Scale invariant boundary Harnack principle at infinity for Feller processes

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Abstract

In this paper we prove a uniform and scale invariant boundary Harnack principle at infinity for a large class of purely discontinuous Feller processes in metric measure spaces.

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1 Introduction

The boundary Harnack principle (BHP) is a result roughly saying that non-negative functions, which are harmonic in an open set and vanish near a portion of the boundary of that open set, have the same boundary decay rate near that portion of the boundary. The BHP was first proved independently in [1, 13, 32] for classical harmonic functions in Lipschitz domains. Since then, it has been extended to more general diffusions and more general domains.

In [3], the BHP was established for harmonic functions of symmetric α-stable processes, $\alpha \in (0, 2)$, in Lipschitz domains. This was the first BHP for discontinuous Markov processes. Since then, the result of [3] has been generalized in various directions. [30] extended it to harmonic functions of symmetric α-stable processes in κ-fat open sets, with the constant depending on the local geometry near the boundary. A uniform version of it was established in [6] for harmonic functions of symmetric α-stable processes in arbitrary open sets. The BHP of [6] is uniform in the sense that the constant does not depend on the open set itself. Note that such uniform version does not hold for Brownian motion.

In another direction, the BHP has been generalized to different classes of discontinuous Markov processes. For example, it was extended to a large class of subordinate Brownian

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motions in [17, 18]. In [20] the uniform BHP was extended to a large class of rotationally symmetric Lévy processes and in [16] it was extended to a class of subordinate Brownian motions including geometric stable processes. The main result of [20] has been extended to a large class of symmetric Lévy processes in [24]. A BHP with explicit decay rate was established in [19, 22] for a large class of subordinate Brownian motions in $C^{1,1}$ open sets. For BHP with respect to subordinate Brownian motions with Gaussian components, see [10, 21].

Recently, a very general BHP for discontinuous Feller processes in metric measure spaces has been proved in [7] under some comparability assumptions on the jump kernel and a Urysohn-type property of the domain of the generator of the process. The main result of [7] is not scale invariant in general. It was shown in [7] that, under a stable-like scaling condition, a scale invariant BHP holds.

All the BHPs mentioned above deal with the decay of harmonic functions near finite boundary points. In the case of symmetric α-stable processes, by using the inversion with respect to spheres, the Kelvin transform and the BHP near finite boundary points, [26] obtained a BHP at infinity for harmonic functions in unbounded open sets. The argument using inversion with respect to spheres and the Kelvin transform does not work for more general Lévy processes. By using a different, more involved argument, a BHP at infinity was established in [23] for a large class of symmetric Lévy processes under a global weak scaling condition on the Lévy exponents.

Motivated by the result and the method from [7], in this paper we prove a uniform and scale invariant BHP at infinity for a class of purely discontinuous Feller processes in metric measure spaces. Even in the special case of symmetric Lévy processes, the BHP at infinity of this paper is more general than that of [23] since we will only assume that the Lévy exponents satisfy a weak scaling condition near the origin. We will also give a uniform and scale invariant BHP near finite boundary points.

We start the paper by recalling the setting and basic assumptions of [7]. Let $(\mathfrak{X}, d)$ be a metric space such that all bounded closed sets are compact and let m be a σ-finite measure on $\mathfrak{X}$ with full support. Let $R_0 \in (0, \infty]$ (the localization radius of $(\mathfrak{X}, d)$) be such that $\mathfrak{X} \setminus B(x, 2r) \neq \emptyset$ for all $x \in \mathfrak{X}$ and all $r < R_0$. We will consider a large class of Feller processes $X = (X_t, t \geq 0; \mathbb{P}_x, x \in \mathfrak{X}; \mathcal{F}_t, t \geq 0)$ on $\mathfrak{X}$ satisfying several assumptions. The first assumption is strong duality and Hunt's hypothesis (H).

A: X is a Hunt process admitting a strong dual process $\widehat{X}$ with respect to the measure m and $\widehat{X}$ is also a Hunt process. The transition semigroups (P_t) and $(\widehat{P}_t)$ of X and $\widehat{X}$ are both Feller and strongly Feller. Every semi-polar set of X is polar.

In the sequel, all objects related to the dual process $\widehat{X}$ will be denoted by a hat. Recall that a set is polar (semi-polar, respectively) for X if and only if it is polar (semi-polar, respectively) for $\widehat{X}$ (see [2, VI. (1.19)]). Under assumption **A** the process X admits a (possibly infinite) Green function $G(x, y)$ serving as a density of the occupation measure: $G(x, A) := \mathbb{E}_x \int_0^\infty \mathbf{1}_{(X_t \in A)} dt = \int_A G(x, y) m(dy)$. Moreover, $G(x, y) = \widehat{G}(y, x)$ for all $x, y \in \mathfrak{X}$, cf. [2, VI.1]. Further, if D is an open subset of $\mathfrak{X}$ and $\tau_D = \inf\{t > 0 : X_t \notin D\}$ the exit time from D, the killed process X^D is defined by $X_t^D = X_t$ if $t < \tau_D$ and $X_t^D = \partial$ where ∂ is an extra point added to $\mathfrak{X}$. Then X^D admits a unique (possibly infinite) Green function (potential kernel) $G_D(x, y)$ such that for every non-negative Borel function f,

$$G_D f(x) := \int_D f(y) G_D(x, y) dy = \mathbb{E}_x \int_0^{\tau_D} f(X_t) dt\,,$$

and $G_D(x, y) = \widehat{G}_D(y, x)$, $x, y \in D$, with $\widehat{G}_D(y, x)$ the Green function of $\widehat{X}^D$. For the details we refer the readers to [7, pp.480–481] and the references therein. We say D is Greenian if the Green function $G_D(x, y)$ is finite for all $x, y \in D$, $x \neq y$. Under this assumption the process X^D is transient in the sense that there exists a non-negative Borel function f on D such that $0 < G_D f < \infty$ (and the same is true for $\widehat{X}$).

Let $C_0(\mathfrak{X})$ stand for the Banach space of bounded continuous functions on $\mathfrak{X}$ vanishing at infinity. Let $\mathcal{A}$ and $\widehat{\mathcal{A}}$ be the generators of (P_t) and $(\widehat{P}_t)$ in $C_0(\mathfrak{X})$ respectively. The second assumption is a Urysohn-type condition.

B: There is a linear subspace $\mathcal{D}$ of $\mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\widehat{\mathcal{A}})$ satisfying the following condition: For any compact K and open D with $K \subset D \subset \mathfrak{X}$, the collection $\mathcal{D}(K, D)$ of functions $f \in \mathcal{D}$ satisfying the conditions (i) $f(x) = 1$ for $x \in K$; (ii) $f(x) = 0$ for $x \in \mathfrak{X} \backslash D$; (iii) $0 \le f(x) \le 1$ for $x \in \mathfrak{X}$, and (iv) the boundary of the set $\{x : f(x) > 0\}$ has zero m measure, is nonempty. We let

$$\rho(K, D) := \inf_{f \in \mathcal{D}(K,D)} \sup_{x \in \mathfrak{X}} \max(\mathcal{A} f(x), \widehat{\mathcal{A}} f(x)).$$

Assumption **B** implies that the jumps of X satisfy the following Lévy system formula: for every stopping time T,

$$\mathbb{E}_x \sum_{s \in (0,T]} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^T \int_{\mathfrak{X}} f(X_s, z) J(X_s, dz) ds. \tag{1.1}$$

Here $f : \mathfrak{X} \times \mathfrak{X} \to [0, \infty]$, $f(x, x) = 0$ for all $x \in \mathfrak{X}$, and J is a kernel on $\mathfrak{X}$ (satisfying $J(x, \{x\}) = 0$ for all $x \in \mathfrak{X}$), called the Lévy kernel of X. As a consequence, the following Ikeda-Watanabe type formula is valid:

$$\mathbb{P}_x(X_{\tau_D} \in E, X_{\tau_D -} \neq X_{\tau_D}, \tau_D < \zeta) = \int_D G_D(x, dy) J(y, E), \quad x \in D, E \subset \mathfrak{X} \setminus D \tag{1.2}$$

where ζ is the life time of X. Furthermore, the Lévy kernel J satisfies

$$Jf(x) := \int_{\mathfrak{X}} f(y) J(x, dy) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x f(X_t)}{t} \tag{1.3}$$

for all bounded continuous function f on $\mathfrak{X}$ and $x \in \mathfrak{X} \setminus \mathrm{supp} f$. The Lévy kernel $\widehat{J}(y, dx)$ of $\widehat{X}$ is defined in a similar manner. By duality, $J(x, dy) m(dx) = \widehat{J}(y, dx) m(dy)$. Further, it follows from (1.3) that if $f \in \mathcal{D}(\mathcal{A})$ and $x \in \mathfrak{X} \setminus \mathrm{supp}(f)$, then $Jf(x) = \mathcal{A}f(x)$. Again, for these facts we refer the reader to [7, p.482] and the reference therein.

Our next assumption is only a part of the corresponding assumption in [7].

C: The Lévy kernels of X and $\widehat{X}$ have the form $j(x,y)m(dy)$ and $\widehat{j}(x,y)m(dy)$ respectively, where $j(x,y) = \widehat{j}(y,x) > 0$ for all $x, y \in \mathfrak{X}, x \neq y$.

For an open set $D \subset \mathfrak{X}$, let

$$P_D(x,z) := \int_D G_D(x,y) j(y,z) m(dy), \qquad x \in D, z \in D^c, \tag{1.4}$$

be the Poisson kernel of D with respect to X. It follows from (1.2) and (1.4) that $P_D(x,z)$ is the exit density of X from D through jumps:

$$\mathbb{P}_x(X_{\tau_D} \in E, X_{\tau_D -} \neq X_{\tau_D}, \tau_D < \zeta) = \int_E P_D(x,z) m(dz), \quad x \in D, E \subset \mathfrak{X} \setminus D\,.$$

Assumptions **A**, **B** and **C** will be in force throughout this paper. In the next section we will assume that the localization radius $R_0 = \infty$, that X and $\widehat{X}$ are conservative, and will add assumptions needed in order to study the behavior of non-negative harmonic functions at infinity. Our main result is a scale invariant approximate factorization of non-negative function harmonic in unbounded open set, Theorem 2.1, from which the scale invariant uniform boundary Harnack principle, Corollary 2.2, immediately follows. Proofs of these results will be given in Section 3. In Section 4, we introduce a different additional set of assumptions and state the scale invariant uniform boundary Harnack principle at a finite boundary point. Proofs of the results in Section 4 are deferred to the Appendix. In Section 5 we discuss examples of processes which satisfy the assumptions of this paper. These include some symmetric and isotropic Lévy processes, strictly stable (not necessarily symmetric) processes in $\mathbb{R}^d$, processes obtained by subordinating a Feller diffusion on unbounded Ahlfors regular n-sets, and some space non-homogeneous processes on $\mathbb{R}^d$. Finally, in Section 6 we study boundary behavior of the Green function G_D at the regular boundary points. These results will be used in subsequent papers.

Notation: We will use the following conventions in this paper. $c, c_0, c_1, c_2, \cdots$ stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. The labeling of the constants $c_0, c_1, c_2, \cdots$ starts anew in the statement of each result. We will use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$. The notation $f \asymp g$ means that the quotient $f(t)/g(t)$ stays bounded between two positive numbers on their common domain of definition. For $x \in \mathfrak{X}$ and $r > 0$ we denote by $B(x,r)$ be the open ball centered at x with radius r and by $\overline{B}(x,r)$ the closure of $B(x,r)$. Further, for $0 < r < R$, let $A(x,r,R) = \{y \in \mathfrak{X} : r < d(x,y) < R\}$ be the open annulus around x, and $\overline{A}(x,r,R)$ the closure of $A(x,r,R)$. Throughout the paper we will adopt the convention that $X_\zeta = \partial$ and $u(\partial) = 0$ for every function u.

2 Additional assumptions and main result

We recall that assumptions **A**, **B** and **C** are in force throughout the paper. In order for the boundary Harnack principle at infinity to make sense the ambient space $\mathfrak{X}$ must be unbounded. Hence in this and the next section we assume that the localization radius $R_0 = \infty$. We further assume that both X and $\widehat{X}$ are conservative processes: For every $t \geq 0$, $P_t 1 = \widehat{P}_t 1 = 1$.

From now on we fix a point z_0 in $\mathfrak{X}$ which will serve as the center of the space. For any $r > 0$, let

$$V(r) = V(z_0, r) := m(B(z_0, r))$$

denote the volume of the ball of radius r centered at z_0. We assume that $V : [0, \infty) \to [0, \infty)$ satisfies the following two properties: (i) The doubling property: There exists $c > 1$ such that

$$V(2r) \leq cV(r), \quad r > 0, \tag{2.1}$$

and (ii) There exist $c > 1$, $r_0 > 0$ and $n_0 \in \mathbb{N}$ with $n_0 \geq 2$ such that

$$V(n_0 r) \geq cV(r), \quad r \geq r_0. \tag{2.2}$$

We further assume the existence of a non-decreasing function $\Phi = \Phi(z_0, \cdot) : [0, \infty) \to [0, \infty)$ which satisfies the doubling property: There exists $c > 1$ such that

$$\Phi(2r) \leq c\,\Phi(r), \quad r > 0. \tag{2.3}$$

The function Φ will play a crucial role in obtaining scale invariant results. Examples of such functions will be given in Section 5. At the moment it suffices to say that in case of isotropic Lévy process in $\mathbb{R}^d$, we have that $\Phi(r) = 1/\Psi(r^{-1})$, where $x \mapsto \Psi(|x|)$ is the Lévy exponent of the process.

Let $C_\infty(\mathfrak{X})$ be the Banach space of continuous functions f on $\mathfrak{X}$ such that f has a limit at infinity. We will use $\|\cdot\|$ to denote the sup norm. It is obvious that any function $f \in C_\infty(\mathfrak{X})$ is the sum of function in $C_0(\mathfrak{X})$ and a constant. It is well known that the semigroup of X being Feller is equivalent to the following conditions: (i) for any $f \in C_\infty(\mathfrak{X})$, $P_t f \in C_\infty(\mathfrak{X})$; (ii) for any $f \in C_\infty(\mathfrak{X})$, $\lim_{t\to 0} \|P_t f - f\| = 0$. We will also use $\mathcal{A}$ (respectively $\widehat{\mathcal{A}}$) to denote the generator of (P_t) (respectively $(\widehat{P}_t)$) in $C_\infty(\mathfrak{X})$. It follows easily from the conservativeness of X that constant functions are in $\mathcal{D}(\mathcal{A})$ and that, for any constant c, $\mathcal{A}c = 0$.

We are now ready for some additions to assumptions **B** and **C** and an additional assumption. In the following assumptions, r_0 is a positive number. Recall the notation $\mathcal{D}(K, D)$ from assumption **B**.

B2-a(z_0, r_0): For any $a \in (1, 2]$, there exists $c = c(z_0, a)$ such that for any $r \geq r_0$,

$$\begin{aligned}\varrho(r) &:= \inf_{f \in \mathcal{D}(\overline{B}(z_0,r), B(z_0,ar))} \sup_{x \in \mathfrak{X}} \max(\mathcal{A}(1-f)(x), \widehat{\mathcal{A}}(1-f)(x)) \\ &= \inf_{f \in \mathcal{D}(\overline{B}(z_0,r), B(z_0,ar))} \sup_{x \in \mathfrak{X}} \max(-\mathcal{A}f(x), -\widehat{\mathcal{A}}f(x)) \leq \frac{c}{\Phi(r)}.\end{aligned}$$

B2-b(z_0, r_0): For any $a \in (1, 2]$, there exists $c = c(z_0, a)$ such that for any $r \ge r_0$ and any $f \in \mathcal{D}(\overline{B}(z_0, r), B(z_0, ar))$,

$$\max(\mathcal{A}f(x), \widehat{\mathcal{A}}f(x)) \le cV(r)j(x, z_0), \quad x \in \overline{A}(z_0, r, (a+1)r).$$

To assumption **C** we add

C2(z_0, r_0): For any $a \in (1, 2]$, there exists $c = c(z_0, a)$ such that for $r \ge r_0$, $x \in B(z_0, r)$ and $y \in \mathfrak{X} \setminus B(z_0, ar)$,

$$c^{-1} j(z_0, y) \le j(x, y) \le c j(z_0, y), \qquad c^{-1} \widehat{j}(z_0, y) \le \widehat{j}(x, y) \le c\widehat{j}(z_0, y), \tag{2.4}$$

and

$$\inf_{y \in \overline{A}(z_0, r, ar)} \min(j(z_0, y), \widehat{j}(z_0, y)) \ge \frac{c}{V(r)\Phi(r)}\,. \tag{2.5}$$

Note that by assumptions **B2-b**(z_0, r_0) and **C2**(z_0, r_0), and the doubling property of V, any $f \in \mathcal{D}(\overline{B}(z_0, r), B(z_0, ar))$ satisfies

$$\max(\mathcal{A}f(x), \widehat{\mathcal{A}}f(x)) \le c\mathbf{1}_{B(z_0, r)^c}(x) V(r) j(x, z_0), \qquad r \ge r_0, \tag{2.6}$$

for a constant $c = c(z_0, a) > 0$. In fact, for $f \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\widehat{\mathcal{A}})$ such that $f(x) = 1$ for $x \in B(z_0, r)$, $f(x) = 0$ for $x \in \mathfrak{X} \setminus \overline{B}(z_0, ar)$ and $0 \le f(x) \le 1$ for $x \in \mathfrak{X}$, we have, using the doubling property of V,

$$\begin{aligned} &\mathcal{A}f(x)\mathbf{1}_{\mathfrak{X}\setminus\overline{B}(z_0,(a+1)r)}(x) = \mathbf{1}_{\mathfrak{X}\setminus\overline{B}(z_0,(a+1)r)}(x) \int_{B(z_0,ar)} f(y) j(x,y) m(dy) \\ &\le c\mathbf{1}_{\mathfrak{X}\setminus\overline{B}(z_0,(a+1)r)}(x) V(ar) j(x, z_0) \le c_2 \mathbf{1}_{\mathfrak{X}\setminus\overline{B}(z_0,(a+1)r)}(x) V(r) j(x, z_0). \end{aligned} \tag{2.7}$$

Combining this with assumption **B2-b**(z_0, r_0), we get (2.6).

Our final assumption concerns Green functions of complements of balls.

D2(z_0, r_0): $\overline{B}(z_0, r_0)^c$ is Greenian. For every $a \in (1, 2)$, there exists a constant $c = c(z_0, a)$ such that for all $r \ge r_0$,

$$\sup_{x \in \overline{B}(z_0, ar)} \sup_{y \in \overline{A}(z_0, 2r, 4r)} \max(G_{\overline{B}(z_0,r)^c}(x, y), \widehat{G}_{\overline{B}(z_0,r)^c}(x, y)) \le c\frac{\Phi(r)}{V(r)}.$$

Recall that a non-negative function $u : \mathfrak{X} \to [0, \infty)$ is said to be regular harmonic in an open set $D \subset \mathfrak{X}$ if

$$u(x) = \mathbb{E}_x \left(u(X_{\tau_D}) \right), \quad \text{for all } x \in D\,.$$

By the strong Markov property, the equality above holds for every stopping time $\tau \le \tau_D$.

Recall also that for an open set $D \subset \mathfrak{X}$, a point $x \in \partial D$ is said to be regular for D^c with respect to X if $\mathbb{P}_x(\tau_D = 0) = 1$. Let D^{reg} denote the set of points $x \in \partial D$ which are regular for D^c with respect to X.

Now we can state our main theorem.

Theorem 2.1 *Assume* $z_0 \in \mathfrak{X}$. *Suppose that, in addition to* **A**, **B** *and* **C**, *assumptions* (2.1)–(2.3), **B2-a**(z_0, r_0), **B2-b**(z_0, r_0), **C2**(z_0, r_0) *and* **D2**(z_0, r_0) *hold true for some* $r_0 > 0$. *For any* $a \in (1, 2)$, *there exists* $C_1 = C_1(z_0, a) > 1$ *such that for any* $r \geq r_0$, *any open set* $D \subset \overline{B}(z_0, r)^c$ *and any non-negative function* u *on* $\mathfrak{X}$ *which is regular harmonic with respect to* X *in* D *and vanishes on* $\overline{B}(z_0, r)^c \cap \left(\overline{D}^c \cup D^{\text{reg}}\right)$, *it holds that*

$$C_1^{-1} P_D(x, z_0) \int_{B(z_0, 2ar)} u(z)\, m(dz) \leq u(x) \leq C_1 P_D(x, z_0) \int_{B(z_0, 2ar)} u(z)\, m(dz) \qquad (2.8)$$

for all $x \in D \cap \overline{B}(z_0, 8r)^c$.

As a consequence of Theorem 2.1, one immediately gets the following scale invariant uniform boundary Harnack principle at infinity.

Corollary 2.2 (Boundary Harnack Principle at Infinity) *Let* $z_0 \in \mathfrak{X}$. *Assume that, in addition to* **A**, **B** *and* **C**, *assumptions* (2.1)–(2.3), **B2-a**(z_0, r_0), **B2-b**(z_0, r_0), **C2**(z_0, r_0) *and* **D2**(z_0, r_0) *hold true for some* $r_0 > 0$. *There exists* $C_2 = C_2(z_0) > 1$ *such that for any* $r \geq 2r_0$, *any open set* $D \subset \overline{B}(z_0, r)^c$ *and any non-negative functions* u *and* v *on* $\mathfrak{X}$ *which are regular harmonic with respect to* X *in* D *and vanish on* $\overline{B}(z_0, r)^c \cap \left(\overline{D}^c \cup D^{\text{reg}}\right)$, *it holds that*

$$C_2^{-1} \frac{u(y)}{v(y)} \leq \frac{u(x)}{v(x)} \leq C_2 \frac{u(y)}{v(y)}, \qquad \textit{for all } x, y \in D \cap \overline{B}(z_0, 8r)^c. \qquad (2.9)$$

Remark 2.3 Note that all our assumptions are symmetric in X and $\widehat{X}$. Therefore, Theorem 2.1 and Corollary 2.2 hold for co-harmonic functions as well.

3 Proofs

Throughout this section, z_0 is a fixed point in $\mathfrak{X}$. We will always assume in this section that assumptions **A**, **B**, **C**, (2.1)–(2.3), **B2-a**(z_0, r_0), **B2-b**(z_0, r_0), **C2**(z_0, r_0) and **D2**(z_0, r_0) hold true for some $r_0 > 0$ and give a proof of Theorem 2.1.

Proposition 3.1 *There exists a constant* $c > 0$ *such that for every* $r \geq r_0$ *and* $D \subset \mathfrak{X} \setminus \overline{B}(z_0, r)$,

$$P_D(x, z_0) \leq c \frac{1}{V(r)}, \quad x \in D.$$

Proof. By (2.4) and the fact that $j(x, y) = \widehat{j}(y, x)$, for every $y \in B(z_0, r/2)$ and $x \in D$, we have

$$P_D(x, z_0) \leq P_{\overline{B}(z_0,r)^c}(x, z_0) = \int_{\overline{B}(z_0,r)^c} G_{\overline{B}(z_0,r)^c}(x, z) j(z, z_0) m(dz)$$
$$\leq c_1 \int_{\overline{B}(z_0,r)^c} G_{\overline{B}(z_0,r)^c}(x, z) j(z, y) m(dz) = c_1 P_{\overline{B}(z_0,r)^c}(x, y).$$

Thus, by integrating over the ball $B(z_0, r/2)$ and using the doubling property of $V(r)$,

$$P_D(x, z_0) \le \frac{c_2}{V(r/2)} \int_{B(z_0,r/2)} P_{\overline{B}(z_0,r)^c}(x,y)m(dy)$$
$$\le \frac{c_3}{V(r)} \int_{B(z_0,r/2)} P_{\overline{B}(z_0,r)^c}(x,y)m(dy) \le \frac{c_3}{V(r)}.$$

□

Let $a \in (1, 2)$. For each $r \ge r_0$, we consider a function $\varphi^{(r)} \in \mathcal{D}(\overline{B}(z_0, r), B(z_0, ar))$, and let $\phi^{(r)} = 1 - \varphi^{(r)}$ and $V^{(r)} = \{x \in \mathfrak{X} : \phi^{(r)}(x) > 0\} = \{x \in \mathfrak{X} : \varphi^{(r)}(x) < 1\}$. Note that, by choosing $\varphi^{(r)}$ appropriately, we can achieve that

$$\delta^{(r)} := \sup_{x \in B(z_0,ar)} \max(\mathcal{A}\phi^{(r)}(x), \widehat{\mathcal{A}}\phi^{(r)}(x)) \le \frac{2c}{\Phi(r)},$$

where $c = c(z_0, a)$ is the constant in assumption **B2-a**(z_0, r_0).

In what follows, our analysis and results are valid for all $r \ge r_0$ with constants depending on $a \in (1, 2)$, but *not* on r. To ease the notation in the remaining part of the section we drop the superscript r from $\varphi^{(r)}$, $\phi^{(r)}$ and $V^{(r)}$ and write simply φ, ϕ and V.

Let

$$\psi(x) = \frac{\max(\mathcal{A}\phi(x), \widehat{\mathcal{A}}\phi(x), \delta(1 - \phi(x)))}{\phi(x)}, \quad x \in \mathfrak{X}, \tag{3.1}$$

with the convention $1/0 = \infty$. Note that $\psi(x) = \infty$ for $x \in V^c$, and $\psi(x) = 0$ for $x \in \overline{B}(z_0, ar)^c$. We define two right-continuous additive functionals by

$$A_t = \lim_{\varepsilon \searrow 0} \int_0^{t+\varepsilon} \psi(X_s)ds \quad \text{and} \quad \widehat{A}_t = \lim_{\varepsilon \searrow 0} \int_0^{t+\varepsilon} \psi(\widehat{X}_s)ds. \tag{3.2}$$

We follow the idea in [7] to mollify the distribution of $X(\tau_{\overline{B}(z_0,r)^c})$ by letting the particle lose mass gradually, with intensity $\psi(X_t)$, before time $\tau_{\overline{B}(z_0,r)^c}$.

We define two right-continuous strong Markov multiplicative functionals $M_t = \exp(-A_t)$ and $\widehat{M}_t = \exp(-\widehat{A}_t)$. By the argument on [7, p. 492] and the references therein, M and $\widehat{M}$ are in fact exact strong Markov multiplicative functionals. As on [7, p. 492], we consider the semigroup of operators $T_t^\psi f(x) = \mathbb{E}_x(f(X_t)M_t)$ associated with the multiplicative functional M and the semigroup of operators $\widehat{T}_t^\psi f(x) = \mathbb{E}_x(f(\widehat{X}_t)\hat{M}_t)$ associated with the multiplicative functional $\widehat{M}$. T_t^ψ is the transition operator of the subprocess X^ψ of X. The subprocess $\widehat{X}^\psi$ of $\widehat{X}$ corresponding to the multiplicative functional $\widehat{M}$ is the dual process of X^ψ. Thus the potential densities of X^ψ and $\widehat{X}^\psi$ satisfy $\widehat{G}^\psi(x, y) = G^\psi(y, x)$ and

$$G^\psi(x, y) \le G_V(x, y) \le G_{\overline{B}(z_0,r)^c}(x, y), \quad (x, y) \in V \times V. \tag{3.3}$$

Let $\tau_a = \inf\{t \ge 0 : A_t \ge a\}$ and

$$\pi^\psi f(x) = -\mathbb{E}_x \int_{[0,\infty)} f(X_t)dM_t = \mathbb{E}_x \left(\int_0^\infty f(X_{\tau_a})e^{-a}da \right) = \int_0^\infty \mathbb{E}_x(f(X_{\tau_a}))e^{-a}da, \tag{3.4}$$

where the second equality follows by substitution. By following the arguments in [7, pp. 492–493] line by line, one can see that $\pi^\psi f$ can be written in the following two ways: If f is nonnegative and vanishes in $\mathfrak{X} \setminus (B(z_0, ar) \cap V)$, then

$$\pi^\psi f(x) = G^\psi(\psi f)(x) = \int_{V\cap B(z_0,ar)} G^\psi(x,y)\psi(y)f(y)m(dy), \quad x \in V, \tag{3.5}$$

and if $f \in \mathcal{D}(\mathcal{A})$ vanishes in V, then for all $x \in V$,

$$\begin{aligned}\pi^\psi f(x) = G^\psi \mathcal{A} f(x) &= \int_V G^\psi(x,y)\mathcal{A}f(y)m(dy)\\ &= \int_V G^\psi(x,y)\int_{\mathfrak{X}\setminus V} f(z)j(y,z)m(dz)m(dy)\\ &= \int_{\mathfrak{X}\setminus V}\left(\int_V G^\psi(x,y)j(y,z)m(dy)\right)f(z)m(dz).\end{aligned} \tag{3.6}$$

Since Lemmas 4.6–4.7 and Corollary 4.8 of [7] and their proofs hold under our setting, we have

$$\pi^\psi(x, \partial V) = 0, \qquad x \in V. \tag{3.7}$$

Repeating the argument of the proof of [7, Lemma 4.10], we get that if f is regular harmonic in $\overline{B}(z_0, r)^c$ with respect to X, then $f(x) = \pi^\psi f(x)$ for all $x \in B(z_0, 2r)^c$. The main step of the proof is to get the correct estimate of $\pi^\psi(x, dy)/m(dy)$.

We recall the following notation and results from [7]. Let U be an open subset of V. For any nonnegative or bounded f and $x \in V$, we let

$$\pi_U^\psi f(x) = \mathbb{E}_x(f(X_{\tau_U})M_{\tau_U -}), \qquad G_U^\psi f(x) = \mathbb{E}_x\int_0^{\tau_U} f(X_t)M_t dt.$$

G_U^ψ admits a density $G_U^\psi(x,y)$, and we have $G_U^\psi(x,y) \le G_U(x,y)$, $G_U^\psi(x,y) \le G^\psi(x,y)$. For any $f \in \mathcal{D}(\mathcal{A})$, we have

$$\pi_U^\psi f(x) = G_U^\psi(\mathcal{A} - \psi)f(x) + f(x), \qquad x \in V. \tag{3.8}$$

In particular, by an approximation argument,

$$\pi_U^\psi(x, E) = \int_U G_U^\psi(x,y)J(y,E)m(dy), \qquad x \in U,\, E \subseteq \mathfrak{X} \setminus \overline{U}. \tag{3.9}$$

By the definition of ψ, we have that $(\mathcal{A} - \psi)\phi(x) \le 0$ for $x \in \mathfrak{X}$ (and, in particular, for all $x \in V$). Thus using (3.8), the proof of the next result is the same as that of [7, Lemma 4.4].

Lemma 3.2 *Let $U = V \cap B(z_0, ar)$. Then*

$$\pi_U^\psi(x, \overline{B}(z_0, ar)^c) \le \phi(x), \qquad x \in U. \tag{3.10}$$

Recall that n_0 is the natural number in (2.2). It follows from the doubling properties of Φ and V that with $c_0 > 1$

$$\Phi(n_0^k r) \le c_0^k \Phi(r) \quad \text{for all } k \ge 1, r > 0, \tag{3.11}$$

$$V(n_0 r/2)\Phi(n_0 r/2) \le c_0 V(r)\Phi(r) \quad \text{for all } r > 0. \tag{3.12}$$

Let $s \ge r_0$. Thus applying (2.5) to each annulus and the monotonicity of Φ and V we have that for $n_0^k s < d(z, z_0) \le n_0^{k+1} s$,

$$j(z_0, z) \ge \frac{c_1}{V(n_0^{k+1}s/2)\Phi(n_0^{k+1}s/2)} \ge \frac{c_1 c_0^{-1}}{V(n_0^k s)\Phi(n_0^k s)}. \tag{3.13}$$

In the second inequality above we have used (3.12). Let $s \ge r_0$. By using (3.13) in the second line, the assumption (2.2) (with the constant $c_2 > 1$) and (3.11) in the last line, we have

$$\begin{aligned}
\int_{\overline{B}(z_0,s)^c} j(z_0, z)m(dz) &= \sum_{k=0}^{\infty} \int_{n_0^k s < d(z,z_0) \le n_0^{k+1} s} j(z_0, z)m(dz) \\
&\ge \sum_{k=0}^{\infty} \int_{n_0^k s < d(z,z_0) \le n_0^{k+1} s} \frac{c_1 c_0^{-1}}{V(n_0^k s)\Phi(n_0^k s)} m(dz) \\
&= c_1 c_0^{-1} \sum_{k=0}^{\infty} \frac{V(n_0^{k+1}s) - V(n_0^k s)}{V(n_0^k s)\Phi(n_0^k s)} \\
&= c_1 c_0^{-1} \sum_{k=0}^{\infty} \left(\frac{V(n_0^{k+1}s)}{V(n_0^k s)} - 1 \right) \frac{1}{\Phi(n_0^k s)} \\
&\ge c_1(c_2 - 1) \sum_{k=0}^{\infty} c_0^{-k-1} \frac{1}{\Phi(s)} =: c_3 \frac{1}{\Phi(s)},
\end{aligned}$$

with $c_2 > 1$ and $c_1, c_3 > 0$ independent of s. In particular, again by using (2.3),

$$\int_{\overline{B}(z_0,br)^c} j(z_0, z)m(dz) \ge c_4 \frac{1}{\Phi(r)}, \quad \text{for every } b \in (1, 2] \text{ and all } r \ge r_0\,, \tag{3.14}$$

with $c_4 > 0$ independent of r and b.

Lemma 3.3 *Let $b \in (a, 2)$ and set $U = V \cap B(z_0, ar)$. There exists a constant $c = c(z_0, b/a) > 0$ such that*

$$\int_U G_U^\psi(x, y)m(dy) \le c\phi(x)\Phi(r), \qquad x \in U. \tag{3.15}$$

Proof. By Lemma 3.2, $\phi(x) \ge \pi_U^\psi(x, \overline{B}(z_0, ar)^c) \ge \pi_U^\psi(x, \overline{B}(z_0, br)^c)$. Thus, using (3.9),

$$\phi(x) \ge \int_{\overline{B}(z_0,br)^c} \int_U G_U^\psi(x, y)j(y, z)m(dy)m(dz)\,.$$

Note that, by (2.4), we have $j(y,z) \ge c_1 j(z_0,z)$ for all $(y,z) \in B(z_0,ar) \times \overline{B}(z_0,br)^c$ with $c_1 = c_1(z_0, b/a)$. Therefore, using (3.14) we conclude that

$$\phi(x) \ge c_1 \int_{\overline{B}(z_0,br)^c} j(z_0,z)m(dz) \int_U G_U^\psi(x,y)m(dy) \ge \frac{c_2}{\Phi(r)} \int_U G_U^\psi(x,y)m(dy)\,.$$

□

The following Lemma is analogous to [7, Lemma 4.5].

Lemma 3.4 *Let $b \in (a,2)$. There exists a constant $c = c(a,b) > 0$ such that*

$$G^\psi(x,y) \le c\frac{\Phi(r)}{V(r)}\phi(x), \quad x \in V \cap \overline{B}(z_0,br),\, y \in \overline{A}(z_0,2r,4r)\,.$$

Proof. If $x \in \overline{A}(z_0,ar,br)$, then $\phi(x) = 1$. Thus, by (3.3) and assumption $\mathbf{D2}(z_0,r_0)$, for $x \in \overline{A}(z_0,ar,br) \subset \overline{B}(z_0,br)$ and $y \in \overline{A}(z_0,2r,4r)$,

$$G^\psi(x,y) \le G_V(x,y) \le G_{\overline{B}(z_0,r)^c}(x,y) \le c_1\frac{\Phi(r)}{V(r)} = c_1\frac{\Phi(r)}{V(r)}\phi(x),$$

with $c_1 = c_1(b)$.

For the remainder of the proof, we assume that $x \in U := V \cap B(z_0,ar)$. Let $f \ge 0$ be supported on $A(z_0,2r,4r)$ with $\int f(w)m(dw) = 1$. Then, by the strong Markov property,

$$G^\psi f(x) = \pi_U^\psi(G^\psi f)(x) = \pi_U^\psi(\mathbf{1}_{\overline{A}(z_0,ar,br)}G^\psi f)(x) + \pi_U^\psi(\mathbf{1}_{B(z_0,br)^c}G^\psi f)(x) =: I + II.$$

First note that by $\mathbf{D2}(z_0,r_0)$ and (3.3), for $y \in \overline{A}(z_0,ar,br)$,

$$G^\psi f(y) \le \int_{\overline{A}(z_0,2r,4r)} G_V(y,w)f(w)m(dw) \le \int_{\overline{A}(z_0,2r,4r)} G_{\overline{B}(z_0,r)^c}(y,w)f(w)m(dw) \le c_2\frac{\Phi(r)}{V(r)}.$$

Thus, combining this with Lemma 3.2 we get

$$I \le \left(\sup_{y\in\overline{A}(z_0,ar,br)} G^\psi f(y)\right) \pi_U^\psi(x,\overline{A}(z_0,ar,br)) \le c_2\frac{\Phi(r)}{V(r)}\pi_U^\psi(x,V\setminus U) \le c_2\frac{\Phi(r)}{V(r)}\phi(x).$$

For II, by using (2.4) and Lemma 3.3, we get that for $z \in \overline{B}(z_0,br)^c$,

$$\begin{aligned} \int_U G_U^\psi(x,y)j(y,z)m(dy) &\le c_3 \int_U G_U^\psi(x,y)m(dy)j(z_0,z) \\ &\le c_4\phi(x)\Phi(r)j(z_0,z), \end{aligned}$$

with $c_4 = c_4(b/a)$. Thus by (3.3) and (3.9),

$$II \le c_4\phi(x)\Phi(r)\int_{\overline{B}(z_0,br)^c} G^\psi f(z)j(z_0,z)m(dz)$$

$$\leq c_4\phi(x)\Phi(r)\int_{A(z_0,2r,4r)}\int_{\overline{B}(z_0,br)^c} G_V(z,y)j(z_0,z)m(dz)f(y)m(dy)$$
$$\leq c_4\phi(x)\Phi(r)\int_{A(z_0,2r,4r)} \widehat{P}_{\overline{B}(z_0,r)^c}(y,z_0)f(y)m(dy).$$

Finally using the dual version of Proposition 3.1, we conclude that

$$II \leq c_5\phi(x)\int_{A(z_0,2r,4r)} f(y)m(dy)\frac{\Phi(r)}{V(r)} = c_5\frac{\Phi(r)}{V(r)}\phi(x)\,.$$

□

Lemma 3.5 *There exists a constant $c = c(a,z_0) > 0$ such that for all $x \in \overline{A}(z_0,2r,4r)$,*

$$\pi^\psi(x,dy)/m(dy) \leq \frac{c}{V(r)}\mathbf{1}_{\overline{B}(z_0,ar)}(y). \tag{3.16}$$

Proof. Let $b := a/2+1$ so that $b \in (a,2)$. First note that ψ vanishes on $\mathfrak{X}\setminus B(z_0,ar)$. Thus

$$\pi^\psi(y,\mathfrak{X}\setminus B(z_0,ar)) = 0, \quad y \in V. \tag{3.17}$$

Fix $x \in \overline{A}(z_0,2r,4r)$. If f is a non-negative function on $\mathfrak{X}$ vanishing in $\mathfrak{X}\setminus(B(z_0,ar)\cap V)$, then by (3.5) and the dual version of Lemma 3.4 (together with $G^\psi(x,y) = \widehat{G}^\psi(y,x)$),

$$\pi^\psi f(x) \leq c_1\frac{\Phi(r)}{V(r)}\int_{V\cap B(z_0,ar)} \phi(y)\psi(y)f(y)m(dy). \tag{3.18}$$

Since for $y \in B(z_0,ar)$ we have $\phi(y)\psi(y) \leq c_2(\Phi(r))^{-1}$ by the definition in (3.1) and assumption **B2-a**(z_0,r_0) (with a constant $c_3 = c_3(a)$ possibly different from the one in **B2-a**(z_0,r_0)), we have

$$\pi^\psi f(x) \leq \frac{c_3}{V(r)}\int_{V\cap B(z_0,ar)} f(y)m(dy). \tag{3.19}$$

On the other hand, if $g \in \mathcal{D}(\mathcal{A})$ vanishes in V then by (3.6),

$$\pi^\psi g(x) = \int_{\mathfrak{X}\setminus V}\left(\int_V G^\psi(x,y)j(y,z)m(dy)\right)g(z)m(dz). \tag{3.20}$$

Assume $z \in \mathfrak{X}\setminus V \subset B(z_0,ar)$ and let

$$I := \int_{V\cap B(z_0,br)} G^\psi(x,y)j(y,z)m(dy) \quad \text{and} \quad II := \int_{\overline{B}(z_0,br)^c} G^\psi(x,y)j(y,z)m(dy).$$

We now consider I and II separately.

By the dual version of Lemma 3.4, assumption **B2-a**(z_0,r_0) and the fact that $\widehat{\mathcal{A}}\phi(z) = \widehat{J}\phi(z)$, for some $c_4 = c_4(a) > 0$,

$$I \leq c_4\frac{\Phi(r)}{V(r)}\int_{V\cap B(z_0,br)} \phi(y)j(y,z)m(dy) = c_4\frac{\Phi(r)}{V(r)}\widehat{\mathcal{A}}\phi(z) \leq \frac{c_4}{V(r)}. \tag{3.21}$$

On the other hand, by assumption $\mathbf{C2}(z_0, r_0)$ and (3.3), for some $c_5 = c_5(a) > 0$,

$$
\begin{aligned}
II &\le c_5 \int_{\overline{B}(z_0,br)^c} G^\psi(x,y) j(y,z_0) m(dy) \\
&\le c_5 \int_{\overline{B}(z_0,br)^c} G_V(x,y) j(y,z_0) m(dy) \le c_5 P_V(x,z_0),
\end{aligned}
\tag{3.22}
$$

which is less than or equal to $c_6 V(r)^{-1}$ by Proposition 3.1. Therefore, (3.21) and (3.22) imply that for all $g \in \mathcal{D}(\mathcal{A})$ vanishing in V we have

$$
\pi^\psi g(x) \le \frac{c_7}{V(r)} \int_{\mathfrak{X}\setminus V} g(z) m(dz). \tag{3.23}
$$

Since $\mathcal{D}(\mathcal{A})$ is dense in $C_0(\mathcal{X})$ we have $\pi_\psi(x,dy)/m(dy) \le c_7/V(r)$ on V^c too. □

Corollary 3.6 *Let f be a non-negative function on $\mathfrak{X}$ and x a point in $\overline{A}(z_0, 2r, 4r)$ such that $f(x) \le \mathbb{E}_x f(X_\tau)$ for every stopping time $\tau \le \tau_{\overline{B}(z_0,r)^c}$. Then*

$$
f(x) \le \frac{c}{V(r)} \int_{\overline{B}(z_0,ar)} f(y) m(dy)\,, \tag{3.24}
$$

where $c = c(a)$ is the constant from Lemma 3.5.

Proof. Recall from (3.4) that $\pi^\psi f(x) = \int_0^\infty \mathbb{E}_x(f(X_{\tau_a}))e^{-a} da$. Since $\tau_a \le \tau_V \le \tau_{\overline{B}(z_0,r)^c}$, we have that $f(x) \le \mathbb{E}_x f(X_{\tau_a})$, and therefore $f(x) \le \pi^\psi f(x)$. Thus by (3.16),

$$
f(x) \le \int f(y) \pi^\psi(x,dy) \le \frac{c}{V(r)} \int_{\overline{B}(z_0,ar)} f(y) m(dy).
$$

□

Lemma 3.7 *There exists $c = c(z_0, a) > 0$ such that for any $r \ge r_0$ and any open set $D \subset \overline{B}(z_0, r)^c$ we have*

$$
\mathbb{P}_x\left(X_{\tau_D} \in B(z_0, r)\right) \le c\,V(r) P_D(x, z_0)\,, \qquad x \in D \cap \overline{B}(z_0, ar)^c\,.
$$

Proof. (2.6) says that for any $f \in \mathcal{D}(\overline{B}(z_0,r), B(z_0,ar))$, we have

$$
\max(\mathcal{A}f(z), \hat{\mathcal{A}}f(z)) \le c \mathbf{1}_{B(z_0,r)^c}(z) V(r) j(z,z_0)
$$

for some $c(z_0, a) > 0$ independent of $r \ge r_0$. Thus, by Dynkin's formula we have

$$
\begin{aligned}
\mathbb{E}_x\left[f(X_{\tau_D})\right] &= \int_D G_D(x,z) \mathcal{A}f(z)\, dz \\
&\le cV(r) \int_D G_D(x,z) j(z,z_0)\, dz = cV(r) P_D(x,z_0).
\end{aligned}
\tag{3.25}
$$

Finally, since $\mathbf{1}_{\overline{B}(z_0,r)} \le f$, $\mathbb{P}_x(X_{\tau_D} \in \overline{B}(z_0,r)) \le \mathbb{E}_x\left[f(X_{\tau_D})\right] \le cV(z_0,r) P_D(x,z_0)$. □

Proposition 3.8 *Let $b \in (a, 2)$. There exists $c = c(z_0, a, b) > 1$ such that for any $r \geq r_0$, any open set $D \subset \overline{B}(z_0, r)^c$ and any non-negative function u on $\mathfrak{X}$ which is regular harmonic with respect to X in D and vanishes on $\overline{B}(z_0, r)^c \cap \left(\overline{D}^c \cup D^{\text{reg}}\right)$, it holds that*

$$
\begin{aligned}
c^{-1} P_{D \cap \overline{B}(z_0, 2br)^c}(x, z_0) \int_{B(z_0, 2ar)} u(z)\, m(dz) &\leq u(x) \\
&\leq c P_{D \cap \overline{B}(z_0, 2br)^c}(x, z_0) \int_{B(z_0, 2ar)} u(z)\, m(dz) \qquad (3.26)
\end{aligned}
$$

for all $x \in D \cap \overline{B}(z_0, 4r)^c$.

Proof. Let $O := D \cap \overline{B}(z_0, 2br)^c$, $D_1 := \overline{A}(z_0, 2ar, 2br)$ and $D_2 := B(z_0, 2ar)$. By the harmonicity of u,

$$
u(x) = \mathbb{E}_x[u(X_{\tau_O})] = \mathbb{E}_x[u(X_{\tau_O}) : X_{\tau_O} \in D_1] + \mathbb{E}_x[u(X_{\tau_O}) : X_{\tau_O} \in D_2]\,, \quad x \in D. \qquad (3.27)
$$

Let D^{irr} be the set of points in ∂D which are irregular for D^c with respect to X. Since u vanishes on $\overline{B}(z_0, r)^c \cap \left(\overline{D}^c \cup D^{\text{reg}}\right)$, it follows that $u(y) \leq \mathbb{E}_y u(X_\tau)$ for every stopping time $\tau \leq \tau_{\overline{B}(z_0, r)^c}$ and every $y \in \overline{B}(z_0, r)^c \setminus D^{\text{irr}}$. Since D^{irr} is polar with respect to X, we see that $X_{\tau_O} \notin D^{\text{irr}}$. It follows from Corollary 3.6 and Lemma 3.7 that for all $x \in D \cap \overline{B}(z_0, 4r)^c$,

$$
\begin{aligned}
\mathbb{E}_x[u(X_{\tau_O}) : X_{\tau_O} \in D_1] &\leq \left(\sup_{y \in D_1 \setminus D^{\text{irr}}} u(y) \right) \mathbb{P}_x(X_{\tau_O} \in D_1) \\
&\leq c_1 V(br) P_O(x, z_0) \frac{c_2}{V(r)} \int_{B(z_0, a^2 r)} u(z)\, m(dz) \qquad (3.28) \\
&\leq c_3 P_O(x, z_0) \int_{B(z_0, 2ar)} u(z)\, m(dz), \qquad (3.29)
\end{aligned}
$$

where $c_3 = c_3(a, b)$. On the other hand, by assumption $\mathbf{C2}(z_0, r_0)$, for all $x \in D \cap \overline{B}(z_0, 4r)^c$,

$$
\begin{aligned}
\mathbb{E}_x[u(X_{\tau_O}) : X_{\tau_O} \in D_2] &= \int_{B(z_0, 2ar)} \int_O G_O(x, y) j(y, z) m(dy) u(z) m(dz) \\
&\asymp \int_{B(z_0, 2ar)} \int_O G_O(x, y) j(y, z_0) m(dy) u(z) m(dz) \\
&= P_O(x, z_0) \int_{B(z_0, 2ar)} u(z) m(dz). \qquad (3.30)
\end{aligned}
$$

The proposition now follows from (3.27)–(3.30). □

Lemma 3.9 *For any $b \in (a, 2)$ there exists $c = c(a, b) > 0$ such that for every $r \geq r_0$ and every open set $D \subset \overline{B}(z_0, r)^c$,*

$$
P_{D \cap \overline{B}(z_0, 2br)^c}(x, z_0) \leq P_D(x, z_0) \leq c P_{D \cap \overline{B}(z_0, 2br)^c}(x, z_0)\,, \qquad x \in D \cap \overline{B}(z_0, 2abr)^c\,.
$$

Proof. First note that, since D is Greenian, by the strong Markov property for all open set $U \subset D$, $G_D(x,y) = G_U(x,y) + \mathbb{E}_x\left[G_D(X_{\tau_U}, y); \tau_U < \infty\right]$ for every $(x,y) \in \mathfrak{X} \times \mathfrak{X}$. Thus

$$\begin{aligned} P_D(x,z_0) &= P_{D\cap\overline{B}(z_0,2br)^c}(x,z_0) \\ &\quad + \mathbb{E}_x[P_D(X_{\tau_{D\cap\overline{B}(z_0,2br)^c}}, z_0) : X_{\tau_{D\cap\overline{B}(z_0,2br)^c}} \in B(z_0,2br)\setminus\overline{B}(z_0,r), \tau_{D\cap\overline{B}(z_0,2br)^c} < \infty]. \end{aligned}$$

By Proposition 3.1, Lemma 3.7 and the doubling property, for $x \in D \cap \overline{B}(z_0, 2abr)^c$,

$$\begin{aligned} &\mathbb{E}_x[P_D(X_{\tau_{D\cap\overline{B}(z_0,2br)^c}}, z_0) :\ X_{\tau_{D\cap\overline{B}(z_0,2br)^c}} \in B(z_0,2br)\setminus\overline{B}(z_0,r)] \\ \leq\ & \left(\sup_{z\in B(z_0,2br)\setminus\overline{B}(z_0,r)} P_D(z,z_0)\right) \mathbb{P}_x(X_{\tau_{D\cap\overline{B}(z_0,2br)^c}} \in \overline{B}(z_0,2br)) \\ \leq\ & c_1 \frac{V(2br)}{V(r)} P_{D\cap\overline{B}(z_0,2br)^c} \leq c_2 P_{D\cap\overline{B}(z_0,2br)^c}(x,z_0)\,. \end{aligned}$$

This finishes the proof. □

Proof of Theorem 2.1 Let $a \in (1,2)$ and choose $b = a/2+1$. Let $D \subset \overline{B}(z_0,r)^c$ and let u be a non-negative function on $\mathfrak{X}$ which is regular harmonic with respect to X in D and vanishes on $\overline{B}(z_0,r)^c \cap \left(\overline{D}^c \cup D^{\mathrm{reg}}\right)$. Since $\overline{B}(z_0,8r)^c \subset \overline{B}(z_0,4r)^c \cap \overline{B}(z_0,2abr)^c$, it follows from Proposition 3.8 and Lemma 3.9 that

$$u(x) \asymp P_D(x,z_0) \int_{B(z_0,2ar)} u(y)m(dy)\,, \qquad x \in D \cap \overline{B}(z_0,8r)^c\,,$$

with a constant depending on a. □

4 Finite boundary point

The goal of this section is to state an analog of Theorem 2.1 for finite boundary points. Again, recall that assumptions **A**, **B** and **C** are in force. Recall that $R_0 \in (0,\infty]$ is the localization radius and that $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are the generators of (P_t) and $(\widehat{P}_t)$ in $C_0(\mathfrak{X})$. The processes X and $\widehat{X}$ are not assumed to be conservative.

Similarly as in Section 2 we fix a point $z_0 \in \mathfrak{X}$ which now serves as a boundary point of an open set. For $r > 0$, we let $V(r) = V(z_0,r) := m(B(z_0,r))$ and assume that the volume function $V : [0,\infty) \to [0,\infty)$ satisfies (2.1) and, instead of (2.2), we assume that there exist $c > 1$ and $r_0 \in (0, R_0]$ and $n_0 \in \mathbb{N}$ with $n_0 \geq 2$ such that

$$V(n_0 r) \geq cV(r), \quad r \leq r_0\,. \tag{4.1}$$

We also assume the existence of an increasing function $\Phi = \Phi(z_0,\cdot) : [0,\infty) \to [0,\infty)$ satisfying the doubling property (2.3) which again will be crucial in obtaining the scale invariant results.

Similarly as in Section 2 we introduce some additional assumptions. Recall the notation $\mathcal{D}(K,D)$ from assumption **B**. In the following assumptions, r_0 is a number in $(0,R_0]$.

B1-a(z_0,r_0): For any $a\in(1/2,1)$, there exists $c=c(z_0,a)$ such that for any $r<r_0$,

$$\rho(r):=\inf_{f\in\mathcal{D}(\overline{B}(z_0,ar),B(z_0,r))}\sup_{x\in\mathfrak{X}}\max(\mathcal{A}f(x),\widehat{\mathcal{A}}f(x))\le\frac{c}{\Phi(r)}.$$

B1-b(z_0,r_0): For any $a\in(1/2,1)$, there exists $c=c(z_0,,a)$ such that for any $r<r_0$ and any $f\in\mathcal{D}(\overline{B}(z_0,ar),B(z_0,r))$,

$$\max(\mathcal{A}f(x),\widehat{\mathcal{A}}f(x))\le cV(r)j(x,z_0),\quad x\in\overline{A}(z_0,ar,(a+1)r).$$

B1-c(z_0,r_0): For any $1/2<b<a<1$, there exists $c=c(z_0,,a,b)$ such that for any $r<r_0$,

$$\inf_{f\in\mathcal{D}(\overline{A}(z_0,br,ar),A(z_0,r/2,r))}\sup_{x\in\mathfrak{X}}\max(\mathcal{A}f(x),\widehat{\mathcal{A}}f(x))\le\frac{c}{\Phi(r)}.$$

To assumption **C** we add

C1(z_0,r_0): For any $a\in(1/2,1)$, there exists $c=c(a,r_0,z_0)$ such that for $r<r_0$, $x\in B(z_0,ar)$ and $y\in\mathfrak{X}\setminus B(z_0,r)$,

$$c^{-1}j(z_0,y)\le j(x,y)\le cj(z_0,y),\qquad c^{-1}\widehat{j}(z_0,y)\le\widehat{j}(x,y)\le c\widehat{j}(z_0,y),\tag{4.2}$$

and

$$\inf_{y\in\overline{A}(z_0,ar,r)}\min(j(z_0,y),\widehat{j}(z_0,y))\ge\frac{c}{V(r)\Phi(r)}.\tag{4.3}$$

Note that by assumption **C1**(z_0,r_0), the function f in assumption **B1-b**(z_0,r_0) satisfies

$$\max(\mathcal{A}f(x),\widehat{\mathcal{A}}f(x))\le c\mathbf{1}_{B(z_0,ar)^c}(x)V(r)j(x,z_0)\tag{4.4}$$

for a constant $c=c(z_0,a)>0$. In fact, for $f\in\mathcal{D}(\mathcal{A})\cap\mathcal{D}(\widehat{\mathcal{A}})$ such that $f(x)=1$ for $x\in B(z_0,ar)$, $f(x)=0$ for $x\in\mathfrak{X}\setminus\overline{B}(z_0,r)$ and $0\le f(x)\le1$ for $x\in\mathfrak{X}$, we have

$$\begin{aligned}\mathcal{A}f(x)\mathbf{1}_{\mathfrak{X}\setminus\overline{B}(z_0,(a+1)r)}(x)&=\mathbf{1}_{\mathfrak{X}\setminus\overline{B}(z_0,(a+1)r)}(x)\int_{B(z_0,r)}f(y)j(x,y)m(dy)\\&\le c\mathbf{1}_{\mathfrak{X}\setminus\overline{B}(z_0,(a+1)r)}(x)V(r)j(x,z_0).\end{aligned}$$

The final assumption concerns Green functions of balls.

D1(z_0,r_0): For any $a\in(1/2,1)$ there exists a constant $c=c(z_0,r_0,a)$ such that for all $r<r_0$,

$$\sup_{x\in\overline{A}(z_0,ar,r)}\sup_{y\in B(z_0,r/2)}\max(G_{B(z_0,r)}(x,y),\widehat{G}_{B(z_0,r)}(x,y))\le c\frac{\Phi(r)}{V(r)}.$$

Theorem 4.1 *Let $z_0 \in \mathfrak{X}$. Assume that, in addition to* **A**, **B** *and* **C**, *the assumptions* (2.1), (4.1), (2.3), **B1-a**(z_0, r_0), **B1-b**(z_0, r_0), **B1-c**(z_0, r_0), **C1**(z_0, r_0) *and* **D1**(z_0, r_0) *hold true for some $r_0 \in (0, R_0]$. For any $a \in (1/2, 1)$, there exists $C_1 = C_1(z_0, r_0, a) > 1$ such that for any $r < r_0/(2n_0)$, any open set $D \subset B(z_0, r)$ and any non-negative function u on $\mathfrak{X}$ which is regular harmonic with respect to X in D and vanishes on $B(z_0, r) \cap \left(\overline{D}^c \cup D^{\mathrm{reg}}\right)$, it holds that*

$$
\begin{aligned}
C_1^{-1}\, \mathbb{E}_x \tau_D \int_{\overline{B}(z_0, ar/2)^c} j(z_0, z) u(z)\, m(dz) &\le u(x) \\
&\le C_1\, \mathbb{E}_x \tau_D \int_{\overline{B}(z_0, ar/2)^c} j(z_0, z) u(z)\, m(dz) \qquad (4.5)
\end{aligned}
$$

for all $x \in D \cap B(z_0, r/8)$.

As a consequence of Theorem 2.1, one immediately gets the following scale invariant uniform boundary Harnack principle.

Corollary 4.2 (Boundary Harnack Principle) *Let $z_0 \in \mathfrak{X}$. Suppose that, in addition to* **A**, **B** *and* **C**, *the assumptions* (2.1), (4.1), (2.3), **B1-a**(z_0, r_0), **B1-b**(z_0, r_0), **B1-c**(z_0, r_0), **C1**(z_0, r_0) *and* **D1**(z_0, r_0) *hold true for some $r_0 \in (0, R_0]$. There exists $C_2 = C_2(z_0, r_0) > 1$ such that for any $r < r_0/(2n_0)$, any open set $D \subset B(z_0, r)$ and any non-negative functions u and v on $\mathfrak{X}$ which are regular harmonic with respect to X in D and vanish on $B(z_0, r) \cap \left(\overline{D}^c \cup D^{\mathrm{reg}}\right)$, it holds that*

$$
C_2^{-1} \frac{u(y)}{v(y)} \le \frac{u(x)}{v(x)} \le C_2 \frac{u(y)}{v(y)}, \qquad \text{for all } x, y \in D \cap B(z_0, r/8). \qquad (4.6)
$$

5 Examples

Example 5.1 Let $X = (X_t, \mathbb{P}_x)$ be a purely discontinuous symmetric Lévy process in $\mathbb{R}^d$ with Lévy exponent $\Psi(\xi)$ so that

$$
\mathbb{E}_x \left[e^{i\xi \cdot (X_t - z_0)} \right] = e^{-t\Psi(\xi)}, \qquad t > 0, x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.
$$

Thus the state space $\mathfrak{X} = \mathbb{R}^d$, the measure m is the d-dimensional Lebesgue measure and the localization radius $R_0 = \infty$. Assume that $r \mapsto j_0(r)$ is a strictly positive and nonincreasing function on $(0, \infty)$ satisfying

$$
j_0(r) \le c j_0(r+1), \qquad r > 1, \qquad (5.1)
$$

for some $c > 1$, and the Lévy measure of X has a density J such that

$$
\gamma^{-1} j_0(|y|) \le J(y) \le \gamma j_0(|y|), \qquad y \in \mathbb{R}^d, \qquad (5.2)
$$

for some $\gamma > 1$. Since $\int_0^\infty j_0(r)(1 \wedge r^2) r^{d-1} dr < \infty$ by (5.2), the function $x \to j_0(|x|)$ is the Lévy density of an isotropic unimodal Lévy process whose characteristic exponent is

$$\Psi_0(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) j_0(|y|) dy. \tag{5.3}$$

The Lévy exponent Ψ can be written as

$$\Psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) J(y) dy$$

and, clearly by (5.2), it satisfies

$$\gamma^{-1} \Psi_0(|\xi|) \le \Psi(\xi) \le \gamma \Psi_0(|\xi|), \quad \text{for all } \xi \in \mathbb{R}^d. \tag{5.4}$$

The function Ψ_0 may not be increasing. However, if we put $\Psi_0^*(r) := \sup_{s \le r} \Psi_0(s)$, then, by [5, Proposition 2] (cf. also [14, Proposition 1]), we have

$$\Psi_0(r) \le \Psi_0^*(r) \le \pi^2 \Psi_0(r).$$

Thus by (5.4),

$$(\pi^2 \gamma)^{-1} \Psi_0^*(|\xi|) \le \Psi(\xi) \le \gamma \Psi_0^*(|\xi|), \quad \text{for all } \xi \in \mathbb{R}^d. \tag{5.5}$$

Under the above assumptions, our process X obviously satisfies Assumptions **A**, **B** and **C**.

Let $\Phi(r) = (\Psi_0^*(r^{-1}))^{-1}$. Since X is a purely discontinuous symmetric Lévy process, we can write down the generator $\mathcal{A}$ of X explicitly in terms of the Lévy density. Using this explicit formula and [14, Corollary 1] one can easily check that Assumptions **B1-a**(z_0, r_0), **B1-b**(z_0, r_0), **B1-c**(z_0, r_0) and **B2-a**(z_0, r_0) are also satisfied for all $z_0 \in \mathbb{R}^d$ and $r_0 > 0$.

Suppose now that Ψ_0 satisfies the following scaling condition at infinity:

H1: There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \left(\frac{t}{s}\right)^{2\delta_1} \le \frac{\Psi_0(t)}{\Psi_0(s)} \le a_2 \left(\frac{t}{s}\right)^{2\delta_2}, \quad t \ge s \ge 1. \tag{5.6}$$

Then by [5, (15) and Corollary 22], for every $R > 0$, there exists $c = c(R) > 1$ such that

$$c^{-1} \frac{\Psi_0(r^{-1})}{r^d} \le j_0(r) \le c \frac{\Psi_0(r^{-1})}{r^d} \quad \text{for } r \in (0, R]. \tag{5.7}$$

Using (5.1), (5.7) and [24, Lemma 2.7], one can easily see that, there exists $r_1 > 0$ such that Assumption **C1**(z_0, r_0) and Assumption **D1**(z_0, r_0) are satisfied for all $z_0 \in \mathbb{R}^d$ and $r_0 \le (0, r_1]$.

Now we assume, instead of **H1**, that Ψ_0 satisfies the following scaling condition at the origin:

H2: There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \left(\frac{t}{s}\right)^{2\delta_3} \le \frac{\Psi_0(t)}{\Psi_0(s)} \le a_4 \left(\frac{t}{s}\right)^{2\delta_4}, \quad s \le t \le 1. \tag{5.8}$$

It follows from [5, Corollary 7] or [15, Theorem 2.2] (see [5, (15)]) and (5.2) that, there exists $c_1 > 1$ such that

$$J(x) \le \gamma j_0(|x|) \le c\gamma \frac{\Psi_0(|x|^{-1})}{r^d} \quad \text{for } x \in \mathbb{R}^d \setminus \{0\}. \tag{5.9}$$

We now prove a matching lower bound for j_0 away from the origin. The proof is similar to that of [5].

Let Y be the isotropic unimodal Lévy process whose characteristic exponent is $\Psi_0(|\xi|)$ and $x \to p_t^0(|x|)$ be its transition density. Let $f_t(r) := \mathbb{P}(|Y_t|^2 > r)$ for $r \ge 0$ and $t > 0$. Then, using [5, Lemma 4 and (13)], $\mathcal{L}f_t$, the Laplace transform of f_t, satisfies that for all $0 < u < v \le 1$,

$$\frac{\mathcal{L}f_t(v)}{\mathcal{L}f_t(u)} \le c_2(v/u)^{-1} \frac{1 - e^{-\pi^2 t\Psi_0(\sqrt{v})}}{1 - e^{-t\Psi_0(\sqrt{u})}} \le c_2(v/u)^{-1} \frac{1 - e^{-\pi^2 t\Psi_0(\sqrt{u}) a_4 (v/u)^{\delta_4}}}{1 - e^{-t\Psi_0(\sqrt{u})}} \le c_3(v/u)^{\delta_4 - 1}.$$

Thus, by [22, Proposition 2.3] and [5, Lemma 4],

$$\mathbb{P}(|Y_t| \ge r) = f_t(r^2) \ge c_4 \mathcal{L}f_t(r^{-2}) \ge 2c_5(1 - e^{-t\Psi_0^*(1/r)}), \qquad r \ge 1. \tag{5.10}$$

Let $a \ge 2$. Since $r \to p_t^0(r)$ is decreasing, we have

$$p_t^0(r) \ge \frac{\mathbb{P}(r \le |Y_t| < ar)}{|B(0, ar) \setminus B(0, r)|} = \frac{c_6}{a^d - 1} r^{-d} (P(|Y_t| \ge r) - \mathbb{P}(|Y_t| \ge ar)). \tag{5.11}$$

Let $r \ge 1$ and $t\Psi_0^*(1/r) \le 1$. Using (5.10), the inequality $s/2 \le 1 - e^{-s} \le s$ for $s \in (0, 1]$, and [5, Corollary 6], we get

$$\begin{aligned} \mathbb{P}(|Y_t| \ge r) - \mathbb{P}(|Y_t| \ge ar) &\ge c_5 t\Psi_0^*(1/r) - \frac{2e}{e-1}(2d+1) t\Psi_0^*(1/ar) \\ &\ge c_5 t\Psi_0^*(1/r)\Big(1 - c_7 \frac{\Psi_0^*(1/ar)}{\Psi_0^*(1/r)}\Big). \end{aligned} \tag{5.12}$$

Choose $a \ge 2$ large enough so that for $ar \ge 1$,

$$c_7 \frac{\Psi_0^*(1/ar)}{\Psi_0^*(1/r)} \le c_7 a_3^{-1} \kappa^{-2\delta_3} \le \frac{1}{2}. \tag{5.13}$$

Then, combining (5.11)–(5.13), we obtain

$$p_t^0(r) \ge c_8 t\Psi_0^*(1/r) r^{-d}, \quad r \ge 1/a, t\Psi_0^*(1/r) \le 1,$$

which, together with (5.2) and the fact that J is the weak limit of p_t^0, implies

$$J(x) \ge \gamma^{-1} j_0(|x|) \ge c_9 \frac{\Psi_0(|x|^{-1})}{|x|^d} \quad \text{for } |x| \ge 1. \tag{5.14}$$

Hence, $\mathbf{C2}(z_0, r_0)$ is valid.

If $d \geq 3$, by [14, (14) and p. 26], the Green function $G(x)$ of X has the following upper bound:

$$\int_{B(0,r)} G(x)dx \leq \frac{c_{10}}{\Psi_0(r^{-1})}.$$

We further assume that there exists a positive constant $c_{11} > 1$ and a non-increasing function $r \to G_0(r)$ such that

$$c_{11}^{-1} G_0(|x|) \leq G(x) \leq c_{11} G_0(|x|), \quad x \in \mathbb{R}^d. \tag{5.15}$$

Then we have that for all $x \in \mathbb{R}^d$,

$$\begin{aligned} G(x) &\leq c_{11} G_0(|x|) \leq c_{12}|x|^{-d} \int_{B(0,|x|)} G_0(|y|)dy \\ &\leq c_{12}c_{11}|x|^{-d} \int_{B(0,|x|)} G(y)dy \leq \frac{c_{12}c_{13}c_{10}}{|x|^d \Psi_0(|x|^{-1})}. \end{aligned} \tag{5.16}$$

It follows immediately from (5.9), (5.14) and (5.16) (when $d \geq 3$) that $\mathbf{C2}(z_0, r_0)$ and $\mathbf{D2}(z_0, r_0)$ are satisfied for all $z_0 \in \mathbb{R}^d$ and all $r_0 \geq 1$. We remark here that, by [20, Lemma 3.3], the upper bound $G(x) \leq \frac{c}{|x|^d \Psi_0(|x|^{-1})}$ holds for $d > 2\delta_4$ when X is a subordinate Brownian motion whose Laplace exponent ϕ is a complete Bernstein function and that $\xi \mapsto \phi(|\xi|^2)$ satisfies Assumption **H2**.

Using Assumption **H2** and the explicit form of the the generator, one can easily check that Assumption **B2-b**(z_0, r_0) is also satisfied for all $z_0 \in \mathbb{R}^d$ and all $r_0 \geq 1$ (e.g. see [23, (3.4)]). Thus, under the assumptions above, **B2-a**(z_0, r_0), **B2-b**(z_0, r_0), $\mathbf{C2}(z_0, r_0)$ and $\mathbf{D2}(z_0, r_0)$ all hold.

Suppose that $\alpha \in (0, 2)$. The subordinate Brownian motion in $\mathbb{R}^d$ via a subordinator with Laplace exponent $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ is called a geometric α-stable process. Let $\phi^{(1)}(\lambda) = \log(1 + \lambda^{\alpha/2})$. For $n > 1$, let $\phi^{(n)}(\lambda) = \phi^{(1)}(\phi^{(n-1)}(\lambda))$. A subordinate Brownian motion in $\mathbb{R}^d$ via a subordinator with Laplace exponent $\phi^{(n)}$ is called an n-iterated geometric α-stable process. It is clear that geometric α-stable and n-iterated geometric α-stable processes satisfy condition **H2** and (5.15) and, again by [20, Lemma 3.3], for $d > 2\alpha$ the upper bound $G(x) \leq \frac{c_{(n)}}{|x|^d \phi^{(n)}(|x|^{-2})}$ holds. Hence, for the geometric α-stable and n-iterated geometric α-stable processes, **B2-a**(z_0, r_0), **B2-b**(z_0, r_0), $\mathbf{C2}(z_0, r_0)$ and $\mathbf{D2}(z_0, r_0)$ all hold.

Example 5.2 Suppose that $\alpha \in (0, 2)$, $d \geq 2$ and that $X = (X_t, \mathbb{P}_x)$ is a strictly α-stable process in $\mathbb{R}^d$. Let S be the unit sphere $S = \{x \in \mathbb{R}^d : |x| = 1\}$. For $\alpha \in (0, 1)$, X is strictly α-stable if and only if X is a Lévy process with Lévy exponent

$$\Psi(\xi) = \int_S \lambda(d\theta) \int_0^\infty (1 - e^{ir\theta\cdot\xi}) r^{-1-\alpha} dr$$

for some finite measure λ on S. For $\alpha = 1$, X is strictly α-stable if and only if X is a Lévy process with Lévy exponent

$$\Psi(\xi) = \int_S \lambda(d\theta) \int_0^\infty (1 - e^{ir\theta\cdot\xi} + ir\theta\cdot\xi 1_{(0,1]}) r^{-2} dr + i\gamma \cdot \xi$$

for some $\gamma \in \mathbb{R}^d$ and some finite measure λ on S satisfying $\int_S \theta\lambda(d\theta) = 0$. For $\alpha \in (1, 2)$, X is strictly α-stable if and only if X is a Lévy process with Lévy exponent

$$\Psi(\xi) = \int_S \lambda(d\theta) \int_0^\infty (1 - e^{ir\theta\cdot\xi} + ir\theta \cdot \xi 1_{(0,1]}) r^{-1-\alpha} dr$$

for some finite measure λ on S.

It follows from [27] that every semipolar set of X is a polar set. We will assume that λ has a density with respect to the surface measure σ on S which is bounded between two positive numbers. Since X is a a strictly α-stable process, it automatically satisfies Assumption **A**. Let $\Phi(r) = r^\alpha$. By our assumption, it is obvious that Assumptions **C**, and **C1**(z_0, r_0) and **C2**(z_0, r_0) are satisfied for all $z_0 \in \mathbb{R}^d$ and $r_0 \in (0, \infty)$. It follows from [31, (4.3)] that **D1**(z_0, r_0) and **D2**(z_0, r_0) are satisfied for all $z_0 \in \mathbb{R}^d$ and $r_0 \in (0, \infty)$. Since the generators of X and its dual can be written out explicitly, one can easily check that Assumptions **B**, **B1-a**(z_0, r_0), **B1-b**(z_0, r_0), **B1-c**(z_0, r_0), **B2-a**(z_0, r_0) and **B2-b**(z_0, r_0) are satisfied for all $z_0 \in \mathbb{R}^d$ and $r_0 \in (0, \infty)$.

Example 5.3 Suppose that $(\mathfrak{X}, d, m)$ is an unbounded Ahlfors regular n-space for some $n > 0$. Assume that d is uniformly equivalent to the shortest-path metric in $\mathfrak{X}$. Suppose that there is a diffusion process Z with a symmetric, continuous transition density $p_t^Z(x, y)$ satisfying the following sub-Gaussian bounds

$$\frac{c_1}{t^{n/d_w}} \exp\left(-c_2\left(\frac{(d(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right) \le p_t^Z(x,y)$$
$$\le \frac{c_3}{t^{n/d_w}} \exp\left(-c_4\left(\frac{(d(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right), \tag{5.17}$$

for all $x, y \in \mathfrak{X}$ and $t > 0$. Here $d_w \ge 2$ is the walk dimension of the space $\mathfrak{X}$. Let T be a subordinator, independent of Z, with Laplace exponent ϕ. We define a process X by $X_t = Z_{T_t}$. Then X is a symmetric Feller process and Assumption **A** is clearly satisfied.

In this example, we will assume that the Laplace exponent ϕ is a complete Bernstein function satisfying the following assumption:

H: There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \left(\frac{t}{s}\right)^{\delta_1} \le \frac{\phi(t)}{\phi(s)} \le a_2 \left(\frac{t}{s}\right)^{\delta_2}, \quad t \ge s > 0\,. \tag{5.18}$$

Under this condition, by using [22, Corollary 2.4 and Proposition 2.5] and repeating the argument of [22, Lemmas 3.1–3.3] we obtain that that the jumping intensity J of X satisfies

$$J(x,y) \asymp d(x,y)^{-n}\phi(d(x,y)^{-d_w}), \qquad x, y \in \mathfrak{X}$$

and that, when $2\delta_2 < n/d_w$, X is transient and its Green function G satisfies

$$G(x,y) \asymp d(x,y)^{-n}(\phi(d(x,y)^{-d_w}))^{-1}, \qquad x, y \in \mathfrak{X}.$$

Using these, one can easily see that Assumptions **C**, and **C1**(z_0, r_0), **C2**(z_0, r_0), **D1**(z_0, r_0) and **D2**(z_0, r_0) are satisfied for all $z_0 \in \mathfrak{X}$ and $r_0 \in (0, \infty)$. By using [7, Proposition A.3] and repeating the argument of [7, Corollary A.4], one can easily show that Assumptions **B**, **B1-a**(z_0, r_0), **B1-b**(z_0, r_0), **B1-c**(z_0, r_0), **B2-a**(z_0, r_0) and **B2-b**(z_0, r_0) are satisfied for all $z_0 \in \mathfrak{X}$ and $r_0 \in (0, \infty)$.

Example 5.4 Suppose that T is a subordinator with Laplace exponent ϕ. In this example, we will assume that the Laplace exponent ϕ is a complete Bernstein function satisfying the assumption **H** in the previous example.

Suppose that W is a Brownian motion in $\mathbb{R}^d$, independent of T, with generator Δ. The process X defined by $X_t = W_{T_t}$ is a subordinate Brownian motion. X is a symmetric Lévy process with Lévy exponent $\phi(|\xi|^2)$ and its Lévy density is given by $J_0(x) = j_0(|x|)$ with

$$j_0(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} \mu(t)dt, \qquad r > 0,$$

where $\mu(t)$ is the Lévy density of T. It follows from [22, Theorem 3.4] that

$$j_0(r) \asymp r^{-d}\phi(r^{-2}), \qquad r > 0. \tag{5.19}$$

Suppose that $k(x, y)$ is a symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ which is bounded between two positive constants. The symmetric form $(\mathcal{E}, C_c^2(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y))k(x, y)j_0(|x - y|)dxdy$$

is closable, and so its minimal extension $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form. Let X be the symmetric Markov process on $\mathbb{R}^d$ associated with $(\mathcal{E}, \mathcal{F})$. It follows from [11] that X is a conservative Feller process and it admits a transition density $p(t, x, y)$ satisfying the following estimates: for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c^{-1}\left(\Phi^{-1}(t)^d \wedge (tj_0(|x - y|))\right) \le p(t, x, y) \le c\left(\Phi^{-1}(t)^d \wedge (tj_0(|x - y|))\right), \tag{5.20}$$

where Φ^{-1} is the inverse of the function

$$\Phi(r) = \frac{1}{\phi(r^{-2})}, \qquad r > 0.$$

When $d > 2\delta_2$, X is transient and its Green function $G(x, y)$ satisfies

$$G(x, y) \asymp \frac{1}{|x - y|^d \phi(|x - y|^{-2})}, \qquad x \ne y \in \mathbb{R}^d.$$

If we further assume that the first partial derivatives of k are bounded and continuous on $\mathbb{R}^d \times \mathbb{R}^d$, then for $f \in C_c^2(\mathbb{R}^d)$, the generator $\mathcal{A}$ of X admits the following expression

$$\mathcal{A}f(x) = \int_{\mathbb{R}^d} (f(x + y) - f(x) - y \cdot \nabla f(x)1_{|y|<1})k(x, x)j_0(|y|)dy$$

$$+\int_{\mathbb{R}^d}(f(x+y)-f(x))(k(x,x+y)-k(x,x))j_0(|y|)dy. \qquad (5.21)$$

Using (5.19)–(5.21), one can easily check that Assumptions **A B**, **B1-a**(z_0,r_0), **B1-b**(z_0,r_0), **B1-c**(z_0,r_0), **B2-a**(z_0,r_0), **B2-b**(z_0,r_0), **C**, **C1**(z_0,r_0), **C2**(z_0,r_0), **D1**(z_0,r_0) and **D2**(z_0,r_0) are satisfied for all $z_0\in\mathbb{R}^d$ and $r_0\in(0,\infty)$.

6 Limit of Green functions at regular boundary points

Suppose that $D\subset\mathfrak{X}$ is an open set. In this section, we will prove that, under some assumptions, the Green function $G_D(x,y)$ of X^D approaches zero when x approaches a point $z\in\partial D$ which is regular for D^c with respect to X.

Proposition 6.1 *Suppose that X is a Hunt process on $\mathfrak{X}$ satisfying both the Feller and the strong Feller property. If z is a regular boundary point of D and f is a bounded Borel function on D^c which is continuous at z, then*

$$\lim_{\overline{D}\ni x\to z}\mathbb{E}_x\left[f(X_{\tau_D});\tau_D<\infty\right]=f(z).$$

Proof. Note that, since z is a regular boundary point, $\mathbb{P}_z(\tau_D=0)=1$. By [12, Lemma 3], for every $s>0$,

$$\limsup_{x\to z}\mathbb{P}_x(\tau_D>s)\le\mathbb{P}_z(\tau_D>s)=0.$$

Thus, for every $s>0$,

$$\lim_{x\to z}\mathbb{P}_x(\tau_D>s)=0. \qquad (6.1)$$

In particular, $\lim_{x\to z}\mathbb{P}_x(\tau_D<\infty)=1$. Thus it is enough to show that,

$$\lim_{\overline{D}\ni x\to z}\mathbb{E}_x\left[|f(X_{\tau_D})-f(z)|:\tau_D<\infty\right]<\varepsilon \qquad (6.2)$$

for arbitrary $\varepsilon>0$.

Given $\varepsilon>0$, choose $\delta>0$ such that

$$|f(w)-f(z)|\le\frac{\varepsilon}{2},\quad\text{for every } w\in B(z,\delta).$$

Now we have for every $s>0$ and $x\in\overline{B(z,\delta/2)}$,

$$\begin{aligned}
&\mathbb{E}_x\left[|f(X_{\tau_D})-f(z)|:\tau_D<\infty\right]\\
&\quad=\ \mathbb{E}_x\left[|f(X_{\tau_D})-f(z)|:\tau_D<\tau_{B(z,\delta)}\right]+\mathbb{E}_x\left[|f(X_{\tau_D})-f(z)|:\tau_{B(z,\delta)}\le\tau_D<\infty\right]\\
&\quad\le\ \frac{\varepsilon}{2}\mathbb{P}_x(\tau_D<\tau_{B(z,\delta)})+2\|f\|_\infty\mathbb{P}_x(\tau_{B(z,\delta)}\le s\text{ or }s<\tau_D)\\
&\quad\le\ \frac{\varepsilon}{2}+2\|f\|_\infty(\mathbb{P}_x(\tau_{B(z,\delta)}\le s)+\mathbb{P}_x(s<\tau_D)).
\end{aligned}$$

It follows from [12, Lemma 2] that there exists an $s > 0$ such that

$$\sup_{x \in \overline{B(z,\delta/2)}} \mathbb{P}_x(\tau_{B(z,\delta/2)} \le s) \le \frac{\varepsilon}{4\|f\|_\infty}.$$

Using (6.1), we get

$$\overline{\lim_{D \ni x \to z}} \mathbb{E}_x \left[|f(X_{\tau_D}) - f(z)| : \tau_D < \infty \right] < \varepsilon + 2\|f\|_\infty \overline{\lim_{D \ni x \to z}} \mathbb{P}_x(s < \tau_D) = \varepsilon.$$

We have proved (6.2). □

The following result will be used in [25].

Proposition 6.2 *Suppose that X is a Hunt process on $\mathfrak{X}$ satisfying both the Feller and the strong Feller property and that for all Greenian open sets V, $x \mapsto G_V(x,y)$ is continuous in $V \setminus \{y\}$. If $D \subset \mathfrak{X}$ is an open set, $z_1 \in \partial D$ is regular for D^c and there exists $r_0 > 0$ such that $D \cup B(z_1, r_0)$ is Greenian, then for all $y \in D$,*

$$\lim_{D \ni x \to z_1} G_D(x,y) = 0.$$

Proof. Fix $y \in D$ and choose $r_1 \le r_0/2$ small enough so that $y \in B(z_1, 4r_1)^c \cap D$. Let $U_1 = D \cup B(z_1, r_1)$ and $U_2 = D \cup B(z_1, 2r_1)$ which are both Greenian. Then, by our assumption, $x \mapsto G_{U_i}(x,y)$ are continuous in $U_i \setminus \{y\}$.

By the strong Markov property we have

$$G_D(x,y) = G_{U_1}(x,y) - \mathbb{E}_x[G_{U_1}(X_{\tau_D}, y)].$$

The function $w \mapsto G_{U_1}(w,y)$ is bounded on D^c. Indeed, using domain-monotonicity of Green functions and continuity of $G_{U_2}(\cdot, y)$ on $U_2 \setminus \{y\}$,

$$\sup_{w \in D^c} G_{U_1}(w,y) \le \sup_{w \in B(z_1,r_1)} G_{U_1}(w,y) \le \sup_{w \in B(z_1,r_1)} G_{U_2}(w,y) < \infty.$$

Since $x \mapsto G_{U_1}(x,y)$ is continuous at z_1, by Proposition 6.1, we have

$$\lim_{D \ni x \to z_1} G_D(x,y) = G_{U_1}(z,y) - \lim_{D \ni x \to z_1} \mathbb{E}_x[G_{U_1}(X_{\tau_D}, y)] = 0.$$

□

The following result is quite general.

Proposition 6.3 *Suppose that X is a Hunt process on $\mathfrak{X}$ satisfying both the Feller and the strong Feller property.*
(a) If $U \subset \mathfrak{X}$ is open and u is a bounded Borel function on U^c, then the function $x \mapsto \mathbb{E}_x[u(X_{\tau_U}); \tau_U < \infty]$ is continuous in U.

(b) Assume further that X satisfies the Harnack principle in the sense that, for any z_1, there exists $r_0 > 0$ and $c > 0$ such that for every $r \in (0, r_0)$ and every function u which is nonnegative on $\mathfrak{X}$ and harmonic in $B(z_1, r)$ with respect to X, it holds that

$$u(x) \leq cu(y), \qquad x, y \in B(z_1, r/2).$$

Then, if h is a nonnegative function on $\mathfrak{X}$ which is harmonic in an open set $D \subset \mathfrak{X}$ with respect to X, then h is continuous in D.

Proof. Part (a) follows from [28, Theorem 3.4]. Note that although [28, Theorem 3.4] is stated for the case $\mathfrak{X} = \mathbb{R}^d$, the argument there works generally. Now one can repeat the argument after [24, Theorem 2.1] to get the conclusion of (b). □

Corollary 6.4 *Suppose that X is a Hunt process on $\mathfrak{X}$ satisfying both the Feller and the strong Feller property and X satisfies the Harnack principle. Assume that $D \subset \mathfrak{X}$ is an open set, that $z_1 \in \partial D$ is regular for D^c and there exists $r_0 > 0$ such that $D \cup B(z_1, r_0)$ is Greenian. Then for all $y \in D$,*

$$\lim_{D \ni x \to z_1} G_D(x, y) = 0.$$

We now weaken the Greenian assumption.

Proposition 6.5 *Suppose that X satisfies Assumption **A** and that, for every $z_1 \in \mathfrak{X}$, there is $r_0 > 0$ such that the conclusion of Corollary 4.2 (BHP) holds. Assume that $D \subset \mathfrak{X}$ is an open Greenian set, that $z_1 \in \partial D$ is regular for D^c and the open balls $B(z_1, r)$ are Greenian for all $r > 0$. Then for all $y \in D$,*

$$\lim_{D \ni x \to z_1} G_D(x, y) = 0.$$

Proof. Fix $y \in D$ and let $r_1 = 2d(z_1, y)$ and $U = D \cap B(z_1, r_1)$ which is Greenian. By the strong Markov property we have

$$G_D(x, y) = G_U(x, y) + \mathbb{E}_x[G_D(X_{\tau_U}, y)].$$

The BHP assumption implies that X satisfies the Harnack principle. Moreover, the open set $U \cup B(z_1, r_1) = B(z_1, r_1)$ is Greenian. Therefore we can apply Corollary 6.4 and get

$$\lim_{D \ni x \to z_1} G_U(x, y) = \lim_{U \ni x \to z_1} G_U(x, y) = 0.$$

Define

$$w(x) = \mathbb{P}_x[X_{\tau_U} \in U^c \setminus B(z_1, r_1); \tau_U < \infty], \qquad x \in U.$$

It follows from Proposition 6.1 that

$$\lim_{U \ni x \to z_1} w(x) = 0. \tag{6.3}$$

Note that $x \to \mathbb{E}_x[G_D(X_{\tau_U}, y)]$ and w are both regular harmonic in U, vanish in $B(z_1, r_1) \cap (\overline{U}^c \cup U^{\mathrm{reg}})$. Now we can combine (6.3) with the BHP to get

$$\lim_{U \ni x \to z_1} \mathbb{E}_x[G_D(X_{\tau_U}, y)] = 0.$$

Therefore

$$\lim_{D \ni x \to z_1} G_D(x, y) = \lim_{U \ni x \to z_1} G_U(x, y) + \lim_{U \ni x \to z_1} \mathbb{E}_x[G_D(X_{\tau_U}, y)] = 0.$$

□

7 Appendix

In this section we present the proof of Theorem 4.1. Throughout this section, z_0 is a fixed point in $\mathfrak{X}$. We will always assume in this section that, in addition to **A**, **B** and **C**, the assumptions (2.1), (4.1), (2.3), **B1-a**(z_0, r_0), **B1-b**(z_0, r_0), **B1-c**(z_0, r_0), **C1**(z_0, r_0) and **D1**(z_0, r_0) hold true for some $r_0 \in (0, R_0]$. Recall that n_0 is the natural number in (4.1) . In the next result we understand $r_0/(2n_0)$ to be ∞ if $r_0 = \infty$.

Proposition 7.1 *There exists a constant $c = c(z_0, r_0) > 0$ such that for all $r < r_0/(2n_0)$ and all $x \in B(z_0, r)$, $\mathbb{E}_x \tau_{B(z_0,r)} \le c\Phi(r)$.*

Proof. Let $r < r_0/(2n_0)$, denote $B = B(z_0, r)$ and let $F(t) := \mathbb{P}_x(\tau_B > t)$. First note that if $y \in B$, then by **C1**(z_0, r_0), $j(y, z) \ge c_1 j(z_0, z)$ for all $z \in \overline{A}(z_0, 2r, 2n_0 r)$. Hence,

$$J(y, \mathfrak{X} \setminus B) \ge J(y, \overline{A}(z_0, 2r, 2n_0 r)) \ge c_1 J(z_0, \overline{A}(z_0, 2r, 2n_0 r)) .$$

In the same way as in [7, Proposition 2.1], $-F'(t) \ge c_1 J(z_0, \overline{A}(z_0, 2r, 2n_0 r))$, implying that $F(t) \le \exp\{-t c_1 J(z_0, \overline{A}(z_0, 2r, 2n_0 r)\}$. Hence,

$$\mathbb{E}_x \tau_{B(z_0,r)} \le \left(c_1 J(z_0, \overline{A}(z_0, 2r, 2n_0 r))\right)^{-1} . \tag{7.1}$$

By using **C1**(z_0, r_0) and the monotonicity of V and Φ in the first line, (2.1) and (2.3) in the second line, (4.1) in the third, we get

$$\begin{aligned} J(z_0, A(z_0, 2r, 2n_0 r)) &= \int_{\overline{A}(z_0,2r,2n_0 r)} j(z_0, z) m(dz) \ge \int_{\overline{A}(z_0,2r,2n_0 r)} \frac{c_2}{V(n_0 r)\Phi(n_0 r)} m(dz) \\ &\ge \int_{\overline{A}(z_0,2r,2n_0 r)} \frac{c_3}{V(2r)\Phi(r)} m(dz) \ge \frac{c_3}{\Phi(r)} \left(\frac{V(2n_0 r)}{V(2r)} - 1 \right) \\ &\ge \frac{c_4}{\Phi(r)}. \end{aligned}$$

Together with (7.1) this proves the claim. □

Let $a \in (1/2, 1)$. For each $r < r_0$, we consider a function $\varphi^{(r)} \in \mathcal{D}(\overline{B}(z_0, ar), B(z_0, r))$, and let $V^{(r)} = \{x \in \mathfrak{X} : \varphi^{(r)}(x) > 0\}$. Note that, by choosing $\varphi^{(r)}$ appropriately, we can achieve that $\delta^{(r)} := \sup_{x\in\mathfrak{X}} \max\left(\mathcal{A}\phi^{(r)}(x), \widehat{\mathcal{A}}\phi^{(r)}(x)\right) \le c/\Phi(r)$, where $c = c(z_0, r_0, a)$ is the constant in assumption **B1-a**(z_0, r_0).

In what follows, our analysis and results are valid for all $r < r_0$ with constants depending on $a \in (1, 2)$, but *not* on r. To ease the notation in the remaining part of the section we drop the superscript r from $\varphi^{(r)}$ and $V^{(r)}$ and write simply φ and V.

Let

$$\psi(x) = \frac{\max(\mathcal{A}\varphi(x), \widehat{\mathcal{A}}\varphi(x), \delta(1 - \varphi(x)))}{\varphi(x)}, \quad x \in \mathfrak{X}, \tag{7.2}$$

with the convention $1/0 = \infty$. Note that $\psi(x) = \infty$ for $x \in V^c$, and $\psi(x) = 0$ for $x \in \overline{B}(z_0, ar)$.

As in Section 3 we define two right-continuous additive functionals A_t and $\widehat{A}_t$ as in (3.2) and define two right-continuous exact strong Markov multiplicative functionals $M_t = \exp(-A_t)$ and $\widehat{M}_t = \exp(-\widehat{A}_t)$. We consider the semigroup of operators $T_t^{\psi} f(x) = \mathbb{E}_x(f(X_t)M_t)$ associated with the multiplicative functional M, which is the transition operator of subprocess X^{ψ} of X and the semigroup of operators $\widehat{T}_t^{\psi} f(x) = \mathbb{E}_x(f(\widehat{X}_t)\hat{M}_t)$ associated with the multiplicative functional $\widehat{M}$, which is the transition operator of subprocess $\widehat{X}^{\psi}$ of $\widehat{X}$. Again, the potential densities of X^{ψ} and $\widehat{X}^{\psi}$ satisfy $\widehat{G}^{\psi}(x, y) = G^{\psi}(y, x)$ and

$$G^{\psi}(x, y) \le G_V(x, y) \le G_{B(z_0,r)}(x, y), \quad (x, y) \in V \times V. \tag{7.3}$$

Let $\tau_a = \inf\{t \ge 0 : A_t \ge a\}$ and

$$\pi^{\psi} f(x) = -\mathbb{E}_x \int_{[0,\infty)} f(X_t) dM_t = \mathbb{E}_x \left(\int_0^{\infty} f(X_{\tau_a}) e^{-a} da \right) = \int_0^{\infty} \mathbb{E}_x(f(X_{\tau_a})) e^{-a} da. \tag{7.4}$$

Recall from [7, pp. 492–493] that $\pi^{\psi} f$ can be written in the following two ways: if f is nonnegative and vanishes in $\mathfrak{X} \setminus (\overline{B}(z_0, ar)^c \cap V)$, then

$$\pi^{\psi} f(x) = G^{\psi}(\psi f)(x) = \int_{V\cap\overline{B}(z_0,ar)^c} G^{\psi}(x, y)\psi(y) f(y) m(dy), \quad x \in V, \tag{7.5}$$

and if $f \in \mathcal{D}(\mathcal{A})$ vanishes in V then for all $x \in V$,

$$\begin{aligned}
\pi^{\psi} f(x) = G^{\psi}\mathcal{A} f(x) &= \int_V G^{\psi}(x, y)\mathcal{A} f(y) m(dy) \\
&= \int_V G^{\psi}(x, y) \int_{\mathfrak{X}\setminus V} f(z) j(y, z) m(dz) m(dy) \\
&= \int_{\mathfrak{X}\setminus V} \left(\int_V G^{\psi}(x, y) j(y, z) m(dy) \right) f(z) m(dz).
\end{aligned} \tag{7.6}$$

By Corollary 4.8 of [7], we have

$$\pi^\psi(x,\partial V)=0, \qquad x\in V. \tag{7.7}$$

By [7, Lemma 4.10], we get that if f is regular harmonic in $B(z_0,r)$ with respect to X, then $f(x)=\pi^\psi f(x)$ for all $x\in\overline{B}(z_0,r/2)$. The main step of the proof is to get the correct estimate of $\pi^\psi(x,dy)/m(dy)$.

Let U be an open subset of V. For any nonnegative or bounded f and $x\in V$ we let

$$\pi^\psi_U f(x)=\mathbb{E}_x(f(X_{\tau_U})M_{\tau_U-}), \qquad G^\psi_U f(x)=\mathbb{E}_x\int_0^{\tau_U} f(X_t)M_t dt.$$

G^ψ_U admits a density $G^\psi_U(x,y)$, and we have $G^\psi_U(x,y)\le G_U(x,y)$, $G^\psi_U(x,y)\le G^\psi(x,y)$. For any $f\in\mathcal{D}(\mathcal{A})$ we have

$$\pi^\psi_U f(x)=G^\psi_U(\mathcal{A}-\psi)f(x)+f(x), \qquad x\in V. \tag{7.8}$$

In particular, by an approximation argument,

$$\pi^\psi_U(x,E)=\int_U G^\psi_U(x,y)J(y,E)m(dy), \qquad x\in U,\ E\subseteq\mathfrak{X}\setminus\overline{U}. \tag{7.9}$$

By the definition of ψ, we have that $(\mathcal{A}-\psi)\varphi(x)\le 0$ for $x\in\mathfrak{X}$ (and, in particular, for all $x\in V$). Thus using (3.8), we have the following.

Lemma 7.2 ([7, Lemma 4.4]) *Let $U=V\cap\overline{B}(z_0,ar)^c$. Then*

$$\pi^\psi_U(x,V\setminus U)\le\varphi(x), \qquad x\in U. \tag{7.10}$$

Lemma 7.3 *Let $b\in(1/2,a)$. There exists a constant $c=c(a,b)>0$ such that*

$$G^\psi(x,y)\le c\frac{\Phi(r)}{V(r)}\varphi(x), \quad x\in V\cap B(z_0,br)^c,\ y\in B(z_0,r/2)\,.$$

Proof. If $x\in\overline{A}(z_0,br,ar)$, then $\varphi(x)=1$. Thus, by Assumption $\mathbf{D1}(z_0,r_0)$, for $x\in\overline{A}(z_0,br,ar)\subset\overline{A}(z_0,br,r)$ and $y\in B(z_0,r/2)$,

$$G^\psi(x,y)\le G_V(x,y)\le G_{B(z_0,r)}(x,y)\le c_1\frac{\Phi(r)}{V(r)}=c_1\frac{\Phi(r)}{V(r)}\varphi(x),$$

with $c_1=c_1(b)$.

For the remainder of the proof we assume that $U:=V\cap\overline{B}(z_0,ar)^c$. Let $f\ge 0$ be supported in $\overline{B}(z_0,r/2)$ with $\int f(w)m(dw)=1$. Then, by the strong Markov property,

$$G^\psi f(x)=\pi^\psi_U(G^\psi f)(x)=\pi^\psi_U(\mathbf{1}_{\overline{A}(z_0,br,ar)}G^\psi f)(x)+\pi^\psi_U(\mathbf{1}_{B(z_0,br)}G^\psi f)(x)=:I+II.$$

First note that by $\mathbf{D1}(z_0, r_0)$ and (7.3), for $y \in \overline{A}(z_0, br, ar)$,

$$G^\psi f(y) \le \int_{\overline{B}(z_0,r/2)} G_V(y,w)f(w)m(dw) \le \int_{\overline{B}(z_0,r/2)} G_{B(z_0,r)}(y,w)f(w)m(dw) \le c_2 \frac{\Phi(r)}{V(r)}.$$

Thus, combining this with Lemma 7.2 we get

$$I \le \left(\sup_{y \in \overline{A}(z_0,br,ar)} G_\psi f(y) \right) \pi_U^\psi(x, \overline{A}(z_0, br, ar)) \le c_2 \frac{\Phi(r)}{V(r)} \pi_U^\psi(x, B(z_0, ar)) \le c_2 \frac{\Phi(r)}{V(r)} \varphi(x).$$

For II, note that by $\mathbf{C1}(z_0, r_0)$, for every $z \in B(z_0, br)$,

$$\int_U G_U^\psi(x,y) j(y,z_0)\, m(dy) \le c_3 \int_U G_U^\psi(x,y) j(y,z)\, m(dy)$$

for a constant $c_3 = c_3(a,b)$. By integrating over the ball $B(z_0, br)$, using the doubling property of V, Lemma 7.2 and (7.9), we obtain that

$$\begin{aligned}
\int_U G_U^\psi(x,y) j(y,z_0) m(dy) &\le \frac{c_4}{V(r)} \int_{B(z_0,br)} \int_U G_U^\psi(x,y) j(y,z)\, m(dy)\, m(dz) \\
&= \frac{c_4}{V(r)} \int_U G_U^\psi(x,y) \left(\int_{B(z_0,br)} j(y,z)\, m(dz) \right) m(dy) \\
&= \frac{c_4}{V(r)} \pi_U^\psi(x, B(z_0, br)) \le \frac{c_4}{V(r)} \varphi(x)\,,
\end{aligned}$$

with $c_4 = c_4(a,b)$. Thus, by using $\mathbf{C1}(z_0, r_0)$ in the third line and the display above in the last line,

$$\begin{aligned}
II &= \int_U G_U^\psi(x,y) \int_{B(z_0,br)} j(y,z) G^\psi f(z)\, m(dz)\, m(dy) \\
&= \int_{B(z_0,br)} G^\psi f(z) \left(\int_U G_U^\psi(x,y) j(y,z)\, m(dy) \right) m(dz) \\
&\le c_5 \int_{B(z_0,br)} G^\psi f(z) \left(\int_U G_U^\psi(x,y) j(y,z_0)\, m(dy) \right) m(dz) \\
&\le \frac{c_6}{V(r)} \varphi(x) \int_{B(z_0,br)} G^\psi f(z)\, m(dz)\,.
\end{aligned}$$

Finally, by using the dual version of Proposition 7.1,

$$\begin{aligned}
\int_{B(z_0,br)} G^\psi f(z)\, m(dz) &\le \int_{B(z_0,r/2)} \left(\int_{B(z_0,r)} G_V(z,y)\, m(dz) \right) f(y)\, m(dy) \\
&\le \int_{B(z_0,r/2)} \left(\int_{B(z_0,r)} G_{B(z_0,r)}(z,y)\, m(dz) \right) f(y)\, m(dy) \\
&= \int_{B(z_0,r/2)} \widehat{\mathbb{E}}_y(\tau_{B(z_0,r)}) f(y)\, m(dy) \le c_7 \Phi(r)\,.
\end{aligned}$$

This completes the proof. □

Lemma 7.4 *There exists a constant $c = c(a, z_0) > 0$ such that for all $x \in B(z_0, r/2)$,*

$$\pi^\psi(x, dy)/m(dy) \le c\Phi(r)j(z_0, y)\mathbf{1}_{\overline{B}(z_0,ar)^c}(y)\,. \tag{7.11}$$

Proof. Let $b := \frac{2a}{1+2a}$ so that $b \in (1/2, a)$. First note that ψ vanishes on $\mathfrak{X} \setminus \overline{B}(z_0, ar)$. Thus

$$\pi^\psi(y, \overline{B}(z_0, ar)) = 0, \quad y \in V. \tag{7.12}$$

Fix $x \in B(z_0, r/2)$. If f is a non-negative function on $\mathfrak{X}$ vanishing in $\mathfrak{X} \setminus (\overline{B}(z_0, ar)^c \cap V)$, then by (7.5) and the dual version of Lemma 7.3 (together with $G^\psi(x, y) = \widehat{G}^\psi(y, x)$),

$$\pi^\psi f(x) \le c_1 \frac{\Phi(r)}{V(r)} \int_{V \cap \overline{B}(z_0,ar)^c} \varphi(y)\psi(y)f(y)m(dy). \tag{7.13}$$

Since for $y \in \overline{B}(z_0, ar)^c$ we have $\varphi(y)\psi(y) \le c_2(\Phi(r))^{-1}$ by the definition in (7.2), assumption **B1-a**(z_0, r_0) and (4.3) (note that $V \cap \overline{B}(z_0, ar)^c \subset \overline{A}(z_0, ar, r)$), we get

$$\pi^\psi f(x) \le \frac{c_3}{V(r)} \int_{V \cap \overline{B}(z_0,ar)^c} f(y)m(dy) \le c_4\Phi(r) \int_{V \cap \overline{B}(z_0,ar)^c} j(z_0, y)f(y)\, m(dy)\,. \tag{7.14}$$

On the other hand, if $g \in \mathcal{D}(\mathcal{A})$ vanishes in V then by (7.6),

$$\pi^\psi g(x) = \int_{\mathfrak{X} \setminus V} \left(\int_V G^\psi(x, y)j(y, z)m(dy) \right) g(z)m(dz). \tag{7.15}$$

Assume $z \in \mathfrak{X} \setminus V \subset \overline{B}(z_0, ar)^c$ and let

$$I := \int_{V \cap \overline{B}(z_0,br)^c} G^\psi(x, y)j(y, z)m(dy) \quad \text{and} \quad II := \int_{B(z_0,br)} G^\psi(x, y)j(y, z)m(dy).$$

We now consider I and II separately.

By the dual versions of Lemma 7.3 and (4.4), and the fact that $\widehat{\mathcal{A}}\varphi(z) = \widehat{J}\varphi(z)$, for $c_4 = c_4(a) > 0$

$$I \le c_4 \frac{\Phi(r)}{V(r)} \int_{V \cap \overline{B}(z_0,br)^c} \varphi(y)j(y, z)m(dy) = c_4 \frac{\Phi(r)}{V(r)} \widehat{\mathcal{A}}\varphi(z) \le cj(z_0, z)\Phi(r)\,. \tag{7.16}$$

On the other hand, by assumption **C1**(z_0, r_0) and (7.3), for $c_6 = c_6(a) > 0$

$$\begin{aligned} II &\le c_5 \int_{B(z_0,br)} G^\psi(x, y)j(z_0, z)m(dy) \\ &\le c_5 j(z_0, z) \int_{B(z_0,r)} G_{B(z_0,r)}(x, y)m(dy) \\ &= c_5 j(z_0, z)\mathbb{E}_x \tau_{B(z_0,r)} \le c_6 j(z_0, z)\Phi(r)\,. \end{aligned} \tag{7.17}$$

Hence,

$$\pi^\psi g(x) \le c_7\Phi(r) \int_{\mathfrak{X} \setminus V} j(z_0, z)g(z)\, m(dz)\,.$$

Together with (7.14) this proves the lemma. □

Corollary 7.5 *Let f be a non-negative function on $\mathfrak{X}$ and $x \in B(z_0, r/2)$ such that $f(x) \le \mathbb{E}_x f(X_\tau)$ for every stopping time $\tau \le \tau_{B(z_0,r)}$. Then*

$$f(x) \le c\Phi(r) \int_{\overline{B}(z_0,ar)^c} j(z_0, y) f(y) m(dy) , \tag{7.18}$$

where $c = c(a)$ is the constant from Lemma 7.4.

Proof. Recall from (7.4) that $\pi^\psi f(x) = \int_0^\infty \mathbb{E}_x(f(X_{\tau_a}))e^{-a} da$. Since $\tau_a \le \tau_V \le \tau_{B(z_0,r)}$, we have that $f(x) \le \mathbb{E}_x f(X_{\tau_a})$, and therefore $f(x) \le \pi_\psi f(x)$. Thus by (7.11),

$$f(x) \le \int f(y)\pi^\psi(x, dy) \le c\Phi(r) \int_{\overline{B}(z_0,ar)^c} j(z_0, y) f(y) m(dy) .$$

□

Lemma 7.6 *For any $b \in (1/2, a)$ there exists $c = c(z_0, r_0, a, b) > 0$ such that for any $r < r_0$ and any open set $D \subset B(z_0, br)$ we have*

$$\mathbb{P}_x \left(X_{\tau_D} \in \overline{A}(z_0, br, ar) \right) \le \frac{c}{\Phi(r)} \mathbb{E}_x \tau_D , \qquad x \in D \cap B(z_0, r/2) .$$

Proof. Let $f \in \mathcal{D}(\overline{A}(z_0, br, ar), A(z_0, r/2, r))$. By assumption **B1-c**$(z_0, r_0)$, $\sup_{y\in\mathfrak{X}} \mathcal{A}f(y) \le \frac{c}{\Phi(r)}$ with $c = c(z_0, a, b)$. By Dynkin's formula, for $x \in D \cap B(z_0, r/2)$,

$$\mathbb{E}_x f(X_{\tau_D}) = \mathbb{E}_x \int_0^{\tau_D} \mathcal{A}f(X_t) dt \le \frac{c}{\Phi(r)} \mathbb{E}_x \tau_D .$$

The claim follows from $\mathbf{1}_{\overline{A}(z_0,br,ar)} \le f$. □

Proposition 7.7 *Let $b \in (1/2, a)$. There exists $c = c(z_0, r_0, a, b) > 1$ such that for any $r < r_0$, any open set $D \subset B(z_0, r)$ and any non-negative function u on $\mathfrak{X}$ which is regular harmonic with respect to X in D and vanishes on $B(z_0, r) \cap \left(\overline{D}^c \cup D^{\mathrm{reg}}\right)$, it holds that*

$$\begin{aligned} c^{-1} \mathbb{E}_x \tau_{D\cap B(z_0,br/2)} \int_{\overline{B}(z_0,ar/2)^c} j(z_0, z) u(z)\, m(dz) &\le u(x) \\ &\le \mathbb{E}_x \tau_{D\cap B(z_0,br/2)} \int_{\overline{B}(z_0,ar/2)^c} j(z_0, z) u(z)\, m(dz) \end{aligned} \tag{7.19}$$

for all $x \in D \cap B(z_0, r/4)$.

Proof. Let $O := D \cap B(z_0, br/2)$, $D_1 := \overline{A}(z_0, br/2, ar/2)$ and $D_2 := B(z_0, ar/2)^c$. By the harmonicity of u,

$$u(x) = \mathbb{E}_x[u(X_{\tau_O})] = \mathbb{E}_x[u(X_{\tau_O}) : X_{\tau_O} \in D_1] + \mathbb{E}_x[u(X_{\tau_O}) : X_{\tau_O} \in D_2] , \quad x \in D. \tag{7.20}$$

Since u vanishes on $B(z_0,r)\cap(\overline{D}^c\cup D^{\mathrm{reg}})$, it follows that $u(y)\le \mathbb{E}_y u(X_\tau)$ for every stopping time $\tau\le\tau_{B(z_0,r)}$ and every $y\in B(z_0,r)\setminus D^{\mathrm{irr}}$. Since D^{irr} is polar with respect to X, we see that $X_{\tau_O}\notin D^{\mathrm{irr}}$. It follows from Corollary 7.5 and Lemma 7.6 that for all $x\in D\cap B(z_0,r/4)$,

$$\begin{aligned}
\mathbb{E}_x[u(X_{\tau_O}):X_{\tau_O}\in D_1] &\le \left(\sup_{y\in D_1\setminus D^{\mathrm{irr}}} u(y)\right)\mathbb{P}_x(X_{\tau_O}\in D_1)\\
&\le \frac{c_1}{\Phi(r)}(\mathbb{E}_x\tau_O)\Phi(r/2)\int_{\overline{B}(z_0,a^2r)^c} j(z_0,z)u(z)\,m(dz) \qquad (7.21)\\
&\le c_1\mathbb{E}_x\tau_O\int_{\overline{B}(z_0,ar/2)^c} j(z_0,z)u(z)\,m(dz) \qquad (7.22)
\end{aligned}$$

with $c_1=c_1(a,b)$. On the other hand, by assumption $\mathbf{C1}(z_0,r_0)$, for all $x\in D\cap B(z_0,r/4)$,

$$\begin{aligned}
\mathbb{E}_x[u(X_{\tau_O}):X_{\tau_O}\in D_2] &= \int_{\overline{B}(z_0,ar/2)^c}\int_O G_O(x,y)j(y,z)m(dy)u(z)m(dz)\\
&\asymp \int_{\overline{B}(z_0,ar/2)^c}\int_O G_O(x,y)j(z_0,z)m(dy)u(z)m(dz)\\
&= \mathbb{E}_x\tau_O\int_{\overline{B}(z_0,ar/2)^c} j(z_0,z)u(z)m(dz). \qquad (7.23)
\end{aligned}$$

The proposition now follows from (7.20)–(7.23). □

Lemma 7.8 *For any $b\in(1/2,a)$ there exists $c=c(a,b)>0$ such that for every $r<r_0/(4n_0)$, and every open set $D\subset B(z_0,2r)$,*

$$\mathbb{E}_x\tau_{D\cap B(z_0,br)}\le \mathbb{E}_x\tau_D\le c\mathbb{E}_x\tau_{D\cap B(z_0,br)}\,,\qquad x\in D\cap B(z_0,abr)\,.$$

Proof. First note that by the strong Markov property,

$$\mathbb{E}_x\tau_D=\mathbb{E}_x\tau_{D\cap B(z_0,br)}+\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_{D\cap B(z_0,br)}}}\tau_D\right].$$

By Proposition 7.1, Lemma 7.6 and doubling property of Φ, for $x\in D\cap B(z_0,abr)$,

$$\begin{aligned}
\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_{D\cap B(z_0,br)}}}\tau_D\right] &\le \left(\sup_{y\in D}\mathbb{E}_y\tau_D\right)\mathbb{P}_x\left(X_{\tau_{D\cap B(z_0,br)}}\in\overline{B}(z_0,br)^c\right)\\
&\le c_1\Phi(2r)\frac{c_2}{\Phi(br)}\mathbb{E}_x\tau_{D\cap B(z_0,br)}\le c_3\mathbb{E}_x\tau_{D\cap B(z_0,br)}\,.
\end{aligned}$$

This finishes the proof. □

Proof of Theorem 4.1 Let $a\in(1/2,1)$ and choose $b:=\frac{2a}{1+2a}$ so that $b\in(1/2,a)$. Let $D\subset B(z_0,r)$ and let u be a non-negative function on $\mathfrak{X}$ which is regular harmonic with respect to X in D and vanishes on $B(z_0,r)\cap(\overline{D}^c\cup D^{\mathrm{reg}})$. Since $B(z_0,r/8)\subset B(z_0,r/4)\cap B(z_0,abr/2)$, it follows from Proposition 7.7 and Lemma 7.8 that

$$u(x)\asymp\mathbb{E}_x\tau_D\int_{\overline{B}(z_0,ar/2)^c} j(z_0,y)u(y)m(dy)\,,\qquad x\in D\cap B(z_0,r/8)\,,$$

with a constant depending on a. □

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