HEAT KERNEL ESTIMATES FOR SUBORDINATE MARKOV PROCESSES AND THEIR APPLICATIONS

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Abstract. In this paper, we establish sharp two-sided estimates for transition densities of a large class of subordinate Markov processes. As applications, we show that the parabolic Harnack inequality and Hölder regularity hold for parabolic functions of such processes, and derive sharp two-sided Green function estimates.

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1. Introduction

Transition densities of Markov processes are of central importance in both probability and analysis. The transition density $p(t,x,y)$ of a Markov process X with generator L is the fundamental solution of the equation $\partial_t u = Lu$. Hence the transition density $p(t,x,y)$ is also known as the heat kernel of L. The heat kernel is rarely known explicitly. Due to the importance of heat kernels, there is a huge amount of literature devoted to estimates of heat kernels.

The purpose of this paper is to study heat kernel estimates for subordinate Markov processes. The main motivation comes from [42], where it was established that the jump kernels of subordinate killed Lévy processes have an unusual form not observed before. It is therefore plausible that the heat kernels of those processes will have some new features. It turns out that this is indeed the case. To illustrate the new features, we explain below the motivating and also the simplest example covered by our results.

Let $D \subset \mathbb{R}^d$, $d \geq 1$, be a bounded $C^{1,1}$ open set. For $x \in D$, let $\delta_D(x)$ denote the distance between x and D^c. Let Y be an isotropic α-stable process in $\mathbb{R}^d$, $\alpha \in (0,2]$ and let Y^D denote the part process of Y killed upon exiting D. When $\alpha = 2$, we further assume that D is connected Sharp two-sided estimates of the heat kernel $p_D(t,x,y)$ of Y^D were obtained in [27, 53] (for $\alpha = 2$) and [9] (for $\alpha < 2$): there exist positive constants c_i, $i = 1, \dots, 8$, such that following estimates hold true. For $(t,x,y) \in (0,1] \times D \times D$,

$$c_1 h(t,x,y)\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big) \leq p_D(t,x,y) \leq c_2 h(t,x,y)\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big), \quad \text{for } \alpha < 2,$$

and

$$c_3 h(t,x,y)\, t^{-d/2} e^{-c_4|x-y|^2/t} \leq p_D(t,x,y) \leq c_5 h(t,x,y)\, t^{-d/2} e^{-c_6|x-y|^2/t}, \quad \text{for } \alpha = 2,$$

where the boundary function $h(t,x,y)$ is given by

$$h(t,x,y) = \Big(1 \wedge \frac{\delta_D(x)^\alpha}{t}\Big)^{1/2}\Big(1 \wedge \frac{\delta_D(y)^\alpha}{t}\Big)^{1/2} = \Big(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\Big)^{\alpha/2}\Big(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\Big)^{\alpha/2}.$$

For $(t,x,y) \in [1,\infty) \times D \times D$,

$$c_7 e^{-\lambda_1 t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} \leq p_D(t,x,y) \leq c_8 e^{-\lambda_1 t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2},$$

where λ_1 is the smallest eigenvalue of the Dirichlet (fractional) Laplacian $(-\Delta)^{\alpha/2}\big|_D$.

Let $S = (S_t)_{t\geq 0}$ be a β-stable subordinator, $\beta \in (0,1)$, independent of Y^D, and let $X = (X_t)_{t\geq 0}$ be the subordinate process: $X_t := Y^D_{S_t}$. The generator of X is equal to (the negative of) $\big((-\Delta)^{\alpha/2}\big|_D\big)^\beta$ –

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the fractional power of the Dirichlet fractional Laplacian. In particular, when $\alpha = 2$, $\big(-\Delta\big|_D\big)^\beta$ is called a spectral fractional Laplacian in the PDE literature, see, for instance, [5] and the references therein. In this respect, the process X bears some similarity with the isotropic $\alpha\beta$-stable process. The heat kernel $q(t,x,y)$ of the subordinate process X is given by

$$q(t,x,y) = \int_0^\infty p_D(s,x,y)\mathbb{P}(S_t \in ds), \quad t>0,\ x,y \in D.$$

Note that the distribution of S_t is not explicitly known, making the above integration rather delicate. To handle this integral, we establish some estimates on the distribution of S_t (in fact, for much more general subordinators than the stable ones). Using these, we can obtain sharp two-sided estimates of $q(t,x,y)$. To present those estimates, we first introduce some notation. Denote $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. The notation $f(s) \simeq g(s)$ means that there exist comparison constants $c_1, c_2 > 0$ such that $c_1 g(s) \le f(s) \le c_2 g(s)$ for specified range of the variable s. For $x,y \in D$, let

$$\begin{aligned} \delta_\vee(x,y) &= \delta_D(x) \vee \delta_D(y), & \delta_\wedge(x,y) &= \delta_D(x) \wedge \delta_D(y),\\ \mathbf{m}_\vee(t,x,y) &= (t^{1/(\alpha\beta)} \vee \delta_\vee(x,y)) \wedge |x-y|, & \mathbf{m}_\wedge(t,x,y) &= (t^{1/(\alpha\beta)} \vee \delta_\wedge(x,y)) \wedge |x-y|. \end{aligned}$$

Our main results, specialized to the present situation, are summarized below.

Theorem 1.1. *(1) Suppose $(t,x,y) \in (0,1] \times D \times D$.*
(i) If $|x-y|^{\alpha\beta} \le t$, then

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta_D(x)}{t^{1/(\alpha\beta)}}\Big)^{\alpha/2}\Big(1 \wedge \frac{\delta_D(y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha/2} t^{-d/(\alpha\beta)}. \tag{1.1}$$

(ii) If $|x-y|^{\alpha\beta} > t$ and $\alpha = 2$, then

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta_D(x)}{|x-y|}\Big)\Big(1 \wedge \frac{\delta_D(y)}{|x-y|}\Big)\frac{t}{|x-y|^{d+2\beta}}. \tag{1.2}$$

(iii) If $|x-y|^{\alpha\beta} > t$ and $\alpha \in (0,2)$, then

$$q(t,x,y) \simeq \begin{cases} \Big(1 \wedge \dfrac{\delta_D(x)}{t^{1/(\alpha\beta)}}\Big)^{\alpha/2}\Big(1 \wedge \dfrac{\delta_D(y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha/2}\Big(\dfrac{\mathbf{m}_\wedge(t,x,y)}{|x-y|}\Big)^{\alpha(1-\beta)}\dfrac{t}{|x-y|^{d+\alpha\beta}} & \beta \in (1/2,1),\\ \Big(1 \wedge \dfrac{\delta_D(x)}{|x-y|}\Big)^{\alpha/2}\Big(1 \wedge \dfrac{\delta_D(y)}{|x-y|}\Big)^{\alpha/2}\Big(\dfrac{\mathbf{m}_\vee(t,x,y)}{|x-y|}\Big)^{-\alpha\beta}\dfrac{t}{|x-y|^{d+\alpha\beta}}, & \beta \in (0,1/2),\\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{|x-y|}\Big)^{\alpha/2}\Big(1 \wedge \dfrac{\delta_\vee(x,y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha/2}\dfrac{t}{|x-y|^{d+\alpha\beta}}\log\Big(e + \dfrac{\mathbf{m}_\vee(t,x,y)}{\mathbf{m}_\wedge(t,x,y)}\Big), & \beta = 1/2. \end{cases}$$

(2) For all $(t,x,y) \in [1,\infty) \times D \times D$,

$$q(t,x,y) \simeq e^{-t\lambda_1^\beta}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2},$$

where λ_1 is the smallest eigenvalue of the Dirichlet (fractional) Laplacian $(-\Delta)^{\alpha/2}\big|_D$.

From Theorem 1.1(1), one can see that, for x,y away from the boundary (in the sense that $\delta_\wedge(x,y) \ge |x-y| \vee t^{1/(\alpha\beta)}$), and for all $\beta \in (0,1)$, it holds that

$$q(t,x,y) \simeq t^{-d/(\alpha\beta)} \wedge \frac{t}{|x-y|^{d+\alpha\beta}}.$$

Recall that the same two-sided estimates are valid for the heat kernel of the isotropic $\alpha\beta$-stable process in the whole space. The novelty of the estimates for $q(t,x,y)$ is in the boundary term, which is quite unusual and involves interplays among $\delta_\vee(x,y)$, $\delta_\wedge(x,y)$ and time t itself. In this respect, the form of the boundary term is very different from the boundary function $h(t,x,y)$ for the underlying process Y^D.

All the estimates in Theorem 1.1 are consequences of Theorems 4.2, 4.7 and Lemma 7.1, see Example 7.2. Integrating the heat kernel estimates, we can obtain sharp two-sided estimates on the Green function of X, see Theorem 5.8 and Example 7.2.

In this paper, we obtain sharp two-sided heat kernel estimates for subordinate Markov processes in a setting which is more general, in several directions, than that of the example above. We allow (i) quite general subordinators, (ii) Markov processes with state space D that is either a bounded or an unbounded subset of a locally compact separable metric space, and (iii) very general form of two-sided estimates of the heat kernel $p_D(t,x,y)$ of the underlying process. In the remaining part of the introduction, we describe some of our assumptions and results and lay out the structure of the paper.

In Section 2, we first introduce the main assumption on the subordinator $S=(S_t)_{t\geq 0}$. Let ν denote its Lévy measure and $w(t):=\nu(t,\infty)$. We assume that there exist constants $R_1\in(0,\infty]$, $c_1,c_2>0$ and $\beta_2\geq\beta_1>0$ such that

$$c_1\Big(\frac{R}{r}\Big)^{\beta_1}\leq\frac{w(r)}{w(R)}\leq c_2\Big(\frac{R}{r}\Big)^{\beta_2}\quad\text{for all }\ 0<r\leq R<R_1.$$

This assumption is quite weak – it implies that the Laplace exponent ϕ of the subordinator S satisfies weak scaling conditions near infinity with lower index β_1 and upper index $\beta_2\wedge 1$. Note that $\beta_2>1$ is allowed. Building upon the results from [22, 23], we show several auxiliary results leading to the important estimate (2.24) saying that $\mathbb{P}(S_t\geq s)\simeq tw(s)$ for $2\phi^{-1}(1/t)^{-1}<s<R_1/2$.

Sections 3–4 are central to the paper. We start with the setup in Section 3: the underlying space is a locally compact separable metric space E with a Radon measure m having full support and satisfying volume doubling conditions. The state space is a proper open subset D of E, bounded or unbounded, and Y^D is a Hunt process living on D. We assume that Y^D admits a transition density $p_D(t,x,y)$. The main assumption on the transition density is given in Definition 3.6. Depending on whether D is bounded or unbounded, the assumptions are somewhat different. Roughly speaking, at least for small times, $p_D(t,x,y)$ is comparable to the product of two parts which we may call the boundary part and the interior part. The latter is specified in terms of the volume and two functions – $\Psi\geq\Phi$ – both enjoying the scaling property, and also includes a Gaussian part. We note that although in most examples it holds that $\Psi\simeq\Phi$, allowing for different functions enlarges the scope of examples. The boundary part is described through a boundary function $h(t,x,y)$ which is not specified but is required to satisfy certain assumptions – see Definition 3.3. To justify our assumptions on the heat kernel $p_D(t,x,y)$, we provide a number of examples from the literature satisfying those assumptions.

The main object of our study is the subordinate process $X_t:=Y^D_{S_t}$. In Subsection 4.1, see Theorem 4.1, we first establish sharp two-sided estimates of the jump kernel $J(x,y)$ of X, thus generalizing [42, Theorem 8.4]. Subsection 4.2 contains sharp two-sided estimates of the heat kernel $q(t,x,y)$ of X which are the main results of the paper. The case of a bounded set D and small time is given in Theorem 4.2, and the case of unbounded D and all time in Theorem 4.3. The near-diagonal estimates have a rather simple form, but the off-diagonal estimates are quite involved, containing four terms which cannot be compared under general assumptions. The form of the estimates is somewhat simplified in case when the upper scaling index $\beta_2<1$ and $\Psi\simeq\Phi$, see Corollary 4.4. Finally, Theorem 4.7 provides the large time estimates in case of bounded D.

In Section 5 we apply our sharp two-sided heat kernel estimates to derive sharp two-sided estimates of the Green function $G(x,y)$ of X. The most general form of the estimates is given in Proposition 5.3. These can be simplified under additional assumptions on the boundary function $h(t,x,y)$. We obtain several forms of the estimates depending on the relationship between the parameters in the volume doubling condition, the scaling indices of the functions w and Φ, and the parameters coming from h. The main results of this section are Theorems 5.8, 5.10 and 5.11.

In Section 6 we show that parabolic functions with respect to X satisfy Hölder regularity and the parabolic Harnack inequality. By using the rough upper estimates and the interior estimates for the heat kernel from Proposition 4.5 and Corollary 4.6 together with the jump kernel estimates from Theorem 4.1, we establish that the process X satisfies all the assumptions from [16, 19] used in the proofs of those results.

Finally, in Section 7, we provide a concrete and explicit example which includes the motivating example from the beginning of this introduction, and, with help of Lemma 7.1, derive the heat kernel estimates using the general Theorems 4.2, 4.3 and 4.7. We also derive the jump kernel estimates and Green function estimates using Theorem 4.1, and Theorems 5.8, 5.10, 5.11 respectively. As an application of these estimates, combined with the main results of [44], we completely determine the region of the parameters where the boundary Harnack principle holds for the process $X_t=Y^D_{S_t}$, where D is the upper half-space, Y^D is the process in Example 3.9 (b-4) and S_t is an independent β-stable subordinator, $\beta\in(0,1)$. At the end we provide an interesting example in which the upper scaling index $\beta_2>1$ and the two scaling function Φ and Ψ are different.

Notations: Values of lower case letters with subscripts c_i, $i=0,1,2,...$ are fixed in each statement and proof, and the labeling of these constants starts anew in each proof. Recall that $a\wedge b:=\min\{a,b\}$,

$a \vee b := \max\{a, b\}$. We use two notations for comparison of functions. First, the notation $f(x) \simeq g(x)$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for specified range of x. On the other hand, the notation $f(x) \asymp g_1(x) + g_2(x)h(cx)$ means that there exist constants $c_3, c_4, c_5, c_6 > 0$ such that $c_3(g_1(x) + g_2(x)h(c_4 x)) \le f(x) \le c_5(g_1(x) + g_2(x)h(c_6 x))$ for specified range of x. We use the convention $1/\infty = 0$.

2. Estimates on distributions of subordinators

Let $S = (S_t)_{t \ge 0}$ be a driftless subordinator (i.e., a non-decreasing pure-jump Lévy process on $\mathbb{R}$ with $S_0 = 0$) with Laplace exponent ϕ given by

$$\phi(\lambda) = -\log \mathbb{E}e^{-\lambda S_1} = \int_0^\infty (1 - e^{-\lambda s})\nu(ds).$$

Let $w(r) := \nu(r, \infty)$. Using that $\phi(\lambda) = \lambda \int_0^\infty e^{-\lambda s} w(s) ds$, it is easy to see (cf. the proof of [22, Lemma 2.1]) that

$$e^{-1}\lambda \int_0^{1/\lambda} w(s)ds \le \phi(\lambda) \le 2\lambda \int_0^{1/\lambda} w(s)ds, \quad \lambda > 0. \tag{2.1}$$

The following is our main assumption on the subordinator S.

(Poly-R_1) There exist constants $R_1 \in (0, \infty]$, $c_1, c_2 > 0$ and $\beta_2 \ge \beta_1 > 0$ such that

$$c_1\Big(\frac{R}{r}\Big)^{\beta_1} \le \frac{w(r)}{w(R)} \le c_2\Big(\frac{R}{r}\Big)^{\beta_2} \quad \text{for all } \ 0 < r \le R < R_1.$$

Suppose that **(Poly-R_1)** holds. Then by [22, Lemma 2.1(ii)], in case $R_1 < \infty$, for every $r_0 > 0$, there exists $c_3 = c_3(r_0) > 0$ such that

$$\frac{\phi(R)}{\phi(r)} \le c_3\Big(\frac{R}{r}\Big)^{\beta_2 \wedge 1}, \quad r_0 < r < R. \tag{2.2}$$

On the other hand, by adapting the proof of [23, Lemma 2.3(3)], we can get that, for every $r_0 > 0$, there exists $c_4 = c_4(r_0) > 0$ such that

$$\frac{\phi(R)}{\phi(r)} \ge c_4\Big(\frac{R}{r}\Big)^{\beta_1}, \quad r_0 < r < R. \tag{2.3}$$

As a consequence of (2.2) and (2.3), we see that $\beta_1 \le 1$ and that ϕ^{-1} enjoys the following scaling: For every $t_0 > 0$, there exist $c_5, c_6 > 0$ depending on t_0 such that

$$c_5\Big(\frac{t}{s}\Big)^{1/(\beta_2 \wedge 1)} \le \frac{\phi^{-1}(t)}{\phi^{-1}(s)} \le c_6\Big(\frac{t}{s}\Big)^{1/\beta_1}, \quad t_0 < s < t. \tag{2.4}$$

In case when **(Poly-∞)** holds, (2.2) and (2.3) are valid for all $0 < r < R$, and (2.4) is valid for all $0 < s < t$.

Lemma 2.1. *Assume* **(Poly-R_1)** *holds. For any $a > 0$, there exists $c_1 = c_1(a) \in (0, 1)$ such that*

$$c_1\phi(\lambda) \le \lambda\phi'(\lambda) \le \phi(\lambda), \quad \lambda > a. \tag{2.5}$$

Further, if **(Poly-∞)** *holds, then* (2.5) *holds for all $\lambda > 0$.*

Proof. The second inequality follows from the fact $1 - e^{-u} - ue^{-u} \ge 0$ for $u \ge 0$. The first inequality follows from [37, Lemma 1.3] and its proof. □

Lemma 2.2. *Assume* **(Poly-R_1)** *holds. For any $a > 0$, there exists $c_1 = c_1(a) > 0$ such that*

$$|\phi''(\lambda)| \le c_1\lambda^{-1}\phi'(\lambda), \quad \lambda > a. \tag{2.6}$$

Further, if **(Poly-∞)** *holds, then* (2.6) *holds for all $\lambda > 0$.*

Proof. The proof is similar to that of [23, Lemma 2.1(3)], where the existence of Lévy density is assumed. We give a detailed proof for the reader's convenience.

Since $e^{-x} \le x^{-2}$ for all $x > 0$, we see that for all $\lambda > 1/R_1$,

$$\lambda|\phi''(\lambda)| = \int_0^{1/\lambda} \lambda y^2 e^{-\lambda y}\nu(dy) + \int_{1/\lambda}^\infty \lambda y^2 e^{-\lambda y}\nu(dy) \le \int_0^{1/\lambda} y\nu(dy) + \lambda^{-1}w(1/\lambda). \tag{2.7}$$

By **(Poly-R_1)**, there exists $\epsilon \in (0, 1/2)$ such that $w(\epsilon/\lambda) \ge 2w(1/\lambda)$ for all $\lambda > 1/R_1$. Hence,

$$\int_0^{1/\lambda} y\nu(dy) \ge \epsilon\lambda^{-1}\int_{\epsilon/\lambda}^{1/\lambda}\nu(dy) \ge \epsilon\lambda^{-1}w(1/\lambda) \quad \text{for all } \lambda > 1/R_1.$$

It follows that

$$\phi'(\lambda) \ge e^{-1}\int_0^{1/\lambda} y\nu(dy) \ge (2e)^{-1}\epsilon\Big(\int_0^{1/\lambda} y\nu(dy) + \lambda^{-1}w(1/\lambda)\Big). \tag{2.8}$$

Combining (2.8) with (2.7), we get that in the case $R_1 = \infty$, (2.6) holds for all $\lambda > 0$ with $c_1 = \epsilon/(2e)$, and in the case $R_1 < \infty$, (2.6) holds with

$$c_1 = \frac{\epsilon}{2e} \vee \sup_{\lambda\in[a,1/R_1]}\big(\lambda|\phi''(\lambda)|/\phi'(\lambda)\big).$$

□

Let $H : (0,\infty) \to (0,\infty)$ be defined by $H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda)$, $\lambda > 0$. The function H is strictly increasing, $H(0+) = 0$, $\lim_{\lambda\to\infty} H(\lambda) = \int_0^\infty \nu(ds) = w(0+)$, and satisfies

$$\frac{H(\lambda)}{\lambda^2} = -\Big(\frac{\phi(\lambda)}{\lambda}\Big)' = \int_0^\infty e^{-\lambda s} s w(s)ds, \quad \lambda > 0. \tag{2.9}$$

Since $1 - e^{-\lambda s} - \lambda s e^{-\lambda} \ge 1 - 2/e$ when $s \ge 1/\lambda$, we see that

$$\phi(\lambda) \ge H(\lambda) \ge \int_{1/\lambda}^\infty (1 - e^{-\lambda s} - \lambda s e^{-\lambda s})\nu(ds) \ge \frac{e-2}{e}w(1/\lambda), \quad \lambda > 0. \tag{2.10}$$

Suppose that **(Poly-R_1)** holds. Then it follows from the proof of [23, Lemma 2.3(3)] that, for every $r_0 > 0$, there exists a constant $c = c(r_0) > 0$ such that

$$\frac{H(R)}{H(r)} \ge c\Big(\frac{R}{r}\Big)^{\beta_1}, \quad r_0 < r \le R. \tag{2.11}$$

As a consequence of (2.11), we have the following upper scaling for the inverse function H^{-1}:

$$\frac{H^{-1}(t)}{H^{-1}(s)} \le c^{-1/\beta_1}\Big(\frac{t}{s}\Big)^{1/\beta_1}, \quad H(r_0) < s \le t. \tag{2.12}$$

In case when **(Poly-∞)** holds, (2.11) and (2.12) hold with $r_0 = 0$. Note that (2.11) implies that $\lim_{\lambda\to\infty} H(\lambda) = +\infty$.

Next we look at the function $b : (0,\infty) \to (0,\infty)$ defined by

$$b(t) := (\phi' \circ H^{-1})(1/t) = \int_0^\infty s e^{-H^{-1}(1/t)s}\nu(ds), \quad t > 0.$$

The function b is strictly increasing, $b(0+) = 0$, and $\lim_{t\to\infty} b(t) = \int_0^\infty s\nu(ds) = \phi'(0+)$. This implies that $t \mapsto tb(t)$ is also strictly increasing and $\lim_{t\to\infty} tb(t) = +\infty$. Moreover, according to [22, Lemma 2.4(ii)], cf. also [23, (2.13)], it holds that

$$\phi^{-1}(7/t)^{-1} \le tb(t) \le \phi^{-1}(1/t)^{-1} \quad \text{for all } t > 0. \tag{2.13}$$

Hence, under **(Poly-R_1)**, we see from the scaling of ϕ^{-1} in (2.4) that, for every $t_0 > 0$, there exists $c_1 = c_1(t_0) > 0$ such that

$$c_1\phi^{-1}(1/t)^{-1} \le tb(t) \le \phi^{-1}(1/t)^{-1} \quad \text{for all } 0 < t < t_0. \tag{2.14}$$

Moreover, if **(Poly-∞)** holds, then (2.14) holds with $t_0 = \infty$.

Finally, we introduce the function $\sigma = \sigma(t,s) : (0,\infty)\times(0,\infty) \to [0,\infty)$ defined by

$$\sigma = \sigma(t,s) := (\phi')^{-1}(s/t)\mathbf{1}_{(0,\phi'(0+))}(s/t), \quad s,t > 0.$$

Note that $s \mapsto \sigma(t,s)$ is decreasing with $\lim_{s\to 0}\sigma(t,s) = \infty$ and $\lim_{s\to\infty}\sigma(t,s) = 0$, while $t \mapsto \sigma(t,s)$ is increasing with $\lim_{t\to 0}\sigma(t,s) = 0$ and $\lim_{t\to\infty}\sigma(t,s) = \infty$. Further, by using the former and the fact that H is increasing, we conclude that

$$t(H\circ\sigma)(t,tb(t)) = 1 \quad \text{and} \quad t(H\circ\sigma)(t,s) < 1 \quad \text{for } s > tb(t). \tag{2.15}$$

The function σ plays a crucial role in estimating the left tail of the subordinator S. We first state a result which follows from [22, Lemma 2.11] and [32, Lemma 5.2].

Proposition 2.3. *There exist constants $c_1, c_2 > 0$ such that for all $t > 0$,*

$$c_1 \exp\big(-c_2 t(H\circ\sigma)(t,s)\big) \le \mathbb{P}(S_t \le s) \le e\exp\big(-t(H\circ\sigma)(t,s)\big).$$

Proof. If $s \le tb(t)$, then it follows from [22, Lemma 2.11] and [32, Lemma 5.2] that there exist $c_1, c_2 > 0$ independent of s and t such that $c_1 \exp\big(-c_2 t(H\circ\sigma)(t,s)\big) \le \mathbb{P}(S_t \le s) \le \exp\big(-t(H\circ\sigma)(t,s)\big)$. (Note that the function $b(t)$ in this paper is the same as $t^{-1}b(t)$ in [22].) In particular, taking $s = tb(t)$ and using (2.15), we get $c_3 := c_1 e^{-c_2} \le \mathbb{P}(S_t \le tb(t)) \le e^{-1}$.

If $s > tb(t)$, then by the second part of (2.15), $\exp(-t(H\circ\sigma)(t,s)) \ge e^{-1}$, and thus $\mathbb{P}(S_t \le s) \le 1 \le e\exp\big(-t(H\circ\sigma)(t,s)\big)$ which gives the desired upper bound. For the desired lower bound,

$$\mathbb{P}(S_t \le s) \ge \mathbb{P}(S_t \le tb(t)) \ge c_3 \ge c_3\exp\big(-c_2 t(H\circ\sigma)(t,s)\big).$$

□

Lemma 2.4. *Suppose* **(Poly-R_1)** *holds. Then, for any $a > 0$, there exists $\delta = \delta(a) > 0$ such that*

$$\frac{\sigma(t,u)}{\sigma(t,s)} \ge 2^{-\delta}\Big(\frac{s}{u}\Big)^{\delta}, \quad 0 < u \le s \le t\phi'(a). \tag{2.16}$$

Moreover, if **(Poly-∞)** *holds, then* (2.16) *holds for all $0 < u \le s < t\phi'(0+)$.*

Proof. Let $a > 0$. For all $0 < 2w \le t\phi'(a)$, it holds that $\sigma(t,2w) \ge a$. By the mean value theorem, the fact that both $|\phi''|$ and $s \mapsto \sigma(t,s)$ are decreasing, and Lemma 2.2, we get

$$\begin{aligned}\frac{w}{t} &= (\phi'\circ\sigma)(t,2w) - (\phi'\circ\sigma)(t,w) \le |(\phi''\circ\sigma)(t,2w)|(\sigma(t,w) - \sigma(t,2w))\\ &\le c_1\frac{(\phi'\circ\sigma)(t,2w)}{\sigma(t,2w)}(\sigma(t,w) - \sigma(t,2w)) = \frac{2c_1 w}{t}\frac{(\sigma(t,w) - \sigma(t,2w))}{\sigma(t,2w)}.\end{aligned} \tag{2.17}$$

Let $\delta = \log_2(1 + 1/(2c_1))$. Then, we see from (2.17) that for all $0 < 2w \le t\phi'(a)$,

$$2^{\delta}\sigma(t,2w) \le \sigma(t,w). \tag{2.18}$$

For any $0 < u \le s \le t\phi'(a)$, let $n = n(u,s)$ be the largest integer such that $2^n u \le s$. Then, by (2.18), we obtain

$$\frac{\sigma(t,u)}{\sigma(t,s)} \ge 2^{\delta n}\frac{\sigma(t,2^n u)}{\sigma(t,s)} \ge 2^{\delta n} \ge 2^{\delta n}2^{-\delta(n+1)}\Big(\frac{s}{u}\Big)^{\delta} = 2^{-\delta}\Big(\frac{s}{u}\Big)^{\delta}.$$

This proves the first assertion.

Assume now that $R_1 = \infty$. Then (2.6) is valid for all $\lambda > 0$ so that (2.17) holds for all $0 < 2w < t\phi'(0+)$. Hence, (2.18) holds for all $0 < 2w < t\phi'(0+)$. We conclude the proof as in the first assertion. □

Lemma 2.5. *Suppose that* **(Poly-R_1)** *holds. Then, for all $\kappa, N > 0$ and $T > 0$, there exists a constant $C = C(T,\kappa,N) > 0$ such that for all $0 < t \le T$ and $0 < s \le \phi^{-1}(1/t)^{-1}$,*

$$\exp\big(-\kappa t(H\circ\sigma)(t,s)\big) \le C(s\phi^{-1}(1/t))^{N}. \tag{2.19}$$

Moreover, if **(Poly-∞)** *holds, then for all $\kappa, N > 0$, there exists a constant $C = C(\kappa,N) > 0$ such that* (2.19) *holds for all $0 < s \le \phi^{-1}(1/t)^{-1}$.*

Proof. Choose an arbitrary $t \in (0,T]$. In view of (2.14), since $e^{-x} \le 1$ for all $x \ge 0$, it suffices to prove (2.19) for $0 < s \le tb(t)$. Recall that $t(H \circ \sigma)(t, tb(t)) = 1$. Hence, by (2.11), Lemma 2.4 and (2.14), we have that, for all $0 < s \le tb(t)$,

$$t(H \circ \sigma)(t,s) = \frac{(H \circ \sigma)(t,s)}{(H \circ \sigma)(t,tb(t))} \ge c_1 \Big(\frac{\sigma(t,s)}{\sigma(t,tb(t))} \Big)^{\beta_1} \ge c_2 \Big(\frac{\phi^{-1}(1/t)^{-1}}{s} \Big)^{\delta\beta_1},$$

where $\delta = \delta(T)$ is the constant from Lemma 2.4. Let $c_3 := \sup_{x>0} x^{N/(\delta\beta_1)} e^{-\kappa x}$. Then

$$\exp\big(-\kappa t(H \circ \sigma)(t,s) \big) \le c_3 \Big(\frac{1}{t(H \circ \sigma)(t,s)} \Big)^{N/(\delta\beta_1)} \le c_2 c_3 \big(s\phi^{-1}(1/t) \big)^N.$$

This proves the first assertion. Moreover, we can see that the second assertion is true by using that (2.12) and (2.14) hold for $r_0 = 0$ and $t_0 = \infty$, respectively and the second assertion of Lemma 2.4. □

Lemma 2.6. *Let $f : (0,\infty) \to (0,\infty)$ be a given function. Assume that* **(Poly-R_1)** *holds and there exist constants $c_1, p > 0$ such that $s^p f(s) \le c_1 t^p f(t)$ for all $0 < s \le t$. Then for every $T > 0$, there exists a constant $C = C(T, c_1, p) > 0$ such that for any $t \in (0,T]$,*

$$\mathbb{E}[f(S_t) : S_t \le r] \le C f(r) \exp\Big(-\frac{t}{2}(H \circ \sigma)(t,r) \Big), \quad 0 < r \le \phi^{-1}(1/t)^{-1}. \tag{2.20}$$

Moreover, if **(Poly-∞)** *holds, then there exists a constant $C = C(c_1,p) > 0$ such that* (2.20) *holds for all $t > 0$.*

Proof. By using Proposition 2.3 in the second and Lemma 2.5 (with $\kappa = 1/2$ and $N = p+1$) in the third inequality below, we get that

$$\begin{aligned}
\mathbb{E}[f(S_t) : S_t \le r] &= \sum_{j=0}^{\infty} \int_{2^{-j-1}r}^{2^{-j}r} f(s)\mathbb{P}(S_t \in ds) \le c_1 2^p \sum_{j=0}^{\infty} f(2^{-j}r)\mathbb{P}(S_t \le 2^{-j}r) \\
&\le c_1^2 2^p f(r) \sum_{j=0}^{\infty} 2^{jp} \exp\Big(-\frac{t}{2}(H \circ \sigma)(t, 2^{-j}r) \Big) \exp\Big(-\frac{t}{2}(H \circ \sigma)(t, 2^{-j}r) \Big) \\
&\le c_2 f(r) \exp\Big(-\frac{t}{2}(H \circ \sigma)(t,r) \Big) \sum_{j=0}^{\infty} 2^{jp} \big(2^{-j} r\phi^{-1}(1/t) \big)^{p+1} \le 2c_2 f(r) \exp\Big(-\frac{t}{2}(H \circ \sigma)(t,r) \Big).
\end{aligned}$$

□

Proposition 2.7. *Suppose that* **(Poly-R_1)** *holds. Then for any $T > 0$, there exist constants $\delta \in (0,1)$ independent of T and $\epsilon = \epsilon(T) \in (0,1)$ such that*

$$\mathbb{P}\big(\epsilon\phi^{-1}(1/t)^{-1} \le S_t \le \phi^{-1}(1/t)^{-1} \big) \ge \delta, \quad t \in (0,T). \tag{2.21}$$

Moreover, if **(Poly-∞)** *holds, then there exist $\epsilon, \delta \in (0,1)$ such that* (2.21) *holds with $T = \infty$.*

Proof. By Proposition 2.3, (2.15) and (2.14), there exists a constant $c_1 \in (0,1)$ such that $\mathbb{P}(S_t \le \phi^{-1}(1/t)^{-1}) \ge c_1$ for all $t > 0$. Let $c_2 = \log(2/c_1)$. Then, using Proposition 2.3 again, we get that for all $t > 0$,

$$\mathbb{P}\big(tb(t/c_2) \le S_t \le \phi^{-1}(1/t)^{-1} \big) \ge c_1 - \mathbb{P}\big(S_t < tb(t/c_2) \big) \ge c_1 - e^{-c_2} = c_1/2. \tag{2.22}$$

Since **(Poly-R_1)** holds, by (2.14) and (2.4), there exists $\epsilon \in (0,1)$ such that

$$tb(t/c_2) \ge \epsilon\phi^{-1}(1/t)^{-1}, \quad t \in (0,T). \tag{2.23}$$

We also see that if **(Poly-∞)** holds, then (2.23) holds with $T = \infty$. Combining this with (2.22), we obtain (2.21). □

Lemma 2.8. *Suppose that* **(Poly-R_1)** *holds. Then, for any $\kappa > 0$, there exists a constant $a_1 = a_1(\kappa) > 0$ such that for all $\phi^{-1}(1/t)^{-1} \le s < R_1/2$, we have $\exp\big(-\kappa s H^{-1}(1/t) \big) \le a_1 t w(s)$.*

Proof. According to [22, Lemma 2.2], there exists $c_1 > 0$ such that $H(1/s)^{1+\beta_2} \le c_1\phi(1/s)^{\beta_2}w(s)$, $s \in (0, R_1/2]$. Note that by (2.9), the map $\lambda \to \lambda^{-2}H(\lambda)$ is decreasing. Let $c_2 := \sup_{x>0} x^{2(1+\beta_2)}e^{-\kappa x}$. Then by using **(Poly-$R_1$)** and the fact that ϕ is increasing, we get that for all $\phi^{-1}(1/t)^{-1} \le s < R_1/2$, since $1/s \le \phi^{-1}(1/t) \le H^{-1}(1/t)$,

$$\begin{aligned}\exp\big(-\kappa s H^{-1}(1/t)\big) &\le c_2\Big(\frac{1/s}{H^{-1}(1/t)}\Big)^{2(1+\beta_2)} \le c_2\Big(\frac{H(1/s)}{1/t}\Big)^{1+\beta_2}\\ &\le c_1c_2t^{1+\beta_2}\phi(1/s)^{\beta_2}w(s) \le c_1c_2t^{1+\beta_2}\phi\big(\phi^{-1}(1/t)\big)^{\beta_2}w(s) = c_1c_2tw(s).\end{aligned}$$

This proves the lemma. □

Proposition 2.9. *Suppose that* **(Poly-R_1)** *holds. Then, for all* $2\phi^{-1}(1/t)^{-1} < s < R_1/2$,

$$\mathbb{P}(S_t \ge s) \simeq tw(s). \tag{2.24}$$

In particular, there exists a constant $M > 1$ *such that for all* $2\phi^{-1}(1/t)^{-1} < s < R_1/(2M)$,

$$\mathbb{P}(S_t \in [s, Ms]) \simeq tw(s).$$

Proof. The lower bound of (2.24) follows from [22, Lemma 2.6] (note that $t\phi(s^{-1}) \le 1$). The upper bound of (2.24) comes from the proof of [22, Proposition 2.7] with a bit of modification. We provide most of the proof for the reader's convenience.

Pick an arbitrary $s \in (2\phi^{-1}(1/t)^{-1}, R_1/2)$. Let $\epsilon = \log(5/4)/2 \in (0,1)$. We set

$$\mu^1 := \mathbf{1}_{(0,\epsilon/H^{-1}(1/t)]}\nu(dx), \quad \mu^2 := \mathbf{1}_{(\epsilon/H^{-1}(1/t),s]}\nu(dx) \ \text{ and } \ \mu^3 := \mathbf{1}_{(s,\infty)}\nu(dx)$$

and denote by S^1, S^2 and S^3 the independent driftless subordinators with Lévy measures μ^1, μ^2 and μ^3, respectively. Then $S_t \le S_t^1 + S_t^2 + S_t^3$ (note that it may happen that $s < \epsilon/H^{-1}(1/t)$) and hence

$$\mathbb{P}(S_t \ge s) \le \mathbb{P}(S_t^1 \ge 3s/4) + \mathbb{P}(S_t^2 \ge s/4) + \mathbb{P}(S_t^3 > 0).$$

Since S^3 is a compound Poisson process, it holds that $\mathbb{P}(S_t^3 > 0) = 1 - e^{-tw(s)} \le tw(s)$. Moreover, by following the proof of [22, Proposition 2.7], one can obtain from [34, Proposition 1 and Lemma 9] that $\mathbb{P}(S_t^2 \ge s/4) \le ctw(s)$. Lastly, by using Markov's inequality and [22, Lemma 2.5], since $s > 2tb(t)$ due to (2.13), we have that

$$\begin{aligned}\mathbb{P}(S_t^1 \ge 3s/4) &\le \mathbb{E}\left[\exp\left(-(3s/4)H^{-1}(1/t) + H^{-1}(1/t)S_t^1\right)\right]\\ &= \exp\Big(-(3s/4)H^{-1}(1/t) + t\int_0^{\epsilon/H^{-1}(1/t)}(e^{H^{-1}(1/t)x}-1)\nu(dx)\Big)\\ &\le \exp\Big(-(3s/4)H^{-1}(1/t) + e^{2\epsilon}tH^{-1}(1/t)\int_0^{\epsilon/H^{-1}(1/t)} xe^{-H^{-1}(1/t)x}\nu(dx)\Big)\\ &\le \exp\big(-(3s/4)H^{-1}(1/t) + (5/4)H^{-1}(1/t)tb(t)\big) \le \exp\big(-2^{-3}sH^{-1}(1/t)\big).\end{aligned}$$

We used the fact that $e^y - 1 \le ye^{-y}e^{2y}$ for all $y \ge 0$ in the third line. Hence, by Lemma 2.8, we get that $\mathbb{P}(S_t^1 \ge 3s/4) \le ctw(s)$ and hence the first assertion holds.

The second assertion follows from **(Poly-R_1)**. □

3. Setup and main assumptions

Let (E, ρ) be a locally compact separable metric space such that all bounded closed sets are compact, and let m a positive Radon measure on E with full support. We use $B(x,r)$ to denote the open ball in (E,ρ) of radius r centered at x, and $V(x,r) := m(B(x,r))$ its volume.

Throughout the remainder of this paper, we assume the following volume doubling and reverse volume doubling properties with localization radius $R_E \in (0,\infty]$: There exist constants $d_2 \ge d_1 > 0$ such that, for every $a \ge 1$, there exists a constant $C_V = C_V(a) \ge 1$ satisfying

$$C_V^{-1}\Big(\frac{R}{r}\Big)^{d_1} \le \frac{V(x,R)}{V(x,r)} \le C_V\Big(\frac{R}{r}\Big)^{d_2} \quad \text{for all } x \in E \ \text{ and } \ 0 < r \le R < aR_E. \tag{3.1}$$

As a consequence of (3.1), we see that for all $R_0, \epsilon, \eta > 0$, there exists a constant $C = C(R_0, \epsilon, \eta) > 0$ such that

$$V(x, r) \le CV(y, \eta r) \quad \text{for all } x, y \in E \ \text{ and } \ \epsilon\rho(x, y) < r \le R_0. \tag{3.2}$$

Indeed, since $B(x, r) \subset B(y, r + \rho(x, y))$, we get from (3.1) that $V(x, r) \le V(y, r + \rho(x, y)) \le V(y, (1 + 1/\epsilon)r) \le c_1 V(y, \eta r)$. Moreover, if the localization radius R_E is infinite, then the above constant C is independent of R_0 and (3.2) holds for $\epsilon\rho(x, y) < r < \infty$.

Let D be a proper open subset of E. We use $\mathrm{diam}(D)$ to denote the diameter of D. If $\mathrm{diam}(D) < \infty$, i.e., D is bounded, then by the assumption on E it holds that $m(D) < \infty$. For $x \in D$, let $\delta_D(x) = \rho(x, E \setminus D)$. In most applications, D will be an open subset of the Euclidean space $\mathbb{R}^d$, $d \ge 1$, and $m(dy)$ will be the Lebesgue measure. For simplicity we write dy instead of $m(dy)$. Let $Y^D = (Y^D_t, \mathbb{P}^x)$ be a Hunt process in D. We assume that the semigroup of Y^D admits a density $p_D(t, x, y)$, which we call the heat kernel of Y^D. Thus, for any non-negative Borel function f on D,

$$\mathbb{E}^x[f(Y^D_t)] = \int_D f(y) p_D(t, x, y)\, dy.$$

Let $S = (S_t)_{t \ge 0}$ be a driftless subordinator independent of Y^D. We will be interested in the subordinate process $X_t := Y^D_{S_t}$. It is well known (cf. [6, p.67, pp. 73–75] and [50]) that X is also a Hunt process and admits a heat kernel $q(t, x, y)$ which is given by the formula

$$q(t, x, y) = \mathbb{E}[p_D(S_t, x, y)] = \int_0^\infty p_D(s, x, y) \mathbb{P}(S_t \in ds). \tag{3.3}$$

Our goal is to find two-sided estimates of $q(t, x, y)$ under certain assumptions on the underlying heat kernel $p_D(t, x, y)$ and the subordinator S. On the subordinator we will impose the assumption **(Poly-R_1)**. Now we explain the assumptions we impose on $p_D(t, x, y)$. These assumptions are motivated by various examples from the literature.

We first introduce two functions $\Phi, \Psi : [0, \infty) \to [0, \infty)$, both strictly increasing and satisfying $\Psi(r) \ge \Phi(r)$ for all $r \ge 0$. Moreover, we always assume that both satisfy global scaling conditions: There exist constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ and $c_1, c_2, c_3, c_4 > 0$ such that for all $R \ge r > 0$,

$$c_1 \Big(\frac{R}{r}\Big)^{\alpha_1} \le \frac{\Phi(R)}{\Phi(r)} \le c_2 \Big(\frac{R}{r}\Big)^{\alpha_2} \quad \text{and} \quad c_3 \Big(\frac{R}{r}\Big)^{\alpha_3} \le \frac{\Psi(R)}{\Psi(r)} \le c_4 \Big(\frac{R}{r}\Big)^{\alpha_4}. \tag{3.4}$$

As an easy consequence we see that, for every $a \ge 1$, there exist two constants $c_1(a) > 0$ and $c_2(a) > 0$ such that, for all $r, R > 0$ satisfying $0 < r \le aR$, it holds that

$$c_1(a) \Big(\frac{R}{r}\Big)^{\alpha_1} \le \frac{\Phi(R)}{\Phi(r)} \le c_2(a) \Big(\frac{R}{r}\Big)^{\alpha_2}. \tag{3.5}$$

The following lemma shows that, without loss of generality, we may replace Φ by a nicer function.

Lemma 3.1. *There exists a strictly increasing differentiable functions $\widetilde{\Phi}$ satisfying the following two properties:*
(P1) $\Phi(r) \simeq \widetilde{\Phi}(r)$ for all $r > 0$ and $\widetilde{\Phi}$ satisfies (3.5);
(P2) $\widetilde{\Phi}'(r) \simeq r^{-1}\widetilde{\Phi}(r)$ and $(\widetilde{\Phi}^{-1})'(t) \simeq t^{-1}\widetilde{\Phi}^{-1}(t)$ for $r, t > 0$.

Proof. According to [8, Lemmas 3.1 and 3.2], for any $\alpha > \alpha_2$, there exists a complete Bernstein function φ such that

$$\Phi(r) \simeq \varphi(r^{-\alpha})^{-1} \quad \text{and} \quad \varphi'(r) \simeq r^{-1}\varphi(r) \quad \text{for all } r > 0,$$

and that φ satisfies the weak scaling conditions with exponents α_1/α and α_2/α. Let $\widetilde{\Phi}(r) := \varphi(r^{-\alpha})^{-1}$, $r > 0$. It is straightforward to check that $\widetilde{\Phi}$ satisfies (3.5) and also that $\widetilde{\Phi}'(r) \simeq r^{-1}\widetilde{\Phi}(r)$. Moreover, by the inverse function theorem, the second comparability in (P2) is also valid. □

Lemma 3.2. *Let $f : (0, \infty) \to (0, \infty)$ be a given function. Assume that there exist constants $c_1, p > 0$ such that $s^p f(s) \le c_1 t^p f(t)$ for all $0 < s \le t$. Then there exists a constant $c_2 = c_2(c_1, p) > 0$ such that for all $r, \kappa > 0$,*

$$\int_0^r f(s) \exp\Big(-\frac{\kappa^2}{\Phi^{-1}(s)^2}\Big) ds \le \frac{c_2 r^{p+1} f(r)}{\Phi(\kappa)^p}. \tag{3.6}$$

Proof. Let $c_3 := \sup_{u>0} u^{p\alpha_2/2} e^{-u}$. Then by the scaling of Φ, we have that

$$\begin{aligned}
&\int_0^r f(s) \exp\Big(-\frac{\kappa^2}{\Phi^{-1}(s)^2}\Big) ds \le c_3 \int_0^{r\wedge\Phi(\kappa)} f(s)\Big(\frac{\Phi^{-1}(s)}{\kappa}\Big)^{p\alpha_2} ds + \int_{r\wedge\Phi(\kappa)}^r f(s)ds \\
&\le c_4 \int_0^{r\wedge\Phi(\kappa)} f(s)\Big(\frac{s}{\Phi(\kappa)}\Big)^p ds + \int_{r\wedge\Phi(\kappa)}^r s^{-p}s^p f(s)ds \\
&\le \frac{c_1c_4 r^p f(r)}{\Phi(\kappa)^p}\int_0^r ds + \frac{c_1 r^p f(r)}{(r\wedge\Phi(\kappa))^p}\int_{r\wedge\Phi(\kappa)}^r ds \le \frac{c_1(c_4+1)r^{p+1}f(r)}{\Phi(\kappa)^p}.
\end{aligned}$$

□

Definition 3.3. We say that a function $h : (0,\infty)\times D\times D \to [0,1]$ is a *boundary function* if it satisfies the following two properties:
(H1) For all fixed $x, y \in D$, the map $s \mapsto h(s,x,y)$ is non-increasing.
(H2) There exist constants $c_1 > 0$, $\gamma \ge 0$ such that

$$s^\gamma h(s,x,y) \le c_1 t^\gamma h(t,x,y), \qquad 0 < s \le t < 4\Phi(\mathrm{diam}(D)) + 1,\ x,y \in D,$$

with $4\Phi(\mathrm{diam}(D)) + 1$ interpreted as ∞ when D is unbounded.

A boundary function h is said to be *regular* if there exists $c_2 > 0$ such that

$$h(t,x,y) \ge c_2, \qquad 0 < t < 4\Phi(\mathrm{diam}(D)) + 1,\ x,y \in D \text{ with } \delta_\wedge(x,y) \ge \Phi^{-1}(t).$$

A regular boundary function h is said to be of *Harnack-type* if there exists $c_3 > 0$ such that for all $x, y, z \in D$ satisfying $\rho(x,z) \le (\rho(x,y) \wedge \delta_D(x))/2$,

$$h(t,x,y) \le c_3 h(t,z,y), \qquad 0 < t < \Phi(\rho(x,y)). \tag{3.7}$$

From now on, $h(t,x,y)$ always denotes a boundary function.

Remark 3.4. Suppose that h is a regular boundary function. Then for every $\epsilon \in (0,1)$, there exists $c_1 = c_1(\epsilon) > 0$ such that

$$h(t,x,y) \ge c_1, \qquad 0 < t < 4\Phi(\mathrm{diam}(D)) + 1,\ x,y \in D \text{ with } \delta_\wedge(x,y) \ge \epsilon\Phi^{-1}(t).$$

Indeed, by (3.4) and (H2), we see that for all $x, y \in D$ with $\delta_\wedge(x,y) \ge \epsilon\Phi^{-1}(t)$,

$$h(t,x,y) \ge c_2 h(\Phi(\epsilon\Phi^{-1}(t)), x, y) \ge c_3.$$

Example 3.5. (a) Let $p, q \ge 0$. For $t > 0$ and $x, y \in D$, define

$$h_{p,q}(t,x,y) := \Big(1 \wedge \frac{\Phi(\delta_D(x))}{t}\Big)^p \Big(1 \wedge \frac{\Phi(\delta_D(y))}{t}\Big)^q, \qquad h_p(t,x,y) := h_{p,p}(t,x,y). \tag{3.8}$$

Then $h_{p,q}(t,x,y)$ is a typical example of a regular boundary function which is also of Harnack-type. Indeed, (H1) and the regularity is clear, while (H2) holds with $c_1 = 1$ and $\gamma = p + q$ since for all $0 < s < t$,

$$t^{p+q} h_{p,q}(t,x,y) = \Big(t \wedge \Phi(\delta_D(x))\Big)^p \Big(t \wedge \Phi(\delta_D(y))\Big)^q \ge s^{p+q} h_{p,q}(s,x,y).$$

Moreover, we see from (3.4) that for all $x, y, z \in D$ satisfying $\rho(x,z) \le (\rho(x,y) \wedge \delta_D(x))/2$, since $\delta_D(z) \ge \delta_D(x) - \rho(x,z) \ge \delta_D(x)/2$,

$$\Big(1 \wedge \frac{\Phi(\delta_D(x))}{t}\Big)^p \le \Big(1 \wedge \frac{\Phi(2\delta_D(z))}{t}\Big)^p \le c_1\Big(1 \wedge \frac{\Phi(\delta_D(z))}{t}\Big)^p.$$

Thus we conclude that $h_{p,q}(t,x,y)$ is of Harnack-type. The boundary function $h_p(t,x,y)$ is very typical when D is a bounded smooth open subset of $\mathbb{R}^d$.
(b) Let $h_p(t,x,y)$ be the function defined in (3.8). Then $h_p(t\wedge 1, x, y)$ is also a regular boundary function of Harnack-type. This is a typical boundary function for smooth exterior open sets.
(c) A quite general example of a boundary function is obtained as follows. Suppose that Y^D admits a dual process $\widehat{Y}^D$. Let ζ and $\widehat{\zeta}$ be the lifetimes of Y^D and $\widehat{Y}^D$ respectively. Assume that the survival probabilities $\mathbb{P}^x(\zeta > t)$ and $\mathbb{P}^y(\widehat{\zeta} > t)$ satisfy the following doubling property: $\mathbb{P}^x(\zeta > t/2) \simeq \mathbb{P}^x(\zeta > t)$ and $\mathbb{P}^y(\widehat{\zeta} > t/2) \simeq \mathbb{P}^y(\widehat{\zeta} > t)$ for all $0 < t < 4\Phi(\mathrm{diam}(D)) + 1$ and $x, y \in D$. Then $h(t,x,y) := \mathbb{P}^x(\zeta > t)\mathbb{P}^y(\widehat{\zeta} > t)$ is a boundary function. Indeed, (H1) is clear, while (H2) follows from the doubling property

of survival probabilities assumed above. The survival probabilities usually satisfy the doubling property, see for instance, [25, Lemma 2.21] and its proof. In fact, by [25, Lemma 2.21] and its proof, one can see that the boundary function h above is often regular.

Moreover, the above $h(t,x,y)$ is of Harnack-type if, in addition, (1) it is regular; (2) Y^D satisfies the (interior elliptic) Harnack inequality and (3) there is $c_1>0$ such that for all $x\in D$ and $\Phi(\delta_D(x))<t<\Phi(\mathrm{diam}(D))$,

$$\mathbb{P}^x(\zeta>t)\simeq \mathbb{P}^x(\zeta>\tau_{U(x,t)})=\mathbb{P}^x(Y^D_{\tau_{U(x,t)}},\in D), \tag{3.9}$$

where $U(x,t):=B(x,c_1\Phi^{-1}(t))\cap D$ and $\tau_V=\inf\{t>0: Y^D_t\notin V\}$.

To see this, we fix $x,z\in D$ satisfying $\rho(x,z)\le \delta_D(x)/2$. If $\delta_D(x)\vee\delta_D(z)\ge (c_1\wedge 2^{-1})\Phi^{-1}(t)$, then we have $\delta_D(x)\wedge\delta_D(z)\ge \delta_D(x)\vee\delta_D(z)-2^{-1}\delta_D(x)\ge 2^{-1}(c_1\wedge 2^{-1})\Phi^{-1}(t)$. By Remark 3.4, it follows that $1\ge \mathbb{P}^x(\zeta>t)\wedge\mathbb{P}^z(\zeta>t)\ge h(t,x,x)\wedge h(t,z,z)\ge c_2$. Hence, we obtain $h(t,x,y)/h(t,z,y)=\mathbb{P}^x(\zeta>t)/\mathbb{P}^z(\zeta>t)\le 1/c_2$. If $\delta_D(x)\vee\delta_D(z)<(c_1\wedge 2^{-1})\Phi^{-1}(t)$, then $B(x,\delta_D(x))\subset U(x,t)$ so that $v\mapsto \mathbb{P}^v(Y^D_{\tau_{U(x,t)}}\in D)$ is harmonic in $B(x,\delta_D(x))$ with respect to Y^D. Using (3.9) twice, we see from the Harnack inequality, (3.4) and (H2) that

$$\mathbb{P}^x(\zeta>t)\le c_3\mathbb{P}^z(Y^D_{\tau_{U(x,t)}}\in D)\le c_3\mathbb{P}^z(Y^D_{\tau_{U(z,\Phi(\Phi^{-1}(t)/2))}}\in D)\le c_4\mathbb{P}^z(\zeta>\Phi(\Phi^{-1}(t)/2))\le c_5\mathbb{P}^z(\zeta>t).$$

The second inequality above is valid since $U(z,\Phi(\Phi^{-1}(t)/2))\subset U(x,t)$. Therefore, we obtain (3.7).

Under the setting and assumptions in [25, Section 2] (Assumptions **A** and **U** in [25]), for the Hunt process Y defined right below [25, (2.27)] on a κ-fat open set D with a critical killing potential $\mu\in\mathbf{K}_1(D)$, by [25, Lemma 2.21], we know that the boundary function $h(t,x,y)=\mathbb{P}^x(\zeta>t)\mathbb{P}^y(\widehat{\zeta}>t)$ is of Harnack-type. (See [25, Definition 2.19] and [25, Definition 2.12] for the definitions of a κ-fat open set and the class $\mathbf{K}_1(D)$, respectively.) See [2, 4, 14] for related work.

For later use, we record the following simple consequence of (H1) and (H2): Let $k>1$ and $s,t>0$ satisfy $k^{-1}s\le t\le ks\le 4\Phi(\mathrm{diam}(D))$. Then for all $x,y\in D$,

$$c_1^{-1}k^{-\gamma}h(s,x,y)\le h(t,x,y)\le c_1k^{\gamma}h(s,x,y), \tag{3.10}$$

where c_1 is the constant from (H2).

Definition 3.6. Let $h(t,x,y)$ be a boundary function.
(a) We say that $\mathbf{HK^h_B}$ holds, if D is bounded and the following estimates hold: (i) there exist $C_0\in\{0,1\}$ and $c_1,c_2,c_3,c_4>0$ such that for all $(t,x,y)\in(0,1]\times D\times D$,

$$\begin{aligned} &c_1h(t,x,y)\left[\frac{1}{V(x,\Phi^{-1}(t))}\wedge\left(\frac{C_0t}{V(x,\rho(x,y))\Psi(\rho(x,y))}+\frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c_2\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big)\right)\right]\\ &\le p_D(t,x,y)\\ &\le c_3h(t,x,y)\left[\frac{1}{V(x,\Phi^{-1}(t))}\wedge\left(\frac{C_0t}{V(x,\rho(x,y))\Psi(\rho(x,y))}+\frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c_4\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big)\right)\right], \end{aligned} \tag{3.11}$$

and (ii) there exists a constant $\lambda_D>0$ such that for all $(t,x,y)\in[1,\infty)\times D\times D$,

$$p_D(t,x,y)\simeq e^{-\lambda_D t}h(1,x,y). \tag{3.12}$$

(b) We say that $\mathbf{HK^h_U}$ holds, if the constant R_E in (3.1) is infinity and (3.11) holds for all $(t,x,y)\in(0,\infty)\times D\times D$.

By using the function $(1\wedge\frac{R_1}{10\Phi(\mathrm{diam}(D))})\Phi(r)$ instead of $\Phi(r)$, we may and do assume that $\Phi(\mathrm{diam}(D))<R_1/8$ whenever **(Poly-R_1)** and $\mathbf{HK^h_B}$ hold.

Remark 3.7. One can easily see that if $\mathbf{HK^h_B}$ holds, then for every $T>0$, there exist constants $c_1,c_2,c_3,c_4>0$ such that (3.11) holds for all $(t,x,y)\in(0,T]\times D\times D$, and (3.12) holds for all $(t,x,y)\in[T,\infty)\times D\times D$.

Remark 3.8. Note that $a\wedge(b+c)\le(a\wedge b)+(a\wedge c)\le 2(a\wedge(b+c))$ for all $a,b,c>0$. Hence (3.11) is equivalent to the statement that for all $(t,x,y)\in(0,1]\times D\times D$,

$$p_D(t,x,y)\asymp h(t,x,y)\left[\left(\frac{1}{V(x,\Phi^{-1}(t))}\wedge\frac{C_0t}{V(x,\rho(x,y))\Psi(\rho(x,y))}\right)+\frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big)\right]. \tag{3.13}$$

Example 3.9. Here are several examples of processes satisfying $\mathbf{HK_B^h}$ or $\mathbf{HK_U^h}$. We will not try to give the most general examples but the reader will see from examples below that our setup is general enough to cover almost all known cases. In all examples below, the boundary functions are of Harnack type.

(a) Suppose that D is a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$.

(1) If D is connected and Y^D is the killed Brownian motion in D, then $\mathbf{HK_B^h}$ is satisfied with $C_0 = 0$, $\Phi(r) = r^2$ and boundary function $h_{1/2}$. See [24] for a more general example.

(2) If $\alpha \in (0, 2)$ and Y^D is a killed isotropic α-stable process in D, then $\mathbf{HK_B^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{1/2}$, cf. [9]. More generally, suppose χ is a complete Bernstein function satisfying global weak scaling conditions with indices $\beta_1, \beta_2 \in (0, 1)$, Y is a subordinate Brownian motion in $\mathbb{R}^d$ via an independent subordinator with Laplace exponent χ, Y^D is the part process of Y in D. Then $\mathbf{HK_B^h}$ is satisfied with $\Phi(r) = \Psi(r) = 1/\chi(r^{-2})$ and boundary function $h_{1/2}$, cf. [14]. See [3, 31, 36] for more general examples.

(3) If D is connected and Y is the independent sum of isotropic α-stable process and Brownian motion, then its part process Y^D in D satisfies $\mathbf{HK_B^h}$ with $\Phi(r) = r^2 \wedge r^\alpha$, $\Psi(r) = r^\alpha$ and boundary function $h_{1/2}$, cf. [11]. More generally, suppose χ is a complete Bernstein function satisfying the conditions in the paragraph above and Y is the independent sum of Brownian motion and a subordinate Brownian motion via a subordinator with Laplace exponent χ, then its part process Y^D in D satisfies $\mathbf{HK_B^h}$ with $\Phi(r) = \Phi_\chi(r) := r^2 \wedge (1/\chi(r^{-2}))$, $\Psi(r) = 1/\chi(r^{-2})$ and boundary function $h_{1/2}$, cf. [15]. Note that since $\lim_{\lambda\to\infty} \chi(\lambda)/\lambda = 0$ (see (2.1)), for every $a > 0$, there are comparability constants depending on a such that $\Phi_\chi(r) \simeq r^2$ for $r \in (0, a)$. We remark here that the estimates in [11, (1.4)] and [15, (1.14)] are comparable to (3.13) since $t \le 1$.

(4) Suppose that χ is a complete Bernstein function such that the function $\lambda \mapsto \chi(\lambda) - \lambda\chi'(\lambda)$ satisfies weak scaling conditions for $\lambda \ge a > 0$ with upper index $\delta < 2$ and lower index $\gamma > 2^{-1}\mathbf{1}_{\{\delta\ge1\}}$, Suppose that Y is a subordinate Brownian motion in $\mathbb{R}^d$ via an independent subordinator with Laplace exponent χ, Y^D is the part process of Y in D. Then $\mathbf{HK_B^h}$ is satisfied with $\Phi(r) = 1/\chi(r^{-2})$, $\Psi(r) = 1/(\chi(r^{-2}) - r^{-2}\chi'(r^{-2}))$ and boundary function $h_{1/2}$, cf. [38].

(5) Let $\alpha \in (1, 2)$ and Y^D be a censored α-stable process in D. Then it follows from [10] that $\mathbf{HK_B^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{(\alpha-1)/\alpha}$.

(6) Let $\alpha \in (0, 2)$ and Z^D be the part process, in D, of a reflected isotropic α-stable process in $\overline{D}$. For any $q \in [\alpha - 1, \alpha) \cap (0, \alpha)$, let Y^D be the process on D corresponding to the Feynman-Kac semigroup of Z^D via the multiplicative functional $\exp(-\int_0^t C(d, \alpha, q)\, \text{dist}(Z_s^D, \partial D)^{-\alpha} ds)$, where the positive constant $C(d, \alpha, q)$ is defined on [25, p. 233]. It follows from [25, Theorem 3.2] that the small time estimates (3.11) holds with $\Phi(r) = \Psi(r) = r^\alpha$ and $h_{q/\alpha}$. Using the small time estimates and the argument in [20, Section 4], one can easily show that the semigroup of Y^D is intrinsically ultracontractive. With this, one can easily check that the large time estimates in Definition 3.6(a)(ii) holds. Thus $\mathbf{HK_B^h}$ holds.

(7) Suppose that D is connected, $d \ge 3$ and $\kappa \ge -\frac{1}{4}$. Let Y^D be the process corresponding to $\Delta|_D - \kappa\delta_D(x)^{-2}$, the Dirichlet Laplacian in D with critical potential $\kappa\delta_D(x)^{-2}$. It follows from [28, (6)] and [29, Corollary 1.8] that the heat kernel of Y^D satisfies $\mathbf{HK_B^h}$ with $C_0 = 0$, $\Phi(r) = r^2$ and boundary function h_p, where $p = \frac{1}{2}(\frac{1}{2} + \sqrt{\frac{1}{4} + \kappa})$.

(8) Suppose that $\alpha \in (1, 2)$ and $d \ge 2$. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ such that $|b|$ is in the Kato class $\mathbb{K}_{d,\alpha-1}$ (see [12, Definition 1.1] for definition). Let Y be an α-stable process with drift b in $\mathbb{R}^d$, that is, a process with generator $-(-\Delta)^{\alpha/2} + b \cdot \nabla$, and let Y^D be the part process of Y in D. By [12, Theorem 1.3], $\mathbf{HK_B^h}$ holds with $\Phi(r) = \Psi(r) = r^\alpha$ and $h_{1/2}$. See also [39].

(9) For general setups in which $\mathbf{HK_B^h}$ is satisfied, see [25, Section 2] and [30].

(b) Suppose that D is an unbounded $C^{1,1}$ open subset of $\mathbb{R}^d$.

(1) If D is the domain above the graph of a bounded Lipschitz function in $\mathbb{R}^{d-1}$, then the killed Brownian motion in D satisfies $\mathbf{HK_U^h}$ with $C_0 = 0$, $\Phi(r) = r^2$ and a boundary function defined in terms of survival probabilities like in Example 3.5(b), which is of Harnack type (cf. [51]).

(2) Suppose that D is a half-space-like $C^{1,1}$ open set in $\mathbb{R}^d$ and $\alpha \in (0, 2)$. Let Y^D be the part process in D of an isotropic α-stable process. Then by [21, Theorem 1.2], $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{1/2}$. More generally, let Y^D be the part process in D of the independent sum of Brownian motion and an isotropic α-stable process. By [13, Theorem 1.4 and Remark 1.5(ii)], $\mathbf{HK_U^h}$ is

satisfied with $\Phi(r) = r^2 \wedge r^\alpha$, $\Psi(r) = r^\alpha$ and boundary function $h_{1/2}$. When D is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$ with $d > \alpha$ and Y^D is part process in D of an isotropic α-stable process, it follows from [21, Theorem 1.2] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{1/2}(t \wedge 1, x, y)$. See [35] for a more general example.

(3) Suppose D is the upper half space in $\mathbb{R}^d$. Let χ be a complete Bernstein function satisfying global weak scaling conditions with indices $\alpha_1, \alpha_2 \in (0,1)$, Y be a subordinate Brownian motion in $\mathbb{R}^d$ via an independent subordinator with Laplace exponent χ, Y^D be the part process of Y in D. It follows from [40, Theorem 5.10] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = 1/\chi(r^{-2})$ and boundary function $h_{1/2}$. See [7] for a more general example.

(4) Suppose that D is the upper half space in $\mathbb{R}^d$ and $\alpha \in (0,2)$. Let Z^D be the part process, in D, of a reflected isotropic α-stable process in $\overline{D}$. For any $q \in [\alpha-1, \alpha) \cap (0, \alpha)$, let Y^D be the process on D corresponding to the Feynman-Kac semigroup of Z^D via the multiplicative functional $\exp(-\int_0^t C(d,\alpha,q)\delta_D(Z_s^D)^{-\alpha} ds)$, where $C(d,\alpha,q)$ is defined on [25, p. 233]. It follows from [25, Theorem 3.2] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{q/\alpha}$.

(5) Suppose that $D = \mathbb{R}^d \setminus \{0\}$ and $\alpha \in (0,2)$. Let Z be an isotropic α-stable process in $\mathbb{R}^d$. For any $q \in (0, \alpha)$, let Y^D be the process on D corresponding to the Feynman-Kac semigroup of Z^D via the multiplicative functional $\exp(-\int_0^t \widetilde{C}(d,\alpha,q)|Z_s^D|^{-\alpha} ds)$, where $\widetilde{C}(d,\alpha,q)$ is defined on [25, p. 250]. It follows from [25, Theorem 3.9] and [33, Theorem 1.1] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{q/\alpha}$.

(6) Suppose that $D = \mathbb{R}^d \setminus \{0\}$, $d \ge 2$ or $D = (0,\infty)$. Let Y^D be a process with generator $\Delta + (a-1)|x|^{-2} \sum_{i,j=1}^d x_i x_j \partial_{ij} + \kappa |x|^{-2} \cdot \nabla - b|x|^{-2}$ for some $a > 0$, $\kappa, b \in \mathbb{R}$ such that

$$\Lambda := \frac{1}{2}\sqrt{\frac{b}{a} + \Big(\frac{d-1+\kappa-a}{2a}\Big)^2} \ge \frac{1}{4a}\big((d-1+\kappa-a) \vee ((2a-1)d+1-\kappa-3a)\big).$$

Note that when $a = 1$ and $\kappa, b \ge 0$, the above inequality is always true. It follows from [46, Proposition 4.14, Theorem 6.2, Corollary 6.4] that $\mathbf{HK_U^h}$ is satisfied with $C_0 = 0$, $\Phi(r) = r^2$ and boundary function $h_{p,q}$ where $p = \Lambda - (d-1+\kappa-a)/(4a)$ and $q = \Lambda - ((2a-1)d+1-\kappa-3a)/(4a)$.

(7) Suppose that $\alpha \in (1,2)$ and $D = \mathbb{R}^d \setminus \{0\}$, $d \ge 3$. Let Y^D be a process with generator $-(-\Delta)^{-\alpha/2} + \kappa|x|^{-\alpha} x \cdot \nabla$ for some $\kappa \in (0,\infty)$. It follows from [45, Theorems 4 and 5] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h = h_{0,\beta/\alpha}$ for $\beta \in (0,\alpha)$ determined by the equation at the beginning of [45, Section 3.2].

We now briefly discuss the term

$$I(t,x,y,C_0) := \frac{1}{V(x,\Phi^{-1}(t))} \wedge \Big(\frac{C_0 t}{V(x,\rho(x,y))\Psi(\rho(x,y))} + \frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c_1\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big)\Big)$$

appearing in (3.11). If $C_0 = 0$, then clearly

$$I(t,x,y,0) = \frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c_1\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big). \tag{3.14}$$

Suppose now that $C_0 = 1$.

Lemma 3.10. *For any $a \ge 1$, there are comparability constants depending on a such that*

$$I(t,x,y,1) \asymp \begin{cases} \dfrac{1}{V(x,\Phi^{-1}(t))}, & t \ge a^{-1}\Phi(\rho(x,y)), \\ \dfrac{t}{V(x,\rho(x,y))\Psi(\rho(x,y))} + \dfrac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\dfrac{c\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big), & t < a\Phi(\rho(x,y)). \end{cases} \tag{3.15}$$

In particular, if $\Psi(r) \simeq \Phi(r)$ for $r \in (0,R_1)$, then

$$I(t,x,y,1) \simeq \frac{1}{V(x,\Phi^{-1}(t))} \wedge \frac{t}{V(x,\rho(x,y))\Phi(\rho(x,y))}, \quad t>0,\ x,y \in D,\ \rho(x,y) < R_1. \tag{3.16}$$

Proof. If $t \ge a^{-1}\Phi(\rho(x,y))$, then by (3.5),

$$\frac{1}{V(x,\Phi^{-1}(t))} \ge I(t,x,y,1) \ge \frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c_1\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big) \ge \frac{c_2(a,c_1)}{V(x,\Phi^{-1}(t))}.$$

Assume that $t < a\Phi(\rho(x,y))$. Set

$$g(t,x,y) := \frac{t}{V(x,\rho(x,y))\Psi(\rho(x,y))} + \frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c_1\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big).$$

Clearly, $I(t,x,y,1) \le g(t,x,y)$. Further, by using that $\Psi \ge \Phi$ and (3.5), we have

$$g(t,x,y) \le \frac{a}{V(x,\Phi^{-1}(t/a))} + \frac{1}{V(x,\Phi^{-1}(t))} \le \frac{c_3(a)+1}{V(x,\Phi^{-1}(t))}.$$

Hence, $I(t,x,y,1) = V(x,\Phi^{-1}(t))^{-1} \wedge g(t,x,y) \ge (c_3(a)+1)^{-1}g(t,x,y)$. Thus, (3.15) holds.

Now, we assume that $\Psi(r) \simeq \Phi(r)$ for $r \in (0,R_1)$. Using (3.1), (3.5) and the fact that $e^{-u} \le k^k u^{-k}$ for all $u,k>0$, we get that for all $t>0$ and $x,y\in D$ satisfying $t < \Phi(\rho(x,y))$ and $\rho(x,y) < R_1$,

$$\begin{aligned}\frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{c_1\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big) &\le \frac{c_4}{V(x,\Phi^{-1}(t))}\Big(\frac{\Phi^{-1}(t)^2}{c_1\rho(x,y)^2}\Big)^{(d_2+\alpha_1)/2}\\ &\le \frac{c_5 t}{V(x,\rho(x,y))\Phi(\rho(x,y))} \le \frac{c_6 t}{V(x,\rho(x,y))\Psi(\rho(x,y))}.\end{aligned}$$

Thus, we can deduce (3.16) from (3.15). □

4. Jump kernel and heat kernel estimates

For a given boundary function h, we define for $(t,x,y) \in [0,\infty)\times D\times D$,

$$\mathcal{B}^*_h(x,y) := \int_0^{\Phi(\rho(x,y))} h(s,x,y)w(s)ds \tag{4.1}$$

and if $\phi^{-1}(1/t)^{-1} \le \Phi(\rho(x,y))$,

$$\mathcal{B}_h(t,x,y) := \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(\rho(x,y))} h(s,x,y)w(s)ds. \tag{4.2}$$

Since $\int_0^r w(s)ds < \infty$ for all $r>0$ (see (2.1)) and $h \le 1$, the integral in (4.1) converges. Note that, by (H1), $\mathcal{B}^*_h(x,y) \simeq \mathcal{B}_h(0,x,y)$ for all $(x,y) \in D\times D$.

4.1. Jump kernel estimates.

The jump kernel of the subordinate process X is given by

$$J(x,y) = \int_0^\infty p_D(s,x,y)\nu(ds), \quad x,y\in D. \tag{4.3}$$

See [6, p.74] and also [48].

Theorem 4.1. *Suppose that either (1)* **(Poly-R_1)** *and* $\mathbf{HK^h_B}$ *hold, or (2)* **(Poly-∞)** *and* $\mathbf{HK^h_U}$ *hold. Then, for $(x,y)\in D\times D$ with $x\ne y$,*

$$J(x,y) \simeq \frac{C_0\mathcal{B}^*_h(x,y)}{V(x,\rho(x,y))\Psi(\rho(x,y))} + h(\Phi(\rho(x,y)),x,y)\frac{w\big(\Phi(\rho(x,y))\big)}{V(x,\rho(x,y))}. \tag{4.4}$$

Proof. Since the proofs are similar, we only give the proof of the case (1), which is more complicated. Fix $x,y\in D$ with $x\ne y$ and let $r := \rho(x,y) > 0$. By Remark 3.7, (3.11) and (3.12) hold with $T := \Phi(2\mathrm{diam}(D))$. Then by (4.3) and (3.15),

$$\begin{aligned}J(x,y) \asymp &\frac{C_0}{V(x,r)\Psi(r)}\int_0^{\Phi(r)} sh(s,x,y)\nu(ds) + \int_0^{\Phi(r)}\frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\nu(ds)\\ &+\int_{\Phi(r)}^T \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\nu(ds) + h(1,x,y)\int_T^\infty e^{-\lambda_D s}\nu(ds) =: C_0J_1+J_2+J_3+J_4.\end{aligned}$$

Since **(Poly-R_1)** holds, there exists a constant $a>1$ such that $w(s/a) \ge 2w(s)$ for all $s<R_1$. Therefore, by (3.10), since we assumed $\Phi(\mathrm{diam}(D)) < R_1/8$,

$$V(x,r)\Psi(r)J_1 = \sum_{i\in\mathbb{N}}\int_{a^{-i}\Phi(r)}^{a^{-i+1}\Phi(r)} sh(s,x,y)\nu(ds)$$

$$\simeq \sum_{i\in\mathbb{N}} a^{-i}\Phi(r)h(a^{-i}\Phi(r),x,y)\big(w(a^{-i}\Phi(r)) - w(a^{-i+1}\Phi(r))\big)$$

$$\simeq \sum_{i\in\mathbb{N}} a^{-i}\Phi(r)h(a^{-i}\Phi(r),x,y)w(a^{-i}\Phi(r)) \simeq \sum_{i\in\mathbb{N}} \int_{a^{-i}\Phi(r)}^{a^{-i+1}\Phi(r)} h(s,x,y)w(s)ds = \mathcal{B}_h^*(x,y).$$

Next, by (H1), the scaling and monotonicity of Φ, we get that

$$J_2 \ge \frac{h(\Phi(r),x,y)}{V(x,r)} \int_{\Phi(r)/a}^{\Phi(r)} \exp\Big(-\frac{c_1 r^2}{\Phi^{-1}(s)^2}\Big)\nu(ds) \ge \frac{c_2 h(\Phi(r),x,y)}{V(x,r)} \int_{\Phi(r)/a}^{\Phi(r)} \nu(ds)$$
$$= \frac{c_2 h(\Phi(r),x,y)}{V(x,r)}\big(w(\Phi(r)/a) - w(\Phi(r))\big) \ge \frac{c_2 h(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}.$$

Hence, we obtain the lower bound in (4.4).

Now, we prove the upper bound in (4.4). Let $\widetilde{\Phi}$ be the function in Lemma 3.1. Since $s\mapsto V(x,\widetilde{\Phi}^{-1}(s))^{-1}$ and $s\mapsto h(s,x,y)$ are non-increasing, using the Leibniz rule for product, integration by parts and the property (P2) of $\widetilde{\Phi}^{-1}$ in Lemma 3.1, we obtain

$$J_2 \le c\int_0^{\Phi(r)} \frac{h(s,x,y)}{V(x,\widetilde{\Phi}^{-1}(s))} \exp\Big(-\frac{c_3 r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)\Big(-\frac{d}{ds}w(s)\Big)$$
$$\le c\int_0^{\Phi(r)} \frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\Big(\frac{d}{ds}\exp\Big(-\frac{c_3 r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)\Big)ds$$
$$\le c\int_0^{\Phi(r)} \frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\frac{r^2}{s\widetilde{\Phi}^{-1}(s)^2} \exp\Big(-\frac{c_3 r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)ds. \tag{4.5}$$

In the second inequality above, we used the following: Since $h\le 1$, $e^{-x}\le k^k x^{-k}$ for all $x,k>0$ and $\lim_{s\to 0} sw(s) = 0$ (because w is the tail of the Lévy mesure ν), by using (3.1) and the scaling of $\widetilde{\Phi}^{-1}$, we have that

$$\lim_{s\to 0}\frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\exp\Big(-\frac{c_3 r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big) \le c\lim_{s\to 0}\frac{w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\Big(\frac{\widetilde{\Phi}^{-1}(s)^2}{r^2}\Big)^{(d_2+\alpha_2)/2}$$
$$\le \frac{c}{r^{d_2+\alpha_2}V(x,\widetilde{\Phi}^{-1}(1))}\lim_{s\to 0} w(s)\widetilde{\Phi}^{-1}(s)^{\alpha_2} \le \frac{c\,\widetilde{\Phi}^{-1}(1)^{\alpha_2}}{r^{d_2+\alpha_2}V(x,\widetilde{\Phi}^{-1}(1))}\lim_{s\to 0} sw(s) = 0.$$

By **(Poly-R_1)**, (H2), (3.1), (3.5) and the fact that $\Phi\simeq\widetilde{\Phi}$, we can use Lemma 3.2 with $f(s) = h(s,x,y)w(s)\ V(x,\widetilde{\Phi}^{-1}(s))^{-1}s^{-1}\widetilde{\Phi}^{-1}(s))^{-2}$ and $p=\gamma+\beta_2+1+(d_2+2)/\alpha_1$ to deduce from (4.5) that

$$J_2 \le c\frac{\Phi(r)^{\gamma+\beta_2+1+(d_2+2)/\alpha_1+1}}{\Phi(r)^{\gamma+\beta_2+1+(d_2+2)/\alpha_1}}\frac{h(\Phi(r),x,y)w(\Phi(r))r^2}{V(x,r)\Phi(r)r^2} = \frac{ch(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}. \tag{4.6}$$

For J_3 and J_4, since $s\mapsto V(x,\Phi^{-1}(s))^{-1}$, $s\mapsto h(s,x,y)$ and $s\mapsto w(s)$ are non-increasing, we have by the boundedness of D that

$$J_3 + J_4 \le \frac{h(\Phi(r),x,y)w(\Phi(r))}{V(x,r)} + h(1,x,y)w(T) \le \frac{ch(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}.$$

This completes the proof. □

Suppose that $\Psi\simeq\Phi$ and $C_0=1$. Then the first term in (4.4) dominates the second. Indeed, by (H1),

$$\begin{aligned}\mathcal{B}_h^*(x,y) &= \int_0^{\Phi(\rho(x,y))} h(s,x,y)w(s)\,ds \ge h(\Phi(\rho(x,y)),x,y)\int_0^{\Phi(\rho(x,y))} w(s)\,ds\\ &\ge h(\Phi(\rho(x,y)),x,y)w(\Phi(\rho(x,y)))\Phi(\rho(x,y)).\end{aligned}$$

Moreover, if $\beta_2<1$, then according to [47, Lemma 2.6, Proposition 2.9] and (2.2), we get that $w(s)\simeq \phi(1/s)$ for all $0<s<R_1/2$. Therefore, it holds that

$$J(x,y) \simeq \frac{1}{V(x,\rho(x,y))\Phi(\rho(x,y))}\int_0^{\Phi(\rho(x,y))} h(s,x,y)\phi(1/s)\,ds.$$

In case the boundary function is equal to $h_{1/2}$, the integral above can be estimated in the same way as in [42, Lemma 8.1], cf. [42, (8.4)].

Suppose that $C_0 = 0$. Then

$$J(x,y) \simeq h(\Phi(\rho(x,y))\frac{w(h(\Phi(\rho(x,y)),x,y)}{V(x,\rho(x,y))}. \tag{4.7}$$

In particular, in the context of Example 3.9(b-1), and assuming $\beta_2 < 1$, the above formula reduces to [41, Theorem 4.4.(1)]. Similarly, if D is an exterior $C^{1,1}$ domain in $\mathbb{R}^d$, the boundary function is equal to $h_{1/2}(t \wedge 1, x, y)$ and $\beta_2 < 1$, then (4.7) reduces to [41, Theorem 4.4.(2)].

4.2. **Heat kernel estimates.** Let

$$\psi(r) := \frac{1}{\phi(1/\Phi(r))}, \quad r > 0. \tag{4.8}$$

Since ϕ and Φ are strictly increasing, ψ is also strictly increasing. Moreover, it follows from (2.2), (2.3) and (3.4) that, for every $R_0 > 0$, there exist $c_1, c_2 > 0$ such that

$$c_1\Big(\frac{R}{r}\Big)^{\alpha_1\beta_1} \le \frac{\psi(R)}{\psi(r)} \le c_2\Big(\frac{R}{r}\Big)^{\alpha_2(\beta_2\wedge 1)}, \quad 0 < r < R < R_0. \tag{4.9}$$

In case when **(Poly-∞)** holds, (4.9) is valid with $R_0 = \infty$. We note that

$$\psi^{-1}(t) = \Phi^{-1}\big(\phi^{-1}(1/t)^{-1}\big), \quad t > 0. \tag{4.10}$$

Recall the definition of the function $\mathcal{B}_h(t,x,y)$ from (4.2).

Theorem 4.2. *Suppose that* **(Poly-R_1)** *and* $\mathbf{HK_B^h}$ *hold. Then for every $T > 0$, the following estimates are valid for all $(t,x,y) \in (0,T] \times D \times D$:*
(i) *If $\psi(\rho(x,y)) \le t$, then*

$$q(t,x,y) \simeq \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}. \tag{4.11}$$

(ii) *If $\psi(\rho(x,y)) \ge t$, then*

$$\begin{aligned} q(t,x,y) \asymp{}& \frac{C_0}{V(x,\rho(x,y))\Psi(\rho(x,y))}\bigg(t\mathcal{B}_h(t,x,y) + \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{\phi^{-1}(1/t)}\bigg) \\ &+ \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c\,\rho(x,y)^2}{\psi^{-1}(t)^2}\Big) + h(\Phi(\rho(x,y)),x,y)\frac{tw\big(\Phi(\rho(x,y))\big)}{V(x,\rho(x,y))}. \end{aligned} \tag{4.12}$$

Proof. Take $x, y \in D$ and let $r := \rho(x,y)$. We start by establishing some relations valid for all $t \in (0,T]$. By Proposition 2.7, there exist constants $\delta, \epsilon \in (0,1)$ such that

$$q(t,x,y) \ge \delta \inf_{s\in[\epsilon\phi^{-1}(1/t)^{-1},\phi^{-1}(1/t)^{-1}]} p_D(s,x,y), \quad t \in (0,T]. \tag{4.13}$$

On the other hand, by Remark 3.7 (with $T = \Phi(\mathrm{diam}(D))$), (3.14), (3.15), (3.12) and the fact that $\exp(-cr^2/\Phi^{-1}(s)^2) \simeq 1$ when $s > \Phi(r)$, we see that

$$\begin{aligned} &q(t,x,y) \\ &\asymp \int_0^{\Phi(r)} h(s,x,y)\bigg(\frac{C_0 s}{V(x,r)\Psi(r)} + \frac{1}{V(x,\Phi^{-1}(s))}\exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\bigg)\mathbb{P}(S_t \in ds) \\ &\quad + \int_{\Phi(r)}^{\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t \in ds) + h(1,x,y)\int_{\Phi(\mathrm{diam}(D))}^{\infty} e^{-\lambda_D s}\,\mathbb{P}(S_t \in ds) \\ &\asymp C_0\int_0^{\Phi(r)} \frac{sh(s,x,y)}{V(x,r)\Psi(r)}\mathbb{P}(S_t \in ds) + \int_0^{\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\mathbb{P}(S_t \in ds) \\ &\quad + h(1,x,y)\int_{\Phi(\mathrm{diam}(D))}^{\infty} e^{-\lambda_D s}\mathbb{P}(S_t \in ds) =: C_0 I_1 + I_2 + I_3. \end{aligned} \tag{4.14}$$

(i) Assume that $\psi(r) \le t$. By Remark 3.7, (3.15), (H1), (3.1), the scaling of Φ^{-1} and (4.10), there exists a constant $c_1 > 0$ such that

$$\inf_{s\in[\epsilon\phi^{-1}(1/t)^{-1},\phi^{-1}(1/t)^{-1}]} p_D(s,x,y) \ge c_1 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}, \quad t \in (0,T].$$

Hence, the lower bound in (4.11) follows from (4.13).

Now, we prove the upper bound in (4.11). First, using Lemma 2.6 in the first inequality below, the assumption $\Psi \ge \Phi$ and Lemma 2.5 with $N = \gamma + d_2/\alpha_1$ in the second, (H2), (4.10) and (3.4) in the third, and (3.1) in the last, we get that

$$\begin{aligned} I_1 &\le \frac{\Phi(r)h(\Phi(r),x,y)}{V(x,r)\Psi(r)} \exp\Big(-\frac{t}{2}(H\circ\sigma)\big(t,\Phi(r)\big)\Big) \\ &\le c_2 \frac{h(\Phi(r),x,y)}{V(x,r)} \big(\Phi(r)\phi^{-1}(1/t)\big)^{\gamma}\big(\Phi(r)\phi^{-1}(1/t)\big)^{d_2/\alpha_1} \\ &\le c_3 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)} \Big(\frac{r}{\psi^{-1}(t)}\Big)^{d_2} \le c_4 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}. \end{aligned}$$

Next, we observe that

$$I_2 \le \int_0^{\phi^{-1}(1/t)^{-1}} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t\in ds) + \int_{\phi^{-1}(1/t)^{-1}}^{\infty} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t\in ds) =: I_{2,1} + I_{2,2}.$$

By (H2), (3.1) and (3.4), we can apply Lemma 2.6 with $f(s) = h(s,x,y)V(x,\Phi^{-1}(s))^{-1}$ and $p = \gamma + d_2/\alpha_1$ to get that

$$I_{2,1} \le c_5 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))} \exp\Big(-\frac{t}{2}(H\circ\sigma)(t,\phi^{-1}(1/t)^{-1})\Big) \le c_5 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}.$$

Moreover, we see from (H1), (4.10) and the monotonicity of ψ^{-1} that

$$I_{2,2} \le \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\mathbb{P}(S_t \ge \phi^{-1}(1/t)^{-1}) \le \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}.$$

Lastly, by using (H1) and (H2), since ϕ and ψ are increasing and $t \le T$, we have that

$$I_3 \le h(1,x,y) \le c_6 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}.$$

Hence, we obtain the upper bound in (4.11) from (4.14).

(ii) Assume that $\psi(r) \ge t$. First we establish the lower bound. From (4.13), Remark 3.7, (3.15), (4.10), (H1), and the scaling and monotonicity of ψ^{-1}, we get that

$$q(t,x,y) \ge c_7 h(\phi^{-1}(1/t)^{-1},x,y)\left[\frac{C_0\phi^{-1}(1/t)^{-1}}{V(x,r)\Psi(r)} + \frac{1}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c_8 r^2}{\psi^{-1}(t)^2}\Big)\right]. \tag{4.15}$$

We also see from Remark 3.7 that

$$q(t,x,y) \ge \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} p_D(s,x,y)\mathbb{P}(S_t\in ds) \ge \frac{c_9C_0}{V(x,r)\Psi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\mathbb{P}(S_t\in ds), \tag{4.16}$$

where in the second inequality we used (3.15) and neglected the second term. Let $M > 1$ be the constant in Proposition 2.9. If $\Phi(r) > M\phi^{-1}(1/t)^{-1}$, then by (H1), (H2) and Proposition 2.9, since we assumed $\Phi(\mathrm{diam}(D)) < R_1/8$,

$$\begin{aligned} &\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\mathbb{P}(S_t\in ds) \ge \sum_{\substack{i\in\mathbb{N}\\ M^i\le 2\Phi(r)\phi^{-1}(1/t)}} \int_{2M^{i-1}\phi^{-1}(1/t)^{-1}}^{2M^i\phi^{-1}(1/t)^{-1}} sh(s,x,y)\mathbb{P}(S_t\in ds) \\ &\ge c_{10}M^{-1}t \sum_{\substack{i\in\mathbb{N}\\ M^i\le 2\Phi(r)\phi^{-1}(1/t)}} 2M^i\phi^{-1}(1/t)^{-1}h(2M^{i-1}\phi^{-1}(1/t)^{-1},x,y)w\big(2M^{i-1}\phi^{-1}(1/t)^{-1}\big) \\ &\ge c_{10}M^{-1}t \sum_{\substack{i\in\mathbb{N}\\ M^i\le 2\Phi(r)\phi^{-1}(1/t)}} \int_{2M^{i-1}\phi^{-1}(1/t)^{-1}}^{2M^i\phi^{-1}(1/t)^{-1}} h(s,x,y)w(s)ds \end{aligned}$$

$$\geq c_{10}M^{-1}t\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)/M} h(s,x,y)w(s)ds \geq c_{10}M^{-1}t\int_{2\Phi(r)/M}^{4\Phi(r)/M} h(s,x,y)w(s)ds$$
$$\geq 2c_{10}M^{-2}t\Phi(r)h(4\Phi(r)/M,x,y)w(4\Phi(r)/M) \geq c_{10}2^{-1}M^{-2}t\int_{4\Phi(r)/M}^{4\Phi(r)} h(s,x,y)w(s)ds.$$

By the fourth and the last inequalities above, we deduce from (4.16) that

$$q(t,x,y) \geq \frac{c_9C_0}{2V(x,r)\Psi(r)}2\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\mathbb{P}(S_t\in ds) \geq \frac{c_9c_{10}}{2M^2}\frac{C_0t\mathcal{B}_h(t,x,y)}{V(x,r)\Psi(r)}.$$

In case when $\Phi(r)\leq M\phi^{-1}(1/t)^{-1}$, we see from (H1), (2.10) and (4.15) that

$$\frac{C_0t\mathcal{B}_h(t,x,y)}{V(x,r)\Psi(r)} \leq \frac{C_0t}{V(x,r)\Psi(r)}\int_{2\phi^{-1}(1/t)^{-1}}^{4M\phi^{-1}(1/t)^{-1}} h(s,x,y)w(s)ds$$
$$\leq \frac{4eMC_0}{e-2}\frac{t\phi^{-1}(1/t)^{-1}}{V(x,r)\Psi(r)}h(\phi^{-1}(1/t)^{-1},x,y)\phi\big(\phi^{-1}(1/t)\big) \leq c_{11}\, q(t,x,y).$$

Hence, it remains to prove that there exists a constant $c_{12}>0$ such that

$$q(t,x,y)\geq c_{12}h(\Phi(r),x,y)\frac{tw(\Phi(r))}{V(x,r)}.$$

Recall that $M>1$ is the constant in Proposition 2.9. By using Proposition 2.9, (3.15), (H1), (3.1) and the scaling of Φ, we get that, if $4\Phi(r)/M > 2\phi^{-1}(1/t)^{-1}$, then

$$q(t,x,y)\geq c_{13}\int_{4\Phi(r)/M}^{4\Phi(r)}\frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t\in ds) \geq c_{14}h(\Phi(r),x,y)\frac{tw(\Phi(r))}{V(x,r)}.$$

If $4\Phi(r)/M\leq 2\phi^{-1}(1/t)^{-1}$, then by (4.10), the scaling of Φ and the assumption that $\psi(r)\geq t$, we get $\psi^{-1}(t)\leq r\leq c_{15}\psi^{-1}(t)$ for some $c_{15}>1$. By (4.10) and (2.10), since w is non-increasing,

$$tw(\Phi(r))\leq tw(\phi^{-1}(1/t)^{-1})\leq e(e-2)^{-1}t\phi\big(\phi^{-1}(1/t)\big) = e(e-2)^{-1}. \tag{4.17}$$

Therefore, by (H1), (4.15) (neglecting the first term) and (3.4), we obtain

$$q(t,x,y)\geq c_{16}\frac{h(\Phi(r),x,y)}{V(x,r)}\exp\Big(-\frac{c_8c_{15}\psi^{-1}(t)^2}{\psi^{-1}(t)^2}\Big)\geq c_{17}h(\Phi(r),x,y)\frac{tw(\Phi(r))}{V(x,r)}.$$

This completes the proof of the lower bound.

Now we prove the upper bound. Recall (4.14). Observe that

$$V(x,r)\Psi(r)I_1 \leq \int_0^{\phi^{-1}(1/t)^{-1}} sh(s,x,y)\mathbb{P}(S_t\in ds) + \int_{\phi^{-1}(1/t)^{-1}}^{2\phi^{-1}(1/t)^{-1}} sh(s,x,y)\mathbb{P}(S_t\in ds)$$
$$+\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\mathbb{P}(S_t\in ds) =: K_1+K_2+K_3.$$

We get from Lemma 2.6 that $K_1\leq c_{18}\phi^{-1}(1/t)^{-1}h(\phi^{-1}(1/t)^{-1},x,y)$. Next, by (H1), we have $K_2\leq 2\phi^{-1}(1/t)^{-1}\ h(\phi^{-1}(1/t)^{-1},x,y)$. To bound K_3, we use integration by parts and Proposition 2.9 to obtain

$$K_3 = \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\frac{d}{ds}\big(-\mathbb{P}(S_t\geq s)\big)$$
$$\leq 2\phi^{-1}(1/t)^{-1}h(\phi^{-1}(1/t)^{-1},x,y) + \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} h(s,x,y)\mathbb{P}(S_t\geq s)ds$$
$$+\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} s\mathbb{P}(S_t\geq s)\frac{dh(s,x,y)}{ds} \leq c_{19}\big(\phi^{-1}(1/t)^{-1}h(\phi^{-1}(1/t)^{-1},x,y)+t\mathcal{B}_h(t,x,y)\big).$$

In the second inequality above, we used the fact that $s\mapsto h(s,x,y)$ is non-increasing (so that $s\mapsto \frac{d}{ds}h(s,x,y)\leq 0$ a.e.).

Now, we estimate I_2. We have

$$I_2\leq \int_0^{2\phi^{-1}(1/t)^{-1}}\frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\mathbb{P}(S_t\in ds)$$

$$
\begin{aligned}
&+\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))} \exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\mathbb{P}(S_t\in ds) + \int_{4\Phi(r)}^{\infty} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t\in ds)\\
&=: L_1+L_2+L_3.
\end{aligned}
$$

By applying Lemma 2.6, we get from (H1), (H2), (3.1), the scaling of Φ and (4.10) that

$$
\begin{aligned}
L_1 &\le \exp\Big(-\frac{cr^2}{\Phi^{-1}(2\phi^{-1}(1/t)^{-1})^2}\Big)\int_0^{2\phi^{-1}(1/t)^{-1}} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t\in ds)\\
&\le c_{20}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c_{21}r^2}{\psi^{-1}(t)^2}\Big).
\end{aligned}
$$

Let $\widetilde{\Phi}$ be the function in Lemma 3.1. By using integration by parts and similar calculations to (4.5), we get that

$$
\begin{aligned}
L_2 &\le c_{22}\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \frac{h(s,x,y)}{V(x,\widetilde{\Phi}^{-1}(s))}\exp\Big(-\frac{c_{23}r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)\frac{d}{ds}\big(-\mathbb{P}(S_t\ge s)\big)\\
&\le c_{24}\Bigg[\frac{h(2\phi^{-1}(1/t)^{-1},x,y)}{V(x,\widetilde{\Phi}^{-1}(2\phi^{-1}(1/t)^{-1}))}\exp\Big(-\frac{c_{23}r^2}{\widetilde{\Phi}^{-1}(2\phi^{-1}(1/t)^{-1})^2}\Big)\mathbb{P}\big(S_t\ge 2\phi^{-1}(1/t)^{-1}\big)\\
&\quad+\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \frac{h(s,x,y)}{V(x,\widetilde{\Phi}^{-1}(s))}\mathbb{P}(S_t\ge s)\frac{r^2}{s\widetilde{\Phi}^{-1}(s)^2}\exp\Big(-\frac{c_{23}r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)ds\Bigg] =: c_{25}\,(L_{2,1}+L_{2,2}).
\end{aligned}
$$

By (H1), (3.1), the scaling of Φ and (4.10), since $\Phi\simeq\widetilde{\Phi}$, we see that

$$
L_{2,1}\le c_{26}h(\phi^{-1}(1/t)^{-1},x,y)V(x,\psi^{-1}(t))^{-1}\exp\big(-c_{27}r^2/\psi^{-1}(t)^2\big).
$$

Also, by using Proposition 2.9 and repeating the calculation in (4.6), we get that

$$
L_{2,2}\le c_{28}t\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\frac{r^2}{s\widetilde{\Phi}^{-1}(s)^2}\exp\Big(-\frac{c_{23}r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)ds \le \frac{c_{29}th(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}.
$$

By (H1) and Proposition 2.9, we obtain

$$
L_3\le h(\Phi(r),x,y)V(x,r)^{-1}\mathbb{P}(S_t\ge 4\Phi(r))\le c_{30}th(\Phi(r),x,y)V(x,r)^{-1}w(\Phi(r)).
$$

Finally, we estimate I_3. By Proposition 2.9, since D is bounded, we get from (H1) and (H2) that

$$
I_3\le c_{31}th(1,x,y)w\big(\Phi(\mathrm{diam}(D))\big)\le c_{32}th(\Phi(r),x,y)V(x,r)^{-1}w(\Phi(r)).
$$

This completes the proof. □

If one assumes $\mathbf{HK^h_U}$, instead of $\mathbf{HK^h_B}$, together with **(Poly-∞)**, the results of Theorem 4.2(i) and (ii) are valid for all time. The proof is analogous to the proof of Theorem 4.2 and hence omitted.

Theorem 4.3. *Suppose that* **(Poly-∞)** *and* $\mathbf{HK^h_U}$ *hold. Then the assertions in Theorem 4.2(i)–(ii) hold for all* $(t,x,y)\in(0,\infty)\times D\times D$.

When the upper scaling index β_2 in **(Poly-R_1)** is strictly less than 1, we can obtain the following simpler form of off-diagonal estimate.

Corollary 4.4. *Suppose that* **(Poly-R_1)** *holds with* $\beta_2<1$ *and* $\Phi(r)\simeq\Psi(r)$ *for* $r\in(0,R_1)$.
(i) If $\mathbf{HK^h_B}$ *holds, then for every* $T>0$, *the following estimates hold for all* $(t,x,y)\in(0,T]\times D\times D$:
(1) If $\psi(\rho(x,y))\le t$, *then* (4.11) *holds.*
(2) If $\psi(\rho(x,y))\ge t$, *then*

$$
q(t,x,y)\simeq\frac{t}{V(x,\rho(x,y))}\times\begin{cases}\dfrac{h(\Phi(\rho(x,y)),x,y)}{\psi(\rho(x,y))} & \text{when } C_0=0,\\[2ex] \dfrac{\mathcal{B}_h(t,x,y)}{\Phi(\rho(x,y))} & \text{when } C_0=1.\end{cases} \tag{4.18}
$$

(ii) If $R_1=\infty$ *and* $\mathbf{HK^h_U}$ *holds, then (1) and (2) above hold for all* $(t,x,y)\in(0,\infty)\times D\times D$.

Proof. According to [47, Lemma 2.6, Proposition 2.9] and (2.2), since $\beta_2 < 1$, we get that $w(s) \simeq \phi(1/s)$ for all $0 < s < R_1/2$.

(i) We only need to deal with the case (2), i.e., the case $\psi(\rho(x,y)) \ge t$. We first assume that $C_0 = 0$. Using $w(s) \simeq \phi(1/s)$, we get that $w(\Phi(\rho(x,y))) \simeq 1/\psi(\rho(x,y))$. Thus by Theorem 4.2, it remains to show that for any given $c_1 > 0$, there exists $c_2 > 0$ such that for all $(t,x,y) \in (0,T] \times D \times D$ satisfying $\psi(\rho(x,y)) \ge t$,

$$\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))} \exp\Big(- \frac{c_1 \rho(x,y)^2}{\psi^{-1}(t)^2}\Big) \le c_2 \frac{th(\Phi(\rho(x,y)),x,y)}{V(x,\rho(x,y))\psi(\rho(x,y))}. \tag{4.19}$$

Let $c_3 := \sup_{u>0} u^{(d_2+\alpha_2\gamma+\alpha_2\beta_2)/2} e^{-u}$. By (3.1), (3.4), (4.9), (4.10) and (H2), we obtain that

$$\begin{aligned}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))} \exp\Big(- \frac{c_1 \rho(x,y)^2}{\psi^{-1}(t)^2}\Big) &\le c_3 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))} \Big(\frac{\psi^{-1}(t)^2}{c_1\rho(x,y)^2}\Big)^{(d_2+\alpha_2\gamma+\alpha_2\beta_2)/2} \\ &\le c_4 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\rho(x,y))} \Big(\frac{\Phi(\psi^{-1}(t))}{\Phi(\rho(x,y))}\Big)^{\gamma} \frac{t}{\psi(\rho(x,y))} \le c_5 \frac{th(\Phi(\rho(x,y)),x,y)}{V(x,\rho(x,y))\psi(\rho(x,y))}.\end{aligned}$$

Now, let $C_0 = 1$. Since $\Phi(r) \simeq \Psi(r)$ for $r \in (0,R_1)$, the first term on the right hand side of (4.12) is comparable with

$$\frac{t\mathcal{B}_h(t,x,y)}{V(x,\rho(x,y))\Phi(\rho(x,y))} + \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\rho(x,y))\Phi(\rho(x,y))\phi^{-1}(1/t)}.$$

By (H2) and **(Poly-R_1)**, it holds that for all $(t,x,y) \in (0,T] \times D \times D$ satisfying $\psi(\rho(x,y)) \ge t$,

$$t\mathcal{B}_h(t,x,y) \ge t \int_{2\Phi(\rho(x,y))}^{4\Phi(\rho(x,y))} h(s,x,y)w(s)ds \ge c_6 th(\Phi(\rho(x,y)),x,y) w\big(\Phi(\rho(x,y))\big)\Phi(\rho(x,y)).$$

Combining this with (4.19), using $w(s) \simeq \phi(1/s)$, one can see that the first term on the right hand side of (4.12) dominates the other two terms. Further, by (H2), (2.10) and (2.2), we see that for all $(t,x,y) \in (0,T] \times D \times D$ satisfying $\psi(\rho(x,y)) \ge t$,

$$\begin{aligned}t\mathcal{B}_h(t,x,y) &\ge t \int_{2\phi^{-1}(1/t)^{-1}}^{4\phi^{-1}(1/t)^{-1}} h(s,x,y)w(s)ds \ge c_7 t \int_{2\phi^{-1}(1/t)^{-1}}^{4\phi^{-1}(1/t)^{-1}} h(s,x,y)\phi(1/s)ds \\ &\ge c_8 \phi^{-1}(1/t)^{-1} h(\phi^{-1}(1/t)^{-1},x,y).\end{aligned}$$

This yields the desired conclusion.

(ii) This can be proved by the same argument as that of (i). We omit the details here. □

In the case when D is a bounded $C^{1,1}$ domain, Y^D is a killed Brownian motion in D and S is an $(\alpha/2)$-stable subordinator, part (i) of the corollary above is equivalent to [49, Theorem 4.7]. In the case when D is an exterior $C^{1,1}$ domain, Y^D is a killed Brownian motion in D and S is an $(\alpha/2)$-stable subordinator, part (ii) of the corollary above corrects [49, Theorem 4.6].

For future use, we note the following rough upper estimates on $q(t,x,y)$.

Proposition 4.5. *(i) Suppose that* **(Poly-R_1)** *and* $\mathbf{HK_B^h}$ *hold. Then for every $T > 0$, there exists a constant $C > 0$ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$q(t,x,y) \le Ch(\phi^{-1}(1/t)^{-1},x,y)\Big(\frac{1}{V(x,\psi^{-1}(t))} \wedge \frac{t}{V(x,\rho(x,y))\psi(\rho(x,y))}\Big). \tag{4.20}$$

(ii) Suppose that **(Poly-∞)** *and* $\mathbf{HK_U^h}$ *hold. Then, there exists a constant $C > 0$ such that (4.20) holds for all $(t,x,y) \in (0,\infty) \times D \times D$.*

Proof. (i) Take $x,y \in D$ and let $r := \rho(x,y)$. If $\psi(r) \le t$, then (4.20) follows from Theorem 4.2(i). Hence, we assume that $\psi(r) \ge t$ and estimate each term in Theorem 4.2(ii) separately.

First, by using (H1), the fact that $\Psi \ge \Phi$ and (2.1), we get

$$\begin{aligned}\frac{t}{V(x,r)\Psi(r)} \int_{2/\phi^{-1}(1/t)}^{4\Phi(r)} h(s,x,y)w(s)ds &\le \frac{th(1/\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)\Phi(r)} \int_0^{4\Phi(r)} w(s)ds \\ &\le 4e \frac{th(1/\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)} \phi\big(1/(4\Phi(r))\big) \le 4e \frac{th(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)\psi(r)}.\end{aligned}$$

Next, we note that, since ϕ is a Bernstein function, the map $u \mapsto \phi(u)/u$ is decreasing so that $\Phi(r)\phi(1/\Phi(r)) \ge \phi^{-1}(1/t)^{-1}\phi(\phi^{-1}(1/t))$. Hence, since $\Psi \ge \Phi$, it holds that

$$\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)\Psi(r)\phi^{-1}(1/t)} \le \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)\psi(r)\phi(\phi^{-1}(1/t))} = \frac{th(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)\psi(r)}.$$

Thirdly, by (3.1) and (4.9), we see that for $c_2 := \sup_{u>0} u^{(d_2+\alpha_2(\beta_2\wedge 1))/2}e^{-u}$,

$$\begin{aligned}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c_1r^2}{\psi^{-1}(t)^2}\Big) &\le c_2\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\Big(\frac{\psi^{-1}(t)^2}{c_1r^2}\Big)^{(d_2+\alpha_2(\beta_2\wedge 1))/2}\\ &\le c_3\frac{th(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)\psi(r)}.\end{aligned}$$

Lastly, by (H1) and (2.10),

$$\frac{th(\Phi(r),x,y)w(\Phi(r))}{V(x,r)} \le \frac{e}{e-2}\frac{th(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)\psi(r)}.$$

(ii) By using Theorem 4.3 instead of Theorem 4.2, we obtain the result by repeating the proof of (i). □

As a corollary to Theorems 4.2 and 4.3, we obtain the following interior estimates on $q(t,x,y)$ in case of a regular boundary function.

Corollary 4.6. *Suppose that $h(t,x,y)$ is a regular boundary function.*
(i) If **(Poly-R_1)** *and* $\mathbf{HK_B^h}$ *hold, then for every $T>0$, the following estimates hold for all $(t,x,y) \in (0,T]\times D\times D$ satisfying $\delta_\wedge(x,y) \ge \rho(x,y)\vee\psi^{-1}(t)$.*

(1) *If $\psi(\rho(x,y)) \le t$, then* $\quad q(t,x,y) \simeq \dfrac{1}{V(x,\psi^{-1}(t))}.$

(2) *If $\psi(\rho(x,y)) \ge t$, then*

$$\begin{aligned}q(t,x,y) \asymp \frac{C_0}{V(x,\rho(x,y))\Psi(\rho(x,y))}\Big(t\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(\rho(x,y))} w(s)ds + \frac{1}{\phi^{-1}(1/t)}\Big)\\ + \frac{1}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c\,\rho(x,y)^2}{\psi^{-1}(t)^2}\Big) + \frac{tw\big(\Phi(\rho(x,y))\big)}{V(x,\rho(x,y))}.\end{aligned}$$

(ii) If **(Poly-∞)** *and* $\mathbf{HK_U^h}$ *hold, then (1) and (2) above hold for all $(t,x,y)\in(0,\infty)\times D\times D$ satisfying $\delta_\wedge(x,y)\ge\rho(x,y)\vee\psi^{-1}(t)$.*

Now we give the large time estimates for $q(t,x,y)$ under $\mathbf{HK_B^h}$.

Theorem 4.7. *Suppose that* **(Poly-R_1)** *and* $\mathbf{HK_B^h}$ *hold. Then for every $T>0$,*

$$q(t,x,y) \simeq e^{-t\phi(\lambda_D)}h(1,x,y), \quad (t,x,y)\in[T,\infty)\times D\times D. \tag{4.21}$$

Proof. Fix $x,y\in D$ and $s_0\in(0,1)$ such that $(H\circ\sigma)(T,s_0) \ge 2\phi(\lambda_D)+1/T$. Since $\lim_{s\to 0}(H\circ\sigma)(T,s) = \infty$, such an s_0 always exists. Then, since H is non-decreasing and ϕ' is non-increasing, we see that

$$(H\circ\sigma)(t,s_0) \ge 2\phi(\lambda_D)+1/T, \quad t\ge T. \tag{4.22}$$

By (H2), (3.1) and (3.4), we can apply Lemma 2.6 with $f(s) = h(s,x,y)V(x,\Phi^{-1}(s))^{-1}$. Using Remark 3.7 (with $T=s_0$), Lemma 2.6 and (4.22), since ϕ is the Laplace exponent of S, we get that, for all $t\ge T$,

$$\begin{aligned}q(t,x,y) &\le c_1\int_0^{s_0}\frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t\in ds) + c_1h(1,x,y)\int_{s_0}^\infty e^{-\lambda_D s}\mathbb{P}(S_t\in ds)\\ &\le c_2\frac{h(s_0,x,y)}{V(x,\Phi^{-1}(s_0))}\exp\Big(-\frac{t}{2}(H\circ\sigma)(t,s_0)\Big) + c_1h(1,x,y)\mathbb{E}[e^{-\lambda_D S_t}] \le c_3h(1,x,y)e^{-t\phi(\lambda_D)}.\end{aligned}$$

On the other hand, we also see from Remark 3.7 that

$$q(t,x,y) \ge c_4h(1,x,y)\int_{s_0}^\infty e^{-\lambda_D s}\mathbb{P}(S_t\in ds) = c_4h(1,x,y)\Big(e^{-t\phi(\lambda_D)} - \int_0^{s_0}e^{-\lambda_D s}\mathbb{P}(S_t\in ds)\Big).$$

According to Proposition 2.3 and (4.22), it holds that for all $t \ge T$,

$$\int_0^{s_0} e^{-\lambda_D s}\mathbb{P}(S_t \in ds) \le \mathbb{P}(S_t \le s_0) \le \exp\big(-2t\phi(\lambda_D)\big).$$

Therefore, we conclude that for all $t \ge T$, $q(t,x,y) \ge c_4(1-e^{-T\phi(\lambda_D)})h(1,x,y)e^{-t\phi(\lambda_D)}$. □

5. Green function estimates

In this section, we always assume that either (1) **(Poly-R_1)** and $\mathbf{HK_B^h}$ hold, or (2) **(Poly-∞)** and $\mathbf{HK_U^h}$ hold. The Green function G_D of X is given by $G_D(x,y) := \int_0^\infty q(t,x,y)dt$.

As an application of the heat kernel estimates obtained in the previous section, we can obtain two-sided estimates on the Green function. To this end, we prove a simple lemma first.

Lemma 5.1. *Let $f : I \to [0,\infty)$ be a function defined on an interval $I \subset [0,\infty)$. Assume that there exist constants $c_1, c_2 > 0$, $p_1, p_2 \in \mathbb{R}$ such that*

$$c_1\Big(\frac{r_2}{r_1}\Big)^{p_1} \le \frac{f(r_2)}{f(r_1)} \le c_2\Big(\frac{r_2}{r_1}\Big)^{p_2} \quad \text{for all } r_1, r_2 \in I,\ 0 < r_1 \le r_2. \tag{5.1}$$

For any $a > 1$, there exists a constant $c_3 > 0$ such that for all $r, R \in I$, $ar \le R$,

$$\int_r^R s^{-1}f(s)ds \ge c_3\big(f(r)+f(R)\big). \tag{5.2}$$

(i) If we assume $p_1 > 0$, then, for any $a > 1$, $\int_r^R s^{-1}f(s)ds \simeq f(R)$ for all $r, R \in I$, $ar \le R$, with comparison constants depending on a.
(ii) If we assume $p_2 < 0$, then, for any $a > 1$, $\int_r^R s^{-1}f(s)ds \simeq f(r)$ for all $r, R \in I$, $0 < ar \le R$, with comparison constants depending on a.

Proof. Suppose $a > 1$. For all $r, R \in I$, $ar \le R$, by (5.1),

$$\int_r^R s^{-1}f(s)ds \ge \int_r^{ar} s^{-1}f(s)ds \vee \int_{R/a}^R s^{-1}f(s)ds \ge c\big(f(r)\vee f(R)\big).$$

Thus we only need to prove the upper bounds in (i) and (ii).

(i) By (5.1), since $p_1 > 0$, we have that

$$\int_r^R s^{-1}f(s)ds = f(R)\int_r^R \frac{f(s)}{sf(R)}ds \le c_1^{-1}f(R)\int_r^R \frac{s^{p_1-1}}{R^{p_1}}ds \le c_1^{-1}p_1^{-1}f(R).$$

(ii) Similarly, by (5.1), since $p_2 < 0$, we have that

$$\int_r^R s^{-1}f(s)ds = f(r)\int_r^R \frac{f(s)}{sf(r)}ds \le c_2 f(r)\int_r^R \frac{s^{p_2-1}}{r^{p_2}}ds \le -c_2p_2^{-1}f(r).$$

□

Recall the definition of ψ in (4.8). The next simple observation will be used in the proof of the next proposition and also later.

Lemma 5.2. *If D is bounded, then there exists $c > 0$ such that for any $x, y \in D$,*

$$\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)}ds \ge c\left(\frac{h(\Phi(\rho(x,y)),x,y)\psi(\rho(x,y))}{V(x,\rho(x,y))} + h(1,x,y)\right).$$

Proof. This follows easily from (5.2), (H1), (H2) and (3.2). □

The following proposition provides the first and most general estimate of the Green function.

Proposition 5.3. *It holds that for $x, y \in D$,*

$$G_D(x,y) \simeq \frac{C_0}{V(x,\rho(x,y))\Psi(\rho(x,y))}\int_0^{\Phi(\rho(x,y))} \frac{h(s,x,y)}{\phi(1/s)}ds + \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)}ds. \tag{5.3}$$

Proof. Since the proofs are similar, we only give the proof when $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ holds, which is more complicated.

Take $x, y \in D$ and let $r := \rho(x, y)$. Set $T_D := 1/\phi\big(1/(2\Phi(\mathrm{diam}(D)))\big)$. By a change of variables and Lemma 2.1, we have that

$$\int_0^{\psi(r)} \frac{h(\phi^{-1}(1/t)^{-1}, x, y)}{\phi^{-1}(1/t)} dt = \int_0^{\Phi(r)} \frac{h(s, x, y)\phi'(1/s)}{s\phi(1/s)^2} ds \simeq \int_0^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds \tag{5.4}$$

and

$$\begin{aligned} \int_{\psi(r)}^{T_D} \frac{h(\phi^{-1}(1/t)^{-1}, x, y)}{V(x, \psi^{-1}(t))} dt &= \int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))} \frac{h(s, x, y)}{V(x, \Phi^{-1}(s))} \frac{\phi'(1/s)}{s^2\phi(1/s)^2} ds \\ &\simeq \int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))} \frac{h(s, x, y)}{sV(x, \Phi^{-1}(s))\phi(1/s)} ds. \end{aligned} \tag{5.5}$$

Combining with Theorem 4.2 (with $T = T_D$), we arrive at the lower bound in (5.3).

By Theorems 4.2 and 4.7 (with $T = T_D$), we have that

$$\begin{aligned} G_D(x, y) \le\, & c_0 \int_0^{\psi(r)} \frac{h(\phi^{-1}(1/t)^{-1}, x, y)}{V(x, \psi^{-1}(t))} \exp\Big(-\frac{c_1 r^2}{\psi^{-1}(t)^2}\Big) dt + c_0 h(\Phi(r), x, y) \frac{w(\Phi(r))}{V(x, r)} \int_0^{\psi(r)} t dt \\ & + c_0 \frac{C_0}{V(x, r)\Psi(r)} \int_0^{\psi(r)} t\mathcal{B}_h(t, x, y) dt + c_0 \frac{C_0}{V(x, r)\Psi(r)} \int_0^{\psi(r)} \frac{h(\phi^{-1}(1/t)^{-1}, x, y)}{\phi^{-1}(1/t)} dt \\ & + c_0 \int_{\psi(r)}^{T_D} \frac{h(\phi^{-1}(1/t)^{-1}, x, y)}{V(x, \psi^{-1}(t))} dt + c_0 h(1, x, y) \int_{T_D}^{\infty} e^{-t\phi(\lambda_D)} dt \\ & =: c_0(G_1 + G_2 + C_0 G_3 + C_0 G_4 + G_5 + G_6). \end{aligned}$$

First note that, following the proof of Lemma 3.2 and with help of (4.9), one can see that under the assumptions of Lemma 3.2, there exists $c_1 > 0$ such that for all $r, \kappa \in (0, T_D)$,

$$\int_0^r f(s) \exp\Big(-\frac{\kappa^2}{\psi^{-1}(s)^2}\Big) ds \le \frac{c_1 r^{p+1} f(r)}{\psi(\kappa)^p}.$$

Applying this inequality with $f(t) = h(\phi^{-1}(1/t)^{-1}, x, y)V(x, \psi^{-1}(t))^{-1}$ and $p := \gamma/\beta_1 + d_2/(\alpha_1\beta_1)$, we get from (H2), (2.4), (3.1) and (4.9) that

$$G_1 \le c_2 \frac{h(\Phi(r), x, y)}{V(x, r)} \frac{\psi(r)^{p+1}}{\psi(r)^p} = c_2 h(\Phi(r), x, y) \frac{\psi(r)}{V(x, r)}.$$

For G_2, we see from (2.10) that

$$G_2 \le \frac{e}{2(e-2)} h(\Phi(r), x, y) \frac{\psi(r)^2}{V(x, r)\psi(r)} = \frac{e}{2(e-2)} h(\Phi(r), x, y) \frac{\psi(r)}{V(x, r)}.$$

For G_3, we use Fubini's theorem to get that

$$\begin{aligned} V(x, r)\Psi(r) G_3 &= \int_0^{\psi(r)} t \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} h(s, x, y) w(s) ds dt \\ &= \int_0^{2\Phi(r)} h(s, x, y) w(s) \int_0^{\phi(2/s)^{-1}} t dt ds + \int_{2\Phi(r)}^{4\Phi(r)} h(s, x, y) w(s) \int_0^{\psi(r)} t dt ds \\ &=: G_{3,1} + G_{3,2}. \end{aligned} \tag{5.6}$$

By (2.10), a change of variables and (H2), we get

$$G_{3,1} \le c_3 \int_0^{2\Phi(r)} \frac{h(s, x, y)\phi(2/s)}{\phi(2/s)^2} ds = 2c_3 \int_0^{\Phi(r)} \frac{h(2s, x, y)}{\phi(1/s)} ds \le c_4 \int_0^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds.$$

On the other hand, we also get from (2.10), (H1) and (2.2) that

$$\begin{aligned} G_{3,2} &\le \frac{e}{2(e-2)} h(\Phi(r), x, y) \phi(1/\Phi(r)) \psi(r)^2 \int_{2\Phi(r)}^{4\Phi(r)} ds = \frac{e}{(e-2)} h(\Phi(r), x, y) \psi(r) \Phi(r) \\ &\le \frac{2e}{(e-2)} \frac{\psi(r)}{\phi(2/\Phi(r))^{-1}} \int_{\Phi(r)/2}^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds \le c_5 \int_{\Phi(r)/2}^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds \le c_5 \int_0^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds. \end{aligned}$$

Clearly, $G_6 \le \phi(\lambda_D)^{-1} h(1,x,y)$.

Recall from (5.4) and (5.5) that

$$G_4 + G_5 \simeq \frac{C_0}{V(x,r)\Psi(r)} \int_0^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)} ds + \int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)} ds.$$

It follows from Lemma 5.2 and the upper bounds above on G_1, G_2, G_6 that G_5 dominates $G_1+G_2+G_6$. Since G_4 dominates G_3, the proof is complete. □

In the remainder of this section, under some additional assumptions on the boundary function, we will obtain Green function estimates in simpler forms. Lemma 5.1 will be a useful tool in all simplifications.

We start with the following condition which is a counterpart of (H2).

(H2*) There exist constants $c_1, \gamma_* > 0$ such that for all $x, y \in D$ and $s, t \ge 0$ with $\Phi(\delta_\vee(x,y)) \le s \le t < 2\Phi(\mathrm{diam}(D))$,

$$s^{\gamma_*} h(s,x,y) \ge c_1 t^{\gamma_*} h(t,x,y).$$

Note that the γ_* above is less than or equal to γ.

Remark 5.4. Suppose that a boundary function $h(t,x,y)$ satisfies (H2*). Then for every $\epsilon \in (0,1)$, there exists $c_2 = c_2(\epsilon) > 0$ such that for all $x, y \in D$ and $s, t \ge 0$ with $\epsilon\Phi(\delta_\vee(x,y)) \le s \le t < 2\Phi(\mathrm{diam}(D))$,

$$s^{\gamma_*} h(s,x,y) \ge c_2 t^{\gamma_*} h(t,x,y).$$

Indeed, let $\epsilon\Phi(\delta_\vee(x,y)) \le s \le \Phi(\delta_\vee(x,y))$ and $s \le t < 2\Phi(\mathrm{diam}(D))$. If $t \le \Phi(\delta_\vee(x,y))$, then $\epsilon t \le s$ so that by (H1),

$$s^{\gamma_*} h(s,x,y) \ge s^{\gamma_*} h(t,x,y) \ge \epsilon^{\gamma_*} t^{\gamma_*} h(t,x,y).$$

If $t > \Phi(\delta_\vee(x,y))$, then by using (H1) in the first inequality below, (H2*) in the second, and the condition that $s \ge \epsilon\Phi(\delta_\vee(x,y))$ in the last inequality, we see that

$$s^{\gamma_*} h(s,x,y) \ge s^{\gamma_*} h(\Phi(\delta_\vee(x,y)),x,y) \ge c_1 \Big(\frac{s}{\Phi(\delta_\vee(x,y))}\Big)^{\gamma_*} t^{\gamma_*} h(t,x,y) \ge c_1 \epsilon^{\gamma_*} t^{\gamma_*} h(t,x,y).$$

Example 5.5. Let $p, q \ge 0$, $p+q > 0$. Recall that the boundary function $h_{p,q}(t,x,y)$ defined in (3.8) satisfies (H2) with $\gamma = p+q$. We claim that $h_{p,q}(t,x,y)$ also satisfies (H2*) with $\gamma_* = \gamma = p+q$. Indeed, for all $x, y \in D$ and $\Phi(\delta_\vee(x,y)) < s < t$,

$$s^{p+q} h_{p,q}(s,x,y) = \Phi(\delta_D(x))^p \Phi(\delta_D(y))^q = t^{p+q} h_{p,q}(t,x,y).$$

In the remainder of this section, we let $d_1, d_2, \gamma, \gamma_*, \beta_1, \beta_2$ and α_1, α_2 be the constants in (3.1), (H2), (H2*), **(Poly-R_1)** and the scaling indices of Φ in (3.4), respectively.

Let

$$\widetilde{G}_D(x,y) := \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)} ds \tag{5.7}$$

denote the second term on the right-hand side of the estimate (5.3).

Lemma 5.6. *The following estimates hold for all $x, y \in D$.*

(i) If $d_1 > \alpha_2(\beta_2 \wedge 1)$, then

$$\widetilde{G}_D(x,y) \simeq h\big(\Phi(\rho(x,y)),x,y\big) \frac{\psi(\rho(x,y))}{V(x,\rho(x,y))}.$$

(ii) If $d_2 < \alpha_1(\beta_1 - \gamma)$, then

$$\widetilde{G}_D(x,y) \simeq \begin{cases} h(1,x,y), & \text{when } \mathbf{HK_B^h} \text{ holds,} \\ \infty, & \text{when } \mathbf{HK_U^h} \text{ holds.} \end{cases}$$

Below, we also assume that $h(t,x,y)$ is regular and (H2) holds.*

(iii) If $\alpha_1\beta_1 > d_2 \ge d_1 > \alpha_2((\beta_2 \wedge 1) - \gamma_*)$, then

$$\widetilde{G}_D(x,y) \simeq h\big(\Phi(\rho(x,y)),x,y\big) \frac{\psi(\rho(x,y) \vee \delta_\vee(x,y))}{V(x,\rho(x,y) \vee \delta_\vee(x,y))}.$$

(iv) If $d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2$, then

$$\widetilde{G}_D(x,y) \simeq h\big(\Phi(\rho(x,y)),x,y\big)\log\Big(e+\frac{\delta_\vee(x,y)}{\rho(x,y)}\Big).$$

(v) If $\alpha_1=\alpha_2$, $\beta_1=\beta_2$, $\gamma=\gamma_*$ and $d_1=d_2=\alpha_1(\beta_1-\gamma)$, then

$$\widetilde{G}_D(x,y) \simeq \begin{cases} h(1,x,y)\log\big(e+\mathrm{diam}(D)(\rho(x,y)\vee\delta_\vee(x,y))^{-1}\big), & \textit{when } \mathbf{HK^h_B} \textit{ holds,}\\ \infty, & \textit{when } \mathbf{HK^h_U} \textit{ holds.}\end{cases}$$

Proof. Take $x,y\in D$. Let $\delta_\wedge := \delta_\wedge(x,y)$ and $\delta_\vee := \delta_\vee(x,y)$. Define

$$g(s) := \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))\phi(1/s)}, \quad s>0.$$

Then

$$\widetilde{G}_D(x,y) = \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{g(s)}{s}ds.$$

By (H1), (H2), (3.1), (3.4), (2.2) and (2.3), there exist $c_1,c_2>0$ such that

$$c_1\Big(\frac{r}{s}\Big)^{-\gamma-d_2/\alpha_1+\beta_1} \le \frac{g(r)}{g(s)} \le c_2\Big(\frac{r}{s}\Big)^{-d_1/\alpha_2+(\beta_2\wedge 1)}, \quad 0<s\le r<2\Phi(\mathrm{diam}(D)). \tag{5.8}$$

If $h(t,x,y)$ is regular, then by Remark 3.4, for every $a>0$, there exists $c_3=c_3(a)>0$ such that

$$c_3\Big(\frac{r}{s}\Big)^{-d_2/\alpha_1+\beta_1} \le \frac{g(r)}{g(s)} \le c_2\Big(\frac{r}{s}\Big)^{-d_1/\alpha_2+(\beta_2\wedge 1)}, \; 0<s\le r<\Phi(a\delta_\wedge)\wedge 2\Phi(\mathrm{diam}(D)); \tag{5.9}$$

if furthermore (H2*) further holds, then by Remark 5.4, there exists $c_4>0$ such that

$$c_1\Big(\frac{r}{s}\Big)^{-\gamma-d_2/\alpha_1+\beta_1} \le \frac{g(r)}{g(s)} \le c_4\Big(\frac{r}{s}\Big)^{-\gamma_*-d_1/\alpha_2+(\beta_2\wedge 1)}, \; \Phi(\delta_\vee/2)<s\le r<2\Phi(\mathrm{diam}(D)). \tag{5.10}$$

(i) By (5.8), since $-d_1/\alpha_2+(\beta_2\wedge 1)<0$, the result follows from Lemma 5.1(ii).

(ii) If D is bounded, then by (5.8) and Lemma 5.1(i), since $-\gamma-d_2/\alpha_1+\beta_1>0$, it holds that $\widetilde{G}_D(x,y)\simeq g(\Phi(\mathrm{diam}(D)))$. By (3.2), there exists a constant $c_5>1$ such that $c_5^{-1}\le V(z,\mathrm{diam}(D))\le c_5$ for all $z\in D$. Hence, by using (H1), (H2) and the definition of g, we get that $G_D(x,y)\simeq h(1,x,y)$. If D is unbounded, then we see from (5.8) and Lemma 5.1(i) that

$$\widetilde{G}_D(x,y) \simeq \lim_{r\to\infty}\int_{\Phi(\rho(x,y))}^{r}\frac{g(s)}{s}ds \simeq \lim_{r\to\infty} g(r) \ge c_1 g(1)\lim_{r\to\infty} r^{-\gamma-d_2/\alpha_1+\beta_1}=\infty.$$

(iii) Suppose that $\delta_\vee\le 2\rho(x,y)$. Since $-\gamma_*-d_1/\alpha_2+(\beta_2\wedge 1)<0$, by (5.10) and Lemma 5.1(ii),

$$\widetilde{G}_D(x,y)\simeq g(\Phi(\rho(x,y))) = \frac{h(\Phi(\rho(x,y)),x,y)\psi(\rho(x,y))}{V(x,\rho(x,y))}.$$

Hence the result follows from (3.1) and (4.9).

Suppose now that $\delta_\vee>2\rho(x,y)$. Then $\delta_\wedge\ge\delta_\vee-\rho(x,y)>\delta_\vee/2>\rho(x,y)$. Since h is regular, we get $h(\Phi(\delta_\vee),x,y)\simeq h(\Phi(\rho(x,y)),x,y)\simeq 1$. Further, since $-d_2/\alpha_1+\beta_1>0$ and $-\gamma_*-d_1/\alpha_2+(\beta_2\wedge 1)<0$, by the scaling of Φ, (5.9), (5.10) and Lemma 5.1(i)-(ii), we get

$$\begin{aligned}\widetilde{G}_D(x,y) &\simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)}\frac{g(s)}{s}ds+\int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))}\frac{g(s)}{s}ds \simeq g(\Phi(\delta_\wedge))+g(\Phi(\delta_\vee))\\ &\simeq g(\Phi(\delta_\vee)) = \frac{h(\Phi(\delta_\vee),x,y)\psi(\delta_\vee)}{V(x,\delta_\vee)} \simeq \frac{\psi(\delta_\vee)}{V(x,\delta_\vee)} \simeq \frac{h(\Phi(\rho(x,y)),x,y)\psi(\delta_\vee)}{V(x,\delta_\vee)}.\end{aligned}$$

This finishes the proof for (iii).

(iv) Since $d_1=d_2$, by (3.1) and (3.2), we see that for every $a>0$, there are comparability constants depending on a such that for all $w,z\in D$ and $0<r<a\,\mathrm{diam}(D)$,

$$V(w,r)\simeq\Big(\frac{r}{\rho(w,z)}\Big)^{d_1}V(w,\rho(w,z))\simeq\Big(\frac{r}{\rho(w,z)}\Big)^{d_1}V(z,\rho(w,z))\simeq V(z,r)\simeq r^{d_1}V(z,1). \tag{5.11}$$

Moreover, since $\beta_1=\beta_2$ and $\alpha_1=\alpha_2$, by (2.2), (2.3) and (3.4), we get that

$$\phi(1/s)^{-1}\simeq s^{\beta_1}, \quad 0<s<2\Phi(\mathrm{diam}(D)) \quad \text{and} \quad \Phi^{-1}(s)\simeq s^{1/\alpha_1}, \quad s>0, \tag{5.12}$$

so that $g(s) \simeq h(s,x,y)$ for all $0 < s < 2\Phi(\mathrm{diam}(D))$. In particular, since h is regular, we see from Remark 3.4 that

$$g(s) \simeq 1, \quad 0 < s < 2\Phi(\delta_\wedge). \tag{5.13}$$

If $\delta_\vee \le 2\rho(x,y)$, then by (5.10) and Lemma 5.1(ii),

$$\widetilde{G}_D(x,y) \simeq g(\Phi(\rho(x,y))) \simeq h(\Phi(\rho(x,y)),x,y) \simeq h(\Phi(\rho(x,y)),x,y)\log\Big(e + \frac{\delta_\vee}{\rho(x,y)}\Big).$$

If $\delta_\vee > 2\rho(x,y)$, then we get $\delta_\wedge > \delta_\vee/2 > \rho(x,y)$ as in (iii), and by (5.13), (5.10) and Lemma 5.1(ii),

$$\widetilde{G}_D(x,y) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} \frac{g(s)}{s}ds + \int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))} \frac{g(s)}{s}ds \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} \frac{ds}{s} + g(\Phi(\delta_\vee)).$$

Note that since $\Phi(s) \simeq s^{\alpha_1}$ for $s > 0$, we have $\int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} s^{-1}ds \simeq \log\Big(e + \frac{\delta_\wedge}{\rho(x,y)}\Big)$ so that

$$g(\Phi(\delta_\vee)) \simeq h(\Phi(\delta_\vee),x,y) \le 1 \le \log\Big(e + \frac{\delta_\wedge}{\rho(x,y)}\Big) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} \frac{ds}{s}.$$

Eventually, since $\delta_\wedge \simeq \delta_\vee$ and $h(\Phi(\rho(x,y)),x,y) \simeq 1$ in this case, we obtain that

$$\widetilde{G}_D(x,y) \simeq \log\Big(e + \frac{\delta_\wedge}{\rho(x,y)}\Big) \simeq h(\Phi(\rho(x,y)),x,y)\log\Big(e + \frac{\delta_\vee}{\rho(x,y)}\Big).$$

(v) By (5.11), (5.12), the regularity of h, (H2), Remark 5.4 and (3.10), we have

$$g(s) \simeq s^\gamma, \quad 0 < s < \Phi(\delta_\wedge) \tag{5.14}$$

and

$$g(s) \simeq s^\gamma h(s,x,y) \simeq t^\gamma h(t,x,y), \qquad \Phi(\delta_\vee/2) < s \le t < 2\Phi(\mathrm{diam}(D)) + 1. \tag{5.15}$$

If $\delta_\vee \le 2\rho(x,y)$, then since $\Phi(s) \simeq s^{\alpha_1}$ for $s > 0$ in this case, we get from (5.15) that

$$\widetilde{G}_D(x,y) \simeq h(1,x,y) \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{ds}{s} \simeq h(1,x,y)\log\Big(e + \frac{\mathrm{diam}(D)}{\rho(x,y)}\Big).$$

If $\delta_\vee > 2\rho(x,y)$, then $\delta_\wedge > \delta_\vee/2 > \rho(x,y)$ as in (iii) and hence by (5.14) and (5.15),

$$\widetilde{G}_D(x,y) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} s^{\gamma-1}ds + h(1,x,y)\int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))} \frac{ds}{s}.$$

Since $\Phi(s) \simeq s^{\alpha_1}$ for $s > 0$, $\delta_\wedge \le \delta_\vee \le 2\delta_\wedge$ and h is regular, by (5.15),

$$\begin{aligned} h(1,x,y)\int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))} \frac{ds}{s} &\simeq h(1,x,y)\log\Big(e + \frac{\mathrm{diam}(D)}{\delta_\vee}\Big) \\ &\ge h(1,x,y) \simeq \Phi(\delta_\wedge)^\gamma h(\Phi(\delta_\wedge),x,y) \simeq \Phi(\delta_\wedge)^\gamma \ge \gamma^{-1}\int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} s^{\gamma-1}ds. \end{aligned}$$

This completes the proof. □

In the next lemma we show that under the additional assumption that $\gamma < \beta_1 + 1$, the first term on the right-hand side of (5.3) is dominated by $\widetilde{G}_D(x,y)$.

Lemma 5.7. *If either $C_0 = 0$ or $\gamma < \beta_1 + 1$, then $G_D(x,y) \simeq \widetilde{G}_D(x,y)$ on $D \times D$.*

Proof. When $C_0 = 0$, the assertion follows from Proposition 5.3. So we now assume $\gamma < \beta_1 + 1$. According to (2.3) and (H2), we have

$$\frac{h(t,x,y)\phi(1/t)^{-1}}{h(s,x,y)\phi(1/s)^{-1}} \ge c\Big(\frac{t}{s}\Big)^{\beta_1-\gamma} \quad \text{for all } 0 < s \le t < \Phi(\mathrm{diam}(D)).$$

Thus, since $\Psi \ge \Phi$, by Lemma 5.1(i), (H1) and (2.2), we get

$$\frac{C_0}{V(x,r)\Psi(r)}\int_0^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)}ds \le c\frac{h(\Phi(r),x,y)\psi(r)\Phi(r)}{V(x,r)\Psi(r)} \le c\frac{h(\Phi(r),x,y)\psi(r)}{V(x,r)}.$$

Combining the above with Lemma 5.2 and Proposition 5.3, we get the assertion. □

Define

$$g_0(x,y) = \begin{cases} \dfrac{\psi(\rho(x,y))}{V(x,\rho(x,y))}, & \text{if } d_1 > \alpha_2(\beta_2 \wedge 1), \\ \log\Big(e + \dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big), & \text{if } d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2, \\ \dfrac{\psi(\rho(x,y) \vee \delta_\vee(x,y))}{V(x,\rho(x,y)\vee\delta_\vee(x,y))}, & \text{if } d_2 < \alpha_1\beta_1. \end{cases} \tag{5.16}$$

By combining Proposition 5.3, Lemma 5.6 and Lemma 5.7 we arrive at the following result.

Theorem 5.8. *Suppose that $C_0 = 0$ or $\gamma < \beta_1 + 1$, $h(t,x,y)$ is regular and (H2*) holds.*

(a) *Suppose also that one of the following holds: (1) $d_1 > \alpha_2(\beta_2 \wedge 1)$ or (2) $d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2$ or (3) $d_2 < \alpha_1\beta_1$. Then it holds that*

$$G_D(x,y) \simeq h(\Phi(\rho(x,y)),x,y)g_0(x,y). \tag{5.17}$$

(b) *If $d_2 < \alpha_1(\beta_1 - \gamma)$, then*

$$G_D(x,y) \simeq \begin{cases} h(1,x,y), & \textit{when } \mathbf{HK^h_B} \textit{ holds}, \\ \infty, & \textit{when } \mathbf{HK^h_U} \textit{ holds}. \end{cases}$$

(c) *If $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma = \gamma_*$ and $d_1 = d_2 = \alpha_1(\beta_1 - \gamma)$, then*

$$G_D(x,y) \simeq \begin{cases} h(1,x,y)\log\Big(e + \mathrm{diam}(D)(\rho(x,y)\vee\delta_\vee(x,y))^{-1}\Big), & \textit{when } \mathbf{HK^h_B} \textit{ holds}, \\ \infty, & \textit{when } \mathbf{HK^h_U} \textit{ holds}. \end{cases}$$

When $C_0 = 1$, Theorem 5.8 only deals with the case $\gamma < \beta_1 + 1$. To cover the case when γ is large, we assume the following condition.

(H2**) There exist constants $c_1 > 0$, $\gamma_{**} \in (0, 1+\beta_1)$ such that for all $x,y \in D$ and $s,t \geq 0$ with $\Phi(\delta_\wedge(x,y)) \leq s \leq t < \Phi(\delta_\vee(x,y))$,

$$s^{\gamma_{**}}h(s,x,y) \leq c_1 t^{\gamma_{**}}h(t,x,y).$$

Example 5.9. For $p,q \geq 0$, let $h_{p,q}(t,x,y)$ be the boundary function defined in (3.8). If $p\vee q < 1+\beta_1$, then $h_{p,q}(t,x,y)$ satisfies (H2**). Indeed, we see that for all $x,y \in D$ and $\Phi(\delta_\wedge(x,y)) \leq s \leq t < \Phi(\delta_\vee(x,y))$,

$$s^{p\vee q}h_{p,q}(s,x,y) = \begin{cases} \Phi(\delta_D(x))^p s^{p\vee q - p}, & \text{if } \delta_D(x) < \delta_D(y) \\ \Phi(\delta_D(y))^q s^{p\vee q - q}, & \text{if } \delta_D(x) > \delta_D(y) \end{cases} \leq t^{p\vee q}h_{p,q}(t,x,y).$$

For a given boundary function h, we define for $x,y \in D$,

$$[h](x,y) := h\big(\Phi(\rho(x,y)\wedge\delta_\vee(x,y)),x,y\big)\left(1 \wedge \frac{\Phi(\delta_\vee(x,y))\psi(\delta_\vee(x,y))}{\Phi(\rho(x,y))\psi(\rho(x,y))}\right).$$

One can see that for all $p,q \geq 0$,

$$[h_{p,q}](x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta_D(x))}{\Phi(\rho(x,y))}\Big)^p\Big(1 \wedge \frac{\Phi(\delta_D(y))}{\Phi(\rho(x,y))}\Big)^q\Big(1 \wedge \frac{\Phi(\delta_\vee(x,y))}{\Phi(\rho(x,y))}\Big)^{1-p-q}\Big(1 \wedge \frac{\psi(\delta_\vee(x,y))}{\psi(\rho(x,y))}\Big). \tag{5.18}$$

Indeed, for $x,y \in D$, if $\delta_\wedge(x,y) \geq \rho(x,y)$, then $[h_{p,q}](x,y) = 1$ and the right-hand side of (5.18) is also equal to 1. If $\delta_\wedge(x,y) < \rho(x,y)$, then $\delta_\vee(x,y) < \rho(x,y) + \delta_\wedge(x,y) \leq \rho(x,y) + \rho(x,y) \leq 2\rho(x,y)$ and hence by (3.10) and the scaling properties of Φ and ψ,

$$\begin{aligned} \frac{\Phi(\rho(x,y))^{1-p-q}\psi(\rho(x,y))}{\Phi(\delta_\vee(x,y))^{1-p-q}\psi(\delta_\vee(x,y))}[h_{p,q}](x,y) &\simeq \frac{\Phi(\delta_\vee(x,y))^{p+q}}{\Phi(\rho(x,y))^{p+q}}h_{p,q}(\Phi(\delta_\vee(x,y),x,y) = \frac{\Phi(\delta_D(x))^p\Phi(\delta_D(y))^q}{\Phi(\rho(x,y))^{p+q}} \\ &\simeq \Big(1 \wedge \frac{\Phi(\delta_D(x))}{\Phi(\rho(x,y))}\Big)^p\Big(1 \wedge \frac{\Phi(\delta_D(y))}{\Phi(\rho(x,y))}\Big)^q. \end{aligned}$$

Recall that $g_0(x,y)$ is defined in (5.16).

Theorem 5.10. *Suppose that $C_0 = 1$, $h(t,x,y)$ is regular, (H2*) holds with $\gamma_* > (\beta_2 \wedge 1)+1$ and (H2**) holds. Suppose also that one of the following holds: (1) $d_1 > \alpha_2(\beta_2 \wedge 1)$ or (2) $d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2$ or (3) $d_2 < \alpha_1\beta_1$. Then it holds that*

$$G_D(x,y) \simeq [h](x,y)\frac{\Phi(\rho(x,y))}{\Psi(\rho(x,y))}\frac{\psi(\rho(x,y))}{V(x,\rho(x,y))} + h(\Phi(\rho(x,y)),x,y)g_0(x,y). \tag{5.19}$$

In particular, if $\Psi \simeq \Phi$, then

$$G_D(x,y) \simeq [h](x,y)g_0(x,y). \tag{5.20}$$

Proof. Take $x,y \in D$ and let $r := \rho(x,y)$ and $\delta_\vee := \delta_\vee(x,y)$. Observe that by (2.2), (2.3), (H1), (H2**) and the regularity of h,

$$c_1\Big(\frac{t}{s}\Big)^{\beta_1-\gamma_{**}} \le \frac{h(t,x,y)/\phi(1/t)}{h(s,x,y)/\phi(1/s)} \le c_2\Big(\frac{t}{s}\Big)^{\beta_2\wedge 1}, \quad 0 < s \le t < \Phi(\delta_\vee). \tag{5.21}$$

Note also that by (2.2), (2.3), (H2), (H2*) and Remark 5.4,

$$c_3\Big(\frac{t}{s}\Big)^{\beta_1-\gamma} \le \frac{h(t,x,y)/\phi(1/t)}{h(s,x,y)/\phi(1/s)} \le c_4\Big(\frac{t}{s}\Big)^{\beta_2\wedge 1-\gamma_*}, \quad \Phi(\delta_\vee)/2 \le s \le t < \Phi(\mathrm{diam}(D)). \tag{5.22}$$

If $\delta_\vee > 2r$, then by Lemma 5.1(i) and (5.21), $\int_0^{\Phi(r)} h(s,x,y)\phi(1/s)^{-1}ds \simeq h(\Phi(r),x,y)\Phi(r)\psi(r)$.

If $\delta_\vee \le 2r$, then by Lemma 5.1(i)-(ii), (5.21), (5.22) and the scaling property of ϕ, since $\beta_1 - \gamma_{**} > -1$ and $\beta_2 \wedge 1 - \gamma_* < -1$, we get that

$$\int_0^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)}ds = \int_0^{\Phi(\delta_\vee)/2} \frac{h(s,x,y)}{\phi(1/s)}ds + \int_{\Phi(\delta_\vee)/2}^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)}ds \simeq h(\Phi(\delta_\vee),x,y)\Phi(\delta_\vee)\psi(\delta_\vee).$$

Therefore, in either case, it holds that

$$\int_0^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)}ds \simeq h(\Phi(r\wedge\delta_\vee),x,y)\left(1 \wedge \frac{\Phi(\delta_\vee)\psi(\delta_\vee)}{\Phi(r)\psi(r)}\right)\Phi(r)\psi(r) = [h](x,y)\Phi(r)\psi(r). \tag{5.23}$$

Combining this with Proposition 5.3 and Lemma 5.6, we get (5.19).

Now we also assume that $\Psi \simeq \Phi$. If $\delta_\vee > r$, then $[h](x,y) = h(\Phi(r),x,y)$. Hence, we see from Lemma 5.2 that in (5.19), the second term dominates the first one so that (5.20) holds. If $\delta_\vee \le r$, then using Lemma 5.1(ii), (5.22) and the condition that $\beta_2 \wedge 1 - \gamma_* < -1$ in the second inequality below, the scaling property of ϕ and (3.10) in the third, and (5.23) in the fourth, we get

$$\begin{aligned}\int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)}ds &\le \frac{1}{V(x,r)\Phi(r)}\int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{\phi(1/s)}ds \\ &\le c_5\frac{h(\Phi(r),x,y)\psi(r)}{V(x,r)} \le \frac{c_6}{V(x,r)\Phi(r)}\int_{\Phi(r)/2}^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)}ds \le c_7[h](x,y)\frac{\psi(r)}{V(x,r)}.\end{aligned}$$

Note that $g_0(x,y) \simeq \psi(r)/V(x,r)$ when $\delta_\vee \le r$. Thus by Proposition 5.3 and (5.23), we get $G_D(x,y) \simeq [h](x,y)\psi(r)/V(x,r) \simeq [h](x,y)g_0(x,y)$ when $\delta_\vee \le r$. This completes the proof for (5.20). □

For completeness, we record the Green function estimates when $C_0 = 1$, $\beta_1 = \beta_2$ and $\gamma_* = \gamma = \beta_1 + 1$.

Theorem 5.11. *Suppose that $C_0 = 1$, $\beta_1 = \beta_2$, $h(t,x,y)$ is regular and (H2*) holds with $\gamma_* = \gamma = \beta_1 + 1$ and (H2**) holds. Suppose also that one of the following holds: (1) $d_1 > \alpha_2(\beta_2 \wedge 1)$ or (2) $d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2$ or (3) $d_2 < \alpha_1\beta_1$. Then it holds that*

$$G_D(x,y) \simeq h(\Phi(\rho(x,y)),x,y)\left[\frac{\Phi(\rho(x,y))}{\Psi(\rho(x,y)}\frac{\Phi(\rho(x,y))^{\gamma-1}}{V(x,\rho(x,y))}\log\Big(e + \frac{\Phi(\rho(x,y))}{\Phi(\delta_\vee(x,y))}\Big) + g_0(x,y)\right] \tag{5.24}$$

In particular, if $\Psi \simeq \Phi$, then

$$G_D(x,y) \simeq h(\Phi(\rho(x,y)),x,y)g_0(x,y)\log\Big(e + \frac{\Phi(\rho(x,y))}{\Phi(\delta_\vee(x,y))}\Big). \tag{5.25}$$

Proof. Take $x, y \in D$ and let $r := \rho(x, y)$ and $\delta_\vee = \delta_\vee(x, y)$. Using Lemma 5.1(i) (which is applicable due to (5.21)), (5.12), the second comparability in (5.15), (3.10) and the scaling property of Φ, we get that, if $\delta_\vee > 2r$, then

$$\int_0^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)} ds \simeq \int_0^{\Phi(r)} s^{\beta_1} h(s,x,y) ds \simeq \Phi(r)^\gamma h(\Phi(r), x, y)$$

and if $\delta_\vee \le 2r$, then

$$\begin{aligned}\int_0^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)} ds &= \int_0^{2^{-1}\Phi(2^{-1}(\delta_\vee))} \frac{h(s,x,y)}{\phi(1/s)} ds + \int_{2^{-1}\Phi(2^{-1}(\delta_\vee))}^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)} ds \\ &\simeq \int_0^{2^{-1}\Phi(2^{-1}(\delta_\vee))} s^{\beta_1} h(s,x,y) ds + \Phi(r)^\gamma h(\Phi(r), x, y) \int_{2^{-1}\Phi(2^{-1}(\delta_\vee))}^{\Phi(r)} s^{\beta_1 - \gamma} ds \\ &\simeq \Phi(\delta_\vee)^\gamma h(\Phi(\delta_\vee), x, y) + \Phi(r)^\gamma h(\Phi(r), x, y) \log\Big(e + \frac{\Phi(r)}{\Phi(\delta_\vee)}\Big) \\ &\simeq \Phi(r)^\gamma h(\Phi(r), x, y) \log\Big(e + \frac{\Phi(r)}{\Phi(\delta_\vee)}\Big).\end{aligned}$$

The last comparability above is valid by the second comparability in (5.15). Therefore, in either case, it holds that

$$\int_0^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)} ds \simeq \Phi(r)^\gamma h(\Phi(r), x, y) \log\Big(e + \frac{\Phi(r)}{\Phi(\delta_\vee)}\Big).$$

Combining this with Proposition 5.3 and Theorem 5.8, we get (5.24).

Now we also assume that $\Psi \simeq \Phi$. Then as in the proof of Theorem 5.10, one can check that in (5.24), if $\delta_\vee > r$, then the second term dominates the first one, and if $\delta_\vee \le r$, then the first term dominates the second one. Combining with the facts that $\psi(r) \simeq \Phi(r)^{\gamma-1}$ for $0 < r < \mathrm{diam}(D)$ since $\beta_1 = \beta_2$ and that $g_0(x, y) \simeq \psi(r)/V(x, r)$ when $\delta_\vee \le r$, we obtain (5.25). □

6. Parabolic Harnack inequality and Hölder regularity

Throughout this section, we assume that $h(t, x, y)$ is a regular boundary function and that either (1) **(Poly-**R_1**)** and $\mathbf{HK_B^h}$ hold, or (2) **(Poly-**∞**)** and $\mathbf{HK_U^h}$ hold.

For $x_0 \in D$ and $r > 0$, let $\tau_{B(x_0,r)} := \inf\{s > 0 : X_s \notin B(x_0, r)\}$ and $X^{B(x_0,r)}$ be the part process of X in $B(x_0, r)$. Denote by $q_{B(x_0,r)}(t, x, y)$ the heat kernel of $X^{B(x_0,r)}$. By the strong Markov property,

$$q_{B(x_0,r)}(t,x,y) = q(t,x,y) - \mathbb{E}_x\big[q(t - \tau_{B(x_0,r)}, X_{\tau_{B(x_0,r)}}, y);\ \tau_{B(x_0,r)} < t\big]. \tag{6.1}$$

Recall the definition of ψ in (4.8).

Lemma 6.1. *There exist constants $C > 0$ and $\epsilon \in (0, 1/4)$ such that for all $x_0 \in D$ and $r \in (0, \delta_D(x_0))$,*

$$q_{B(x_0,r)}(t,x,y) \ge \frac{C}{V(x_0, \psi^{-1}(t))} \quad \text{for all } t \in (0, \psi(\epsilon r)] \text{ and } x, y \in B\big(x_0, \epsilon\psi^{-1}(t)\big).$$

Proof. Since the proofs are similar, we only give the proof in case (1).

Fix $x_0 \in D$ and $r \in (0, \delta_D(x_0))$. Let $\epsilon \in (0, 1/4)$ to be chosen later. Let $0 < t \le \psi(\epsilon r)$ and $x, y \in B(x_0, \epsilon\psi^{-1}(t))$. Clearly, $x, y \in B(x_0, \epsilon^2 r)$. Further, $\delta_\wedge(x, y) > \delta_D(x_0) - \epsilon^2 r > r - r/4 > \psi^{-1}(t) > 2\epsilon\psi^{-1}(t) > \rho(x, y)$. Therefore, we have $\delta_\wedge(x, y) > \psi^{-1}(t) = \rho(x, y) \vee \psi^{-1}(t)$. From Corollary 4.6(i) (with $T = \psi(\mathrm{diam}(D))$) and (3.2), it follows that there exist constants $c_1, c_2 > 0$ (independent of t, x_0, x, y) such that

$$q(t,x,y) \ge \frac{c_1}{V(x, \psi^{-1}(t))} \ge \frac{c_2}{V(x_0, \psi^{-1}(t))}. \tag{6.2}$$

Let $z \in D \setminus B(x_0, r)$. Then since $t \le \psi(\epsilon r)$ and $\epsilon < 1/4$,

$$\rho(z,y) \ge \rho(z,x_0) - \rho(y,x_0) \ge (1-\epsilon^2)\rho(z,x_0) \ge (1-\epsilon^2) r \ge (2\epsilon)^{-1}\psi^{-1}(t).$$

Then according to Proposition 4.5(i), (3.1) and (3.2), since $h \le 1$, there exist constants $c_3, c_4, c_5 > 0$ (independent of t, x_0, z, y) such that for every $0 < s < t$,

$$q(s,z,y) \le \frac{c_3 s}{V(z, \rho(z,y))\psi(\rho(z,y))} \le \frac{c_3}{V(z, \rho(z,y))} \le \frac{c_4}{V(x_0, \rho(z,y))} \le \frac{c_5 \epsilon^{d_1}}{V(x_0, \psi^{-1}(t))}. \tag{6.3}$$

Combining (6.1) with (6.2) and (6.3), we get that

$$q_{B(x_0,r)}(t,x,y) \geq \frac{c_2 - c_5\epsilon^{d_1}}{V(x_0,\psi^{-1}(t))}.$$

Now we finish the proof by choosing $\epsilon = (c_2/(2c_5))^{1/d_1}$ so that $c_2 - c_5\epsilon^{d_1} = c_2/2$. □

Remark 6.2. By using (4.9) we may replace $\psi(\epsilon r)$ and $\epsilon\psi^{-1}(r)$ in the statement of Lemma 6.1 with $\epsilon\psi(r)$ and $\psi^{-1}(\epsilon r)$ respectively, cf. [19, p.3758].

Lemma 6.3. *There exists a constant $C > 1$ such that for all $x \in D$ and $r \in (0, \delta_D(x))$,*

$$C^{-1}\psi(r) \leq \mathbb{E}^x[\tau_{B(x,r)}] \leq C\psi(r). \tag{6.4}$$

Proof. Fix $x \in D$ and $r \in (0, \delta_D(x))$. Let $\epsilon \in (0, 1/4)$ be as in the statement of Lemma 6.1. Then by Lemma 6.1, we have that

$$q_{B(x,r)}(\psi(\epsilon r), x, y) \geq \frac{c_1}{V(x,\epsilon r)}, \quad y \in B(x, \epsilon^2 r).$$

By (3.1), this implies that

$$\mathbb{P}^x(\tau_{B(x,r)} > \psi(\epsilon r)) \geq \int_{B(x,\epsilon^2 r)} q_{B(x,r)}(\psi(\epsilon r), x, y)\, dy \geq \frac{c_1 V(x,\epsilon^2 r)}{V(x,\epsilon r)} \geq c_2.$$

Hence, by Markov's inequality and (4.9), we get that

$$\mathbb{E}^x[\tau_{B(x,r)}] \geq \psi(\epsilon r)\mathbb{P}^x(\tau_{B(x,r)} > \psi(\epsilon r)) \geq c_2\psi(\epsilon r) \geq c_3\psi(r).$$

To obtain the upper bound in (6.4), we first assume that $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ holds. We claim that there exists a constant $A > 1$ such that

$$\sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) \leq \frac{1}{2}. \tag{6.5}$$

Indeed, according to Proposition 4.5(i) and Theorem 4.7, since $h \leq 1$, there exists $c_4 > 1$ such that

$$q(t,z,y) \leq c_4\left(V(z,\psi^{-1}(t))^{-1}\mathbf{1}_{\{t\leq 1\}} + e^{-\phi(\lambda_D)t}\mathbf{1}_{\{t>1\}}\right), \quad z,y \in B(x,r). \tag{6.6}$$

Further, by (3.1), there is $c_5 > 1$ such that for all $z \in B(x,r)$,

$$V(z, c_5 r) \geq V(x, (c_5-1)r) \geq 2c_4 V(x,r). \tag{6.7}$$

Let $A > c_5$ be a constant such that

$$\exp\left(\phi(\lambda_D)\psi(Ac_5^{-1}\psi^{-1}(1))\right) \geq 2m(D). \tag{6.8}$$

In case when $r \leq c_5^{-1}\psi^{-1}(1)$, we see from (6.6) and (6.7) that for all $z \in B(x,r)$,

$$\mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) \leq \mathbb{P}^z(\tau_{B(x,r)} > \psi(c_5 r)) \leq \int_{B(x,r)} q(\psi(c_5 r), z, y)dy \leq \frac{c_4 V(x,r)}{V(z,c_5 r)} \leq \frac{1}{2}.$$

On the other hand, if $r > c_5^{-1}\psi^{-1}(1)$, then since $B(x,r) \subset B(x,\delta_D(x)) \subset D$, we get from (6.6) and (6.8) that for all $z \in B(x,r)$,

$$\begin{aligned}\mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) &\leq \mathbb{P}^z\big(\tau_{B(x,r)} > \psi(Ac_5^{-1}\psi^{-1}(1))\big) \leq V(x,r)\exp\big(-\phi(\lambda_D)\psi(Ac_5^{-1}\psi^{-1}(1))\big)\\ &\leq m(D)\exp\big(-\phi(\lambda_D)\psi(Ac_5^{-1}\psi^{-1}(1))\big) \leq \frac{1}{2}.\end{aligned}$$

Hence, (6.5) holds.

Now, by (6.5) and Markov's property, we see that for all $n \geq 2$,

$$\begin{aligned}&\sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > n\psi(Ar)) = \sup_{z\in B(x,r)} \mathbb{P}^z\big(\tau_{B(x,r)} > n\psi(Ar),\, \tau_{B(x,r)} > \psi(Ar)\big)\\ &\leq \sup_{z\in B(x,r)} \mathbb{P}^z\big(\mathbb{P}^{X_{\psi(Ar)}}(\tau_{B(x,r)} > (n-1)\psi(Ar)),\, \tau_{B(x,r)} > \psi(Ar)\big)\\ &\leq \sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > (n-1)\psi(Ar)) \sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar))\end{aligned}$$

$$\leq \cdots \leq \Big(\sup_{z \in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) \Big)^n \leq 2^{-n}.$$

Therefore, we get from (4.9) that

$$\begin{aligned}
\mathbb{E}^x[\tau_{B(x,r)}] &\leq \sum_{n=1}^{\infty} n\psi(Ar)\mathbb{P}^x\big(\tau_{B(x,r)} \in ((n-1)\psi(Ar), n\psi(Ar)]\big) \\
&\leq c_6 A^{\alpha_2(\beta_2 \wedge 1)}\psi(r) \sum_{n=1}^{\infty} n2^{-(n-1)} = 4c_6 A^{\alpha_2(\beta_2\wedge 1)}\psi(r).
\end{aligned}$$

Similarly, by using Proposition 4.5(ii), we can obtain the upper bound in (6.4) when **(Poly-∞)** and $\mathbf{HK_U^h}$ hold. □

Recall that the jump kernel $J(x,y)$ is given in (4.3).

Lemma 6.4. *There exists a constant $C > 1$ such that for all $x \in D$ and $r \in (0, \delta_D(x))$,*

$$\int_{D \setminus B(x,r)} J(x,y)dy \leq \frac{C}{\psi(r)}. \tag{6.9}$$

Proof. By Theorem 4.1 and the fact that $h \leq 1$, we have that

$$\begin{aligned}
\int_{D\setminus B(x,r)} J(x,y)dy &\leq c_1 \left(\int_{D\setminus B(x,r)} \frac{\int_0^{\Phi(\rho(x,y))} w(s)ds}{V(x,\rho(x,y))\Psi(\rho(x,y))}dy + \int_{D\setminus B(x,r)} \frac{w\big(\Phi(\rho(x,y))\big)}{V(x,\rho(x,y))}dy \right) \\
&=: c_1(I_1 + I_2).
\end{aligned}$$

By the first inequality in (2.1), since $\Psi \geq \Phi$, we see that

$$I_1 \leq \int_{D\setminus B(x,r)} \frac{e\Phi(\rho(x,y))\phi(1/\Phi(\rho(x,y)))}{V(x,\rho(x,y))\Psi(\rho(x,y))}dy \leq \int_{D\setminus B(x,r)} \frac{e\, dy}{V(x,\rho(x,y))\psi(\rho(x,y))}.$$

It follows from (2.10) that $I_2 \leq \frac{e}{e-2}\int_{D\setminus B(x,r)} V(x,\rho(x,y))^{-1}\psi(\rho(x,y))^{-1}dy$. Hence, by (3.1), (4.9) and the proof of [18, Lemma 2.1], we conclude that $I_1 + I_2 \leq c_2/\psi(r)$. □

Let $Z := (V_s, X_s)_{s\geq 0}$ be the time-space process corresponding to X, where $V_s = V_0 - s$. The augmented filtration of Z will be denoted by $(\widetilde{\mathcal{F}}_s)_{s\geq 0}$. The law of the time-space process $s \mapsto Z_s$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. For every open subset B of $[0,\infty)\times D$, define $\tau_B = \inf\{s > 0 : Z_s \notin B\}$ and $\sigma_B = \tau_{B^c}$.

Recall that a Borel measurable function $u : [0,\infty)\times D \to \mathbb{R}$ is parabolic (or caloric) on $(a,b)\times B(x_0,r)$ with respect to the process X if for every relatively compact open set $U \subset (a,b)\times B(x_0,r)$ it holds that $u(t,x) = \mathbb{E}^{(t,x)}u(Z_{\tau_U})$ for all $(t,x)\in U$.

We denote by $dt \otimes m$ the product of the Lebesgue measure on $[0,\infty)$ and m on E.

Lemma 6.5. *Let $\epsilon \in (0,1/4)$ be the constant from Lemma 6.1. For every $\delta \in (0,\epsilon]$, there exists a constant $C_1 > 0$ such that for all $x \in D$, $r \in (0,\delta_D(x))$, $t \geq \delta\psi(r)$, and any compact set $A \subset [t-\delta\psi(r), t-\delta\psi(r)/2] \times B(x, \psi^{-1}(\epsilon\delta\psi(r)/2))$,*

$$\mathbb{P}^{(t,x)}(\sigma_A < \tau_{[t-\delta\psi(r),t]\times B(x,r)}) \geq C_1 \frac{dt\otimes m(A)}{V(x,r)\psi(r)}. \tag{6.10}$$

Proof. Let $\tau_r = \tau_{[t-\delta\psi(r),t]\times B(x,r)}$ and $A_s = \{y \in D : (s,y)\in A\}$. For any $t,r>0$ and $x\in D$ such that $B(x,r)\subset D$,

$$\begin{aligned}
\delta\psi(r)\mathbb{P}^{(t,x)}(\sigma_A < \tau_r) &\geq \int_0^{\delta\psi(r)} \mathbb{P}^{(t,x)}\left(\int_0^{\tau_r} \mathbf{1}_A(t-s,X_s)ds > 0\right)du \\
&\geq \int_0^{\delta\psi(r)} \mathbb{P}^{(t,x)}\left(\int_0^{\tau_r} \mathbf{1}_A(t-s,X_s)ds > u\right)du = \mathbb{E}^{(t,x)}\left[\int_0^{\tau_r} \mathbf{1}_A(t-s,X_s)ds\right].
\end{aligned} \tag{6.11}$$

For any $t \geq \delta\psi(r)$,

$$\mathbb{E}^{(t,x)}\left[\int_0^{\tau_r}\mathbf{1}_A(t-s,X_s)ds\right] \geq \int_{\delta\psi(r)/2}^{\delta\psi(r)} \mathbb{P}^{(t,x)}\left((t-s, X_s^{B(x,r)}) \in A\right)ds$$

$$= \int_{\delta\psi(r)/2}^{\delta\psi(r)} \mathbb{P}^x(X_s^{B(x,r)} \in A_{t-s})ds = \int_{\delta\psi(r)/2}^{\delta\psi(r)} ds \int_{A_{t-s}} q_{B(x,r)}(s,x,y)\,dy.$$

Let $s \in [\delta\psi(r)/2, \delta\psi(r)]$ and $y \in B(x, \psi^{-1}(\epsilon\delta\psi(r)/2))$. Then $s \le \epsilon\psi(r)$ and $\psi^{-1}(\epsilon\delta\psi(r)/2) \le \psi^{-1}(\epsilon s)$ so that $y \in B(x, \psi^{-1}(\epsilon s))$. Hence, by (3.1), (4.9), Lemma 6.1 and Remark 6.2,

$$q_{B(x,r)}(s,x,y) \ge c_1 V(x, \psi^{-1}(s))^{-1} \ge c_2 V(x,r)^{-1}.$$

Therefore

$$\mathbb{E}^{(t,x)}\left[\int_0^{\tau_r} \mathbf{1}_A(t-s, X_s)ds\right] \ge \frac{c_2}{V(x,r)} \int_{\delta\psi(r)/2}^{\delta\psi(r)} ds \int_{A_{t-s}} dy = c_2 \frac{dt \otimes m(A)}{V(x,r)}.$$

Combining with (6.11), we arrive at (6.10). □

Theorem 6.6. *There exists a constant $\eta \in (0,1]$ such that for all $\delta \in (0,1)$, there exists a constant $C = C(\delta) > 0$ so that for every $x_0 \in D$, $r \in (0, \delta_D(x_0))$, $t_0 \ge 0$, and any function u on $(0,\infty) \times D$ which is parabolic in $(t_0, t_0 + \psi(r)) \times B(x_0, r)$ and bounded in $(t_0, t_0 + \psi(r)) \times D$, we have*

$$|u(s,x) - u(t,y)| \le C\Big(\frac{\psi^{-1}(|s-t|) + \rho(x,y)}{r}\Big)^{\eta} \operatorname*{ess\,sup}_{[t_0, t_0+\psi(r)] \times D} |u|, \tag{6.12}$$

for every $s,t \in (t_0 + \psi(r) - \psi(\delta r), t_0 + \psi(r))$ and $x, y \in B(x_0, \delta r)$.

Proof. Using (3.1), (4.9) and Lemmas 6.3, 6.4 and 6.5, the result can be proved using the same argument as in the proof of [17, Theorem 4.14] (see also the proof of [19, Proposition 3.8]). We omit details here. □

Lemma 6.7. *Let $\epsilon \in (0, 1/4)$ be the constant from Lemma 6.1 and let $\delta \in (0, \epsilon/4)$ be such that $4\delta\psi(2r) \le \epsilon\psi(r)$ for all $r > 0$. Then there exists a constant $C_2 > 0$ such that for all $x_0 \in D$, $R \in (0, \delta_D(x_0))$, $r \in (0, \psi^{-1}(\epsilon\delta\psi(R)/2)/2]$, $\delta\psi(R)/2 \le t - s \le 4\delta\psi(2R)$, $x \in B(x_0, \psi^{-1}(\epsilon\delta\psi(R)/2)/2)$, and $z \in B(x_0, \psi^{-1}(\epsilon\delta\psi(R)/2))$,*

$$\mathbb{P}^{(t,z)}(\sigma_{\{s\}\times B(x,r)} \le \tau_{[s,t]\times B(x_0,R)}) \ge C_2 \frac{V(x,r)}{V(x,R)}. \tag{6.13}$$

Proof. The left-hand side of (6.13) is equal to

$$\mathbb{P}^z(X_{t-s}^{B(x_0,R)} \in B(x,r)) = \int_{B(x,r)} q_{B(x_0,R)}(t-s, z, y)\,dy. \tag{6.14}$$

Note that $t - s \le 4\delta\psi(2R) \le \epsilon\psi(R)$ by the choice of δ. Next, if $z \in B(x,r)$, then $\rho(z, x_0) \le \rho(z,x) + \rho(x, x_0) \le r + \rho(x,x_0) \le \psi^{-1}(\epsilon\delta\psi(R)/2) \le \psi^{-1}(\epsilon(t-s))$, implying that $z \in B(x_0, \psi^{-1}(\epsilon(t-s)))$ (with the same conclusion for $y \in B(x,r)$). Thus it follows from Lemma 6.1, Remark 6.2 and (3.2) that $q_{B(x_0,R)}(t-s,z,y) \ge c_1 V(x_0,R)^{-1} \ge c_2 V(x,R)^{-1}$. By inserting in (6.14) we obtain (6.13). □

In the remainder of this section, we further assume that h is of Harnack-type.

Suppose that $x,y,z \in D$ are such that $\rho(x,z) \le \rho(x,y)/2$. Then

$$\frac{2}{3}\rho(x,y) \le \rho(z,y) \le \frac{3}{2}\rho(x,y). \tag{6.15}$$

As a consequence, by the scalings of Φ and Ψ, there exists $a > 1$ such that

$$a^{-1}\Phi(\rho(x,y)) \le \Phi(\rho(z,y)) \le a\Phi(\rho(x,y)), \quad a^{-1}\Psi(\rho(x,y)) \le \Psi(\rho(z,y)) \le a\Psi(\rho(x,y)). \tag{6.16}$$

Proposition 6.8. *Suppose that h is of Harnack-type. Then there exists a constant $C > 0$ such that for all $x,y,z \in D$ satisfying $\rho(x,z) \le (\rho(x,y) \wedge \delta_D(x))/2$, it holds that*

$$J(x,y) \le CJ(z,y).$$

Proof. This follows from Theorem 4.1, (3.7), the scaling property of w, (3.10) and (6.16). □

Corollary 6.9. *Suppose that h is of Harnack-type. Then there exists a constant $C>0$ such that for all $x,y\in D$ and $0<r\le (\rho(x,y)\wedge \delta_D(x))/2$, it holds that*

$$J(x,y)\le \frac{C}{V(x,r)}\int_{B(x,r)} J(z,y)dz.$$

Proof. Let $x,y\in D$ and $r>0$ be as in the statement. If $z\in B(x,r)$, then $\rho(x,z)<r\le \rho(x,y)/2$. Therefore, by Proposition 6.8, $J(x,y)\le c_1 J(z,y)$, whence the claim immediately follows. □

Lemma 6.10. *Suppose that h is of Harnack-type. Let $\epsilon\in(0,1/4)$ be the constant from Lemma 6.1, and $\theta\ge 1/2$. Further, let $0<\delta_0<\epsilon$, and $0<\delta_1<\delta_2<\delta_3<\delta_4$ be such that $(\delta_3-\delta_2)\psi(r)\ge\psi(\delta_0 r)$ and $\delta_4\psi(r)\le\psi(\epsilon r)$ for all $r\in(0,\mathrm{diam}(D))$. For $x_0\in D$, $t_0\ge 0$ and $r\in(0,\delta_D(x))$, define*

$$Q_1=(t_0,t_0+\delta_4\psi(r))\times B(x_0,\delta_0^2 r), \qquad Q_2=(t_0,t_0+\delta_4\psi(r))\times B(x_0,r),$$
$$Q_3=[t_0+\delta_1\psi(r),t_0+\delta_2\psi(r)]\times B(x_0,\delta_0^2 r/2), \qquad Q_4=[t_0+\delta_3\psi(r),t_0+\delta_4\psi(r)]\times B(x_0,\delta_0^2 r/2).$$

Let $f:(t_0,\infty)\times D\to[0,\infty)$ be bounded and supported in $(t_0,\infty)\times(D\setminus B(x_0,(1+\theta)r))$. Then there exists a constant $C_2>0$ such that

$$\mathbb{E}^{(t_1,y_1)} f(Z_{\tau_{Q_1}})\le C_2\mathbb{E}^{(t_2,y_2)} f(Z_{\tau_{Q_2}}) \quad \textit{for all } (t_1,y_1)\in Q_3, (t_2,y_2)\in Q_4.$$

Proof. Without loss of generality we may assume that $t_0=0$. Fix $x_0\in D$. For $s>0$, we set $B_s=B(x_0,s)$. Let $(t_1,y_1)\in Q_3$ and $(t_2,y_2)\in Q_4$. By the Lévy system formula for the time-space process Z we have

$$\begin{aligned}
&\mathbb{E}^{(t_2,y_2)} f(Z_{\tau_{Q_2}})=\mathbb{E}^{(t_2,y_2)} f(t_2-(\tau_{B_r}\wedge t_2),X_{\tau_{B_r}\wedge t_2})\\
&= \mathbb{E}^{(t_2,y_2)}\left[\int_0^{t_2}\mathbf{1}_{t\le\tau_{B_r}}\,dt\int_{D\setminus B(x_0,(1+\theta)r)} f(t_2-t,v)J(X_t,v)\,dv\right]\\
&= \int_0^{t_2}dt\int_{D\setminus B(x_0,(1+\theta)r)} f(t_2-t,v)\mathbb{E}^{(t_2,y_2)}\left[\mathbf{1}_{t\le\tau_{B_r}}J(X_t,v)\right]dv\\
&= \int_0^{t_2}ds\int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)\mathbb{E}^{(t_2,y_2)}\left[\mathbf{1}_{t_2-s\le\tau_{B_r}}J(X_{t_2-s},v)\right]dv\\
&= \int_0^{t_2}ds\int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)dv\int_{B_r} q_{B_r}(t_2-s,y_2,z)J(z,v)dz \qquad (6.17)\\
&\ge \int_0^{t_1}ds\int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)dv\int_{B_{\delta_0^2 r}} q_{B_r}(t_2-s,y_2,z)J(z,v)dz.
\end{aligned}$$

For $s\in[0,t_1]$ it holds that $\psi(\delta_0 r)\le(\delta_3-\delta_2)\psi(r)\le t_2-t_1\le t_2-s\le\delta_4\psi(r)\le\psi(\epsilon r)$, hence $\delta_0^2 r\le\epsilon\delta_0 r\le\epsilon\psi^{-1}(t_2-s)$. Therefore, for any $z\in B_{\delta_0^2 r}$, by Lemma 6.1 and (3.2), $q_{B_r}(t_2-s,y_2,z)\ge c_1 V(x_0,\psi^{-1}(t_2-s))^{-1}\ge c_1 V(x_0,\epsilon r)^{-1}$. We conclude that

$$\mathbb{E}^{(t_2,y_2)} f(Z_{\tau_{Q_2}})\ge\frac{c_1}{V(x_0,\epsilon r)}\int_0^{t_1}ds\int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)dv\int_{B_{\delta_0^2 r}} J(z,v)dz. \qquad (6.18)$$

Now, by using the Lévy system formula again, similar to (6.17) we obtain that

$$\begin{aligned}
&\mathbb{E}^{(t_1,y_1)} f(Z_{\tau_{Q_1}})=\int_0^{t_1}ds\int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)dv\int_{B_{\delta_0^2 r}} q_{B_{\delta_0^2 r}}(t_1-s,y_1,z)J(z,v)dz\\
&= \int_0^{t_1}ds\int_{B_{\delta_0^2 r}} q_{B_{\delta_0^2 r}}(t_1-s,y_1,z)dz\int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)J(z,v)dv\\
&= \int_0^{t_1}ds\left(\int_{B_{\delta_0^2 r}\setminus B_{3\delta_0^2 r/4}} q_{B_{\delta_0^2 r}}(t_1-s,y_1,z)dz\int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)J(z,v)dv\right.
\end{aligned}$$

$$+\int_{B_{3\delta_0^2 r/4}} q_{B_{\delta_0^2 r}}(t_1-s,y_1,z)dz \int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)J(z,v)dv\Bigg) =: \int_0^{t_1} ds(I_1+I_2).$$

Let $z \in B_{\delta_0^2 r}\setminus B_{3\delta_0^2 r/4}$. Since $y_1 \in B_{\delta_0^2 r/2}$, we have that $2\rho(x_0,y_1) \le \delta_0^2 r/4 \le \rho(y_1,z) \le 3\delta_0^2 r/2$, which implies by (4.9) that $\psi(\rho(y_1,z)) \ge c_2\psi(r) \ge c_2\delta_2^{-1}t_1$. Hence by Proposition 4.5, (3.1) and (3.2), we get that for any $s>0$,

$$\begin{aligned} q_{B_{\delta_0^2 r}}(t_1-s,y_1,z) &\le q(t_1-s,y_1,z) \le \frac{c_3 t_1}{V(y_1,\rho(y_1,z))\psi(\rho(y_1,z))} \le \frac{c_2^{-1}c_3\delta_2}{V(y_1,\rho(y_1,z))} \\ &\le \frac{c_4}{V(x_0,\rho(y_1,z))} \le \frac{c_4}{V(x_0,\delta_0^2 r/4)} \le \frac{c_5}{V(x_0,\epsilon r)}. \end{aligned}$$

Therefore,

$$I_1 \le \frac{c_5}{V(x_0,\epsilon r)} \int_{B_{\delta_0^2 r}\setminus B_{3\delta_0^2 r/4}} dz \int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)J(z,v)dv.$$

Let $z \in B_{3\delta_0^2 r/4}$. Then by (3.2), we have

$$V(z,\delta_0^2 r/4) \ge c_6 V(x_0,\epsilon r). \tag{6.19}$$

We also have that $\delta_D(z) \ge r - 3\delta_0^2 r/4 \ge (1-\delta_0^2)r$, and for $v \in D\setminus B(x_0,(1+\theta)r)$,

$$\rho(z,v) \ge \rho(x_0,v)-\rho(x_0,z) \ge (1+\theta)r - \frac{3\delta_0^2 r}{4} \ge \frac{3r}{2} - \frac{3\delta_0^2 r}{4} \ge (1-\delta_0^2)r.$$

Thus, for any $w \in B(z,\delta_0^2 r/4)$, since $\delta_0^2 < 2/5$,

$$\frac{1}{2}\left(\rho(z,v)\wedge\delta_D(z)\right) \ge \frac12(1-\delta_0^2)r \ge \frac{\delta_0^2 r}{4} \ge \rho(z,w).$$

From Proposition 6.8, we get that $J(z,v) \le c_7 J(w,v)$. Therefore, by (6.19),

$$\begin{aligned} I_2 &= \int_{B_{3\delta_0^2 r/4}} q_{B_{\delta_0^2 r}}(t_1-s,y_1,z)dz \int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)J(z,v)dv \\ &\le \int_{B_{3\delta_0^2 r/4}} q_{B_{\delta_0^2 r}}(t_1-s,y_1,z)dz \left(\frac{c_7}{V(z,\delta_0^2 r/4)}\int_{B(z,\delta_0^2 r/4)} J(w,v)dw \int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)dv\right) \\ &\le \frac{c_6^{-1}c_7}{V(x_0,\epsilon r)} \int_{B(z,\delta_0^2 r/4)} dw \int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)J(w,v)dv. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}^{(t_1,y_1)} f(Z_{\tau_{Q_1}}) &= \int_0^{t_1} ds(I_1+I_2) \le \frac{c_8}{V(x_0,\epsilon r)} \int_0^{t_1} ds \int_{B_{\delta_0^2 r}} dz \int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)J(z,v)dv \\ &= \frac{c_8}{V(x_0,\epsilon r)} \int_0^{t_1} ds \int_{D\setminus B(x_0,(1+\theta)r)} f(s,v)dv \int_{B_{\delta_0^2 r}} J(z,v)dz \overset{(6.18)}{\le} c_9 \mathbb{E}^{(t_2,y_2)} f(Z_{\tau_{Q_2}}). \end{aligned}$$

□

Theorem 6.11. *Suppose that h is of Harnack-type. Then there exist constants $\delta>0$, $C>1$ and $K\ge 1$ such that for all $t_0 \ge 0$, $x_0 \in D$ and $R \in (0,R_1)$ with $B(x_0,CR)\subset D$, and any non-negative function u on $(0,\infty)\times D$ which is parabolic on $Q := (t_0, t_0+4\delta\psi(CR))\times B(x_0,CR)$, we have*

$$\sup_{(t_1,y_1)\in Q_-} u(t_1,y_1) \le K \inf_{(t_2,y_2)\in Q_+} u(t_2,y_2), \tag{6.20}$$

where $Q_- = [t_0+\delta\psi(CR), t_0+2\delta\psi(CR)]\times B(x_0,R)$ and $Q_- = [t_0+3\delta\psi(CR), t_0+4\delta\psi(CR)]\times B(x_0,R)$.

Proof. Using (3.1), (4.9), Lemmas 6.1, 6.3, 6.5, 6.7, 6.10 and Corollary 6.9 the result can be proved using the same argument as in the proof of [16, Lemma 5.3] (see also the proof of [19, Lemma 4.1]). We omit details here. □

7. Examples

Recall the definition of $\mathcal{B}_h(t,x,y)$ from (4.2). For $p,q\ge 0$, we let $\mathcal{B}_{p,q}(t,x,y):=\mathcal{B}_{h_{p,q}}(t,x,y)$ where $h_{p,q}(t,x,y)$ is the boundary function defined in (3.8). We remind the reader that this is the typical and most important boundary function. Recall that $\psi(r)=1/\phi(1/\Phi(r))$, $\delta_\vee(x,y)=\delta_D(x)\vee\delta_D(y)$ and $\delta_\wedge(x,y)=\delta_D(x)\wedge\delta_D(y)$. Define $\mathbf{m}^t_{x,y}(r)=\big(\psi^{-1}(t)\vee r\big)\wedge\rho(x,y)$. For simplicity, we will use $\delta(x)$ and $\delta(y)$ instead of $\delta_D(x)$ and $\delta_D(y)$, respectively.

The following lemma provides the list of estimates of $\mathcal{B}_{p,q}(t,x,y)$ depending on the relationship between the parameters p,q,β_1,β_2. The list is not exhaustive, but it suffices for our purpose. The proof of the lemma is rather technical and consists of carefully estimating the integral defining $\mathcal{B}_{p,q}(t,x,y)$.

Lemma 7.1. *Let $q\ge p\ge 0$, $p+q>0$. Suppose that* **(Poly-R_1)** *holds with $\beta_1,\beta_2\in(0,1)$. Then, the following estimates hold for all $x,y\in D$, $0<t\le\psi(\rho(x,y))$ such that $\Phi(\rho(x,y))<R_1/8$.*

(i) If $\beta_2<1-p-q$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(x))}{\Phi(\rho(x,y))}\Big)^p\Big(1\wedge\frac{\Phi(\delta(y))}{\Phi(\rho(x,y))}\Big)^q\frac{\Phi(\rho(x,y))}{\psi(\rho(x,y))}.$$

(ii) If $1-p-q<\beta_1\le\beta_2<1-q$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(x))}{\Phi(\rho(x,y))}\Big)^p\Big(1\wedge\frac{\Phi(\delta(y))}{\Phi(\rho(x,y))}\Big)^q\frac{\Phi(\rho(x,y))^{p+q}\Phi(\mathbf{m}^t_{x,y}(\delta_\vee(x,y)))^{1-p-q}}{\psi(\mathbf{m}^t_{x,y}(\delta_\vee(x,y)))}. \tag{7.1}$$

(iii) If $1-q<\beta_1\le\beta_2<1-p$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(x))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}\Big)^p\Big(1\wedge\frac{\Phi(\delta(y))}{\phi^{-1}(1/t)^{-1}}\Big)^q\frac{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}{\psi(\mathbf{m}^t_{x,y}(\delta(y)))}. \tag{7.2}$$

(iv) If $1-p<\beta_1\le\beta_2<1$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(x))}{\phi^{-1}(1/t)^{-1}}\Big)^p\Big(1\wedge\frac{\Phi(\delta(y))}{\phi^{-1}(1/t)^{-1}}\Big)^q\frac{\Phi(\mathbf{m}^t_{x,y}(\delta_\wedge(x,y)))}{\psi(\mathbf{m}^t_{x,y}(\delta_\wedge(x,y)))}. \tag{7.3}$$

(v) If $\beta_1=\beta_2=1-p-q$ and $p>0$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(x))}{\Phi(\rho(x,y))}\Big)^p\Big(1\wedge\frac{\Phi(\delta(y))}{\Phi(\rho(x,y))}\Big)^q\Phi(\rho(x,y))^{p+q}\log\Big(e+\frac{\Phi(\rho(x,y))}{\Phi(\mathbf{m}^t_{x,y}(\delta_\vee(x,y)))}\Big). \tag{7.4}$$

(vi) If $\beta_1=\beta_2=1-q$ and $p=0$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(y))}{\Phi(\rho(x,y))}\Big)^q\Phi(\rho(x,y))^q\log\Big(e+\frac{\Phi(\rho(x,y))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}\Big). \tag{7.5}$$

(vii) If $\beta_1=\beta_2=1-q$ and $q=p$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p\Big(1\wedge\frac{\Phi(\delta_\vee(x,y))}{\phi^{-1}(1/t)^{-1}}\Big)^p\Phi(\rho(x,y))^p\log\Big(e+\frac{\Phi(\mathbf{m}^t_{x,y}(\delta_\vee(x,y)))}{\Phi(\mathbf{m}^t_{x,y}(\delta_\wedge(x,y)))}\Big). \tag{7.6}$$

(viii) If $\beta_1=\beta_2=1-q$ and $q>p>0$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(x))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}\Big)^p\Big(1\wedge\frac{\Phi(\delta(y))}{\Phi(\rho(x,y))}\Big)^q\Phi(\rho(x,y))^q\log\Big(e+\frac{\Phi(\mathbf{m}^t_{x,y}(\delta(x)))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}\Big). \tag{7.7}$$

(ix) If $\beta_1=\beta_2=1-p$ and $q>p>0$, then

$$\mathcal{B}_{p,q}(t,x,y)\simeq\Big(1\wedge\frac{\Phi(\delta(x))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}\Big)^p\Big(1\wedge\frac{\Phi(\delta(y))}{\phi^{-1}(1/t)^{-1}}\Big)^q\Phi(\mathbf{m}^t_{x,y}(\delta(y)))^p\log\Big(e+\frac{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}{\Phi(\mathbf{m}^t_{x,y}(\delta(x)))}\Big). \tag{7.8}$$

Proof. Fix $x,y\in D$ such that $r:=\rho(x,y)<\Phi^{-1}(R_1/8)$, and $t\in(0,\psi(r)]$. Let $\delta_\wedge:=\delta_\wedge(x,y)$ and $\delta_\vee:=\delta_\vee(x,y)$, and note that $\delta_\vee\le r+\delta_\wedge$. We also note that since $\beta_2<1$, by [47, Lemma 2.6, Proposition 2.9] and (2.2),

$$w(s)\simeq\phi(1/s),\quad s\in(0,R_1/2), \tag{7.9}$$

which is equivalent to that $w(\Phi(s))\simeq\psi(s)^{-1}$ for $s\in(0,\Phi^{-1}(R_1/2)$.

If $\delta_\wedge \ge r$, then by Lemma 5.1(i), **(Poly-R_1)** and (7.9), we have

$$\mathcal{B}_{p,q}(t,x,y) \simeq \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} w(s)ds \simeq \frac{\Phi(r)}{\psi(r)}. \tag{7.10}$$

(i) This follows immediately from Lemma 5.1(i) and **(Poly-R_1)**.
(ii) First assume $\Phi(\delta_\vee) > \phi^{-1}(1/t)^{-1}$, which is equivalent to $\delta_\vee > \psi^{-1}(t)$. If $\delta_\wedge > r$, then $\mathbf{m}^t_{x,y}(\delta_\vee) = r$ so that (7.1) follows from (7.10). If $\delta_\wedge = \delta(x) \le r$, then $\delta(y) \le 2r$ so that $\mathbf{m}^t_{x,y}(\delta_\vee) \simeq \delta(y)$ and by Lemma 5.1(i)-(ii) and **(Poly-R_1)**,

$$\begin{aligned}\mathcal{B}_{p,q}(t,x,y) &\simeq \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta(y))} \Big(1 \wedge \frac{\Phi(\delta(x))}{s}\Big)^p w(s)ds + \Phi(\delta(x))^p\Phi(\delta(y))^q \int_{3\Phi(\delta(y))}^{4\Phi(r)} \frac{w(s)}{s^{p+q}}ds \\ &\simeq \Phi(\delta(x))^p\Phi(\delta(y))^{1-p}\psi(\delta(y))^{-1}. \end{aligned}\tag{7.11}$$

Similarly, if $\delta_\wedge = \delta(y) \le r$, then $\mathbf{m}^t_{x,y}(\delta_\vee) \simeq \delta(x)$ and

$$\begin{aligned}\mathcal{B}_{p,q}(t,x,y) &\simeq \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta(x))} \Big(1 \wedge \frac{\Phi(\delta(y))}{s}\Big)^q w(s)ds + \Phi(\delta(x))^p\Phi(\delta(y))^q \int_{3\Phi(\delta(x))}^{4\Phi(r)} \frac{w(s)}{s^{p+q}}ds \\ &\simeq \Phi(\delta(x))^{1-q}\Phi(\delta(y))^{q}\psi(\delta(x))^{-1}. \end{aligned}\tag{7.12}$$

Thus in the case $\Phi(\delta_\vee) > \phi^{-1}(1/t)^{-1}$, $\mathcal{B}_{p,q}(t,x,y)$ is comparable with the right hand side of (7.1).

If $\Phi(\delta_\vee) \le \phi^{-1}(1/t)^{-1}$, then by Lemma 5.1(ii), **(Poly-R_1)** and (7.9), we obtain

$$\mathcal{B}_{p,q}(t,x,y) \simeq \Phi(\delta(x))^p\Phi(\delta(y))^q \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} s^{-p-q}w(s)ds \simeq \Phi(\delta(x))^p\Phi(\delta(y))^q\, \frac{\phi^{-1}(1/t)^{-(1-p-q)}}{t}.$$

Since $\mathbf{m}^t_{x,y}(\delta_\vee) = \psi^{-1}(t)$ in this case, we conclude (7.1).
(iii) First assume $\Phi(\delta(y)) > \phi^{-1}(1/t)^{-1}$. If $\delta_\wedge > r$, then $\mathbf{m}^t_{x,y}(\delta(y)) = r$ so that (7.2) follows from (7.10). If $\delta_\wedge = \delta(x) \le r$, then (7.11) holds by Lemma 5.1(i)-(ii) and **(Poly-R_1)**. If $\delta_\wedge = \delta(y) \le r$, then

$$\mathcal{B}_{p,q}(t,x,y) \simeq \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta(y))} w(s)ds + \Phi(\delta(y))^q \int_{3\Phi(\delta(y))}^{4\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta(x))}{s}\Big)^p \frac{w(s)}{s^q}ds \simeq \frac{\Phi(\delta(y))}{\psi(\delta(y))}.$$

Thus when $\Phi(\delta(y)) > \phi^{-1}(1/t)^{-1}$, since $\mathbf{m}^t_{x,y}(\delta(y)) \simeq \delta(y)$, $\mathcal{B}_{p,q}(t,x,y)$ is comparable with the right hand side of (7.2).

If $\Phi(\delta(y)) \le \phi^{-1}(1/t)^{-1}$, then by Lemma 5.1(ii), **(Poly-R_1)** and (7.9), we get

$$\begin{aligned}\mathcal{B}_{p,q}(t,x,y) &\simeq \Phi(\delta(y))^q \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta(x))}{s}\Big)^p s^{-q}w(s)ds \\ &\simeq \Big(1 \wedge \frac{\Phi(\delta(x))}{\phi^{-1}(1/t)^{-1}}\Big)^p \Big(\frac{\Phi(\delta(y))}{\phi^{-1}(1/t)^{-1}}\Big)^q \frac{\phi^{-1}(1/t)^{-1}}{t}.\end{aligned}$$

Since $\mathbf{m}^t_{x,y}(\delta(y)) = \psi^{-1}(t)$ in this case, we conclude (7.2).
(iv) If $\Phi(\delta_\wedge) > \Phi(r)$, then (7.3) follows from (7.10). If $\Phi(r) \ge \Phi(\delta_\wedge) > \phi^{-1}(1/t)^{-1}$ and $\delta_\wedge = \delta(x)$, then by Lemma 5.1(i)–(ii) and **(Poly-R_1)**, we have

$$\mathcal{B}_{p,q}(t,x,y) \simeq \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta_\wedge)} w(s)ds + \Phi(\delta_\wedge)^p \int_{3\Phi(\delta_\wedge)}^{4\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta(y))}{s}\Big)^q s^{-p}w(s)ds \simeq \frac{\Phi(\delta_\wedge)}{\psi(\delta_\wedge)}.$$

Similarly, if $\Phi(r) \ge \Phi(\delta_\wedge) > \phi^{-1}(1/t)^{-1}$ and $\delta_\wedge = \delta(y)$, then

$$\mathcal{B}_{p,q}(t,x,y) \simeq \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta_\wedge)} w(s)ds + \Phi(\delta_\wedge)^q \int_{3\Phi(\delta_\wedge)}^{4\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta(x))}{s}\Big)^p s^{-q}w(s)ds \simeq \frac{\Phi(\delta_\wedge)}{\psi(\delta_\wedge)}.$$

If $\Phi(\delta_\wedge) \le \phi^{-1}(1/t)^{-1}$, then by considering the cases $\delta_\wedge = \delta(x)$ and $\delta_\wedge = \delta(y)$ separately, we see from Lemma 5.1(ii) and **(Poly-R_1)** that

$$\mathcal{B}_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta(x))}{\phi^{-1}(1/t)^{-1}}\Big)^p \Big(1 \wedge \frac{\Phi(\delta(y))}{\phi^{-1}(1/t)^{-1}}\Big)^q \frac{\Phi(\psi^{-1}(t))}{t}.$$

(v) Note that $w(s) \simeq s^{p+q-1}$ for $s \in (0, R_1)$ in this case. If $\Phi(\delta_\vee) > \phi^{-1}(1/t)^{-1}$, then by Lemma 5.1(i), we get in the case $\delta_\vee = \delta(x)$,

$$\begin{aligned}\mathcal{B}_{p,q}(t,x,y) &\simeq \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r\wedge\delta(x))} (s\wedge \Phi(\delta(y)))^q s^{-1+p} ds + \Phi(\delta(x))^p\Phi(\delta(y))^q \int_{4\Phi(r\wedge\delta_\vee)}^{4\Phi(r)} s^{-1}ds \\ &\simeq \Phi(r\wedge\delta(x))^p\Phi(r\wedge\delta(y))^q + \Phi(\delta(x))^p\Phi(\delta(y))^q \log\frac{\Phi(r)}{\Phi(\mathbf{m}^t_{x,y}(\delta_\vee))}.\end{aligned}$$

Similarly, when $\delta_\vee = \delta(y)$, we also get the same conclusion. One can check that the last line in the above is comparable with the right hand side of (7.4). If $\Phi(\delta_\vee) \le \phi^{-1}(1/t)^{-1}$, then $\mathcal{B}_{p,q}(t,x,y) \simeq \Phi(\delta(x))^p\Phi(\delta(y))^q \log\big(2\Phi(r)\phi^{-1}(1/t)\big)$ and $\Phi(\mathbf{m}^t_{x,y}(\delta_\vee)) = \phi^{-1}(1/t)^{-1}$. Hence, (7.4) is valid.

(vi) Note that $w(s) \simeq s^{q-1}$ for $s \in (0, R_1/2)$ and that $\psi(s) \simeq \Phi(s)^{1-q}$ for $s \in (0, R_1/2)$ in this case. If $\delta(y) > r$, then (7.5) follows from (7.10). If $r \ge \delta(y) = \mathbf{m}^t_{x,y}(\delta(y)) > \phi^{-1}(1/t)^{-1}$, then

$$\mathcal{B}_{p,q}(t,x,y) \simeq \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta(y))} s^{q-1}ds + \Phi(\delta(y))^q\int_{3\Phi(\delta(y))}^{4\Phi(r)} s^{-1}ds \simeq \Phi(\delta(y))^q \log\Big(e + \frac{\Phi(r)}{\Phi(\delta(y))}\Big).$$

If $\delta(y) \le \phi^{-1}(1/t)^{-1}$, then $\Phi(\mathbf{m}^t_{x,y}(\delta(y))) = \phi^{-1}(1/t)^{-1}$ and

$$\mathcal{B}_{p,q}(t,x,y) \simeq \Phi(\delta(y))^q \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} s^{-1}ds \simeq \Phi(\delta(y))^q \log\big(e + \Phi(r)\phi^{-1}(1/t)\big).$$

(vii) Let $F_1(t,x,y)$ be the function given in the right hand side of (7.6). Then, we have

$$F_1(t,x,y) \simeq \begin{cases} \Phi(\delta_\wedge)^p\Phi(\delta_\vee)^p\phi^{-1}(1/t)^p, & \text{if } \Phi(\delta_\vee) \le \phi^{-1}(1/t)^{-1}, \\ \Phi(\delta_\wedge)^p\log\big(e+\Phi(\delta_\vee)\phi^{-1}(1/t)\big), & \text{if } \Phi(\delta_\wedge) \le \phi^{-1}(1/t)^{-1} < \Phi(\delta_\vee), \\ \Phi(\delta_\wedge)^p\log\big(e+\Phi(\delta_\vee)/\Phi(\delta_\wedge)\big), & \text{if } \Phi(\delta_\wedge) > \phi^{-1}(1/t)^{-1},\ \delta_\vee \le 2r, \\ \Phi(r)^p, & \text{if } \Phi(\delta_\wedge) > \phi^{-1}(1/t)^{-1},\ \delta_\vee > 2r. \end{cases} \tag{7.13}$$

The last comparison in (7.13) above holds since $\Phi(\delta_\wedge) \ge \Phi(\delta_\vee - r) \ge \Phi(r)$ in such case.

Note that $w(s) \simeq s^{p-1}$ for $s \in (0, R_1)$ in this case. If $\Phi(\delta_\vee) \le \phi^{-1}(1/t)^{-1}$, then

$$\mathcal{B}_{p,p}(t,x,y) \simeq \Phi(\delta_\wedge)^p\Phi(\delta_\vee)^p \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} s^{-p-1}ds \simeq \Phi(\delta_\wedge)^p\Phi(\delta_\vee)^p\phi^{-1}(1/t)^p.$$

If $\Phi(\delta_\vee) > \phi^{-1}(1/t)^{-1} \ge \Phi(\delta_\wedge)$, then by (3.4), $\Phi(\delta_\vee) \le \Phi(\delta_\wedge + r) \le \Phi(2r) \le c\Phi(r)$ so that

$$\begin{aligned}\mathcal{B}_{p,p}(t,x,y) &\simeq \Phi(\delta_\wedge)^p \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r\wedge\delta_\vee)} s^{-1}ds + \Phi(\delta_\wedge)^p\Phi(\delta_\vee)^p\int_{4\Phi(r\wedge\delta_\vee)}^{4\Phi(r)} s^{-p-1}ds \\ &\simeq \Phi(\delta_\wedge)^p\left[\log\big(2\Phi(r\wedge\delta_\vee)\phi^{-1}(1/t)\big) + \frac{\Phi(\delta_\vee)^p}{\Phi(r\wedge\delta_\vee)^p}\right] \simeq \Phi(\delta_\wedge)^p\log\big(e+\Phi(\delta_\vee)\phi^{-1}(1/t)\big).\end{aligned}$$

If $\Phi(\delta_\wedge) > \phi^{-1}(1/t)^{-1}$, then

$$\begin{aligned}\mathcal{B}_{p,p}(t,x,y) &\simeq \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r\wedge\delta_\wedge)} s^{p-1}ds + \Phi(\delta_\wedge)^p\int_{4\Phi(r\wedge\delta_\wedge)}^{4\Phi(r\wedge\delta_\vee)} s^{-1}ds + \Phi(\delta_\wedge)^p\Phi(\delta_\vee)^p\int_{4\Phi(r\wedge\delta_\vee)}^{4\Phi(r)} s^{-p-1}ds \\ &\simeq \Phi(r\wedge\delta_\wedge)^p + \Phi(\delta_\wedge)^p\log\frac{\Phi(r\wedge\delta_\vee)}{\Phi(r\wedge\delta_\wedge)} + \Phi(\delta_\wedge)^p\frac{\Phi(\delta_\vee)^p}{\Phi(r\wedge\delta_\vee)^p}.\end{aligned}$$

Now, by considering each case separately, (7.6) follows from (7.13).

(viii) If $\delta(y) \ge 2r$ (so that $\delta(x) \ge r$), then (7.7) follows from (7.10). If $2r > \delta(y) \ge \psi^{-1}(t)$ and $\delta(y) \ge \delta(x)$, then by the scaling property of Φ and Lemma 5.1(i),

$$\begin{aligned}\mathcal{B}_{p,q}(t,x,y) &\simeq \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r\wedge\delta(y))} (s\wedge\Phi(\delta(x)))^p s^{q-p-1}ds + \Phi(\delta(x))^p\Phi(\delta(y))^q\int_{4\Phi(r\wedge\delta(y))}^{4\Phi(r)} s^{-p-1}ds \\ &\simeq \Phi(r\wedge\delta(x)\wedge\delta(y))^p\Phi(r\wedge\delta(y))^{q-p} + \Phi(\delta(x))^p\Phi(\delta(y))^q\Phi(r\wedge\delta(y))^{-p} \\ &\simeq \Phi(\delta(x))^p\Phi(\delta(y))^{q-p} \simeq \Phi(\delta(x))^p\Phi(\mathbf{m}^t_{x,y}(\delta(y)))^{-p}\Phi(\delta(y))^q.\end{aligned}$$

If $2r > \delta(y) \ge \psi^{-1}(t)$ and $\delta(y) < \delta(x)$, then $\delta(x) \ge \mathbf{m}^t_{x,y}(\delta(y))$ and $\delta(x) \le \delta(y) + r < 3r$ so that by the scaling property of Φ, we get

$$\begin{aligned}
&\mathcal{B}_{p,q}(t,x,y)\\
&\simeq \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r\wedge\delta(y))} s^{q-1}ds + \Phi(\delta(y))^q \int_{4\Phi(r\wedge\delta(y))}^{4\Phi(r\wedge\delta(x))} s^{-1}ds + \Phi(\delta(x))^p\Phi(\delta(y))^q \int_{4\Phi(r\wedge\delta(x))}^{4\Phi(r)} s^{-p-1}ds\\
&\simeq \Phi(\delta(y))^q + \Phi(\delta(y))^q \log \frac{\Phi(\mathbf{m}^t_{x,y}(\delta(x)))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))} + \Phi(\delta(y))^q\\
&\simeq \Big(1 \wedge \frac{\Phi(\delta(x))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}\Big)^p \Phi(\delta(y))^q \log\Big(e + \frac{\Phi(\mathbf{m}^t_{x,y}(\delta(x)))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))}\Big).
\end{aligned}$$

If $\delta(y) < \psi^{-1}(t)$, then $\delta(x) < \delta(y) + r < 2r$ so that $\mathbf{m}^t_{x,y}(\delta(x)) \simeq \psi^{-1}(t) \vee \delta(x)$ and hence

$$\begin{aligned}
\mathcal{B}_{p,q}(t,x,y) &\simeq \Phi(\delta(y))^q \int_{2\phi^{-1}(1/t)^{-1}}^{2\Phi(\psi^{-1}(t)\vee\delta(x))} s^{-1}ds + \Phi(\delta(x))^p\Phi(\delta(y))^q \int_{2\Phi(\psi^{-1}(t)\vee\delta(x))}^{4\Phi(r)} s^{-p-1}ds\\
&\simeq \Phi(\delta(y))^q \log \frac{\Phi(\mathbf{m}^t_{x,y}(\delta(x)))}{\Phi(\mathbf{m}^t_{x,y}(\delta(y)))} + \Phi(\delta(x))^p\Phi(\delta(y))^q\Phi(\mathbf{m}^t_{x,y}(\delta(x)))^{-p}.
\end{aligned}$$

By considering the cases $\delta(x) \ge \psi^{-1}(t)$ and $\delta(x) < \psi^{-1}(t)$ separately, we conclude (7.7).
(ix) Let $F_2(t,x,y)$ be the function given in the right hand side of (7.6). Then, we have

$$F_2(t,x,y) \simeq \begin{cases} \Phi(r)^p, & \text{if } \delta(x) \ge 2r,\\ \Phi(\delta(y))^p & \text{if } 2r > \delta(x) \ge \delta(y) \ge \psi^{-1}(t),\\ \Phi(\delta(y))^q\phi^{-1}(1/t)^{q-p} & \text{if } 2r > \delta(x) \ge \psi^{-1}(t) \ge \delta(y),\\ \Phi(\delta(x))^p \log\big(e + \Phi(\delta(y))/\Phi(\delta(x))\big), & \text{if } 2r > \delta(x) \ge \psi^{-1}(t),\ \delta(x) \le \delta(y),\\ \Phi(\delta(x))^p \log\big(e + \Phi(\delta(y))\phi^{-1}(1/t)\big), & \text{if } \delta(x) < \psi^{-1}(t) \le \delta(y),\\ \Phi(\delta(x))^p\Phi(\delta(y))^q\phi^{-1}(1/t)^q, & \text{if } \delta(x)\vee\delta(y) < \psi^{-1}(t). \end{cases}$$

Now by considering each case separately, using similar arguments to the ones given in the proof of (viii), we conclude (7.8). □

Example 7.2. Let $d, \alpha > 0$, $\beta \in (0,1)$ and $p_1, p_2 \ge 0$ such that $p_1 + p_2 > 0$. Suppose that for every $r_0 \ge 1$, there are comparability constants such that

$$V(x,r) \simeq r^d, \quad x \in D,\ 0 < r < r_0. \tag{7.14}$$

Let Y^D be a Hunt process in D and $S = (S_t)_{t\ge0}$ be an independent driftless subordinator with Laplace exponent ϕ. Suppose that the tail w of the Lévy measure of S satisfies

$$w(r) \simeq r^{-\beta}, \quad 0 < r < r_1, \tag{7.15}$$

for some $r_1 > 0$. Suppose that the heat kernel $p_D(t,x,y)$ of Y^D satisfies either $\mathbf{HK}_{\mathbf{B}}^{\mathbf{h}_{\mathbf{p_1,p_2}}}$ or $\mathbf{HK}_{\mathbf{U}}^{\mathbf{h}_{\mathbf{p_1,p_2}}}$ with $\Phi(r) = \Psi(r) = r^\alpha$ where the boundary function h_{p_1,p_2} is defined as (3.8). When $\mathbf{HK}_{\mathbf{U}}^{\mathbf{h}_{\mathbf{p_1,p_2}}}$ is satisfied, we also assume that (7.14) and (7.15) hold for all $r > 0$. See Example 3.9 for concrete examples of Y^D. By switching the roles of x and y if needed, without loss of generality, we assume that $p_2 \ge p_1$.

Let $q(t,x,y)$, $J(x,y)$ and $G_D(x,y)$ be the heat kernel, the jump kernel and the Green function of the subordinate process $X_t := Y^D_{S_t}$ respectively. Using our theorems in Sections 4 and 5, and Lemma 7.1, we get explicit estimates on $q(t,x,y)$, $J(x,y)$ and $G_D(x,y)$. We list them in terms of the range of $p_1 + p_2$, similar to the format of the Green function estimates for Dirichlet forms degenerate at the boundary in [44].

In particular, by putting $p_1 = p_2 = 1/2$, we get Theorem 1.1.

We first give the Green function estimates. Define

$$\mathfrak{g}(x,y) := \Big(1\wedge \frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha p_1}\Big(1\wedge \frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha p_2} \times \begin{cases} \dfrac{1}{\rho(x,y)^{d-\alpha\beta}}, & d>\alpha\beta,\\ \log\Big(e+\dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big), & d=\alpha\beta,\\ [\rho(x,y)\vee\delta_\vee(x,y)]^{\alpha\beta-d}, & d<\alpha\beta. \end{cases}$$

When $C_0=0$, by Theorem 5.8 and Example 5.5, for all $x,y\in D$, if $d>\alpha(\beta-p_1-p_2)$, then

$$G_D(x,y)\simeq \mathfrak{g}(x,y) \tag{7.16}$$

and if $d\le \alpha(\beta-p_1-p_2)$, then

$$G_D(x,y)\simeq \begin{cases} \delta(x)^{\alpha p_1}\delta(y)^{\alpha p_2}\log\Big(e+\frac{\mathrm{diam}(D)}{\rho(x,y)\vee\delta_\vee(x,y)}\Big), & d=\alpha(\beta-p_1-p_2) \text{ and } \mathbf{HK}_{\mathbf{B}}^{\mathbf{h}_{\mathbf{p_1},\mathbf{p_2}}} \text{ holds},\\ \delta(x)^{\alpha p_1}\delta(y)^{\alpha p_2}, & d<\alpha(\beta-p_1-p_2) \text{ and } \mathbf{HK}_{\mathbf{B}}^{\mathbf{h}_{\mathbf{p_1},\mathbf{p_2}}} \text{ holds},\\ \infty, & d\le\alpha(\beta-p_1-p_2) \text{ and } \mathbf{HK}_{\mathbf{U}}^{\mathbf{h}_{\mathbf{p_1},\mathbf{p_2}}} \text{ holds}. \end{cases} \tag{7.17}$$

Now assume that $C_0=1$. If $p_1+p_2<\beta+1$, then using Theorem 5.8 and Example 5.5 again, we see that (7.16) and (7.17) also hold. If $p_1+p_2=\beta+1$ and $p_2<\beta+1$ (so that (H2**) holds, cf. Example 5.9), then by Theorem 5.11, for all $x,y\in D$,

$$G_D(x,y)\simeq \mathfrak{g}(x,y)\log\Big(e+\frac{\rho(x,y)}{\delta_\vee(x,y)}\Big). \tag{7.18}$$

If $p_1+p_2>\beta+1$ and $p_2<\beta+1$ (so, again, (H2**) holds), then by Theorem 5.10 and (5.18), for all $x,y\in D$,

$$G_D(x,y)\simeq \Big(1\wedge\frac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{-\alpha(p_1+p_2-\beta-1)}\mathfrak{g}(x,y). \tag{7.19}$$

The unusual form of the estimates in (7.18)-(7.19) should be compared with similar estimates of the Green function obtained in a different context in [44, Theorem 1.1 (2),(3)]. Such estimates lead to anomalous boundary behavior of the corresponding Green potential, cf. [1].

Next, we obtain estimates on $J(x,y)$ from Theorem 4.1. When $C_0=0$, using the fact that

$$\Big(1\wedge\frac{\delta(x)}{s}\Big)\Big(1\wedge\frac{\delta(y)}{s}\Big)=\Big(1\wedge\frac{\delta_\wedge(x,y)}{s}\Big)\Big(1\wedge\frac{\delta_\vee(x,y)}{s}\Big),\quad s>0,\ x,y\in D, \tag{7.20}$$

we see that for all $x,y\in D$,

$$J(x,y)\simeq\Big(1\wedge\frac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p_1}\Big(1\wedge\frac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p_1}\Big(1\wedge\frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha(p_2-p_1)}\frac{1}{\rho(x,y)^{d+\alpha\beta}}.$$

Now assume that $C_0=1$. Using Lemma 7.1, the fact that $\mathcal{B}_h^*(x,y)\simeq\mathcal{B}_h(0,x,y)$ and (7.20) several times, we see that for all $x,y\in D$, if $p_1+p_2<1-\beta$, then

$$J(x,y)\simeq\Big(1\wedge\frac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p_1}\Big(1\wedge\frac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p_1}\Big(1\wedge\frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha(p_2-p_1)}\frac{1}{\rho(x,y)^{d+\alpha\beta}},$$

if $p_1+p_2=1-\beta$, then

$$J(x,y)$$
$$\simeq\frac{1}{\rho(x,y)^{d+\alpha\beta}}\begin{cases}\Big(1\wedge\dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p_1}\Big(1\wedge\dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p_1}\Big(1\wedge\dfrac{\delta(y)}{\rho(x,y)}\Big)^{\alpha(p_2-p_1)}\log\Big(e+\dfrac{\rho(x,y)}{\delta_\vee(x,y)}\Big), & p_1>0,\\ \Big(1\wedge\dfrac{\delta(y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)}\log\Big(e+\dfrac{\rho(x,y)}{\delta(y)}\Big), & p_1=0,\end{cases}$$

and if $p_1+p_2>1-\beta$, then

$$J(x,y)\simeq\frac{1}{\rho(x,y)^{d+\alpha\beta}}\times$$

$$\times \begin{cases} \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)}, & p_2 \ge p_1 > 1-\beta, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)} \log\Big(e + \dfrac{\delta(y) \wedge \rho(x,y)}{\delta(x)}\Big), & p_2 > p_1 = 1-\beta, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p_1} \Big(1 \wedge \dfrac{\delta(y)}{\rho(x,y)}\Big)^{\alpha(1-p_1-\beta)}, & p_2 > 1-\beta > p_1, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)} \log\Big(e + \dfrac{\delta_\vee(x,y) \wedge \rho(x,y)}{\delta_\wedge(x,y)}\Big), & p_2 = p_1 = 1-\beta, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p_1} \Big(1 \wedge \dfrac{\delta(y)}{\rho(x,y)}\Big)^{\alpha(1-p_1-\beta)} \log\Big(e + \dfrac{\delta(x) \wedge \rho(x,y)}{\delta(y)}\Big), & p_2 = 1-\beta > p_1 > 0, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p_1} \Big(1 \wedge \dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta-p_2)} \Big(1 \wedge \dfrac{\delta(y)}{\rho(x,y)}\Big)^{\alpha(p_2-p_1)}, & p_2 < 1-\beta. \end{cases}$$

Below, we also assume that $p_1 = p_2 = p$ for simplicity, and obtain heat kernel estimates from Lemma 7.1, Corollary 4.4, Theorem 4.7 and (7.20).

(1) The following estimates hold for all $(t,x,y) \in (0,1] \times D \times D$.

(i) The case of $\rho(x,y)^{\alpha\beta} \le t$. For all $p > 0$,

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta_\wedge(x,y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha p} \Big(1 \wedge \frac{\delta_\vee(x,y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha p} t^{-d/(\alpha\beta)}.$$

(ii) The case $\rho(x,y)^{\alpha\beta} > t$. If $C_0 = 0$, then

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p} \Big(1 \wedge \frac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p} \frac{t}{\rho(x,y)^{d+\alpha\beta}}$$

and if $C_0 = 1$, then

$$q(t,x,y) \simeq \frac{t}{\rho(x,y)^{d+\alpha\beta}} \times$$

$$\times \begin{cases} \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha p} \Big(1 \wedge \dfrac{\delta_\vee(x,y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha p} \Big(1 \wedge \dfrac{t^{1/(\alpha\beta)} \vee \delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)}, & p > 1-\beta, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p} \Big(1 \wedge \dfrac{\delta_\vee(x,y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha p} \log\left(e + \dfrac{(t^{1/(\alpha\beta)} \vee \delta_\vee(x,y)) \wedge \rho(x,y)}{t^{1/(\alpha\beta)} \vee \delta_\wedge(x,y)}\right), & p = 1-\beta, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p} \Big(1 \wedge \dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p} \Big(1 \wedge \dfrac{t^{1/(\alpha\beta)} \vee \delta_\vee(x,y)}{\rho(x,y)}\Big)^{-\alpha(2p+\beta-1)}, & p \in (\frac{1-\beta}{2}, 1-\beta), \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p} \Big(1 \wedge \dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p} \log\Big(e + \dfrac{\rho(x,y)}{t^{1/(\alpha\beta)} \vee \delta_\vee(x,y)}\Big), & p = \dfrac{1-\beta}{2}, \\ \Big(1 \wedge \dfrac{\delta_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p} \Big(1 \wedge \dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p}, & p \in (0, \dfrac{1-\beta}{2}). \end{cases} \tag{7.21}$$

(2) For all $(t,x,y) \in [1,\infty) \times D \times D$, if $\mathbf{HK_B^{h_p}}$ holds, then

$$q(t,x,y) \simeq e^{-t\phi(\lambda_D)} \delta_\wedge(x,y)^{\alpha p} \delta_\vee(x,y)^{\alpha p} = e^{-t\phi(\lambda_D)} \delta(x)^{\alpha p} \delta(x)^{\alpha p},$$

and if $\mathbf{HK_U^{h_p}}$ holds, then (i) and (ii) above hold for all $(t,x,y) \in (0,\infty) \times D \times D$.

Example 7.3. Let $D := \{x \in \mathbb{R}^d : x_d > 0\}$ be the upper half space in $\mathbb{R}^d$ and $q \in [\alpha-1, \alpha) \cap (0,\alpha)$. We recall the process Y^D from Example 3.9 (b-4), which corresponds to the Feynman-Kac semigroup of the part process Z^D, in D, of the reflected isotropic α-stable process in $\overline{D}$ via the multiplicative functional $\exp(-\int_0^t C(d,\alpha,q)(Z_s^D)_d^{-\alpha} ds)$, where the positive constants $C(d,\alpha,q)$ is defined on [25, p. 233]. It is easy to see that Y^D satisfies the scaling property and horizontal translation invariance, more precisely, for any $\lambda > 0$, the transition density p_D of Y^D satisfies

$$p_D(\lambda^\alpha, \lambda x, \lambda y) = p_D(t,x,y)\lambda^{-d}, \quad t > 0, x, y \in D$$

and

$$p_D(t, x + (\widetilde{z}, 0), y + (\widetilde{z}, 0)) = p_D(t,x,y), \quad t > 0, x, y \in D, \widetilde{z} \in \mathbb{R}^{d-1}.$$

Let $S = (S_t)_{t\ge 0}$ be a β-stable subordinator independent of the process Y, $\beta \in (0,1)$. Then the process $X_t := Y^D_{S_t}$ falls into the framework of the present paper and thus we can get sharp two-sided estimates on the jump kernel, heat kernel and Green functions of X. By using the scaling property and horizontal translation invariance of p_D above, we can show that the killing function $\kappa(x)$ of X is given by

$$\kappa(x) = Cx_d^{-\alpha\beta}, \quad x \in D$$

for some constant $C \in (0,\infty)$. One can check, by following arguments at the end of [43, Section 2], that the jump kernel of the subordinate process $X_t := Y^D_{S_t}$ satisfies assumptions in [44, (A1)-(A4)] except in the case $q = \alpha(1-\beta)/2$.

Moreover, by comparing the Green function estimates in [44, Theorem 1.1] with the Green function function estimates in Example 7.2, one can see that the value of the constant in the critical killing potential is indeed related to the power of the decay correctly. Thus, instead of computing the constant C of the killing function $\kappa(x)$, we see that the exponent p in [44] should be q (and the constant α in [44] is equal to $\alpha\beta$ in the present case) and we can use [44, Theorems 1.2 and 1.3] directly. Therefore, by checking the range of $p = q/\alpha$ in the jump kernel estimates in Example 7.2 and [44, Theorems 1.2 and 1.3], from [44, Theorems 1.2 and 1.3] we obtain the following corollary. See [44, Theorem 1.2] for the precise statement of the scale-invariant boundary Harnack principle.

Corollary 7.4. *Suppose $d > \alpha$ and Y^D is the process defined above. Let $\beta_{1/2} := \beta \vee 1/2$. If $q \in [\alpha-1,\alpha) \cap ((\alpha\beta-1)_+, \alpha\beta_{1/2})$, then the scale-invariant boundary Harnack principle is valid for Y^D. If $q \in [\alpha\beta_{1/2} \vee (\alpha-1), \alpha)$, then the non-scale-invariant boundary Harnack principle is not valid for Y^D.*

Note that constants (β_1,β_2) in [44] are $(\alpha-\alpha\beta, 0)$ for $\alpha(1-\beta) \le q < \alpha$, $(q, \alpha-q-\alpha\beta)$ for $\alpha(1-\beta)/2 < q < \alpha(1-\beta)$ and (q,q) for $0 \le q < \alpha(1-\beta)/2$. The case $q = \alpha(1-\beta)/2$ does not fit exactly in the framework of [43, 44], but by a slight modification of the boundary term, one can cover this case as well. We omit the details.

The next example illustrates that any of the four terms in (4.12) can not dominates all the other terms in general.

Example 7.5. Let $0 < \beta_1 < 1 < \beta_2$ and S be a subordinator with tail Lévy measure w satisfying

$$w(r) \simeq r^{-\beta_1} \wedge r^{-\beta_2}, \quad r > 0.$$

Let $\alpha_2 \ge \alpha_1 > 0$. Suppose that the heat kernel $p_D(t,x,y)$ of Y^D enjoys the estimate $\mathbf{HK}_{\mathbf{U}}^{\mathbf{h_{1/2}}}$ with

$$\Phi(r) = r^{\alpha_1}, \qquad \Psi(r) = r^{\alpha_1} \vee r^{\alpha_2}, \quad r > 0.$$

Examples of such Y^D can be found in [38, Theorem 1.4] where Dirichlet heat kernel estimates for a large class of subordinate Brownian motions are treated. Recall that $\psi(r) := \phi(1/\Phi(r))^{-1}$. Note that **(Poly-$\infty$)** holds, and by (2.1),

$$\phi(1/r) \simeq \begin{cases} r^{-\beta_1}, & \text{if } r \le 1 \\ r^{-1}, & \text{if } r > 1, \end{cases} \qquad \psi(r) \simeq \begin{cases} r^{\alpha_1\beta_1}, & \text{if } r \le 1 \\ r^{\alpha_1}, & \text{if } r > 1, \end{cases}$$

$$\phi^{-1}(1/r)^{-1} \simeq \begin{cases} r^{1/\beta_1}, & \text{if } r \le 1 \\ r, & \text{if } r > 1, \end{cases} \qquad \psi^{-1}(r) \simeq \begin{cases} r^{1/(\alpha_1\beta_1)}, & \text{if } r \le 1 \\ r^{1/\alpha_1}, & \text{if } r > 1. \end{cases}$$

Let $q(t,x,y)$ be the heat kernel of the subordinate process $X_t := Y^D_{S_t}$. We obtain global estimates on $q(t,x,y)$ from Theorem 4.3 and Lemma 7.1. Write the terms on the right-hand side of (4.12) respectively as

$$xA_1 = \frac{C_0 t\mathcal{B}_h(t,x,y)}{V(x,\rho(x,y))\Psi(\rho(x,y))}, \qquad A_2 = \frac{C_0\phi^{-1}(1/t)^{-1}h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\rho(x,y))\Psi(\rho(x,y))},$$

$$A_3 = \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-c\frac{\rho(x,y)^2}{\psi^{-1}(t)^2}\Big), \qquad A_4 = \frac{th(\Phi(\rho(x,y)),x,y)w(\Phi(\rho(x,y)))}{V(x,\rho(x,y))}.$$

(1) The following estimates hold for all $(t,x,y) \in (0, 1/\phi(4)] \times D \times D$.
(i) If $\rho(x,y) < t^{1/(\alpha_1\beta_1)}$, then

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta(x)}{t^{1/(\alpha_1\beta_1)}}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{t^{1/(\alpha_1\beta_1)}}\Big)^{\alpha_1/2}\frac{1}{V(x,t^{1/(\alpha_1\beta_1)})}.$$

(ii) Let $C_0 = 0$ and $\rho(x,y) \ge t^{1/(\alpha_1\beta_1)}$. It is easy to see that A_4 dominates A_3. Hence, we have that

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha_1/2} \frac{t}{V(x,\rho(x,y))[\rho(x,y)^{\alpha_1\beta_1} \vee \rho(x,y)^{\alpha_1\beta_2}]}.$$

(iii) Let $C_0 = 1$ and $t^{1/(\alpha_1\beta_1)} \le \rho(x,y) < 1$. First observe that $tw(\phi^{-1}(1/t)^{-1}) \simeq 1$. Thus it follows from (5.2), **(Poly-R_1)** and (3.10) that A_1 dominates A_2 and A_4 (hence also A_3). Further, note that for any $R_1 > 0$, w satisfies **(Poly-R_1)** with both the upper and the lower index equal to $\beta_1 \in (0,1)$. Therefore we see that (7.21) holds with $\alpha = \alpha_1$, $\beta = \beta_1$ and $p = 1/2$, after multiplying $\rho(x,y)^d V(x,\rho(x,y))^{-1}$ in each case.
(iv) Let $C_0 = 1$ and $\rho(x,y) \ge 1$. Then A_1 dominates A_2, and A_4 dominates A_3. Observe that

$$\mathcal{B}_{1/2}(t,x,y) \simeq \int_{2\phi^{-1}(1/t)^{-1}}^{1} h_{1/2}(s,x,y)s^{-\beta_1}ds + \int_1^{4\rho(x,y)^{\alpha_1}} h_{1/2}(s,x,y)s^{-\beta_2}ds$$

and for all $u > 0$,

$$\int_u^{\rho(x,y)^{\alpha_1}} h_{1/2}(s,x,y)s^{-\beta_2}ds \le h_{1/2}(u,x,y)\int_u^\infty s^{-\beta_2}ds \le \frac{u^{1-\beta_2}}{\beta_2 - 1} h_{1/2}(u,x,y). \tag{7.22}$$

Hence, if $\beta_1 > 1/2$, then

$$\begin{aligned} q(t,x,y) \simeq \frac{t}{V(x,\rho(x,y))\rho(x,y)^{\alpha_2}} \times \Bigg[&\Big(1 \wedge \frac{\delta(x)}{t^{1/(\alpha_1\beta_1)}}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{t^{1/(\alpha_1\beta_1)}}\Big)^{\alpha_1/2}(1 \wedge \mathbf{m}^t_{x,y}(\delta_\wedge))^{\alpha_1(1-\beta_1)} \\ &+ \Big(1 \wedge \frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha_1/2} \rho(x,y)^{\alpha_2 - \alpha_1\beta_2}\Bigg], \end{aligned}$$

else if $\beta_1 < 1/2$, then

$$\begin{aligned} q(t,x,y) \simeq \frac{t}{V(x,\rho(x,y))\rho(x,y)^{\alpha_2}} \times \Bigg[&\Big(1 \wedge \delta(x)\Big)^{\alpha_1/2}\Big(1 \wedge \delta(y)\Big)^{\alpha_1/2}(1 \wedge \mathbf{m}^t_{x,y}(\delta_\vee))^{-\alpha_1\beta_1} \\ &+ \Big(1 \wedge \frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha_1/2} \rho(x,y)^{\alpha_2 - \alpha_1\beta_2}\Bigg], \end{aligned}$$

otherwise if $\beta_1 = 1/2$, then

$$\begin{aligned} q(t,x,y) \simeq \frac{t}{V(x,\rho(x,y))\rho(x,y)^{\alpha_2}} \times \Bigg[&\Big(1 \wedge \delta_\wedge(x,y)\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta_\vee(x,y)}{t^{1/(\alpha_1\beta_1)}}\Big)^{\alpha_1/2} \log\Big(e + \frac{\mathbf{m}^t_{x,y}(\delta_\vee) \wedge 1}{\mathbf{m}^t_{x,y}(\delta_\wedge) \wedge 1}\Big) \\ &+ \Big(1 \wedge \frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha_1/2} \rho(x,y)^{\alpha_2 - \alpha_1\beta_2}\Bigg]. \end{aligned}$$

(2) The following estimates hold for all $(t,x,y) \in [1/\phi(4),\infty) \times D \times D$.
(i) If $\rho(x,y) < t^{1/\alpha_1}$, then

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta(x)}{t^{1/\alpha_1}}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{t^{1/\alpha_1}}\Big)^{\alpha_1/2} \frac{1}{V(x,t^{1/\alpha_1})}. \tag{7.23}$$

(ii) Let $\rho(x,y) \ge t^{1/\alpha_1}$. Then by (7.22), we see that A_2 dominates A_1 and

$$\begin{aligned} q(t,x,y) &\asymp \Big(1 \wedge \frac{\delta(x)}{t^{1/\alpha_1}}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{t^{1/\alpha_1}}\Big)^{\alpha_1/2}\Big[\frac{C_0 t}{V(x,\rho(x,y))d(x,y)^{\alpha_2}} + \frac{1}{V(x,t^{1/\alpha_1})}\exp\Big(-c\frac{\rho(x,y)^2}{t^{2/\alpha_1}}\Big)\Big] \\ &\quad + \Big(1 \wedge \frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha_1/2} \frac{t}{V(x,\rho(x,y))\rho(x,y)^{\alpha_1\beta_2}} \\ &\asymp p_D(ct,x,y) + \Big(1 \wedge \frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha_1/2}\Big(1 \wedge \frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha_1/2} \frac{t}{V(x,\rho(x,y))\rho(x,y)^{\alpha_1\beta_2}}. \end{aligned} \tag{7.24}$$

Note that, in case of $C_0 = 1$ and $\beta_2 \ge \alpha_2/\alpha_1$, we see from (7.23) and (7.24) that

$$q(t,x,y) \asymp p_D(ct,x,y), \quad t \ge 1/\phi(4), \ x,y \in D.$$

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