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# GEOMETRY OF IM-QUASIGROUPS

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# GEOMETRY OF IM-QUASIGROUPS

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*Abstract.* In a case of idempotent medial quasigroups the results of [8] can be completed with some new »geometrical« results. A »geometric« proof of the representation theorem of IM-quasigroups will be given.

A quasigroup $(Q, \cdot)$ is said to be IM-*quasigroup* iff it has the properties of *idempotency* and *mediality*, i. e. iff it satisfies the identities

$$aa = a, \tag{1}$$

$$ab \cdot cd = ac \cdot bd. \tag{2}$$

*Example* 1. Let $(G, +)$ be a commutative group with an automorphism $\varphi$ such that the mapping $\alpha$, defined by $\psi(x) = x - \varphi(x)$, is an automorphism of $(G, +)$ too. If $\cdot$ is a binary operation on the set $G$ defined by

$$ab = a + \varphi(b - a),$$

then $(G, \cdot)$ is an IM-quasigroup. Let us prove this statement. For every $a, b \in G$ the equation $ax = b$, i. e. the equation $a + \varphi(x - a) = b$, has the unique solution $x = a + \varphi^{-1}(b - a)$, and the equation $ya = b$, i. e. $y + \varphi(a - y) = b$, has the unique solution $y = \psi^{-1}[b - \varphi(a)]$, because of $y - \varphi(y) = \psi(y)$. The identity (1) is obvious and the identity (2) is a consequence of the fact that the product

$$ab \cdot cd = ab + \varphi(cd - ab) = a + \varphi(b - a) + \varphi[c + \varphi(d - c) -$$

$$- a - \varphi(b - a)] = a + \varphi(b) - \varphi(a) + \varphi(c) + (\varphi \circ \varphi)(d) -$$

$$- (\varphi \circ \varphi)(c) - \varphi(a) - (\varphi \circ \varphi)(b) + (\varphi \circ \varphi)(a)$$

is a symmetric function of $b$ and $c$.

*Example* 2. Let $(F +, \cdot)$ be any field, $q \in F$ and $*$ a binary operation on the set $F$ defined by

$$a * b = (1 - q)a + qb. \tag{3}$$

The identities $\varphi(a) = qa$, $\psi(a) = (1 - q)a$ define obviously two automorphisms $\varphi$ and $\psi$ of the commutative group $(F, +)$ such that $\varphi(a) + \psi(a) = a$. The equality (3) can be written in the form $a * b = a + \varphi(b - a)$ and because of Example 1 it follows that $(F, *)$ is an IM-quasigroup.

*Example* 3. Let $(C, +, \cdot)$ be the field of the complex numbers, $q \in C$ and $*$ the operation on $C$ defined by (3). Because of Example 2 it follows that $(C, *)$ is an IM-quasigroup. This quasigroup has a beautiful geometrical interpretation which motivates the study of the IM-quasigroups. Let us regard the complex numbers as the points of the *Euclid*ean plane. For any two different points $a$ and $b$ the equality (3) can be written in the form

$$\frac{a * b - a}{b - a} = \frac{q - 0}{1 - 0},$$

which means that the points $a$, $b$, $a * b$ are the vertices of a triangle directly similar to the triangle with the vertices 0, 1, $q$, i. e. the vertices of a triangle of a given form. (If $q$ is a real number, the triangle is degenerate.) Every identity in the IM-quasigroup $(C, *)$ can be interpreted as a geometrical theorem, which of course can be proved directly, but the theory of IM-quasigroups gives a better insight into the mutual relations of such theorems.

In an IM-quasigroup $(Q, \cdot)$ all results of [8] are valid and here we shall prove some new statements, for proof of which the identity (1) is necessary. The definitions of necessary »geometric« notions are given in [8] and this notions agree with Example 3. Theorem $n$ from [8] we designate by $n'$. The elements of the set $Q$ is said to be *points*.

If we put $c = d = a$, then $d = c$ and finally $b = a$ (with substitutions $c \to b$ and $d \to c$) in (2), then according to (1) we obtain the properties of *elasticity* and *right* and *left distributivity* i. e. the identities

$$ab \cdot a = a \cdot ba, \tag{4}$$

$$ab \cdot c = ac \cdot bc, \tag{5}$$

$$a \cdot bc = ab \cdot ac. \tag{6}$$

THEOREM 1. *From $ab = e$, $ca = b$ and $bd = a$ it follows $cd = e$.*

*Proof.* We have successively

$$cd \cdot ad \overset{(5)}{=} ca \cdot d = bd \overset{(1)}{=} bd \cdot bd = a \cdot bd \overset{(6)}{=} ab \cdot ad = e \cdot ad,$$

i. e. $cd = e$.

In the case of the quasigroup $(C, *)$ from Example 3 this theorem represents the following statement of *Euclid*ean geometry in a plane:

If *abe*, *cab* and *bda* are directly similar triangles constructed on a given segment *ab*, then the triangle *cde* is directly similar to them.

THEOREM 2. *From* $ab = c$, $ap = c'$, $a'b = p$ and $pb' = c$ *it follows* $a'b' = c'$.

*Proof.* We have

$$c' \cdot a'b' = ap \cdot a'b' \overset{(2)}{=} aa' \cdot pb' = aa' \cdot ab \overset{(6)}{=} a \cdot a'b = ap = c' \overset{(1)}{=} c'c',$$

i. e. $a'b' = c'$.

In the case of the quasigroup $(C, *)$ this theorem proves the following result from [1]:

Let $abc$ be a triangle and $p$ a point. If $a'$, $b'$, $c'$ are points such that the triangles $apc'$, $bpa'$, $cpb'$ are directly similar to the triangles $abc$, $bca$, $cab$, respectively, then the triangle $a'b'c'$ is directly similar to the triangle $abc$.

THEOREM 3. *From* Par $(a, b, c, d)$, $ea = b$ *and* $af = d$ *it follows* $ef = c$.

*Proof.* By Theorems 23′ and 20′ we have Par $(a, a, e, e)$ and Par $(f, a, a, f)$, wherefrom by Theorem 25′ it follows Par $(af, aa, ea, ef)$, i. e. Par $(d, a, b, ef)$. On the other hand, by Theorem 20′ we have Par $(d, a, b, c)$ and by Theorem 21′ it follows $ef = c$.

In the case of the quasigroup $(C, *)$ we have the following statement, a special case of which is equivalent to a result of [7]:

If $abcd$ is a parallelogram and $eab$ and $afd$ two directly similar triangles, then the triangle $efc$ is directly similar to them.

THEOREM 4. *If* $ac = b$, *then the statements* Par $(a, b, c, d)$ *and* $ca = d$ *are equivalent.*

*Proof.* By Theorem 23′ and 20′ we have Par $(a, a, c, c)$ and Par $(c, a, a, c)$, wherefrom by Theorem 25′ it follows Par $(ac, aa, ca, cc)$, i. e. Par $(a, b, c, ca)$ because of Theorem 20′. Now, the statement of our theorem is a consequence of Theorem 21′.

THEOREM 5. *If* $ab = c$, *then the statements* Par $(a, b, c, d)$ *and* $cd = a$ *are equivalent.*

*Proof.* By Theorem 23′ we have Par $(a, a, c, c)$ and by Theorem 21′ there is one and only one point $e$ such that Par $(a, b, c, e)$. Because of Theorem 25′ it follows Par $(aa, ab, cc, ce)$, i. e. Par $(a, c, c, ce)$. On the other hand, by Theorem 23′ and 20′ have Par $(a, c, c, a)$ and by Theorem 21′ it follows $ce = a$. Now, it is obvious that any of two statements Par $(a, b, c, d)$ and $cd = a$ is equivalent to $d = e$.

THEOREM 6. *If* $ab = d$, *then the statements* Par $(a, b, c, d)$ *and* $cd = b$ *are equivalent.*

*Proof.* By Theorem 21′ there is one and only one point $e$ such that Par $(d, a, b, e)$ and by Theorem 23′ and 20′ we have Par $(d, b, b, d)$. Because of Theorem 25′ it follows Par $(dd, ab, bb, ed)$, i. e. Par $(d, d, b, ed)$. On the other hand, by Theorem 23′ we have Par $(d, d, b, b)$ and by Theorem 21′ it follows $ed = b$. Any of two statements Par $(a, b, c, d)$ and $cd = b$ is equivalent to $c = e$.

THEOREM 7. *For any points* $a, b, c$ *we have* P $(a, ca, cb, ab)$ *and* Par $(a, ac, bc, ba)$.

*Proof*. Because of Theorem 23′ and 20′ we have Par $(a, c, c, a)$ and Par $(a, a, b, b)$ wherefrom by Theorem 25′ it follows Par $(aa, ca, cb, ab)$ and Par $(aa, ac, bc, ba)$, i. e. Par $(a, ca, cb, ab)$ and Par $(a, ac, bc, ba)$.

In the case of the quasigroup $(C, *)$ this theorem proves the following result from [2]:

If $bca'$, $cab'$, $abc'$, $cba''$, $acb''$, $bac''$ are directly similar triangles constructed on the sides of a given triangle $abc$, then $ab'a''c'$ and $ab''a'c''$ are parallelograms.

THEOREM 8. *If we have* Par $(a, f, d, e)$, Par $(b, d, e, f)$ *and* Par $(c, e, f, d)$ *then any two of four equalities*

$$de = f, \qquad af = e, \qquad fb = d, \qquad ed = c \tag{7}$$

*are mutually equivalent and any of these equalities implies the equality* $ab = c$.

*Proof*. By Theorem 6 from Par $(a, f, d, e)$ and $af = e$ it follows $de = f$ and analogously from Par $(d, e, a, f)$ and $de = f$ it follows $af = e$, i. e. we have proved the equivalence of the first and the second equality (7). Analogously by Theorem 5 from Par $(d, e, f, b)$ it follows the equivalence of the first and the third equality (7) and by Theorem 4 from Par $(e, c, d, f)$ if follows the equivalence of the first and the fourth equality (7). Now. let we have the equalities (7). From Par $(f, d, e, a)$ and Par $(f, e, d, b)$ by Theorem 25′ we obtain Par $(ff, de, ed, ab)$, i. e. Par $(f, f, c, ab)$. On the other hand by Theorem 23′ we have Par $(f, f, c, c)$ and because of Theorem 21′ it follows $ab = c$.

THEOREM 9. *Any of the six statements* Par $(o, a, d, g)$, Par $(o, b, e, h)$. Par $(o, c, f, i)$, $ab = c$, $de = f$, $gh = i$ is a consequence of the other five statements.

*Proof*. This is an immediate consequence of Theorem 25′ because of Theorem 20′ and 21′.

In the case of the quasigroup $(C, *)$ this theorem proves the following result from [6]:

If $abc$ and $def$ are directly similar triangles and $og$, $oh$, $oi$ three segments equipolent to the segments $ad$, $be$, $cf$, respectively, then the triangle $ghi$ is directly similar to the triangles $abc$ and $def$.

THEOREM 10. *From* Par $(o, a, g, b)$, Par $(o, c, h, d)$, Par $(o, e, i, f)$, $ao = f$, $bc = o$, *and* $od = e$ *it follows* $gh = i$.

*Proof*. From Par $(o, f, i, e)$, $ao = f$, $od = e$ by Theorem 3 it follows $ad = i$. From Par $(o, b, g, a)$ and Par $(o, c, h, d)$ by Theorem 25′ it follows Par $(oo, bc, gh, ad)$, i. e. Par $(o, o, gh, i)$, and because of Theorem 23′ we obtain $gh = i$.

In the case of the quasigroup $(C, *)$ this theorem proves the following result from [3] and [4]:

If $oagb$, $ochd$, $oeif$ are parallelograms and $aof$, $bco$, $ode$ are directly similar triangles, then $ghi$ is a triangle directly similar to these triangles.

In [8] the notion of a parallelogram is characterized by Corollary 1′ in the following way:

$$\text{Par}\,(a, b, c, d) \Leftrightarrow (\exists\, p, q \in Q)\ ap = bq,\ dp = cq.$$

But, in our case of an IM-quasigroup the notion of a parallelogram can be characterized by only one equality without auxiliary points, because of following two theorems.

THEOREM 11. *The statement* Par $(a, b, c, d)$ *is equivalent to the equality* $ab \cdot da = ac \cdot a$.

*Proof.* Let $q$ be the point such that $ad = bq$. Because of Corollaries 1′ and 3′ the statement Par $(a, b, c, d)$ is equivalent to the equality $dd = cq$, i. e. $d = cq$. Moreover, we have successively

$$ad \cdot ba \stackrel{(1)}{=} (ad \cdot ba)(ad \cdot ba),$$

$$(a \cdot cq) \cdot ba \stackrel{(6)}{=} (ac \cdot aq) \cdot ba \stackrel{(2)}{=} (ac \cdot b)(aq \cdot a) \stackrel{(4)}{=} (ac \cdot b)(a \cdot qa) \stackrel{(2)}{=}$$

$$\stackrel{(2)}{=} (ac \cdot a)(b \cdot qa) \stackrel{(6)}{=} (ac \cdot a)(bq \cdot ba) = (ac \cdot a)(ad \cdot ba),$$

and by (2) we have $ad \cdot ba = ab \cdot da$. Therefore, we obtain

$$\text{Par}(a, b, c, d) \Leftrightarrow d = cq \Leftrightarrow ad \cdot ba = (a \cdot cq) \cdot ba \Leftrightarrow (ad \cdot ba)(ad \cdot ba) =$$

$$= (ac \cdot a)(ad \cdot ba) \Leftrightarrow ad \cdot ba = ac \cdot a \Leftrightarrow ab \cdot da = ac \cdot a.$$

THEOREM 12. *The statement* Par $(a, b, c, d)$ *is equivalent to the equality* $ab \cdot dc = ac$.

*Proof.* Let $p$ and $q$ be two points such that $ap = bq$. According to Corollaries 1′ and 3′ the statement Par $(a, b, c, d)$ is equivalent to the equality $dp = cq$. On the other hand, we have

$$ap \cdot (dp \cdot cq) \stackrel{(4)}{=} (ap \cdot dp)(ap \cdot cq) = (ap \cdot dp)(bq \cdot cq) \stackrel{(5)}{=}$$

$$\stackrel{(5)}{=} (ad \cdot p)(bc \cdot q) \stackrel{(2)}{=} (ad \cdot bc) \cdot pq \stackrel{(2)}{=} (ab \cdot dc) \cdot pq,$$

$$ap \cdot (cq \cdot cq) \stackrel{(1)}{=} ap \cdot cq \stackrel{(2)}{=} ac \cdot pq,$$

and therefore it follows

$$\text{Par}(a, b, c, d) \Leftrightarrow dp = cq \Leftrightarrow ap \cdot (dp \cdot cq) = ap \cdot (cq \cdot cq) \Leftrightarrow$$

$$\Leftrightarrow (ab \cdot dc) \cdot pq = ac \cdot pq \Leftrightarrow ab \cdot dc = ac.$$

In the case of the quasigroup $(C, *)$ this theorem proves Theorem 2 from [5].

THEOREM 13. *If the pairs of points $b$, $c$, and $c$, $a$ have midpoints $d$ and $e$, then there are midpoints $f$ and $g$ of the pairs of points $bc$, $ca$ and $ab$, $c$ and $de = f$, $ed = g$ and* Par $(d, f, e, g)$ *holds.*

*Proof.* From M $(b, d, c)$ and M $(c, e, a)$ it follows by Theorem 33′ M $(bc, de, ca)$, i. e. M $(bc, f, ca)$, where $f = de$. Analogously, from M $(a, e, c)$ and M $(b, d, c)$ it follows M $(ab, ed, cc)$, i. e. M $(ab, g, c)$ ,where $g = ed$. Because of Theorem 4 from $de = f$ and $ed = g$ we obtain Par $(d, f, e, g)$.

According to (1). every element of the IM-quasigroup is the left and the right unit of its own. Let $O$ be a given point and let $+$ be an addition of points defined as in [8] by

$$c = a + b \Leftrightarrow \text{Par}\,(O, a, c, b).$$

Because of (1) and $l_O = r_O = O$, Theorems 46′ and 47′ imply immediately the following theorem.

THEOREM 14. *For any points a and b we have the equality*

$$ab = aO + Ob, \tag{8}$$

*i. e.*

$$ab = \varrho_O(a) + \lambda_O(b), \tag{9}$$

*where $\lambda_O$ and $\varrho_O$ are two automorphisms of a commutative group $(Q, +)$ such that $\lambda_O \circ \varrho_O = \varrho_O \circ \lambda_O$.*

Owing to (1) and (8) we get

$$a = aa = aO + Oa = \varrho_O(a) + \lambda_O(a),$$

i. e. $aO = a - Oa$, which substituted in (8) implies $ab = a + Ob - Oa$. But, $\lambda_O$ is an automorphism of $(Q, +)$ and we obtain

$$Ob - Oa = \lambda_O(b) - \lambda_O(a) = \lambda_O(b - a).$$

Therefore:

THEOREM 15. *For any points a and b we have the equality*

$$ab = a + \lambda_O(b - a),$$

*and for any point a we have $\lambda_O(a) + \varrho_O(a) = a$, where $\lambda_O$ and $\varrho_O$ are two automorphisms of the commutative group $(Q, +)$.*

From Theorem 15 it follows that every IM-quasigroup can be obtained as in Example 1, i. e. we have the following theorem.

THEOREM 16. *There is an* IM-*quasigroup $(Q, \cdot)$ iff there is a commutative group $(Q, +)$ and two of its automorphisms $\varphi$ and $\psi$ such that $\varphi(a) + \psi(a) = a$ for every $a \in Q$. If the commutative gruop $(Q, +)$ and the automorphisms $\varphi$ and $\psi$ are given, then the operation $\cdot$ is defined by*

$$ab = a + \varphi(b - a),$$

*and if the* IM-*quasigroup $(Q, \cdot)$ and an element $O \in Q$ are given, then the operation $+$ is defined by*

$$a + b = \varrho_O^{-1}(a) \cdot \lambda_O^{-1}(b), \tag{10}$$

*where $O$ is the neutral element of the group $(Q, +)$ and $\varphi = \lambda_O$ and $\psi = \varrho_O$ are a left and a right translation of $(Q, \cdot)$.*

At the end let us prove one more theorem.

THEOREM 17. *Let $P \subset Q$. Two following statements are equivalent:*

$1^\circ$ *$(P, \cdot)$ is a subquasigroup of $(Q, \cdot)$ and $O \in P$;*

$2^\circ$ *$(P, +)$ is a subgroup of $(Q, +)$, which is invariant with respect to the mappings $\lambda_O$ and $\varrho_O$.*

*Proof.* $1^\circ \Rightarrow 2^\circ$: For any $a \in P$ there is one and only one element $u \in Q$ such that $uO = a$, i. e. $u = \varrho_O^{-1}(a)$, and from $O, a \in P$ because of $1^\circ$ we obtain $u \in P$. Analogously, it can be proved that $b \in P$ implies $\lambda_O^{-1}(b) \in P$. If we make the substitutions $a \to \varrho_O^{-1}(a)$ and $b \to \lambda_O^{-1}(b)$, then the equality (9) takes the form (10). For any two elements, $a, b \in P$ we have $\varrho_O^{-1}(a), \lambda_O^{-1}(b) \in P$ and by (10) because of $1^\circ$ it follows $a + b \in P$. For every $a \in P$ there is one and only one element $-a \in \in Q$ such that $a + (-a) = O$, i. e. $\varrho_O^{-1}(a) \cdot \lambda_O^{-1}(-a) = O$ because of (10). Therefore, $O, \varrho_O^{-1}(a) \in P$ imply by $1^\circ$ $\lambda_O^{-1}(-a) \in P$ and because of $O \in P$ we obtain $O \cdot \lambda_O^{-1}(-a) \in P$, i. e. $-a = \lambda_O(\lambda_O^{-1}(-a)) = O \cdot \lambda_O^{-1}(-a) \in P$. Therefore, $(P, +)$ is a subgroup of the group $(Q, +)$. For every $x \in P$ we have by $1^\circ$ $Ox \in P$, i. e. $\lambda_O(x) \in P$, and hence $\lambda_O(P) \subseteq P$. For every $y \in P$ there is an element $x \in P$ (because of $1^\circ$) such that $Ox = y$, i. e. $\lambda_O(x) = y$, and hence $P \subseteq \subseteq \lambda_O(P)$. Therefore $\lambda_O(P) = P$ and analogously $\lambda_O(P) = P$.

$2^\circ \Rightarrow 1^\circ$: For every $a, b \in P$ we have by $2^\circ$ $\varrho_O(a), \lambda_O(b) \in P$, i. e. by (9) and $2^\circ$ $ab \in P$. For every $a, c \in P$ there is one and only one element $b \in Q$ such that $ab = c$, i. e. $c = \varrho_O(a) + \lambda_O(b)$, and therefore $\lambda_O(b) = c - \varrho_O(a)$. But, $c \in P$ and from $a \in P$ it follows $\varrho_O(a) \in P$. Therefore, by $2^\circ$ we get $\lambda_O(b) \in P$ and $\lambda_O(P) = = P$ implies $b \in P$. Analogously, we can prove that for every $b, c \in P$ there is one and only one element $a \in P$ such that $ab = c$. This means that $(P, \cdot)$ is a subquasigroup of $(Q, \cdot)$. From $2^\circ$ it follows immediately $O \in P$.

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## Geometrija IM-kvazigrupa

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*Sadržaj*

U kvazigrupi $(Q, \cdot)$ sa svojstvima idempotentnosti i medijalnosti, izra
nim identitetima $aa = a$ i $ab \cdot cd = ac \cdot bd$, vrijede sva svojstva iz [8], ali idem
tentnost daje mogućnost da se dobije i niz novih »geometrijskih« rezultata. D
se i »geometrijski« dokaz teorema o reprezentaciji IM-kvazigrupa.

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