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GEOMETRY OF MEDIAL QUASIGROUPS

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Abstract. Introduced will be the »geometrical« notions of transfer, parallelogram and midpoint in a medial quasigroup, and then a »geometrical« proof of the representation theorem of medial quasigroups will be given.

1. INTRODUCTION

A quasigroup $(Q, \cdot)$ is said to be *medial* iff for every $a, b, c, d \in Q$ it holds the equality

$$ab \cdot cd = ac \cdot bd. \tag{1}$$

For any $a \in Q$ there can be defined a *left translation* λ_a and a *right translation* ϱ_a of any quasigroup $(Q, \cdot)$ as the bijections of the set Q defined by $\lambda_a : x \mapsto ax$ and $\varrho_a : x \mapsto xa$.

The medial quasigroups can be obtained from the commutative groups because it holds:

THEOREM 1. *Let $(Q, +)$ be a commutative group with the neutral element 0, further φ and ψ two automorphisms of $(Q, +)$ such that $\varphi \circ \psi = \psi \circ \varphi$ and $h \in Q$. If the operation $\cdot$ is defined on the set Q by*

$$ab = \varphi(a) + \psi(b) + h, \tag{2}$$

then $(Q, \cdot)$ is a medial quasigroup and it holds $\varphi = \varrho_{r_0}, \psi = \lambda_{l_0}, h = 00$, where l_0 and r_0 are a left resp. right unit of the element 0 in the quasigroup $(Q, \cdot)$, i. e. it holds the equalities $l_0 0 = 0$, $0 r_0 = 0$.

Proof. For any $a, b, c, d \in Q$ we have successively

$$ab \cdot cd = \varphi[\varphi(a) + \psi(b) + h] + \psi[\varphi(c) + \psi(d) + h] + h =$$
$$= (\varphi \circ \varphi)(a) + (\varphi \circ \psi)(b) + \varphi(h) + (\psi \circ \varphi)(c) + (\psi \circ \psi)(d) + \psi(h) + h =$$
$$= (\varphi \circ \varphi)(a) + (\varphi \circ \psi)(c) + \varphi(h) + (\psi \circ \varphi)(b) + (\psi \circ \psi)(d) + \psi(h) + h =$$
$$= \varphi[\varphi(a) + \psi(c) + h] + \psi[\varphi(b) + \psi(d) + h] + h = ac \cdot bd$$

and it holds the identity (1). Because of (2) the equation $ax = b$ resp. $ya = b$ has the unique solution $x = \psi^{-1}[b - \varphi(a) - h]$ resp. $y = \varphi^{-1}[b - \psi(a) - h]$

and $(Q, \cdot)$ is a quasigroup. From (2) we obtain

$$00 = \varphi(0) + \psi(0) + h = h$$

and then we have

$$0 = l_0 0 = \varphi(l_0) + \psi(0) + h = \varphi(l_0) + h,$$

$$\lambda_{l_0}(x) = l_0 x = \varphi(l_0) + \psi(x) + h,$$

wherefrom for any $x \in Q$ it follows $\lambda_{l_0}(x) = \psi(x)$, i. e. it holds $\psi = \lambda_{l_0}$. Analogously, the equality $\varphi = \varrho_{r_0}$ can be proved.

Later we shall prove that any medial quasigroups can be obtained from a commutative group as in Theorem 1.

2. TRANSFERS

Further, let $(Q, \cdot)$ be any medial quasigroup.

For the sake of simplicity the elements of the considered set Q are said to be *points*.

For any points a, b the bijections $\lambda_{a,b} = \lambda_b^{-1} \circ \lambda_a$ and $\varrho_{a,b} = \varrho_b^{-1} \circ \varrho_a$ are said to be a *left transfer* resp. a *right transfer*. From these definitions it follows at once the equivalences such that:

$$\lambda_{a,b}(p) = q \Leftrightarrow ap = bq, \tag{3}$$

$$\varrho_{a,b}(p) = q \Leftrightarrow pa = qb. \tag{4}$$

The equation $ap = bq$ resp. $pa = qb$ has the unique solution a or b if the remaining three elements are given. Therefore, we have:

THEOREM 2. *If the points a, p, q are given, then there is the unique point b, and if the points b, p, q are given, then there is the unique point a, such that it holds the equality $\lambda_{a,b}(p) = q$ resp. $\varrho_{a,b}(p) = q$.*

Let us prove some more statements about the transfers (left and right).

THEOREM 3. *Let $\lambda_{a,b} = \lambda_{c,d}$ or $\varrho_{a,b} = \varrho_{c,d}$. Then the equalities $a = c$ and $b = d$ are equivalent.*

Proof. Let $\lambda_{a,b} = \lambda_{c,d}$, let p be a given point and further let $q = \lambda_{a,b}(p) = \lambda_{c,d}(p)$. Because of (3) we have now the equalities $ap = bq$, $cp = dq$. If $a = c$, then we have $bq = dq$, wherefrom it follows $b = d$, and if $b = d$, then we have $ap = cp$, wherefrom it follows $a = c$. We can repeat the similar conclusions if $\varrho_{a,b} = \varrho_{c,d}$.

THEOREM 4. *If for a point p it holds $\lambda_{a,b}(p) = \varrho_{c,d}(p)$, then $\lambda_{a,b} = \varrho_{c,b}$.*

Proof. Let $q = \lambda_{a,b}(p) = \varrho_{c,d}(p)$. If x is any point, let $y = \lambda_{a,b}(x)$. By (3) and (4) we have $ap = bq$, $pc = qd$, $ax = by$. Because of (1) from the second and third equalities we have successively

$$ap \cdot xc = ax \cdot pc = by \cdot qd = bq \cdot yd,$$

wherefrom by the first equality it follows $xc = yd$. By (4) we obtain $\varrho_{c,d}(x) = y$ and therefore we have $\lambda_{a,b}(x) = y = \varrho_{c,d}(x)$, i. e. $\lambda_{a,b} = \varrho_{c,d}$.

THEOREM 5. *For any three of the points a, b, c, d there is one and only one fourth point such that $\lambda_{a,b} = \varrho_{c,d}$.*

Proof. Let a, b be given points. Let p be any point and $q = \lambda_{a,b}(p)$. By Theorem 2 there is a point d if the point c is given, resp. there is a point c if the point d is given, such that in both cases it holds the equality $\varrho_{c,d}(p) = q$. Therefore, we have $\lambda_{a,b}(p) = \varrho_{c,d}(p)$, i. e. by Theorem 4 it holds $\lambda_{a,b} = \varrho_{c,d}$. Now, if $\lambda_{a,b} = \varrho_{c,d'}$ or $\lambda_{a,b} = \varrho_{c',d}$ then we have $\varrho_{c,d'} = \varrho_{c,d}$ resp. $\varrho_{c',d} = \varrho_{c,d}$, wherefrom by Theorem 3 it follows $d' = d$ resp. $c' = c$. We have similar conclusions if the points a, c, d or b, c, d are given.

THEOREM 6. *If for a point p it holds $\lambda_{a,b}(p) = \lambda_{c,d}(p)$, then $\lambda_{a,b} = \lambda_{c,d}$.*

Proof. Let $q = \lambda_{a,b}(p) = \lambda_{c,d}(p)$ and let e be a given point. Because of Theorem 1 there is a point f such that $\varrho_{e,f}(p) = q$. From $\lambda_{a,b}(p) = \varrho_{e,f}(p) = \lambda_{c,d}(p)$ by Theorem 4 it follows $\lambda_{a,b} = \varrho_{e,f} = \lambda_{c,d}$.

Similarly, it can be proved

THEOREM 7. *If for a point p it holds $\varrho_{a,b}(p) = \varrho_{c,d}(p)$, then $\varrho_{a,b} = \varrho_{c,d}$.*

We have proven Theorem 5 by Theorems 2, 4 and 3, and by Theorems 2, 6 and 3 and a similar procedure we can prove as follows:

THEOREM 8. *For any three of the points a, b, c, d there is one and only one fourth point such that $\lambda_{a,b} = \lambda_{c,d}$.*

Analogously, we obtain

THEOREM 9. *For any three of the points a, b, c, d there is one and only one fourth point such that $\varrho_{a,b} = \varrho_{c,d}$.*

Further, we have:

THEOREM 10. *The equalities $\lambda_{a,b}(c) = d$ and $\varrho_{c,d}(a) = b$ are equivalent.*

Proof. Because of (3) and (4) both equalities are equivalent with the same equality $ac = bd$.

THEOREM 11. *The equalities $\lambda_{a,b} = \lambda_{c,d}$ and $\varrho_{a,b} = \varrho_{c,d}$ are equivalent.*

Proof. Suppose that $\lambda_{a,b} = \lambda_{c,d}$. Let p, r be any points and $q = \lambda_{a,b}(p) = \lambda_{c,d}(p)$, $s = \varrho_{a,b}(r)$. From these equalities by Theorem 10 we obtain the equalities $\varrho_{p,q}(a) = b$, $\varrho_{p,q}(c) = d$, $\lambda_{r,s}(a) = b$. The first and third of these equalities imply by Theorem 4 $\varrho_{p,q} = \lambda_{r,s}$ and hence from the second equality it follows $\lambda_{r,s}(c) = d$. Therefore, by Theorem 10 we obtain $\varrho_{c,d}(r) = s$, which together with $\varrho_{a,b}(r) = s$ owing to Theorem 7 gives finally $\varrho_{a,b} = \varrho_{c,d}$. The converse can be proved analogously.

THEOREM 12. *The equalities $\lambda_{a,b} = \lambda_{c,d}$, $\varrho_{a,b} = \varrho_{c,d}$, $\lambda_{a,c} = \lambda_{b,d}$, $\varrho_{a,c} = \varrho_{b,d}$ are mutually equivalent.*

Proof. Because of Theorem 11 it is sufficient to prove the equivalence of the first and the fourth equalities. Suppose that it holds the first equality. Let p, q be given points and let $q = \lambda_{a,b}(p) = \lambda_{c,d}(p)$, $s = \varrho_{a,c}(r)$. From these equalities by (3) and (4) we get $ap = bq$, $cp = dq$, $ra = sc$ and together with (1) we obtain successively

$$rb \cdot cq = rc \cdot bq = rc \cdot ap = ra \cdot cp = sc \cdot dq = sd \cdot cq.$$

Therefore, it follows $rb = sd$, i. e. because of (4) $\varrho_{b,d}(r) = s$. Together with $\varrho_{a,c}(r) = s$ this equality implies finally, by Theorem 7, $\varrho_{a,c} = \varrho_{b,d}$. Analogously, the converse can be proved.

Owing to Theorem 5 any left transfer is also a right transfer and any right transfer is a left transfer, too. Therefore, we shall call both left transfers and right transfers simply *transfers*.

THEOREM 13. *If T is the set of all transfers, then $(T, \circ)$ is a commutative group, which acts sharply transitively on the set Q.*

Proof. Let τ_1, τ_2 be any transfers and a any given point. By Theorem 5 or Theorem 8 there are the points b, c such that $\tau_1 = \lambda_{a,b}$, $\tau_2 = \lambda_{a,c}$ and then a point d such that $\tau_1 = \lambda_{c,d}$. Therefore $\lambda_{a,b} = \lambda_{c,d}$, wherefrom by Theorem 12 it follows $\lambda_{a,c} = \lambda_{b,d}$ and hence $\tau_2 = \lambda_{b,d}$. Now, we have

$$\tau_2 \circ \tau_1 = \lambda_{b,d} \circ \lambda_{a,b} = \lambda_d^{-1} \circ \lambda_b \circ \lambda_b^{-1} \circ \lambda_a = \lambda_d^{-1} \circ \lambda_a = \lambda_{a,d}$$

and hence $\tau_2 \circ \tau_1$ is a transfer. On the other hand, we have similarly

$$\tau_1 \circ \tau_2 = \lambda_{c,d} \circ \lambda_{a,c} = \lambda_{a,d} = \tau_2 \circ \tau_1.$$

For any point a, $\lambda_{a,a}$ is the identity. For any points a, b the inverse of the transfer $\lambda_{a,b}$ is the transfer $\lambda_{b,a}$ because of $\lambda_{b,a} \circ \lambda_{a,b} = \lambda_{a,a}$. For any points p, q there is one and only one transfer τ such that $\tau(p) = q$. Indeed, owing to Thereom 2 for a given point a there is one and only one point b such that $\lambda_{a,b}(p) = q$ and if τ is any transfer such that $\tau(p) = q$, then by Theorems 4 and 6 it follows $\tau = \lambda_{a,b}$.

From the above proof and by analogy it holds:

THEOREM 14. *For any points a, b, c it holds the equalities*

$$\lambda_{a,b} \circ \lambda_{b,c} = \lambda_{b,c} \circ \lambda_{a,b} = \lambda_{a,c}, \qquad \varrho_{a,b} \circ \varrho_{b,c} = \varrho_{b,c} \circ \varrho_{a,b} = \varrho_{a,c}.$$

THEOREM 15. *From any two of three equalities*

$$\lambda_{a,c}(p) = q, \ \lambda_{b,d}(r) = s, \ \lambda_{ab,cd}(pr) = qs$$

the remaining equality follows. It holds also a similar statement for right transfers instead of left transfers.

Proof. By (3) our equalities are equivalent respectively with the equalities

$$ap = cq, \ br = ds, \ ab \cdot pr = cd \cdot qs.$$

Because of (1), the last equality is equivalent to $ap \cdot br = cq \cdot ds$ and the statement is now obvious.

THEOREM 16. *From any two of three equalities*

$$\lambda_{a,c} = \lambda_{e,g}, \tag{5}$$

$$\lambda_{b,d} = \lambda_{f,h}, \tag{6}$$

$$\lambda_{ab,cd} = \lambda_{ef,gh} \tag{7}$$

the remaining equality follows. It holds the analogous statement for right transfers.

Proof. Let the equalities (5) and (6) hold true and let p, r be two given points. Put $q = \lambda_{a,c}(p) = \lambda_{e,g}(p)$, $s = \lambda_{b,d}(r) = \lambda_{f,h}(r)$. From the equalities $\lambda_{a,c}(p) =$

$= q$, $\lambda_{b,d}(r) = s$ resp. $\lambda_{e,g}(p) = q$, $\lambda_{f,h}(r) = s$ by Theorem 15 we obtain $\lambda_{ab,cd}(pr) = qs$ resp. $\lambda_{ef,gh}(pr) = qs$. But, the equality $\lambda_{ab,cd}(pr) = \lambda_{ef,gh}(pr)$ implies by Theorem 6 the equality (7). Now, let the equalities (5) and (7) hold true and let p, u be given points. Put $q = \lambda_{a,c}(p) = \lambda_{e,g}(p)$, $v = \lambda_{ab,cd}(u) = \lambda_{ef,gh}(u)$. There are the points r, s such that $pr = u$, $qs = v$. From the equalities $\lambda_{a,c}(p) = q$, $\lambda_{ab,cd}(pr) = qs$ resp. $\lambda_{e,g}(p) = q$, $\lambda_{ef,gh}(pr) = qs$ by Theorem 15 it follows $\lambda_{b,d}(r) = s$ resp. $\lambda_{f,h}(r) = s$. But, the equality $\lambda_{b,d}(r) = \lambda_{f,h}(r)$ implies by Theorem 6 the equality (6). Analogously, the implication (6) & (7) $\Rightarrow$ (5) can be proved.

THEOREM 17. *From any four of five equalities*

$$\lambda_{p,q}(a) = d,\ \lambda_{r,s}(b) = e,\ \lambda_{pr,qs}(c) = f,\ ab = c,\ de = f \tag{8}$$

the remaining equality follows. An analogous statement holds for right transfers.

Proof: Consider the first two equalities (8) and the equality

$$\lambda_{pr,qs}(ab) = de. \tag{9}$$

Any two of these three equalities imply the remaining equality on the basis of Theorem 15. But, if we consider the last three equalities (8) and the equality (9), then it is obvious that from any three of these four equalities the remaining equality follows. Therefore, the statement of our theorem can be easily proven in any of five possible cases.

3. PARALLELOGRAMS

Let us say that the points a, b, c, d form a *parallelogram* and write Par (a, b, c, d) iff it holds the equality $\lambda_{a,b} = \lambda_{d,c}$.

Let us prove some statements about the parallelograms.

THEOREM 18. *The statement* Par (a, b, c, d) *holds iff there is a transfer* τ *such that* $\tau(a) = b$, $\tau(d) = c$.

Proof. Let p be a given point and $q = \lambda_{a,b}(p)$. By Theorem 6 we have the equivalences

$$\text{Par}(a, b, c, d) \Leftrightarrow \lambda_{a,b} = \lambda_{d,c} \Leftrightarrow \lambda_{a,b}(p) = \lambda_{d,c}(p).$$

Because of (3) the equalities $\lambda_{a,b}(p) = q$, $\lambda_{d,c}(p) = q$ are equivalent with the equalities $ap = bq$, $dp = cq$, i. e. with the equalities $\varrho_{p,q}(a) = b$, $\varrho_{p,q}(d) = c$ according to (4). Therefore, the transfer $\tau = \varrho_{p,q}$ has the properties for which we are looking.

COROLLARY 1. *The statement* Par (a, b, c, d) *holds iff there are two points* p, q *such that* $ap = bq$, $dp = cq$.

COROLLARY 2. *Let it hold* Par (a, b, c, d). *For any point* p *there is the unique point* q *and for any point* q *there is the unique point* p *such that* $ap = bq, dp = cq$

COROLLARY 3. *Let it hold* Par (a, b, c, d). *The equalities* $ap = bq$ *and* $dp = cq$ *are equivalent.*

According to Theorem 11 it follows:

THEOREM 19. *The statement* Par (a, b, c, d) *holds iff it holds the equality* $\varrho_{a,b} = \varrho_{d,c}$.

By analogy we immediately have:

COROLLARY 4. *The statement* Par (a, b, c, d) *holds iff there are two points* p, q *such that* $pa = qb$, $pd = qc$.

COROLLARY 5. *Let it hold* Par (a, b, c, d). *For any point* p *there is the unique point* q *and for any point* q *there is the unique point* p *such that* $pa = qb$, $pd = qc$.

COROLLARY 6. *Let it hold* Par (a, b, c, d). *The equalities* $pa = qb$ *and* $pd = qc$ *are equivalent.*

THEOREM 20. *If* (e, f, g, h) *is any cyclic permutation of* (a, b, c, d) *or of* (d, c, b, a), *then* Par (a, b, c, d) *implies* Par (e, f, g, h).

Proof. Let it hold Par (a, b, c, d). Then $\lambda_{a,b} = \lambda_{d,c}$. It is sufficient to prove the statements Par (d, c, b, a), Par (b, c, d, a), i. e. the equalities $\lambda_{d,c} = \lambda_{a,b}$, $\lambda_{b,c} = \lambda_{a,d}$. The second equality follows by Theorem 12.

Because of Theorem 8 we immediately obtain:

THEOREM 21. *For any three points* a, b, c *there is one and only one point* d *such that it holds* Par (a, b, c, d).

THEOREM 22. *From* Par (a, b, c, d) *and* Par (c, d, e, f) *it follows* Par (a, b, f, e).

Proof. We have $\lambda_{a,b} = \lambda_{d,c}$ and $\lambda_{c,d} = \lambda_{f,e}$, i. e. $\lambda_{d,c} = \lambda_{e,f}$. Therefore, it follows $\lambda_{a,b} = \lambda_{e,f}$.

Because of Theorems 20—22 the structure (Q, Par) is a special case of a *Desargues* system in the terminology of *D. Vakarelov* [6] and (Q, P) is a parallelogram space in the terminology of *F. Ostermann* and *J. Schmidt* [2], where

$$P(a, b, c, d) \Leftrightarrow \mathrm{Par}\,(a, b, d, c).$$

For the sake of simplicity, we shall also call (Q, Par) the *parallelogram space*.

In the set Q we define a ternary operation $(\quad)$ by

$$(abc) = d \Leftrightarrow \mathrm{Par}\,(a, b, c, d).$$

In [6] it has been proven that it holds the identities

$$((abc)\,de) = (a\,(bcd)\,e) = (ab\,(cde)),$$

$$(abc) = (cba),$$

$$(abb) = a,$$

i. e. $(Q, (\quad))$ is a laterally commutative heap in the terminology of *V. V. Vagner* [4]. Also $(Q, (\ ))$ is an ordinary ternary group in the terminology of *D. Vakarelov* [5].

In the set Q^2 let us define a binary relation $\sim$ by

$$(a, b) \sim (c, d) \Leftrightarrow \mathrm{Par}\,(a, b, d, c),$$

which is an equivalence relation ([2], [6]). The elements of the set $\mathbf{Q} = Q^2/\sim$ are said to be *vectors*. The vector with the representative (a, b) will be denoted by $[a, b]$. By Theorem 21 for any point a and any vector $\mathbf{a}$ there is one and only one point b that $\mathbf{a} = [a, b]$.

Now, let O be a given point. The addition of vectors is defined by

$$[O, a] \dotplus [O, b] = [O, c] \Leftrightarrow \text{Par}(O, a, c, b).$$

In [2] and [6] it has been proven that the addition of vectors is independent of the choice of the point O, that $(\mathbf{Q}, \dotplus)$ is a commutative group and that for every three points a, b, c holds the equality

$$[a, b] \dotplus [b, c] = [a, c].$$

Let us prove two more theorems, which hold in any parallelogram space (Q, Par).

THEOREM 23. *For any points* a, b, c *it holds* $\text{Par}(a, a, b, c)$ *iff* $b = c$.

This is a consequence of the equivalence $\lambda_{a,a} = \lambda_{b,c} \Leftrightarrow b = c$.

THEOREM 24. *From* $\text{Par}(a, b, d, e)$ *and* $\text{Par}(b, c, e, f)$ *it follows* $\text{Par}(c, d, f, a)$.

Proof. Let g be the point such that $\text{Par}(a, b, c, g)$. By Theorem 22 we successively have the following implications

$$\text{Par}(c, g, a, b) \;\&\; \text{Par}(a, b, d, e) \Rightarrow \text{Par}(c, g, e, d),$$

$$\text{Par}(g, a, b, c) \;\&\; \text{Par}(b, c, e, f) \Rightarrow \text{Par}(g, a, f, e),$$

$$\text{Par}(c, d, e, g) \;\&\; \text{Par}(b, g, a, f) \Rightarrow \text{Par}(c, d, f, a).$$

Here we have applied Theorem 20 several times.

The statement of Theorem 24 can be proven directly by means of the medial quasigroup $(Q, \cdot)$. By the hypothesis we have $\lambda_{a,b} = \lambda_{e,d}$ and $\lambda_{b,c} = \lambda_{f,e}$, wherefrom it follows by Theorem 14

$$\lambda_{a,c} = \lambda_{a,b} \circ \lambda_{b,c} = \lambda_{e,d} \circ \lambda_{f,e} = \lambda_{f,d},$$

i. e. by Theorem 12 further $\lambda_{a,f} = \lambda_{c,d}$ and hence it holds $\text{Par}(c, d, f, a)$.

Now, let us prove some more statements about the parallelograms in the media quasigroup $(Q, \cdot)$.

By Theorem 16 it follows:

THEOREM 25. *Any two of three statements* $\text{Par}(a, b, c, d)$, $\text{Par}(e, f, g, h)$, $\text{Par}(ae, bf, cg, dh)$ *imply the remaining statement.*

For any point p we have obviously $\text{Par}(p, p, p, p)$ and from Theorem 25 it follows further:

THEOREM 26. *For any point* p *the statements* $\text{Par}(a, b, c, d)$, $\text{Par}(pa, pb, pc, pd)$, $\text{Par}(ap, bp, cp, dp)$ *are mutually equivalent.*

THEOREM 27. *From* $\text{Par}(a, b, c, d)$ *it follows* $\text{Par}(ab, bc, cd, da)$, $\text{Par}(ac, bd, ca, db)$, $\text{Par}(ad, ba, cb, dc)$, $\text{Par}(ad, bc, cb, da)$.

Proof. It follows by Theorem 20 and Theorem 25.

THEOREM 28. *For any points* a, b, c, d *it holds* $\text{Par}(ab, ad, cd, cb)$.

Proof. Because of Theorem 23 and Theorem 20 it holds $\text{Par}(a, a, c, c)$ and $\text{Par}(b, d, d, b)$ and by Theorem 25 it follows $\text{Par}(ab, ad, cd, cb)$.

For any point p there are the points l_p, r_p such that

$$l_p p = p, \; pr_p = p, \tag{10}$$

i. e. l_p, r_p are the left resp. right unit of the element p in the quasigroup $(Q, \cdot)$.

THEOREM 29. *If l_p, r_p are the left resp. right units of a point p and a, b any two points, then it holds* Par $(p, l_p a, ba, bp)$, Par (p, ar_p, ab, pb).

Proof. By Theorem 28 and Theorem 20 it holds Par $(l_p p, l_p a, ba, bp)$, Par (pr_p, ar_p, ab, pb), wherefrom by (10) the statement of the theorem follows.

4. MIDPOINTS

We shall say that c is a *midpoint* of the pair of points a and b and write $M(a, c, b)$ iff it holds Par (a, c, b, c), i. e. iff it holds the equality $\lambda_{a,c} = \lambda_{c,b}$, i. e. iff there is a transfer τ such that $\tau(a) = c$, $\tau(c) = b$.

As from $\lambda_{a,c} = \lambda_{c,b}$ it immediately follows $\lambda_{b,c} = \lambda_{c,a}$, so we have

THEOREM 30. *From $M(a, c, b)$ it follows $M(b, c, a)$.*

Obviously, we have:

THEOREM 31. *For any point a it holds $M(a, a, a)$.*

Because of Theorem 8 for any two points a, c there is one and only one point b such that $\lambda_{a,c} = \lambda_{c,b}$ and we obtain:

THEOREM 32. *For any two points a, c there is one and only one point b such that it holds $M(a, c, b)$.*

From Theorems 25 and 26, the next two theorems follow.

THEOREM 33. *Any two of three statements $M(a, e, c)$, $M(b, f, d)$, $M(ab, ef, cd)$ imply the remaining statement.*

THEOREM 34. *For any point p the statements $M(a, c, b)$, $M(pa, pc, pb)$, $M(ap, cp, bp)$ are mutually equivalent.*

Further, we have:

THEOREM 35. *If there is a midpoint of the pair of points a, c, and if it holds $M(ab, g, cd)$, then it holds $M(ad, g, cb)$.*

Proof. Let e be a point such that it holds $M(a, e, c)$. There is a point f such that $ef = g$. According to Theorem 33 from $M(a, e, c,)$ and $M(ab, ef, cd)$ it follows $M(b, f, d)$, i. e. $M(d, f, b)$. But, from $M(a, e, c)$ and $M(d, f, b)$ by Theorem 33 we obtain $M(ad, ef, cb)$, i. e. $M(ad, g, cb)$.

By the substitutions $b \to c$, $c \to b$, $d \to c$ from Theorem 27 it follows:

THEOREM 36. *From $M(a, c, b)$ it follows* Par (ac, cb, bc, ca).

Let us prove:

THEOREM 37. *Let it hold $M(a, p, c)$. The statements* Par (a, b, c, d) *and $M(b, p, d)$ are equivalent.*

Proof. By the hypothesis we have Par (a, p, c, p). If it holds Par (a, b, c, d), then by Theorem 20 it follows Par (b, a, d, c), which together with Par (a, p, c, p)

implies by Theorem 24 Par (p, d, p, b), i. e. Par (b, p, d, p) or $M(b, p, d)$. Conversely, if it holds $M(b, p, d)$, then we have Par (d, p, b, p), which together with *Par* (p, a, p, c) by Theorem 24 implies *Par* (a, b, c, d).

THEOREM 38. *Any two of three statements* $M(a, c, b)$, Par (a, c, d, e), Par (b, c, e, d) *imply the remaining statement.*

Proof. From $M(a, c, b)$ it follows Par (b, c, a, c), which together with Par (a, c, d, e) by Theorem 22 implies Par (b, c, e, d). Analogously, from $M(a, c, b)$ it follows Par (a, c, b, c), which together with Par (b, c, e, d) implies *Par* (a, c, d, e). Finally, from Par (a, c, d, e) and Par (d, e, c, b) by Theorem 22 Par (a, c, b, c), i. e. $M(a, c, b)$ follows.

With $e = b$ from Theorem 38 we obtain:

COROLLARY 7. *Any two of three statements* $M(a, c, b)$, $M(c, b, d)$, Par (a, c, d, b) *imply th: remaining statemen .*

Let us prove:

THEOREM 39. *Let it hold* $M(b, d, c)$ *and* $M(c, e, a)$. *The statements* Par (a, f, d, e), Par (b, d, e, f), Par (c, e, f, d) *are mutually equivalent and if these statements holds, then it holds* $M(a, f, b)$ *too.*

Proof. As it holds $M(a, e, c)$, so by Theorem 38 with the substitutions $b \to c$, $c \to e$, $e \to f$ it follows that the statements Par (a, f, d, e), Par (c, e, f, d) are equivalent. As it holds $M(b, d, c)$, so by the same theorem, with the substitutions $a \to b$, $b \to c$, $c \to d$, $d \to e$, $e \to f$, the equivalence of the statements Par (b, d, e, f), Par (c, e, f, d) follows. Now, if it holds Par $(a, f, d\ e)$, Par (b, f, e, d), then by Theorem 38, with the substitution $c \to f$, $M(a, f, b)$ follows.

According to Theorem 21 from Theorem 39 it immediately follows

THEOREM 40. *If the pairs of points b, c, and c, a have the midpoints d and e, then the pair of the points a, b has a midpoint f such that it holds* Par (a, f, d, e), Par (b, d, e, f), Par (c, e, f, d).

If we repeat the last conclusion in the proof of Theorem 39 twice again, then we obtain:

THEOREM 41. *Let d, e, f be any points and a, b, c the points such that it holds* Par (f, d, e, a), Par (d, e, f, b), Par (e, f, d, c). *Then it holds* $M(b, d, c)$ $M(c, e, a)$, $M(a, f, b)$.

THEOREM 42. *If the pairs of points a, b; b, c; c, d have the midpoints e, f, g, then pair of points d, a has a midpoint h such that it holds* Par (e, f, g, h).

Proof. It holds $M(a, e, b)$, $M(b, f, c)$ and by Theorem 40 there is a point i such that it holds $M(c, i, a)$, Par (a, e, f, i). Now, it holds $M(a, i, c)$ $M(c, g, d)$ and by the same theorem there is a point h such that it holds $M(d, h, a)$, Par (a, i, g, h). But, from Par (e, f, i, a) Par (i, a, h, g) by Theorem 22 it follows *Par* (e, f, g, h).

5. ADDITION OF POINTS

Let 0 be a given point. For any two points a, b we define the sum $a + b$ by the equivalence

$$c = a + b \Leftrightarrow \text{Par}(0, a, c, b). \qquad (11)$$

This means that for any points a, b it holds Par $(0, a, a + b, b)$.

Let us consider the mapping $\varphi: T \to Q$ defined by $\varphi : \lambda_{0,a} \mapsto a$. By Theorems 5 and 8 for any transfer τ there is one and only one point a such that $\tau = \lambda_{0,a}$ and hence φ is a bijection. Let $\lambda_{0,a}$ and $\lambda_{0,b}$ be any two transfers. As it holds Par $(0, a, a + b, b)$, so we have the equality $\lambda_{0,a} = \lambda_{b,a+b}$. Therefore, by Theorem 14 we successively have

$$\varphi(\lambda_{0,a} \circ \lambda_{0,b}) = \varphi(\lambda_{b,a+b} \circ \lambda_{0,b}) = \varphi(\lambda_{0,a+b}) = a + b = \varphi(\lambda_{0,a}) + \varphi(\lambda_{0,b})$$

and φ is an i omorphism of the groupoids $(T, \circ)$ and $(Q, +)$. According to Theorem 13 it immediately follows

THEOREM 43. *$(Q, +)$ is a commutative group with the neutral element* 0, *which is isomorphic with the group $(T, \circ)$.*

Now, let $0'$ be any other given point. Then the equivalence

$$c = a +' b \Leftrightarrow \text{Par}(0', a, c, b) \tag{12}$$

defines also a commutative group $(Q, +')$. But, by Theorem 43 we obtain:

THEOREM 44. *The groups $(Q, +)$ and $(Q, +')$ defined by* (11) *and* (12) *are isomorphic.*

THEOREM 45. *The statements* Par (a, b, c, d) *and* $a + c = b + d$ *are equivalent.*

Proof. From Par $(0, a, a + c, c)$, Par (a, b, c, d) by Theorem 24 it follows Par $(b, a + c, d, 0)$, i. e. Par $(b, 0, d, a + c)$. But, from Par $(0, b, b + d, d)$ we obtain Par $(b, 0, d, b + d)$. Therefore, by Theorem 21 it follows $a + c = b + d$. Conversely, if $a + c = b + d$, then we have Par $(a, 0, c, a + c)$ and Par $(0, b, a + c, d)$, wherefrom by Theorem 24 it follows Par (b, c, d, a), i. e. Par (a, b, c, d).

COROLLARY 8. *The statements $M(0, b, a)$ and $b + b = a$ are equivalent.*

Now, let the points l_0 and r_0 be the left and right units of the point O, i. e. let it hold the equalities

$$l_0 0 = 0,\ 0 r_0 = 0. \tag{13}$$

THEOREM 46. *The translations λ_{l_0}* and *ϱ_{r_0} of the quasigroup $(Q, \cdot)$ are two automorphisms of the group $(Q, +)$ and it holds the equality*

$$\lambda_{l_0} \circ \varrho_{r_0} = \varrho_{r_0} \circ \lambda_{l_0}. \tag{14}$$

Proof. Because of (1) and (13) for any point x we successively have $(l_0 \cdot x r_0) \cdot 00 = l_0 0 \cdot (x r_0 \cdot 0) = 0 (x r_0 \cdot 0) = 0 r_0 \cdot (x r_0 \cdot 0) = (0 \cdot x r_0) \cdot r_0 0 = (l_0 0 \cdot x r_0) \cdot r_0 0 = (l_0 x \cdot 0 r_0) \cdot r_0 0 = (l_0 x \cdot 0) \cdot r_0 0 = (l_0 x \cdot r_0) \cdot 00$, wherefrom it follows $l_0 \cdot x r_0 = l_0 x \cdot r_0$, i. e. $(\lambda_{l_0} \circ \varrho_{r_0})(x) = (\varrho_{r_0} \circ \lambda_{l_0})(x)$ and the equality (14) is proved. From Par $(0, a, a + b, b)$ and first equality (13) by Theorem 26 it follows Par $(0, l_0 a, l_0 (a + b), l_0 b)$, i. e. $l_0 (a + b) = l_0 a + l_0 b$ or $\lambda_{l_0}(a + b) = \lambda_{l_0}(a) + \lambda_{l_0}(b)$. Analogously, we can prove that ϱ_{r_0} is also an automorphism of the group $(Q, +)$.

THEOREM 47. *For any two points a, b it holds the equality*

$$ab = a r_0 + l_0 b + 00. \tag{15}$$

Proof. According to Theorem 29 it holds Par $(0, ar_0, ab, 0b)$ and Par $(0, l_0 b, 0b, 00)$, i. e. by (11) we have the equalities $ab = ar_0 \dotplus 0b$, $0b = l_0 b \dotplus 00$, wherefrom (15) follows.

The equality (15) can be written in the form $ab = \varrho_{r_0}(a) \dotplus \lambda_{l_0}(b) \dotplus 00$, where it holds the equality (14). This result agrees with the well-known *Toyoda*'s theorem and its more precise version proved in [1] (Lemma 6).

6. CONJUGATED MEDIAL QUASIGROUPS

Let $(Q, \cdot)$ be the considered medial quasigroup and $\blacksquare$ the operation on the set Q defined by the equivalence

$$a \blacksquare b = c \Leftrightarrow ac = b. \tag{16}$$

Obviously $(Q, \blacksquare)$ is also a quasigroup, which is medial by Theorem 12.1 from [3]. Therefore, in this new quasigroup we can also define the notions of transfer, parallelogram and addition of points. We shall investigate the relations of these notions with the corresponding notions defined in the quasigroup $(Q, \cdot)$. If we denote by $\lambda_a^\blacksquare$, $\lambda_{a,b}^\blacksquare$, $\mathrm{Par}^\blacksquare$ the notions defined in the quasigroup $(Q, \blacksquare)$ analogous to the notions denoted by λ_a, $\lambda_{a,b}$, Par in the quasigroup $(Q, \cdot)$, then it holds the next two theorems:

THEOREM 48. *For any point a it holds the equality $\lambda_a^\blacksquare = \lambda_a^{-1}$.*

Proof. For any point x let it be $y = \lambda_a^\blacksquare(x) = a \blacksquare x$. Then by (16) it follows $x = ay = \lambda_a(y)$, i. e. $y = \lambda_a^{-1}(x)$. Therefore, we have $\lambda_a^\blacksquare(x) = \lambda_a^{-1}(x)$, i. e. $\lambda_a^\blacksquare = \lambda_a^{-1}$.

THEOREM 49. *For any points a, b, c, d it holds the equivalence*

$$\mathrm{Par}^\blacksquare(a, b, c, d) \Leftrightarrow \mathrm{Par}(a, b, c, d).$$

Proof. Let p be a given point and

$$\lambda_{a,b}^\blacksquare(p) = q. \tag{17}$$

Because of the statement $(3)^\blacksquare$ analogous to the statement (3) we have then the equality $a \blacksquare p = b \blacksquare q$. Let us denote this point by r. Then from $a \blacksquare p = r$, $b \blacksquare q = r$ by (16) follow the equalities:

$$ar = p, \tag{18}$$

$$br = q. \tag{19}$$

Let it be

$$\lambda_{a,d}(r) = s, \tag{20}$$

which together with (3) implies the equality $ar = ds$. Then by (18) it follows $ds = p$ or by (16) further

$$d \blacksquare p = s. \tag{21}$$

Now, we have the equivalences

$$\mathrm{Par}^\blacksquare(a, b, c, d) \Leftrightarrow \lambda_{a,b}^\blacksquare = \lambda_{d,c}^\blacksquare \overset{T6^\blacksquare}{\Leftrightarrow} \lambda_{a,b}^\blacksquare(p) = \lambda_{d,c}^\blacksquare(p) \overset{(17)}{\Leftrightarrow}$$

$$\Leftrightarrow \lambda_{d,c}^\blacksquare(p) = q \overset{(3)^\blacksquare}{\Leftrightarrow} d \blacksquare p = c \blacksquare q \overset{(21)}{\Leftrightarrow} c \blacksquare q = s \overset{(16)}{\Leftrightarrow} cs = q \overset{(19)}{\Leftrightarrow}$$

$$\Leftrightarrow br = cs \overset{(3)}{\Leftrightarrow} \lambda_{b,c}(r) = s \overset{(20)}{\Leftrightarrow} \lambda_{a,d}(r) = \lambda_{b,c}(r) \overset{T6}{\Leftrightarrow}$$

$$\Leftrightarrow \lambda_{a,d} = \lambda_{b,c} \Leftrightarrow \mathrm{Par}\,(a, d, c, b) \Leftrightarrow \mathrm{Par}\,(a, b, c, d),$$

where $T6^{\blacksquare}$ is the statement for the quasigroup $(Q, \blacksquare)$ analogous to the statement T6 of Theorem 6 in the quasigroup $(Q, \cdot)$.

Now, let $\square$ be the operation on the set Q defined by the equivalence

$$a \square b = c \Leftrightarrow cb = a.$$

Then $(Q, \square)$ is also a medial quasigroup and with the corresponding notations it holds the next two theorems analogous to Theorems 48 and 49.

THEOREM 50. *For any point a it holds the equality* $\varrho_a^{\square} = \varrho_a^{-1}$.

THEOREM 51. *For any points a, b, c, d it holds the equivalence*

$$\mathrm{Par}^{\square}\,(a, b, c, d) \Leftrightarrow \mathrm{Par}\,(a, b, c, d).$$

If (i, j, k) is any permutation of the set $\{1, 2, 3\}$ and $\bullet$ the operation on the set Q defined by the equivalence

$$a_i \bullet a_j = a_k \Leftrightarrow a_1\, a_2 = a_3,$$

then by Theorem 12.1 in [3] $(Q, \bullet)$ is a medial quasigroup. By a combination of the results of Theorems 49 and 51 we obtain with the corresponding notations:

THEOREM 52. *For any points a, b, c, d it holds the equivalence*

$$\mathrm{Par}^{\bullet}\,(a, b, c, d) \Leftrightarrow \mathrm{Par}\,(a, b, c, d).$$

The quasigroups $(Q, \cdot)$ and $(Q, \bullet)$ are conjugated quasigroups in the terminology of *S. K. Stein* [3]. In the quasigroup $(Q, \bullet)$ we can define, with a given point O, also an addition of points by the equivalence

$$a + b = c \Leftrightarrow \mathrm{Par}^{\bullet}\,(0, a, c, b). \tag{22}$$

According to Theorem 52 it immediately follows:

THEOREM 53. *If* $(Q, \cdot)$ *and* $(Q, \bullet)$ *are conjugated medial quasigroups, then by the equivalences* (11) *and* (22) *the same commutative group* $(Q, +)$ *is defined.*

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Geometrija medijalnih kvazigrupa

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Sadržaj

U kvazigrupi $(Q, \cdot)$ sa svojstvom medijalnosti, izraženim identitetom $ab \cdot cd = ac \cdot bd$, definiraju se »geometrijski« pojmovi: točka, pomak, paralelogram, vektor, polovište para točaka i zbrajanje točaka. Dokazuje se da ti pojmovi imaju neka standardna svojstva. Daje se »geometrijski« dokaz preciznije verzije Toyodina teorema o reprezentaciji medijalnih kvazigrupa, koji pokazuje da se svaka medijalna kvazigrupa može dobiti iz neke komutativne grupe.

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