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REGULAR TRIANGLES IN HEXAGONAL QUASIGROUPS

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Vladimir Volenec

Abstract. The "geometrical" notions of parallelogram, midpoint and regular triangle are defined and investigated in a general hexagonal quasigroup.

A quasigroup $(Q, \cdot)$ is said to be hexagonal if it is idempotent, medial and semisymmetric, i.e. if identically

$$aa = a, \tag{1}$$

$$ab \cdot cd = ac \cdot bd, \tag{2}$$

$$ab \cdot a = a \cdot ba = b. \tag{3}$$

The first equality in (3) expresses the elasticity. In a hexagonal quasigroup we have also the left and right distributivity

$$a \cdot bc = ab \cdot ac, \tag{4}$$

$$ab \cdot c = ac \cdot bc \tag{5}$$

and the identity

$$(ab \cdot c)d = b(c \cdot da). \tag{6}$$

The identities (3) can be represented in the form of an equivalence

$$ab = c \Leftrightarrow a = bc. \tag{7}$$

Hexagonal quasigroups were studied in [5] and because of mediality we can apply all results of [4].

Example. Let C be the set of all points of an *Euclid*ean plane and $\cdot$ an operation on C such that $aa = a$ for any $a \in C$ and for any $a, b \in C$ $(a \neq b)$ let a, b, ab be successively the vertices of a positively oriented regular triangle. In [5] is proved that $(C, \cdot)$ is a hexagonal quasigroup. This quasigroup can give a motivation for the definition of "geometric" notions and the proving of "geometric" properties of a general hexagonal quasigroup. In this quasigroup $(C, \cdot)$ we can illustrate (by figures) the properties of general hexagonal quasigroups.

In the sequel let $(Q, \cdot)$ be any hexagonal quasigroup. The elements of Q are said to be points.

The points a, b, c, d are said to be the vertices of a parallelogram and we write $\mathrm{Par}\,(a,b,c,d)$ if there are two points p and q such that $ap = bq$ and $dp = cq$ [4, Cor. 1].

In [4] it was proved that $(Q, \mathrm{Par}\,)$ is a parallelogram space and all the properties of the quaternary relations Par proved in [4] hold. In [5] it was shown that the notion of parallelogram can be given by an alternative definition

$$\mathrm{Par}\,(a,b,c,d) \Leftrightarrow d = bc \cdot ab. \tag{8}$$

In Q^2 we define a binary relation $\sim$ by

$$(a,b) \sim (c,d) \Leftrightarrow \mathrm{Par}\,(a,b,d,c).$$

Then $\sim$ is relation of equivalence. The elements of the set $Q^2/_{\sim}$ are said to be vectors. A vector with the representant (a,b) is designed by $[a,b]$. If $\mathbf{v}$ is a given vector, then for any point a there is one and only one point b resp. for any point b there is one and only one point a such that $\mathbf{v} = [a,b]$.

From (7) it follows that anyone of three equalities

$$ab = c, \quad bc = a, \quad ca = b \tag{9}$$

implies the other two equalities. We say in this case that a, b, c are the vertices of a (positively oriented) regular triangle and we write $\mathrm{Tr}(a,b,c)$. The statements $\mathrm{Tr}(a,b,c)$, $\mathrm{Tr}(b,c,a)$ and $\mathrm{Tr}(c,a,b)$ are mutually equivalent.

THEOREM 1. *Anyone of the six statements* $\mathrm{Tr}(a_{i1}, a_{i2}, a_{i3})$ $(i = 1,2,3)$ *and* $\mathrm{Tr}(a_{1j}, a_{2j}, a_{3j})$ $(j = 1,2,3)$ *is a consequence of the other five equalities (Fig. 1).*

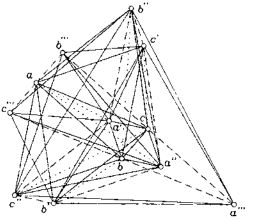

Fig. 1.

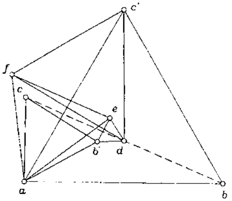

Fig. 2.

Proof. Let the first five statements be satisfied. Then we have $a_{i1}a_{i2} = a_{i3}$ $(i = 1,2,3)$ and $a_{1j}a_{2j} = a_{3j}$ $(j = 1,2)$. Therefore

$$a_{13}a_{23} = a_{11}a_{12} \cdot a_{21}a_{22} \overset{(2)}{=} a_{11}a_{21} \cdot a_{12}a_{22} = a_{31}a_{32} = a_{33},$$

i.e. we have the sixth statement. By cyclic permutations the proofs of other cases follow.

THEOREM 2. *If* $\mathrm{Tr}(b,c,a')$, $\mathrm{Tr}(c,a,b')$, $\mathrm{Tr}(c,b,a'')$, $\mathrm{Tr}(a,c,b'')$, *then there is a point* c''' *such that* $\mathrm{Tr}(b',a',c''')$, $\mathrm{Tr}(a'',b'',c''')$, $\mathrm{Par}\,(b,c,a,c''')$ *(Fig. 2).*

Proof. We have

$$b'a' = ca \cdot bc \overset{(2)}{=} cb \cdot ac = a''b''$$

and with $c''' = b'a'$ we have $c''' = a''b''$, $c''' = ca \cdot bc$.

THEOREM 3. *From* $\mathrm{Tr}(b,b',b'')$, $\mathrm{Tr}(c'',c',c)$, $\mathrm{Tr}(c',b',a)$, $\mathrm{Tr}(a,b'',c'')$ *it follows* $\mathrm{Par}\,(b',a,b'',c)$, $\mathrm{Par}\,(c'',a,c',b)$ *(Fig. 3).*

Proof. We have

$$ab'' \cdot b'a = c''c' = c,$$

$$ac' \cdot c''a = b'b'' = b.$$

In the case of the quasigroup $(C,\cdot)$ the statements $\mathrm{Par}\,(b',a,b'',c)$ and $\mathrm{Par}\,(c'',a,c',b)$ means that the segments $\{b',b''\}$, $\{a,c\}$ resp. $\{c'',c'\}$, $\{a,b\}$ have the same midpoint. Therefore the triangles (b,b',b''), (c'',c',c) (a,b,c) have the same centroid. But, if the given triangle (a,b,c) and the regular triangles (b,b',b''), (c'',c',c) have the same centroid, then these regular triangles are uniquely determined by this condition. Therefore we have (in the case of the quasigroup $(C,\cdot)$) the following statement proved in [1]:

If (b,b',b''), (c,c',c'') are two oppositely oriented regular triangles with the same centroid as the given triangle (a,b,c), then (a,b'',c''), (a,b',c') are oppositely oriented regular triangles.

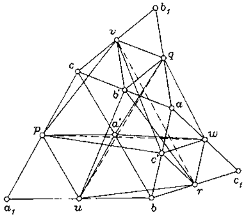

Fig. 3.

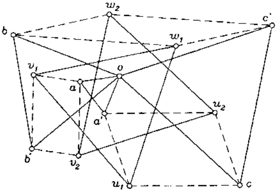

Fig. 4.

We say that the vectors $\mathbf{a} = [a,b]$, $\mathbf{c} = [c,d]$, $\mathbf{e} = [e,f]$ define a regular triangle and write $\mathrm{Tr}(\mathbf{a},\mathbf{c},\mathbf{e})$ if there are three points u, v, w such that $\mathrm{Tr}(u,v,w)$ and $\mathbf{a} = [v,w]$, $\mathbf{c} = [w,u]$, $\mathbf{e} = [u,v]$, i.e. $\mathrm{Par}\,(a,b,w,v)$, $\mathrm{Par}\,(c,d,u,w)$, $\mathrm{Par}\,(e,f,v,u)$ (Fig. 4).

THEOREM 4. *The statement* $\mathrm{Tr}([a,b],[c,d],[e,f])$ *holds iff any two (and then all three) of three equalities*

$$da = cb, \quad fc = ed, \quad be = af \tag{10}$$

are valid (Fig. 4).

COROLLARY 1. *From* $\mathrm{Tr}([a,b],[c,d],[e,f])$ *it follows* $\mathrm{Tr}([b,a],[d,c],[f,e])$.

For the proof of Theorem 4 we need a lemma.

LEMMA. *Let* $[a,b] = [v,w]$, $[c,d] = [w,u]$. *The statement* $\mathrm{Tr}(u,v,w)$ *is equivalent to the equality* $da = cb$ *(Fig. 4).*

Proof. We have $\mathrm{Par}\,(d,c,w,u)$, $\mathrm{Par}\,(a,b,w,v)$ and by [4, Th. 25] it follows $\mathrm{Par}\,(da,cb,w,uv)$ because of (1). Therefore by [4, Th. 23] and [4, Th. 20] equality $da = cb$ is equivalent to equality $uv = w$, i.e. to the statement $\mathrm{Tr}(u,v,w)$.

Proof of Theorem 4. If we have $\mathrm{Tr}(u,v,w)$ and $[a,b] = [v,w]$, $[c,d] = [w,u]$, $[e,f] = [u,v]$ (Fig. 4) then by Lemma the equalities (10) follow. Conversely, let two first equalities (10) hold. Let v be any point and let w and u be two points such that $[a,b] = [v,w]$, $[c,d] = [w,u]$ (Fig. 4). Then by Lemma we have $\mathrm{Tr}(u,v,w)$. Let v' be the point such that $[e,f] = [u,v']$. The equalities $[c,d] = [w,u]$, $[e,f] = [u,v']$, $fc = ed$ imply by Lemma the statement $\mathrm{Tr}(w,u,v')$. Therefore, we have $v' = wu = v$ and finally $[e,f] = [u,v]$.

A consequence of this proof is that any two of three equalities (10) imply the remaining equality (10). But, this fact can be proved directly. Indeed, from $(10)_2$ we obtain by (7) $f = c \cdot ed$ and then (3), $(10)_1$ and (6) imply

$$be = (cb \cdot c)e = (da \cdot c)e = a(c \cdot ed) = af,$$

i.e. $(10)_1$ and $(10)_2$ imply $(10)_3$.

THEOREM 5. *By the hypothesis of Theorem 1 we have* $\mathrm{Tr}([a_{12},a_{21}],[a_{23},a_{32}],[a_{31},a_{13}])$ *(Fig. 1).*

Proof. We have

$$a_{32}a_{12} = a_{22} = a_{23}a_{21},$$

$$a_{21}a_{31} = a_{11} = a_{12}a_{13}$$

and the statement follows by Theorem 4.

THEOREM 6. *By the hypothesis of Theorem 2 we have* $\mathrm{Tr}([a,b],[a',b''],[a'',b'])$ *(Fig. 2).*

Proof. We have

$$b''a = c = a'b, \quad ba'' = c = ab'$$

and the statement follows by Theorem 4.

THEOREM 7. $\mathrm{Tr}([p,a],[p,b],[p,c])$ *implies* $\mathrm{Tr}(a,b,c)$.

Proof. By Theorem 4 we have $cp = pb$, $ap = pc$ and therefore successively

$$ab \overset{(3)}{=} (p \cdot ap)(pb \cdot p) = (p \cdot pc)(cp \cdot p) \overset{(2)}{=} (p \cdot cp)(pc \cdot p) \overset{(3)}{=} cc \overset{(1)}{=} c.$$

THEOREM 8. *From* $\mathrm{Tr}(b,c,a')$, $\mathrm{Tr}(c,a,b')$, $\mathrm{Tr}(a,b,c')$, $\mathrm{Tr}(c,b,a'')$, $\mathrm{Tr}(a,c,b'')$, $\mathrm{Tr}(b,a,c'')$ *it follows* $\mathrm{Tr}([c,c'],[b,b'],[a,a'])$, $\mathrm{Tr}([a,a''],[b,b''],[c,c''])$, $\mathrm{Par}\,(c',a,b',a'')$, $\mathrm{Par}\,(a',b,c',b'')$, $\mathrm{Par}\,(b',c,a',c'')$, $\mathrm{Par}\,(b'',a,c'',a')$, $\mathrm{Par}\,(c'',b,a'',b')$, $\mathrm{Par}\,(a'',c,b'',c')$, $\mathrm{Tr}([a,b],,[a',b''],[a'',b'])$, $\mathrm{Tr}([b,c],[b',c''],[b'',c'])$, $\mathrm{Tr}([c,a],[c',a''],[c'',a'])$. *There exist points* a''', b''', c''' *such that* $\mathrm{Tr}(c',b',a''')$, $\mathrm{Tr}(a',c',b''')$, $\mathrm{Tr}(b',a',c''')$, $\mathrm{Tr}(b'',c'',a''')$, $\mathrm{Tr}(c'',a'',b''')$, $\mathrm{Tr}(a'',b'',c''')$, $\mathrm{Par}\,(c,a,b,a''')$, $\mathrm{Par}\,(a,b,c,b''')$, $\mathrm{Par}\,(b,c,a,c''')$, $\mathrm{Par}\,(a'',a,a',a''')$, $\mathrm{Par}\,(b'',b,b',b''')$, $\mathrm{Par}\,(c'',c,c',c''')$ *(Fig. 5).*

Proof. The first three hypotheses imply

$$b'c = a = bc', \quad a'b = c = ab'$$

and by Theorem 4 we have $\mathrm{Tr}([c,c'],[b,b'],[a,a'])$. Analogously, the following three hypotheses imply $\mathrm{Tr}([a,a''],[b,b''],[c,c''])$. By Theorem 6 we obtain $\mathrm{Tr}([a,b],[a',b''],[a'',b'])$ and by Theorem 2 there is a point c''' such that $\mathrm{Tr}(b',a',c''')$, $\mathrm{Tr}(a'',b'',c''')$, $\mathrm{Par}\,(b,c,a,c''')$ Further, we have

$$ca' \cdot b'c = ba = c'', \quad cb'' \cdot a''c = ab = c',$$

$$cc' \cdot c''c = (ab' \cdot ab)(ba \cdot a'b) \overset{(4)}{=} (a \cdot b'b)(ba \cdot a'b) \overset{(2)}{=}$$

$$= (a \cdot ba)(b'b \cdot a'b) \overset{(3)}{=} b(b'b \cdot a'b) \overset{(5)}{=} b(b'a' \cdot b) \overset{(3)}{=} b'a' = c'''$$

and by (8) the statements $\mathrm{Par}\,(b',c,a',c''')$, $\mathrm{Par}\,(a'',c,b'',c')$, $\mathrm{Par}\,(c'',c,c',c''')$ follow. Analogously, the remaining statements can be proved.

A pairs of points is said to be a segment. We say that b is a midpoint of the segment $\{a,c\}$ and we write $M(a,b,c)$ or $M(c,b,a)$ if $\mathrm{Par}\,(a,b,c)$ holds.

For any a obviously $M(a,a,a)$ holds. In the quasigroup $(C,\cdot)$ any segment has the unique midpoint, but this is not generally true.

THEOREM 9. *By the hypothesis of Theorem 8 we have* $M(b''',a,'c''')$, $M(c''',b,a''')$, $M(a''',c,b''')$ *(Fig. 5).*

Proof. It follows from $\mathrm{Par}\,(c,a,b,a''')$, $\mathrm{Par}\,(a,b,c,b''')$, $\mathrm{Par}\,(b,c,a,c''')$ by [4, Th. 41].

THEOREM 10. *From* $M(b,d,c)$, $\mathrm{Tr}(c,a,b')$, $\mathrm{Tr}(a,b,c')$, $\mathrm{Tr}(b',d,e)$, $\mathrm{Tr}(d,c',f)$ *it follows* $\mathrm{Tr}(a,e,f)$ *(Fig. 6).*

Proof. We have $c = bd \cdot db$, $b' = ca$, $c' = ab$, $e = b'd$, $f = dc'$ and therefore

$$ef = b'd \cdot dc' = (ca \cdot d)(d \cdot ab) = [(bd \cdot db)a \cdot d](d \cdot ab) \overset{(6)}{=}$$

$$= [db \cdot a(d \cdot bd)](d \cdot ab) \overset{(3)}{=} (db \cdot ab)(d \cdot ab) \overset{(5)}{=} (db \cdot d) \cdot ab \overset{(3)}{=} b \cdot ab \overset{(3)}{=} a.$$

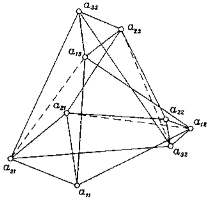

Fig. 5.

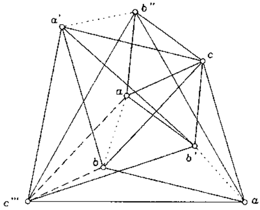

Fig. 6.

THEOREM 11. $\mathrm{Tr}(a_1, b_1, c_1)$, $\mathrm{Tr}(a_2, b_2, c_2)$, $\mathrm{Tr}(a, b, c)$, $M(a_1, a, a_2)$, $M(b_1, b, b_2)$ *imply* $M(c_1, c, c_2)$.

Proof. From $M(a_1, a, a_2)$ and $M(b_1, b, b_2)$ it follows $M(a_1 b_1, ab, a_2 b_2)$ by [4, Th. 23], i.e. we have $M(c_1, c, c_2)$ because of $a_1 b_1 = c_1$, $ab = c$, $a_2 b_2 = c_2$.

Using Theorem 11 the statements $\mathrm{Tr}(b, c, a)$, $\mathrm{Tr}(c, a, b)$, $\mathrm{Tr}(d, e, f)$, $M(b, d, c)$, $M(c, e, a)$ imply:

COROLLARY 2. $\mathrm{Tr}(a, b, c)$, $\mathrm{Tr}(d, e, f)$, $M(b, d, c)$, $M(c, e, a)$ *imply* $M(a, f, b)$.

THEOREM 12. *If* $\mathrm{Tr}(b, c, a_1)$, $\mathrm{Tr}(c, a, b_2)$, $\mathrm{Tr}(a, b, c_3)$, $M(b, a', c)$, $M(c, b', a)$, $M(a, c', b)$, $M(c, p, a_1)$ *and if* u, v, q, r, w *are points such that* $\mathrm{Tr}(a', p, u)$, $\mathrm{Tr}(p, c', v)$, $\mathrm{Tr}(v, b', q)$, $\mathrm{Tr}(b', u, r)$, $\mathrm{Tr}(q, a', w)$, *then* $\mathrm{Tr}(c', r, w)$, $M(a, q, b_1)$, $M(b, r, c_1)$, $M(b, u, a_1)$, $M(c, v, b_1)$, $M(a, w, c_1)$ *and* $\mathrm{Tr}([r, v], [q, u], [p, w])$ *(Fig. 7)*.

Proof. We have $pu = a'$, $vp = c'$, $b'u = r$, $vb' = q$, $qa' = w$ and it follows

$$c'r = vp \cdot b'u \overset{(2)}{=} vb' \cdot pu = qa' = w,$$

i.e. $\mathrm{Tr}(c', r, w)$. From $\mathrm{Tr}(a_1, b, c)$, $\mathrm{Tr}(a', p, u)$, $M(b, a', c)$, $M(c, p, a_1)$ it follows by Corollary 2 $M(a_1, u, b)$. By Theorem 11 $\mathrm{Tr}(c, a, b_1)$, $\mathrm{Tr}(a_1, b, c)$, $\mathrm{Tr}(p, c', v)$, $M(c, p, a_1)$, $M(a, c', b)$ imply $M(b_1, v, c)$ and $\mathrm{Tr}(c, a_1, b)$, $\mathrm{Tr}(a, b, c_1)$, $\mathrm{Tr}(b', u, r)$, $M(c, b', a)$, $M(a_1, u, b)$ imply $M(b, r, c_1)$. By Corollary 2 $\mathrm{Tr}(c_1, a, b)$, $\mathrm{Tr}(c', r, w)$, $M(a, c', b)$, $M(b, r, c_1)$ imply $M(c_1, w, a)$ and $\mathrm{Tr}(a, b_1, c)$, $\mathrm{Tr}(v, b', q)$, $M(b_1, v, c)$, $M(c, b', a)$ imply $M(a, q, b_1)$. The equalities $ur = b'$, $qv = b'$, $wq = a'$, $pu = a'$ imply $ur = qv$, $wq = pu$ and by Theorem 4 we obtain $\mathrm{Tr}([r, v], [q, u], [p, w])$.

THEOREM 13. *If* o, a, u_1, u_2, v_1, v_2 *are any points,* w_1, w_2, a' *the points such that* $\mathrm{Tr}(v_1, u_1, w_1)$, $\mathrm{Tr}(v_2, u_2, w_2)$, $\mathrm{Tr}(o, a, a')$, *further* b', c *points such that* $\mathrm{Par}\,(v_1, a, v_2, b')$, $\mathrm{Par}\,(u_1, a', u_2, c)$ *and* b, c' *points such that* $\mathrm{Tr}(b', o, b)$, $\mathrm{Tr}(o, c, c')$, *then* $\mathrm{Par}\,(w_1, b, w_2, c')$ *(Fig. 8)*.

Proof. We have the equalities

$$v_1 u_1 = w_1, \quad v_2 u_2 = w_2, \tag{11}$$

$$a = a'o, \quad b = b'o, \tag{12}$$

$$oc = c'. \tag{13}$$

From $\mathrm{Par}(u_1, c, u_2, a')$, $\mathrm{Par}(v_1, a, v_2, b')$ it follows $cu_2 \cdot u_1c = a'$, $av_2 \cdot v_1a = b'$, i.e. by (12)

$$a = (cu_2 \cdot u_1c)o, \tag{14}$$

$$b = (av_2 \cdot v_1a)o. \tag{15}$$

Now, we obtain successively

$$b_1w_1 \cdot w_2b \overset{(11)}{=} (b \cdot v_1u_1)(v_2u_2 \cdot b) \overset{(15)}{=}$$

$$= [(av_2 \cdot v_1a)o \cdot v_1u_1][v_2u_2 \cdot (av_2 \cdot v_1a)o] \overset{(2)}{=}$$

$$= [(av_2 \cdot v_1a)v_1 \cdot ou_1][v_2(av_2 \cdot v_1a) \cdot u_2o] \overset{(6)}{=}$$

$$= [v_2(v_1a \cdot v_1a) \cdot ou_1][(av_2 \cdot av_2)v_1 \cdot u_2o] \overset{(1)}{=}$$

$$= (v_2 \cdot v_1a)(ou_1) \cdot (av_2 \cdot v_1)(u_2o) \overset{(2)}{=} (v_2 \cdot v_1a)(av_2 \cdot v_1) \cdot (ou_1 \cdot u_2o) \overset{(2)}{=}$$

$$= (v_2 \cdot av_2)(v_1a \cdot v_1) \cdot (ou_1 \cdot u_2o) \overset{(3)}{=} aa \cdot (ou_1 \cdot u_2o) \overset{(1)}{=} a(ou_1 \cdot u_2o) \overset{(14)}{=}$$

$$= (cu_2 \cdot u_1c)o \cdot (ou_1 \cdot u_2o) \overset{(2)}{=} (cu_2 \cdot u_1c)(ou_1) \cdot (o \cdot u_2o) \overset{(3)}{=}$$

$$= (cu_2 \cdot u_1c)(ou_1) \cdot u_2 \overset{(2)}{=} (cu_2 \cdot o)(u_1c \cdot u_1) \cdot u_2 \overset{(3)}{=}$$

$$= (cu_2 \cdot o)c \cdot u_2 \overset{(6)}{=} u_2(o \cdot cc) \cdot u_2 \overset{(3)}{=} o \cdot cc \overset{(1)}{=} oc \overset{(13)}{=} c',$$

i.e. $\mathrm{Par}(w_2, b, w_1, c')$.

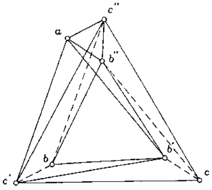

Fig. 7.

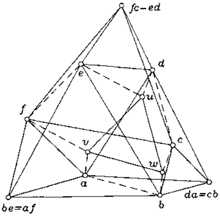

Fig. 8.

If $u_1 = u_2$, $v_1 = v_2$ then (11) implies $w_1 = w_2$ and we get:

COROLLARY 3. *If o, a, u, v are any points, w, a' the points such that* $\mathrm{Tr}(v, u, w)$, $\mathrm{Tr}(o, a, a')$, *further b', c points such that $M(a, v, b')$, $M(a', u, c)$ and b, c' points such that* $\mathrm{Tr}(b', o, b)$, $\mathrm{Tr}(o, c, c')$ *then $M(b, w, c')$ (Fig. 9).*

If we substitute the points u, v, w, b, c, b', c' by the points y, x, z, c, b, c', b' we have:

COROLLARY 4. *If o, a, x, y are any points, z, a' the points such that* $\mathrm{Tr}(x, y, z)$, $\mathrm{Tr}(o, a, a')$, *further c', b points such that $M(a, x, c')$, $M(a', y, b)$ and c, b' points such that* $\mathrm{Tr}(c', o, c)$, $\mathrm{Tr}(o, b, b')$ *then $M(c, z, b')$ (Fig. 9).*

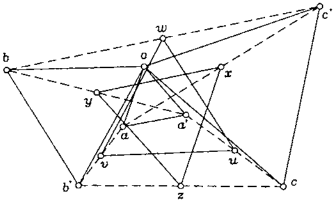

Fig. 9.

In the case of the quasigroup $(C, \cdot)$ the Corollary 4 can be stated as a result from [2] (with our symbolism):

If $\mathrm{Tr}(o, a, a')$, $\mathrm{Tr}(o, b, b')$, $\mathrm{Tr}(o, c, c')$, $M(a, x, c')$, $M(b, y, a')$, $M(c, z, a')$ then $\mathrm{Tr}(x, y, z)$ (Fig. 9).

The Corollaries 3 and 4 prove in the quasigroup $(C, \cdot)$ a result of [3]:

A rotation about the point o through the angle $\frac{\pi}{3}$ maps the given points a, b, c into the points a', b', c'. If $M(a, x, c')$, $M(b, y, a')$, $M(c, z, b')$, $M(c, u, a')$, $M(a, v, b')$, $M(b, w, c')$ then $\mathrm{Tr}(x, y, z)$, $\mathrm{Tr}(w, v, u)$ (Fig. 9).

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Pravilni trokuti u šesterokutnim kvazigrupama

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Sadržaj

Kvazigrupa $(Q, \cdot)$ je šesterokutna ako u njoj vrijede identiteti $aa = a$, $ab \cdot cd = ac \cdot bd$, $ab \cdot a = a \cdot ba = b$. Elemente skupa Q zovemo točkama. Točke a, b, c, d tvore paralelogram i pišemo $\mathrm{Par}(a, b, c, d)$, ako vrijedi $d = bc \cdot ab$. Tada je struktura (Q, Par) paralelogramski prostor. Kaže se da je c polovište para točaka $\{a, b\}$ ako vrijedi $\mathrm{Par}(a, c, b, c)$. Na uobičajeni način može se uvesti i pojam vektora. Točke a, b, c su vrhovi pravilnog trokuta ako vrijede jednakosti $ab = c$, $bc = a$, $ca = b$, koje su inače međusobno ekvivalentne. Kažemo da vektori $[a, b]$, $[c, d]$, $[e, f]$ određuju pravilan trokut ako postoji pravilan trokut s vrhovima u, v, w tako da vrijedi $\mathrm{Par}(a, b, w, v)$, $\mathrm{Par}(c, d, u, w)$, $\mathrm{Par}(e, f, v, u)$. Nužan i dovoljan uvjet za to su jednakosti $da = cb$, $fc = ed$, $bc = af$, od kojih je svaka posljedica ostalih dviju. U radu se dokazuje niz tvrdnji o odnosima definiranih geometrijskih pojmova u bilo kojoj šesterokutnoj kvazigrupi.

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