisfies

$$\begin{aligned}
h^*(s) &= \int_0^\infty \mathbf{1}_{\{h > \lambda\}^*}(s) d\lambda \leq \int_0^\infty \mathbf{1}_{[0, 2(g^{-1}(\lambda) - r)^+]}(s) d\lambda = \int_0^\infty \mathbf{1}_{[0, g(r+s/2)]}(\lambda) d\lambda \\
&= \int_0^{g(r+s/2)} d\lambda \leq g(r + s/2) .
\end{aligned}$$

Therefore, by the rearrangement inequality (see [24, Chapter 3]),

$$\int_0^M f(s)g(r + |s - x(r)|) ds = \int_0^M f(s)h(s) ds \leq \int_0^M f(s)h^*(s) ds \leq \int_0^M f(s)g(r + s/2) ds .$$

Finally,

$$\begin{aligned}
&\int_0^M \int_0^M f(s)g(r + |s - x(r)|) dr ds \leq \int_0^M \int_0^M f(s)g(r + s/2) dr ds \\
&\leq \int_0^{3M} \int_{s/2}^{3M/2} f(s)g(u) du ds = \int_0^{3M/2} \left(\int_0^{2u} f(s) ds \right) g(u) du \\
&= \int_0^{3M/2} F(2u)g(u) du \leq 2 \int_0^{3M/2} F(u)g(u) du .
\end{aligned}$$

□

Lemma 4.5 *Let D be an open set in \mathbb{R}^d satisfying the interior and exterior ball conditions with radius 1. Fix $x \in D$ with $\delta_D(x) < 1/8$ and let $x_0 \in \partial D$ be such that $\delta_D(x) = |x - x_0|$ and CS_{x_0} be a coordinate system such that $x = (\tilde{0}, x_d)$ and $x_d > 0$. There exists $c > 0$ independent of D and x such that for every positive non-increasing functions ν and ϑ on $(0, \infty)$ and $\Theta(r) = \int_0^r \vartheta(s) ds$*

$$\int_{B(x, 1/8)} |\Theta(\delta_D(z)) - \Theta(\delta_{H^+}(z))| \frac{\nu(|z - x|)}{|z - x|^d} dz \leq c \int_0^1 \Theta(2r)\nu(r) dr , \quad (4.1)$$

where $H^+ := \{z = (\tilde{z}, z_d) \text{ in } CS_{x_0} : z_d > 0\}$.

Proof. In this proof we assume that $d \geq 2$, the case $d = 1$ being simpler. By the interior and exterior ball conditions with radius 1,

$$\{z = (\tilde{z}, z_d) \in B(0, 1/2) : z_d \geq \psi(|\tilde{z}|\}) \subset B(0, 1/2) \cap D \subset \{z = (\tilde{z}, z_d) \in B(0, 1/2) : z_d \geq -\psi(|\tilde{z}|\})\},$$

where $\psi(r) := 1 - \sqrt{1 - r^2}$.

Define

$$\begin{aligned} A &:= \{z = (\tilde{z}, z_d) \in (D \cup H^+) \cap B(x, 1/8) : -\psi(|\tilde{z}|) \leq z_d < \psi(|\tilde{z}|)\}, \\ F &:= \{z \in B(x, \frac{1}{8}) : z_d > \psi(|\tilde{z}|)\}. \end{aligned}$$

Then

$$\begin{aligned} &\int_{B(x, 1/8)} |\Theta(\delta_D(z)) - \Theta(\delta_{H^+}(z))| \frac{\nu(|z - x|)}{|z - x|^d} dz \\ &\leq \int_A \Theta(\delta_D(z)) + \Theta(\delta_{H^+}(z)) \frac{\nu(|z - x|)}{|z - x|^d} dz + \int_F |\Theta(\delta_D(z)) - \Theta(\delta_{H^+}(z))| \frac{\nu(|z - x|)}{|z - x|^d} dz =: I + II. \end{aligned}$$

Let $E = B((\tilde{0}, -1), 1)^c$. Then

$$I \leq 2 \int_0^{\frac{1}{8}} \int_{|\tilde{z}|=r} \mathbf{1}_{\{z=(\tilde{z}, z_d): |\tilde{z}|=r, -\psi(r) \leq z_d < \psi(r)\}}(z) \Theta(\delta_E(z)) \frac{\nu(\sqrt{r^2 + |z_d - x_d|^2})}{(r^2 + |z_d - x_d|^2)^{d/2}} m_{d-1}(dz) dr,$$

where m_{d-1} is the surface measure, that is, the $(d-1)$ -dimensional Lebesgue measure. Noting that $1 - \sqrt{1 - |\tilde{z}|^2} \leq |\tilde{z}|^2 = r^2$ for $|\tilde{z}| = r$, we obtain

$$m_{d-1}(\{z = (\tilde{z}, z_d) : |\tilde{z}| = r, -1 + \sqrt{1 - r^2} \leq z_d < 1 - \sqrt{1 - r^2}\}) \leq c_1 r^d \quad \text{for } r \leq \frac{1}{8}.$$

Since Θ is increasing and $1 - \sqrt{1 - |\tilde{z}|^2} \leq |\tilde{z}|^2 \leq |\tilde{z}|$ we deduce $\Theta(\delta_E(z)) \leq \Theta(2\psi(|\tilde{z}|)) \leq \Theta(2|\tilde{z}|)$. By using that ν is decreasing we get

$$\begin{aligned} I &\leq \int_0^{\frac{1}{4}} \int_{|\tilde{z}|=r} \mathbf{1}_{\{z: |\tilde{z}|=r, -1 + \sqrt{1 - r^2} \leq z_d < 1 - \sqrt{1 - r^2}\}}(z) \Theta(2r) \nu(r) r^{-d} m_{d-1}(dz) dr \\ &\leq c_2 \int_0^{\frac{1}{4}} \Theta(2r) \nu(r) dr. \end{aligned}$$

In order to estimate II , we consider two cases. First, if $0 < z_d = \delta_{H^+}(z) \leq \delta_D(z)$, then using the exterior ball condition and the fact that $\delta_D(z)$ is smaller than the vertical distance from z to the exterior of the ball which is equal to $z_d + 1 - \sqrt{1 - |\tilde{z}|^2} \leq z_d + |\tilde{z}|^2$, we get that

$$\Theta(\delta_D(z)) - \Theta(\delta_{H^+}(z)) \leq \Theta(z_d + |\tilde{z}|^2) - \Theta(z_d) = \int_{z_d}^{z_d + |\tilde{z}|^2} \vartheta(t) dt \leq |\tilde{z}|^2 \vartheta(z_d), \quad (4.2)$$

since ϑ is decreasing.

If $z_d = \delta_{H^+}(z) > \delta_D(z)$ and $z \in F$, using the fact that $\delta_D(z)$ is greater than or equal to the distance between z and the graph of ψ (by the interior ball condition) and

$$z_d - 1 + \sqrt{|\tilde{z}|^2 + (1 - z_d)^2} = \frac{|\tilde{z}|^2}{\sqrt{|\tilde{z}|^2 + (1 - z_d)^2 + (1 - z_d)}} \leq \frac{|\tilde{z}|^2}{2(1 - z_d)} \leq |\tilde{z}|^2, \quad \forall z \in F,$$

we obtain

$$\Theta(\delta_{H^+}(z)) - \Theta(\delta_D(z)) \leq \int_{1-\sqrt{|\tilde{z}|^2+(1-z_d)^2}}^{z_d} \vartheta(t) dt \leq |\tilde{z}|^2 \vartheta\left(1 - \sqrt{|\tilde{z}|^2 + (1-z_d)^2}\right). \quad (4.3)$$

By (4.2) and (4.3),

$$II \leq \int_F |\tilde{z}|^2 \left(\vartheta(z_d) \vee \vartheta(1 - \sqrt{|\tilde{z}|^2 + (1-z_d)^2}) \right) \frac{\nu(|z-x|)}{|z-x|^d} dz.$$

Since

$$F \subset \{z = (\tilde{z}, z_d) \in \mathbb{R}^d : |\tilde{z}| < 1/8 \text{ and } \psi(r) < z_d \leq 1/4\},$$

switching to polar coordinates for \tilde{z} and reversing the order of integration we get

$$II \leq c_3 \int_0^{\frac{1}{8}} \int_{\psi(r)}^{\psi(r)+1/4} (\vartheta(z_d) \vee \vartheta(1 - \sqrt{r^2 + (1-z_d)^2})) \frac{r^d \nu(\sqrt{r^2 + |z_d - x_d|^2})}{(r^2 + |z_d - x_d|^2)^{d/2}} dz_d dr.$$

Writing $s = z_d - \psi(r)$ gives

$$II \leq c_3 \int_0^{\frac{1}{8}} \int_0^{1/4} (\vartheta(\psi(r) + s) \vee \vartheta(1 - \sqrt{1 - 2s\sqrt{1-r^2} + s^2})) r^d \frac{\nu(\sqrt{r^2 + |s + \psi(r) - x_d|^2})}{(r^2 + |s + \psi(r) - x_d|^2)^{d/2}} ds dr.$$

Since

$$1 - \sqrt{1 - 2s\sqrt{1-r^2} + s^2} \geq (2s\sqrt{1-r^2} - s^2)/2 = (2\sqrt{1-r^2} - s)s/2 \geq s/2$$

and $\sqrt{r^2 + |z_d - x_d|^2} \geq (r + |z_d - x_d|)/2$, we have

$$II \leq c_3 \int_0^{\frac{1}{8}} \int_0^{1/4} \vartheta(s/2) \nu((r + |s + \psi(r) - x_d|)/2) ds dr.$$

Thus by Lemma 4.4,

$$II \leq 2c_3 \int_0^{3/8} \Theta(r/2) \nu(r/2) dr \leq c_4 \int_0^1 \Theta(r) \nu(r) dr.$$

□

Proposition 4.6 *Let D be an open set in \mathbb{R}^d satisfying the interior and exterior ball conditions with radius R . Fix $Q \in \partial D$ and define*

$$h(y) := V(\delta_D(y)) \mathbf{1}_{D \cap B(Q, R/2)}(y).$$

There exists $C_1 = C_1(\phi) > 0$ independent of Q and R (and D) such that $h \in \mathfrak{D}_x(\mathcal{A})$ for every $x \in D \cap B(Q, R/8)$ and

$$|\mathcal{A}h(x)| \leq C_1 \sqrt{\phi(R^{-2})} \quad \text{for all } x \in D \cap B(Q, R/8). \quad (4.4)$$

Proof. In this proof we assume that $d \geq 2$, the case $d = 1$ being simpler. We fix $x \in D \cap B(Q, R/8)$ and let $x_0 \in \partial D$ be such that $\delta_D(x) = |x - x_0|$. We may assume, without loss of generality, that $x_0 = 0$, $x = (\tilde{0}, x_d)$ and $x_d > 0$. Note that, since $|y - Q| \leq |y - x| + |x - Q| \leq R/4$ for $y \in B(x, R/8)$, we have

$$B(x, R/8) \cap D \subset B(Q, R/4) \cap D. \quad (4.5)$$

Let $h_x(y) := V(\delta_{\mathbb{H}}(y))$. Note that $h_x(x) = h(x)$. Since $\delta_{\mathbb{H}}(y) = (y_d)^+$, it follows from Theorem 4.3 that $\mathcal{A}h_x$ is well defined in \mathbb{H} and

$$\mathcal{A}h_x(y) = 0, \quad \forall y \in \mathbb{H}. \quad (4.6)$$

We show now that $\mathcal{A}(h - h_x)(x)$ is well defined. For each small $\varepsilon > 0$ we have that

$$\begin{aligned} & \int_{\{y \in D: |y-x| > \varepsilon\}} |h(y) - h_x(y)| j(|y-x|) dy \\ & \leq \int_{B(x, \frac{R}{8})^c} (h(y) + h_x(y)) j(|y-x|) dy + \int_{B(x, \frac{R}{8})} |h(y) - h_x(y)| j(|y-x|) dy =: I_1 + I_2. \end{aligned}$$

We claim that

$$I_1 + I_2 \leq C_1 \sqrt{\phi(R^{-2})} \quad (4.7)$$

for some positive constant $C_1 > 0$. Since (4.7) implies that

$$\begin{aligned} & \mathbf{1}_{\{y \in D \cup \mathbb{H}: |y-x| > \varepsilon\}} |h(y) - h_x(y)| j(|y-x|) \\ & \leq \mathbf{1}_{B(x, \frac{R}{8})^c} (h(y) + h_x(y)) j(|y-x|) + \mathbf{1}_{B(x, \frac{R}{8})} |h(y) - h_x(y)| j(|y-x|) \in L^1(\mathbb{R}^d), \end{aligned}$$

by the dominated convergence theorem the limit

$$\lim_{\varepsilon \downarrow 0} \int_{\{y \in D \cup \mathbb{H}: |y-x| > \varepsilon\}} (h(y) - h_x(y)) j(|y-x|) dy$$

exists, and hence $\mathcal{A}(h - h_x)(x)$ is well defined and $|\mathcal{A}(h - h_x)(x)| \leq C_1 \sqrt{\phi(R^{-2})}$. By linearity and (4.6), we get that $\mathcal{A}h(x)$ is well defined and $|\mathcal{A}h(x)| \leq C_1 \sqrt{\phi(R^{-2})}$. Therefore, it remains to prove (4.7).

Since $h(y) = 0$ for $y \in B(Q, R)^c$, it follows that

$$\begin{aligned} I_1 & \leq \int_{B(x, \frac{R}{8})^c} V(y_d) j(|y-x|) dy + V(R) \int_{B(x, \frac{R}{8})} j(|y-x|) dy \\ & \leq \sup_{z \in \mathbb{R}^d: 0 < z_d < R} \int_{B(z, \frac{R}{8})^c \cap \mathbb{H}} V(y_d) j(|z-y|) dy + V(R) \int_{B(0, \frac{R}{8})^c} j(|y|) dy \\ & =: K_1 + V(R) K_2. \end{aligned}$$

By Proposition 4.2 we have that $K_1 \leq c \sqrt{\phi(R^{-2})}$. Moreover, by Lemma 3.1, (3.20) and (2.11),

$$V(R) \int_{B(0, \frac{R}{8})^c} j(|y|) dy \leq c_2 V(R) \int_{R/8}^{\infty} r^{-1} \phi(r^{-2}) dr \leq c_3 V(R) \phi(R^{-2}/64) \leq c_4 \sqrt{\phi(R^{-2})}.$$

For I_2 , we use scaling. Let $x^R = R^{-1}x$ and $\widehat{D} := \{z : Rz \in D\}$. Then by (3.14) and (3.19) we have

$$\begin{aligned} I_2 &= \int_{\{y \in B(x, \frac{R}{8}) : y_d > R - \sqrt{R^2 - |y|^2}\}} |V(\delta_D(y)) - V(\delta_{\mathbb{H}}(y))| j(|y - x|) dy \\ &= \sqrt{\phi(R^{-2})} \int_{B(x^R, 1/8)} |V^R(\delta_{\widehat{D}}(z)) - V^R(\delta_{\mathbb{H}}(z))| j^R(|z - x^R|) dz =: \sqrt{\phi(R^{-2})} \widehat{I}_2 . \end{aligned}$$

Using (3.15) and (3.20),

$$\widehat{I}_2 \leq c_5 \int_{B(x^R, 1/8)} |V^R(\delta_{\widehat{D}}(z)) - V^R(\delta_{\mathbb{H}}(z))| \frac{\phi^R(|z - x^R|^{-2})}{|z - x^R|^d} dz .$$

Finally by Lemma 2.1, Lemma 4.5, (3.20) and (2.6),

$$\widehat{I}_2 \leq c_6 \int_0^1 V^R(2r) \phi^R(r^{-2}) \leq c_7 \int_0^1 \sqrt{\frac{\phi(R^{-2}(2r)^{-2})}{\phi(R^{-2})}} dr \leq c_8 \int_0^1 r^{-(\delta_2 \vee \delta_4)} dr < \infty .$$

□

Theorem 4.7 (a) *There exist $a = a(\phi) \in (0, 1)$ and $c_1 = c_1(\phi) > 0$ such that for every open set D satisfying the interior and exterior ball conditions with radius $R > 0$, any $r \leq aR$ and $Q \in \partial D$,*

$$\mathbb{E}_x [\tau_{D \cap B(Q, r)}] \leq c_1 V(r) V(\delta_D(x)), \quad \text{for every } x \in D \cap B(Q, r) . \quad (4.8)$$

(b) *There exists $c_2 = c_2(\phi) > 0$ such that for every open set D satisfying the interior and exterior ball conditions with radius $R > 0$, $r \in (0, R]$, $Q \in \partial D$ and any nonnegative function u in \mathbb{R}^d which is harmonic in $D \cap B(Q, r)$ with respect to X and vanishes continuously on $D^c \cap B(Q, r)$, we have*

$$\frac{u(x)}{u(y)} \leq c_2 \sqrt{\frac{\phi(\delta_D(y)^{-2})}{\phi(\delta_D(x)^{-2})}} , \quad \text{for every } x, y \in D \cap B(Q, \frac{r}{2}) . \quad (4.9)$$

Proof. Without loss of generality, we assume $Q = 0$. Define

$$h(y) := V(\delta_D(y)) \mathbf{1}_{B(0, R/2) \cap D}(y) .$$

Let f be a non-negative smooth radial function such that $f(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} f(y) dy = 1$. For $k \geq 1$, define $f_k(y) = 2^{kd} f(2^k y)$ and

$$h^{(k)}(z) := (f_k * h)(z) := \int_{\mathbb{R}^d} f_k(y) h(z - y) dy ,$$

and for $\lambda \geq 8$ let $B_k^\lambda := \{y \in D \cap B(0, \lambda^{-1}R) : \delta_{D \cap B(0, \lambda^{-1}R)}(y) \geq 2^{-k}\}$. Since $h^{(k)}$ is a C^∞ function, $\mathcal{A}h^{(k)}$ is well defined everywhere. Then by the same argument as that in [18, Lemma 4.5], we have for large k

$$-C_1 \sqrt{\phi(R^{-2})} \leq \mathcal{A}h^{(k)} \leq C_1 \sqrt{\phi(R^{-2})} \quad \text{on } B_k^\lambda , \quad (4.10)$$

where C_1 is the constant from Proposition 4.6.

Since $h^{(k)}$ is in $C_c^\infty(\mathbb{R}^d)$ and that \mathcal{A} restricted to C_c^∞ coincides with the infinitesimal generator \mathcal{L} of the process X , by Dynkin's formula, with $\sigma(\lambda, k) := \tau_{B_k^\lambda}$

$$\mathbb{E}_x \int_0^{\sigma(\lambda, k)} \mathcal{A}h^{(k)}(X_t)dt = \mathbb{E}_x[h^{(k)}(X_{\sigma(\lambda, k)})] - h^{(k)}(x). \quad (4.11)$$

Using (4.10)–(4.11) and then letting $k \rightarrow \infty$ we obtain that for all $\lambda \geq 8$ and $x \in D \cap B(0, \lambda^{-1}R)$,

$$V(\delta_D(x)) = h(x) \geq \mathbb{E}_x \left[h \left(X_{\tau_{D \cap B(0, \lambda^{-1}R)}} \right) \right] - C_1 \sqrt{\phi(R^{-2})} \mathbb{E}_x [\tau_{D \cap B(0, \lambda^{-1}R)}] \quad (4.12)$$

and

$$V(\delta_D(x)) - C_1 \sqrt{\phi(R^{-2})} \mathbb{E}_x [\tau_{D \cap B(0, \lambda^{-1}R)}] \leq \mathbb{E}_x \left[h \left(X_{\tau_{D \cap B(0, \lambda^{-1}R)}} \right) \right]. \quad (4.13)$$

Since

$$j(|y - z|) \geq j(|y| + |z|) \geq j(2|y|) \geq c_1 j(|y|), \quad \forall (z, y) \in (D \cap B(0, \lambda^{-1}R)) \times B(0, \lambda^{-1}R)^c,$$

we get

$$\begin{aligned} & \int_{(B(0, R) \setminus B(0, \lambda^{-1}R)) \cap D} \int_{D \cap B(0, \lambda^{-1}R)} G_{D \cap B(0, \lambda^{-1}R)}(x, z) j(|z - y|) dz V(\delta_D(y)) dy \\ & \geq c_1 \mathbb{E}_x [\tau_{D \cap B(0, \lambda^{-1}R)}] \int_{(B(0, R) \setminus B(0, \lambda^{-1}R)) \cap D} j(|y|) V(\delta_D(y)) dy. \end{aligned} \quad (4.14)$$

The remainder of the proof is written for $d \geq 2$. The interpretation in the case $d = 1$ is obvious.

By the interior ball condition with radius R , we may assume, without loss of generality, that

$$B(0, R) \cap D \supset \{y = (\tilde{y}, y_d) \in B(0, R) : R - \sqrt{R^2 - |\tilde{y}|^2} < y_d\}.$$

For $y \in B(0, R)$ with $2|\tilde{y}| < y_d$ we have

$$\begin{aligned} \delta_D(y) & \geq R - \sqrt{|\tilde{y}|^2 + (R - y_d)^2} \geq R - \sqrt{R^2 - 2Ry_d + (5/4)y_d^2} \\ & \geq (2Ry_d - (5/4)y_d^2)/(2R) = y_d(1 - (5/8)y_d/R) \geq \frac{3y_d}{8} \geq \frac{3|y|}{4\sqrt{5}}. \end{aligned} \quad (4.15)$$

Thus, by changing into polar coordinates and using (4.15), we have

$$\begin{aligned} \int_{(B(0, R) \setminus B(0, \lambda^{-1}R)) \cap D} j(|y|) V(\delta_D(y)) dy & \geq \int_{\{(\tilde{y}, y_d) : 2|\tilde{y}| < y_d, \lambda^{-1}R < |y| < R\}} j(|y|) V\left(\frac{3|y|}{4\sqrt{5}}\right) dy \\ & \geq c_2 \int_{\lambda^{-1}R}^R j(r) V\left(\frac{3r}{4\sqrt{5}}\right) r^{d-1} dr. \end{aligned}$$

By Lemma 2.1, (2.3), (3.10) and (3.20), we have that for $r < R$,

$$j(r) V\left(\frac{3r}{4\sqrt{5}}\right) r^{d-1} \geq c_3 r^{-3} \frac{\phi'(r^{-2})}{\phi(r^{-2})^{1/2}} = c_4 \frac{d}{dr} (-\phi(r^{-2})^{1/2}).$$

Thus

$$\int_{(B(0,R) \setminus B(0,\lambda^{-1}R)) \cap D} j(|y|)V(\delta_D(y))dy \geq c_4(\phi(R^{-2}\lambda^2)^{1/2} - \phi(R^{-2})^{1/2}). \quad (4.16)$$

Now combining (4.14) and (4.16), we get

$$\mathbb{E}_x \left[h \left(X_{\tau_{D \cap B(0,\lambda^{-1}R)}} \right) \right] \geq c_5 \left(\phi(R^{-2}\lambda^2)^{1/2} - \phi(R^{-2})^{1/2} \right) \mathbb{E}_x \left[\tau_{D \cap B(0,\lambda^{-1}R)} \right]. \quad (4.17)$$

Thus by (4.12) for every $x \in D \cap B(0,\lambda^{-1}R)$,

$$V(\delta_D(x)) \geq \left(c_5\phi(R^{-2}\lambda^2)^{1/2} - (c_5 + C_1)\phi(R^{-2})^{1/2} \right) \mathbb{E}_x[\tau_{D \cap B(0,\lambda^{-1}R)}]. \quad (4.18)$$

Without loss of generality we assume $a_5 < 1$ (the constant in (2.6)). Let $\lambda_0 := (2a_5^{-1/2}(1 + C_1/c_5))^{1/(\delta_1 \wedge \delta_3)} \vee 8$. Recall from (2.6) that

$$\phi(t) \geq a_5 s^{\delta_1 \wedge \delta_3} \phi(s^{-1}t) \quad \text{for every } s \geq 1 \quad \text{and } t > 0. \quad (4.19)$$

Applying this with $t = \lambda_0^2 R^{-2}$ and $s = \lambda_0^2 \geq 1$, we get that for $\lambda \geq \lambda_0$,

$$\phi(R^{-2}\lambda^2) \geq \phi(R^{-2}\lambda_0^2) \geq a_5(\lambda_0^2)^{\delta_1 \wedge \delta_3} \phi(R^{-2}) \geq 4(1 + C_1/c_5)^2 \phi(R^{-2}).$$

Hence for every $\lambda \geq \lambda_0$

$$c_5\phi(R^{-2}\lambda^2)^{1/2} - (c_5 + C_1)\phi(R^{-2})^{1/2} \geq \frac{c_5}{2}\phi(R^{-2}\lambda^2)^{1/2}. \quad (4.20)$$

Combining (4.18) and (4.20), we have proved part (a) of the theorem with $a = \lambda_0^{-1}$.

To prove (b), we first consider estimates on h first. Combining (4.13) and (4.18), we get

$$\begin{aligned} \mathbb{E}_x \left[h \left(X_{\tau_{D \cap B(0,\lambda^{-1}R)}} \right) \right] &\geq V(\delta_D(x)) - C_1 \sqrt{\phi(R^{-2})} \mathbb{E}_x \left[\tau_{D \cap B(0,\lambda^{-1}R)} \right] \\ &\geq V(\delta_D(x)) \left(1 - C_1 \frac{\sqrt{\phi(R^{-2})}}{c_5\phi(R^{-2}\lambda^2)^{1/2} - (c_5 + C_1)\phi(R^{-2})^{1/2}} \right). \end{aligned} \quad (4.21)$$

Let

$$\lambda_1 := ((3C_1 + c_5)/(c_5 a_5^{1/2}))^{1/(\delta_1 \wedge \delta_3)} \vee \lambda_0.$$

Applying (4.19) with $t = \lambda_1^2 R^{-2}$ and $s = \lambda_1^2 \geq 1$, we get that for $\lambda \geq \lambda_1$,

$$\phi(R^{-2}\lambda^2) \geq \phi(R^{-2}\lambda_1^2) \geq a_5(\lambda_1^2)^{\delta_1 \wedge \delta_3} \phi(R^{-2}) = (3C_1 + c_5)^2 c_5^{-2} \phi(R^{-2}).$$

Hence for every $\lambda \geq \lambda_1$,

$$2C_1 \sqrt{\phi(R^{-2})} \leq c_5\phi(R^{-2}\lambda^2)^{1/2} - (c_5 + C_1)\phi(R^{-2})^{1/2}. \quad (4.22)$$

Combining (4.21)–(4.22), we have for every $x \in D \cap B(0,\lambda^{-1}R)$,

$$\mathbb{E}_x \left[h \left(X_{\tau_{D \cap B(0,\lambda^{-1}R)}} \right) \right] \geq \frac{1}{2} V(\delta_D(x)). \quad (4.23)$$

Moreover, by (4.12), (4.18) and (4.20), for every $\lambda \geq \lambda_0$ and $x \in D \cap B(0, \lambda^{-1}R)$,

$$\begin{aligned} \mathbb{E}_x \left[h \left(X_{\tau_{D \cap B(0, \lambda^{-1}R)}} \right) \right] &\leq V(\delta_D(x)) + C_1 \sqrt{\phi(R^{-2})} \mathbb{E}_x \left[\tau_{D \cap B(0, \lambda^{-1}R)} \right] \\ &\leq V(\delta_D(x)) \left(1 + \frac{2C_1}{c_5} \sqrt{\frac{\phi(R^{-2})}{\phi(R^{-2}\lambda^2)}} \right) \leq (1 + 2C_1/c_5)V(\delta_D(x)). \end{aligned} \quad (4.24)$$

Now we assume $r \in (0, R]$. Let u be a nonnegative function in \mathbb{R}^d which is harmonic in $D \cap B(0, r)$ with respect to X and vanishes continuously on $D^c \cap B(0, r)$. Note that $0 < r/(2\lambda_1) < r/\lambda_1 = R(R\lambda_1/r)^{-1}$. Thus by applying Theorem 3.12 to u and $v(x) := \mathbb{E}_x[h(X_{\tau_{D \cap B(0, \lambda_0^{-1}r)}})]$ first and then by applying (4.23)–(4.24) (with $\lambda = R\lambda_1/r$), we obtain that for every $x, y \in D \cap B(0, r/(2\lambda_1))$,

$$\frac{u(x)}{u(y)} \leq c_6 \frac{v(x)}{v(y)} \leq c_7 \frac{V(\delta_D(x))}{V(\delta_D(y))} \leq c_8 \sqrt{\frac{\phi(\delta_D(y)^{-2})}{\phi(\delta_D(x)^{-2})}}.$$

When x or y in $D \cap (B(0, r/2) \setminus B(0, r/(2\lambda_1)))$, we first use the standard chain argument and then apply the above result. \square

5 Heat kernel estimates in the half-space

Recall that $p(t, x, y)$ is the transition density of X and Φ stands for the function $\Phi(r) = 1/\phi(r^{-2})$, $r > 0$. We use $\Phi^{-1}(r)$ to denote the inverse function of Φ . Since X satisfies [12, (1.4), (1.13) and (1.14)], by [12, Theorem 1.2] the following estimates for $p(t, x, y)$ are valid: there exists $c_1 > 0$ such that for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_1^{-1} \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x, y) \right) \leq p(t, x, y) \leq c_1 \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x, y) \right). \quad (5.1)$$

It is known (see [12]) that the killed process X^D has a transition density $p_D(t, x, y)$ with respect to the Lebesgue measure that is jointly Hölder continuous. In a recent preprint [11], sharp two-sided estimates on $p_D(t, x, y)$ for bounded open sets have been established for subordinate Brownian motions under weaker conditions.

The goal of this section is to get sharp two-sided estimates for $p_{\mathbb{H}}(t, x, y)$, and, as a consequence, sharp two-sided estimates of the Green function $G_{\mathbb{H}}(x, y)$.

Lemma 5.1 *There exists $c = c(\phi) > 1$ such that for every $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}$,*

$$p_{\mathbb{H}}(t, x, y) \leq c(\Phi^{-1}(t))^{-d} \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right).$$

Proof. Let $c(t) := \sup_{z, w \in \mathbb{R}^d} p(t/3, z, w)$. By the semigroup property and symmetry,

$$p_{\mathbb{H}}(t, x, y) = \int_{\mathbb{H}} \int_{\mathbb{H}} p_{\mathbb{H}}(t/3, x, z) p_{\mathbb{H}}(t/3, z, w) p_{\mathbb{H}}(t/3, w, y) dz dw \leq c(t) \mathbb{P}_x(\tau_{\mathbb{H}} > t/3) \mathbb{P}_y(\tau_{\mathbb{H}} > t/3).$$

Now the lemma follows from Lemma 2.1, (5.1) and [23, Theorem 4.6]. \square

The next lemma and its proof are given in [8] (also see [5, Lemma 2] and [7, Lemma 2.2]).

Lemma 5.2 *Suppose that U_1, U_3, E are open subsets of \mathbb{R}^d with $U_1, U_3 \subset E$ and $\text{dist}(U_1, U_3) > 0$. Let $U_2 := E \setminus (U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for all $t > 0$,*

$$p_E(t, x, y) \leq \mathbb{P}_x \left(X_{\tau_{U_1}} \in U_2 \right) \left(\sup_{s < t, z \in U_2} p_E(s, z, y) \right) + \mathbb{E}_x [\tau_{U_1}] \left(\sup_{u \in U_1, z \in U_3} J(u, z) \right). \quad (5.2)$$

Lemma 5.3 *There exists $c = c(\phi) > 0$ such that for every $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}$,*

$$p_{\mathbb{H}}(t, x, y) \leq c \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x, y) \right).$$

Proof. By (5.1), (3.10) and Lemma 5.1, it suffices to prove that

$$p_{\mathbb{H}}(t, x, y) \leq c_1 \sqrt{t} \sqrt{\Phi(\delta_{\mathbb{H}}(x))} J(x, y) \quad \text{when } \delta_{\mathbb{H}}(x) \leq \Phi^{-1}(t) \leq |x - y|. \quad (5.3)$$

We assume $\delta_{\mathbb{H}}(x) \leq \Phi^{-1}(t) \leq |x - y|$ and let $x_0 = (\tilde{x}, 0)$, $U_1 := B(x_0, 8^{-1}\Phi^{-1}(t)) \cap \mathbb{H}$, $U_3 := \{z \in \mathbb{H} : |z - x| > |x - y|/2\}$ and $U_2 := \mathbb{H} \setminus (U_1 \cup U_3)$. Note that, by Lemma 2.1 and Theorem 4.7(a), we have

$$\mathbb{E}_x[\tau_{U_1}] \leq c_2 \sqrt{t} \sqrt{\Phi(\delta_{\mathbb{H}}(x))}. \quad (5.4)$$

Since $U_1 \cap U_3 = \emptyset$ and $|z - x| > 2^{-1}|x - y| \geq 2^{-1}\Phi^{-1}(t)$ for $z \in U_3$, we have for $u \in U_1$ and $z \in U_3$,

$$|u - z| \geq |z - x| - |x_0 - x| - |x_0 - u| \geq |z - x| - 4^{-1}\Phi^{-1}(t) \geq \frac{1}{2}|z - x| \geq \frac{1}{4}|x - y|. \quad (5.5)$$

Thus, by (3.13),

$$\sup_{u \in U_1, z \in U_3} J(u, z) \leq \sup_{(u, z): |u - z| \geq \frac{1}{4}|x - y|} J(u, z) \leq c_3 j(|x - y|). \quad (5.6)$$

If $z \in U_2$,

$$\frac{3}{2}|x - y| \geq |x - y| + |x - z| \geq |z - y| \geq |x - y| - |x - z| \geq \frac{|x - y|}{2} \geq 2^{-1}\Phi^{-1}(t). \quad (5.7)$$

Thus, by (3.13), (5.1) and (5.7),

$$\sup_{s \leq t, z \in U_2} p(s, z, y) \leq c_4 \sup_{|x - y|/2 \leq |z - y|} tJ(z, y) \leq c_5 t j(|x - y|). \quad (5.8)$$

Applying Lemma 5.2, (5.4), (5.6) and (5.8), we obtain,

$$p_{\mathbb{H}}(t, x, y) \leq c_6 \mathbb{E}_x[\tau_{U_1}] j(|x - y|) + c_6 \mathbb{P}_x \left(X_{\tau_{U_1}} \in U_2 \right) t j(|x - y|)$$

$$\leq c_7 \sqrt{t} \sqrt{\Phi(\delta_{\mathbb{H}}(x))} j(|x-y|) + c_6 \mathbb{P}_x(X_{\tau_{U_1}} \in U_2) t j(|x-y|).$$

Finally, applying Lemma 3.11 and then (5.4), we have

$$\mathbb{P}_x(X_{\tau_{U_1}} \in U_2) \leq \mathbb{P}_x(X_{\tau_{U_1}} \in B(x_0, 8^{-1}\Phi^{-1}(t))^c) \leq c_8 \frac{1}{t} \mathbb{E}_x[\tau_{U_1}] \leq c_9 t^{-1/2} \sqrt{\Phi(\delta_{\mathbb{H}}(x))}.$$

Thus we have proved (5.3). \square

Proposition 5.4 *There exists $c = c(\phi) > 0$ such that for all $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}$,*

$$p_{\mathbb{H}}(t, x, y) \leq c \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x, y) \right).$$

Proof. By Lemma 5.3 and the lower bound of $p(t, x, y)$ in (5.1), there exists $c_1 > 0$ so that for every $z, w \in \mathbb{H}$, $p_{\mathbb{H}}(t/2, x, z) \leq c_1 (\sqrt{\Phi(\delta_{\mathbb{H}}(x))/t} \wedge 1) p(t/2, x, z)$. Thus, by the semigroup property and the upper bound of $p(t, x, y)$ in (5.1),

$$\begin{aligned} p_{\mathbb{H}}(t, x, y) &= \int_{\mathbb{H}} p_{\mathbb{H}}(t/2, x, z) p_{\mathbb{H}}(t/2, z, y) dz \\ &\leq c_1^2 \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \int_{\mathbb{H}} p(t/2, x, z) p(t/2, y, z) dz \\ &\leq c_1^2 \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) p(t, x, y) \\ &\leq c_2 \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x, y) \right). \end{aligned}$$

\square

Lemma 5.5 *There exists $c = c(\phi) > 0$ such that for any $t > 0$ and $y \in \mathbb{R}^d$,*

$$\mathbb{P}_y(\tau_{B(y, 8^{-1}\Phi^{-1}(t))} > t/3) \geq c.$$

Proof. By [12, Proposition 4.9], there exists $\varepsilon = \varepsilon(\phi) > 0$ such that for every $t > 0$,

$$\inf_{y \in \mathbb{R}^d} \mathbb{P}_y(\tau_{B(y, 16^{-1}\Phi^{-1}(t))} > \varepsilon t) \geq \frac{1}{2}.$$

Suppose $\varepsilon < \frac{1}{3}$, then by the parabolic Harnack inequality in [12],

$$c_1 p_{B(y, 8^{-1}\Phi^{-1}(t))}(\varepsilon t, y, w) \leq p_{B(y, 8^{-1}\Phi^{-1}(t))}(t/3, y, w) \quad \text{for } w \in B(y, 16^{-1}\Phi^{-1}(t)),$$

where the constant $c_1 = c_1(\phi) > 0$ is independent of $y \in \mathbb{R}^d$. Thus

$$\mathbb{P}_y(\tau_{B(y, 8^{-1}\Phi^{-1}(t))} > t/3) = \int_{B(y, 8^{-1}\Phi^{-1}(t))} p_{B(y, 8^{-1}\Phi^{-1}(t))}(t/3, y, w) dw$$

$$\geq c_1 \int_{B(y, 16^{-1}\Phi^{-1}(t))} p_{B(y, 8^{-1}\Phi^{-1}(t))}(\varepsilon t, y, w) dw \geq \frac{c_1}{2}.$$

□

The next result holds for any symmetric discontinuous Hunt process that possesses a transition density and whose Lévy system admits a jumping density kernel. The proof is the same as that of [9, Lemma 3.3] and so it is omitted here.

Lemma 5.6 *Suppose that U_1, U_2, U are open subsets of \mathbb{R}^d with $U_1, U_2 \subset U$ and $\text{dist}(U_1, U_2) > 0$. If $x \in U_1$ and $y \in U_2$, then for all $t > 0$,*

$$p_U(t, x, y) \geq t \mathbb{P}_x(\tau_{U_1} > t) \mathbb{P}_y(\tau_{U_2} > t) \inf_{u \in U_1, z \in U_2} J(u, z). \quad (5.9)$$

Lemma 5.7 *There exists $c = c(\phi) > 0$ such that for all $t > 0$ and $u, v \in \mathbb{R}^d$ with $|u - v| \geq \Phi^{-1}(t)/2$,*

$$p_{B(u, \Phi^{-1}(t)) \cup B(v, \Phi^{-1}(t))}(t/3, u, v) \geq ctj(|u - v|).$$

Proof. Let $U = B(u, \Phi^{-1}(t)) \cup B(v, \Phi^{-1}(t))$, $U_1 = B(u, \Phi^{-1}(t)/8)$, $U_2 = B(v, \Phi^{-1}(t)/8)$ and $K = \inf_{w \in U_1, z \in U_2} j(|w - z|)$. We have by Lemma 5.6 that

$$p_U(t/3, u, v) \geq 3^{-1} K t \mathbb{P}_u(\tau_{U_1} > t/3) \mathbb{P}_v(\tau_{U_2} > t/3).$$

Moreover, for $(w, z) \in U_1 \times U_2$, $|w - z| \leq |u - v| + |w - u| + |z - v| \leq |u - v| + \Phi^{-1}(t)/4 \leq \frac{3}{2}|u - v|$. Hence by (3.13) $K \geq c_1 j(|u - v|)$. Thus by Lemma 5.5,

$$p_U(t/3, u, v) \geq 3^{-1} K t (\mathbb{P}_0(\tau_{B(0, \Phi^{-1}(t)/8)} > t/3))^2 \geq c_2 t j(|u - v|).$$

□

Lemma 5.8 *Suppose that D is an open subset of \mathbb{R}^d and $(t, x, y) \in (0, \infty) \times D \times D$ with $\delta_D(x) \geq \Phi^{-1}(t) \geq 2|x - y|$. Then there exists $c = c(\phi) > 0$ such that*

$$p_D(t, x, y) \geq c(\Phi^{-1}(t))^{-d}. \quad (5.10)$$

Proof. Let $t < \infty$ and $x, y \in D$ with $\delta_D(x) \geq \Phi^{-1}(t) \geq 2|x - y|$. By the parabolic Harnack inequality ([12, Theorem 4.12]), there exists $c_1 = c_1(\phi) > 0$ such that

$$p_D(t/2, x, w) \leq c_1 p_D(t, x, y) \quad \text{for every } w \in B(x, 2\Phi^{-1}(t)/3).$$

This together with Lemma 5.5 yields that

$$\begin{aligned} p_D(t, x, y) &\geq \frac{1}{c_1 |B(x, \Phi^{-1}(t)/2)|} \int_{B(x, \Phi^{-1}(t)/2)} p_D(t/2, x, w) dw \\ &\geq c_2 (\Phi^{-1}(t))^{-d} \mathbb{P}_x(\tau_{B(x, \Phi^{-1}(t)/2)} > t/2) \geq c_3 (\Phi^{-1}(t))^{-d}, \end{aligned}$$

where $c_i = c_i(\phi) > 0$ for $i = 2, 3$. □

For any $x \in \mathbb{H}$ and $a, t > 0$, we define $Q_x(a, t) := B((\tilde{x}, 0), a\Phi^{-1}(t)) \cap \mathbb{H}$.

Lemma 5.9 *There exists $c = c(\phi) > 0$ such that for all $(t, x) \in (0, \infty) \times \mathbb{H}$ with $\delta_{\mathbb{H}}(x) < \Phi^{-1}(t)/2$,*

$$\mathbb{P}_x(\tau_{Q_x(2,t)} > t/3) \geq c \frac{\sqrt{\Phi(\delta_{\mathbb{H}}(x))}}{\sqrt{t}}.$$

Proof. We fix $(t, x) \in (0, \infty) \times \mathbb{H}$ with $\delta_{\mathbb{H}}(x) < \Phi^{-1}(t)/2$. The constants c_1, \dots, c_8 below are independent of t and x . Without loss of generality we assume that $\tilde{x} = \tilde{0}$ and let $Q(a, t) := Q_0(a, t)$, $x_1 := (\tilde{0}, \frac{3}{2}\Phi^{-1}(t))$ and $x_2 := (\tilde{0}, \frac{1}{4}\Phi^{-1}(t))$. Note that, by Lévy system and (3.13),

$$\begin{aligned} \mathbb{P}_{x_2}\left(X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) &\geq \mathbb{P}_{x_2}\left(X_{\tau_{B(x_2, 4^{-1}\Phi^{-1}(t))}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) \\ &= \int_{B(x_1, 4^{-1}\Phi^{-1}(t))} \int_{B(x_2, 4^{-1}\Phi^{-1}(t))} G_{B(x_2, 4^{-1}\Phi^{-1}(t))}(x_1, y) dy J(y, z) dz \\ &\geq c_1 \mathbb{E}_0[\tau_{B(0, 4^{-1}\Phi^{-1}(t))}] \int_{B(x_1, 4^{-1}\Phi^{-1}(t))} J(z) dz. \end{aligned}$$

Applying Theorem 3.4(a) and Lemmas 2.1 and 3.8(b) to the above display, we get

$$\mathbb{P}_{x_2}\left(X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) \geq c_2 t |B(x_1, 4^{-1}\Phi^{-1}(t))| t^{-1} \Phi^{-1}(t)^d \geq c_3.$$

Thus, by Theorem 4.7(b),

$$\mathbb{P}_x\left(X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) \geq c_4 \mathbb{P}_{x_2}\left(X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) \frac{\sqrt{\Phi(\delta_{\mathbb{H}}(x))}}{\sqrt{\Phi(\delta_{\mathbb{H}}(x_2))}} \geq c_5 \frac{\sqrt{\Phi(\delta_{\mathbb{H}}(x))}}{\sqrt{t}}.$$

Now, using this, Lemma 5.5 and the strong Markov property,

$$\begin{aligned} \mathbb{P}_x\left(\tau_{Q(2,t)} > t/3\right) &\geq \mathbb{P}_x\left(\tau_{Q(2,t)} > t/3, X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) \\ &\geq \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_{Q(1,t)}}}\left(\tau_{Q(2,t)} > t/3\right) : X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right] \\ &\geq \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_{Q(1,t)}}}\left(\tau_{B(X_{\tau_{Q(1,t)}}, 4^{-1}\Phi^{-1}(t))} > t/3\right) : X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right] \\ &= \mathbb{P}_0\left(\tau_{B(0, 4^{-1}\Phi^{-1}(t))} > t/3\right) \mathbb{P}_x\left(X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) \\ &\geq c_7 \mathbb{P}_x\left(X_{\tau_{Q(1,t)}} \in B(x_1, 4^{-1}\Phi^{-1}(t))\right) \geq c_8 \frac{\sqrt{\Phi(\delta_{\mathbb{H}}(x))}}{\sqrt{t}}. \end{aligned}$$

This proves the lemma. □

Recall that e_d denotes the unit vector in the positive direction of the x_d -axis in \mathbb{R}^d . Now we are ready to prove the main result of this section

Theorem 5.10 *There exists $c = c(\phi) > 1$ such that for all $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}$,*

$$\begin{aligned} c^{-1} \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x, y) \right) \\ \leq p_{\mathbb{H}}(t, x, y) \leq c \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x, y) \right). \end{aligned}$$

Proof. By Proposition 5.4, we only need to show the lower bound of $p_{\mathbb{H}}(t, x, y)$ in the theorem. Fix $x, y \in \mathbb{H}$. Let $x_0 = (\tilde{x}, 0)$, $y_0 = (\tilde{y}, 0)$, $\xi_x := x + 32\Phi^{-1}(t)e_d$ and $\xi_y := y + 32\Phi^{-1}(t)e_d$. If $\delta_{\mathbb{H}}(x) < \Phi^{-1}(t)/2$, by Lemmas 5.5, 5.6 and 5.9,

$$\begin{aligned}
& \int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) du \\
& \geq t \mathbb{P}_x (\tau_{Q_x(2,t)} > t/3) \left(\inf_{\substack{v \in Q_x(2,t) \\ w \in B(\xi_x, 4\Phi^{-1}(t))}} J(v, w) \right) \int_{B(\xi_x, 2\Phi^{-1}(t))} \mathbb{P}_u (\tau_{B(\xi_x, 4\Phi^{-1}(t))} > t/3) du \\
& \geq c_1 t \mathbb{P}_x (\tau_{Q_x(2,t)} > t/3) t^{-1} (\Phi^{-1}(t))^{-d} \mathbb{P}_0 (\tau_{B(0, 8^{-1}\Phi^{-1}(t))} > t/3) |B(\xi_x, 2\Phi^{-1}(t))| \\
& \geq c_2 \mathbb{P}_x (\tau_{Q_x(2,t)} > t/3) \geq c_3 \frac{\sqrt{\Phi(\delta_{\mathbb{H}}(x))}}{\sqrt{t}}.
\end{aligned}$$

On the other hand, if $\delta_{\mathbb{H}}(x) \geq \Phi^{-1}(t)/2$, by Lemmas 5.5 and 5.6,

$$\begin{aligned}
& \int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) du \\
& \geq t \mathbb{P}_x (\tau_{B(x, 8^{-1}\Phi^{-1}(t)) \cap \mathbb{H}} > t/3) \left(\inf_{\substack{v \in B(x_0, 2\Phi^{-1}(t)) \cap \mathbb{H} \\ w \in B(\xi_x, 4\Phi^{-1}(t))}} J(v, w) \right) \int_{B(\xi_x, 2\Phi^{-1}(t))} \mathbb{P}_u (\tau_{B(\xi_x, 4\Phi^{-1}(t))} > t/3) du \\
& \geq c_4 t \mathbb{P}_x (\tau_{B(x, 8^{-1}\Phi^{-1}(t))} > t/3) t^{-1} (\Phi^{-1}(t))^{-d} \mathbb{P}_0 (\tau_{B(0, 8^{-1}\Phi^{-1}(t))} > t/3) |B(\xi_x, 2\Phi^{-1}(t))| \\
& \geq c_5 \mathbb{P}_0 (\tau_{B(0, 8^{-1}\Phi^{-1}(t))} > t/3)^2 \geq c_6.
\end{aligned}$$

Thus

$$\int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) du \geq c_7 \left(1 \wedge \frac{\sqrt{\Phi(\delta_{\mathbb{H}}(x))}}{\sqrt{t}} \right), \quad (5.11)$$

and similarly,

$$\int_{B(\xi_y, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, y, u) du \geq c_7 \left(1 \wedge \frac{\sqrt{\Phi(\delta_{\mathbb{H}}(y))}}{\sqrt{t}} \right). \quad (5.12)$$

Now we deal with the cases $|x - y| \geq 5\Phi^{-1}(t)$ and $|x - y| < 5\Phi^{-1}(t)$ separately.

Case 1: Suppose that $|x - y| \geq 5\Phi^{-1}(t)$. Note that by the semigroup property and Lemma 5.7,

$$\begin{aligned}
& p_{\mathbb{H}}(t, x, y) \\
& \geq \int_{B(\xi_y, 2\Phi^{-1}(t))} \int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) p_{\mathbb{H}}(t/3, u, v) p_{\mathbb{H}}(t/3, v, y) dudv \\
& \geq \int_{B(\xi_y, 2\Phi^{-1}(t))} \int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) p_{B(u, \Phi^{-1}(t)) \cup B(v, \Phi^{-1}(t))}(t/3, u, v) p_{\mathbb{H}}(t/3, v, y) dudv \\
& \geq c_8 t \left(\inf_{(u,v) \in B(\xi_x, 2\Phi^{-1}(t)) \times B(\xi_y, 2\Phi^{-1}(t))} j(|u - v|) \right)
\end{aligned}$$

$$\times \int_{B(\xi_y, 2\Phi^{-1}(t))} \int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) p_{\mathbb{H}}(t/3, v, y) dudv.$$

It then follows from (5.11)–(5.12) that

$$p_{\mathbb{H}}(t, x, y) \geq c_9 t \left(\inf_{(u,v) \in B(\xi_x, 2\Phi^{-1}(t)) \times B(\xi_y, 2\Phi^{-1}(t))} j(|u-v|) \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right). \quad (5.13)$$

Using the assumption $|x-y| \geq 5\Phi^{-1}(t)$ we get that, for $u \in B(\xi_x, 2\Phi^{-1}(t))$ and $v \in B(\xi_y, 2\Phi^{-1}(t))$, $|u-v| \leq 4\Phi^{-1}(t) + |x-y| \leq 2|x-y|$. Hence

$$\inf_{(u,v) \in B(\xi_x, 2\Phi^{-1}(t)) \times B(\xi_y, 2\Phi^{-1}(t))} j(|u-v|) \geq c_{10} j(|x-y|). \quad (5.14)$$

By (5.13) and (5.14), we conclude that for $|x-y| \geq 5\Phi^{-1}(t)$

$$p_{\mathbb{H}}(t, x, y) \geq c_{11} \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) t j(|x-y|).$$

Case 2: Suppose $|x-y| < 5\Phi^{-1}(t)$. In this case, for every $(u, v) \in B(\xi_x, 2\Phi^{-1}(t)) \times B(\xi_y, 2\Phi^{-1}(t))$, $|u-v| \leq 9\Phi^{-1}(t)$. Thus, using the fact that $\delta_{\mathbb{H}}(\xi_x) \wedge \delta_{\mathbb{H}}(\xi_y) \geq 32\Phi^{-1}(t)$, there exists $w_0 \in \mathbb{H}$ such that

$$B(\xi_x, 2\Phi^{-1}(t)) \cup B(\xi_y, 2\Phi^{-1}(t)) \subset B(w_0, 6\Phi^{-1}(t)) \subset B(w_0, 12\Phi^{-1}(t)) \subset \mathbb{H}. \quad (5.15)$$

Now, by the semigroup property and (5.15), we get

$$\begin{aligned} & p_{\mathbb{H}}(t, x, y) \\ & \geq \int_{B(\xi_y, 2\Phi^{-1}(t))} \int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) p_{B(w_0, 12\Phi^{-1}(t))}(t/3, u, v) p_{\mathbb{H}}(t/3, v, y) dudv \\ & \geq \left(\inf_{u,v \in B(w_0, 6\Phi^{-1}(t))} p_{B(w_0, 12\Phi^{-1}(t))}(t/3, u, v) \right) \int_{B(\xi_y, 2\Phi^{-1}(t))} \int_{B(\xi_x, 2\Phi^{-1}(t))} p_{\mathbb{H}}(t/3, x, u) p_{\mathbb{H}}(t/3, v, y) dudv. \end{aligned}$$

It then follows from (5.11)–(5.12) and Lemmas 2.1 and 5.8 that

$$p_{\mathbb{H}}(t, x, y) \geq c_{12} \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) (\Phi^{-1}(t))^{-d}.$$

Combining these two cases, we have proved the theorem. \square

Note that by using Theorem 3.4 we can express the sharp two-sided estimates for $p_{\mathbb{H}}(t, x, y)$ solely in terms of the Laplace exponent ϕ .

By integrating out time t from the estimates in the preceding theorem, we can obtain sharp two-sided estimates of the Green function. Since the calculations are long and somewhat cumbersome, we only state the result and omit the proof. We refer the readers to [11] for similar calculations.

Theorem 5.11 (i) For all $d \geq 1$ there exists $c_1 = c_1(d, \phi) > 0$ such that for all $(x, y) \in \mathbb{H} \times \mathbb{H}$,

$$G_{\mathbb{H}}(x, y) \geq c_1 \frac{\Phi(|x - y|)}{|x - y|^d} \left(1 \wedge \frac{\Phi(\delta_{\mathbb{H}}(x))^{1/2}}{\Phi(|x - y|)^{1/2}} \right) \left(1 \wedge \frac{\Phi(\delta_{\mathbb{H}}(y))^{1/2}}{\Phi(|x - y|)^{1/2}} \right).$$

(ii) If $d > (\delta_2 \vee \delta_4)$, then for all $(x, y) \in \mathbb{H} \times \mathbb{H}$,

$$G_{\mathbb{H}}(x, y) \asymp \frac{\Phi(|x - y|)}{|x - y|^d} \left(1 \wedge \frac{\Phi(\delta_{\mathbb{H}}(x))^{1/2}}{\Phi(|x - y|)^{1/2}} \right) \left(1 \wedge \frac{\Phi(\delta_{\mathbb{H}}(y))^{1/2}}{\Phi(|x - y|)^{1/2}} \right).$$

(iii) There exists $c_2 = c_2(d, \phi) > 0$ such that for all $(x, y) \in \mathbb{H} \times \mathbb{H}$ with $\Phi(\delta_{\mathbb{H}}(x))\Phi(\delta_{\mathbb{H}}(y)) \leq \Phi(|x - y|)^2$,

$$G_{\mathbb{H}}(x, y) \leq c_2 \frac{\Phi(\delta_{\mathbb{H}}(x))^{1/2}\Phi(\delta_{\mathbb{H}}(y))^{1/2}}{|x - y|^d}.$$

(iv) If $d = 1$ and $\delta_1 \wedge \delta_3 > 1/2$, then for all $(x, y) \in \mathbb{H} \times \mathbb{H}$,

$$G_{\mathbb{H}}(x, y) \asymp \left(\frac{\Phi(\delta_{\mathbb{H}}(x))^{1/2}\Phi(\delta_{\mathbb{H}}(y))^{1/2}}{\Phi^{-1}(\Phi(\delta_{\mathbb{H}}(x))^{1/2}\Phi(\delta_{\mathbb{H}}(y))^{1/2})} \wedge \frac{\Phi(\delta_{\mathbb{H}}(x))^{1/2}\Phi(\delta_{\mathbb{H}}(y))^{1/2}}{|x - y|} \right).$$

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Panki Kim

Department of Mathematical Sciences and Research Institute of Mathematics,
Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 151-747, Republic of
Korea

E-mail: pkim@snu.ac.kr

Renming Song

Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

E-mail: rsong@math.uiuc.edu

Zoran Vondraček

Department of Mathematics, University of Zagreb, Zagreb, Croatia

Email: vondra@math.hr