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ADMISSIBLE IDENTITIES IN COMPLEX IM-QUASIGROUPS II.

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Abstract. The mutual relations of primitive admissible identities (1)—(149) from [7] and of corresponding complementary identities were found. On this basis the characterizations of some special classes of IM-quasigroups were given.

In [7], wherefrom we shall use the symbols and notions, all primitive admissible identities (1)—(149) (with the corresponding complementary identities) of the form $\langle m\rangle' = \langle n\rangle$ in complex IM-quasigroups are found, where $m, n \in \in \{1, 2 3, 4\}$, $m \geqslant n$. Obviously two such identities are equivalent in a complex IM-quasigroup $(C, \circ)$ iff they hold in the same quasigroup $C(q)$.

But, it is a question which mutual relations these identities have in any IM-quasigroup or even in a general (perhaps in an idempotent) quasigroup $(Q, \cdot)$. The answer to this question will be given here. So we shall also obtain the characterizations of some special IM-quasigroups.

Instead of the quasigroup $(C, \circ)$ we shall consider here any IM-quasigroup $(Q, \cdot)$, i. e. for the sake of simplicity the operation is denoted as a multiplication because the multiplication of complex numbers is not necessary here (it will appear only in the symbols of the quasigroup $C(q)$). In IM-quasigroup $(Q, \cdot)$ it hold the identities

$$aa = a, \tag{II}$$

$$ab \cdot cd = ac \cdot bd, \tag{III}$$

$$ab \cdot a = a \cdot ba, \tag{IV}$$

$$ab \cdot c = ac \cdot bc, \tag{V}$$

$$a \cdot bc = ab \cdot ac, \tag{VI}$$

which express respectively the properties of idempotency, mediality, elasticity and right and left distributivity.

The study of mutual relations of the identities (1)—(149) and corresponding complementary identities will be realized so that identities will be collected with regard to the quasigroups $C(q)$ in which these identities hold.

1. $C\left(\frac{1}{2}\right)$

$$ab = ba, \tag{1}$$

$$a \cdot ab = ba \cdot a, \tag{16}$$

$$a \cdot bc = cb \cdot a, \tag{17}$$

(94) $$a(ab \cdot b) = (ab \cdot b)a, \quad a(b \cdot ba) = (b \cdot ba)a, \tag{94'}$$

(98) $$a(ab \cdot b) = (b \cdot ab)a, \quad a(ba \cdot b) = (b \cdot ba)a, \tag{98)'}$$

$$a(ab \cdot b) = (b \cdot ba)a, \tag{99}$$

$$a(ab \cdot c) = (c \cdot ba)a, \tag{101}$$

$$a(ba \cdot b) = (b \cdot ab)a, \tag{105}$$

$$a(ba \cdot c) = (c \cdot ab)a, \tag{107}$$

$$a(bc \cdot c) = (c \cdot cb)a, \tag{113}$$

$$a(a \cdot ab) = (ba \cdot a)a, \tag{115}$$

$$a(b \cdot ac) = (ca \cdot b)a, \tag{117}$$

$$a(b \cdot ba) = (ab \cdot b)a, \tag{120}$$

$$a(a \cdot bc) = (cb \cdot a)a, \tag{123}$$

$$a(b \cdot bc) = (cb \cdot b)a, \tag{124}$$

$$ab \cdot ba = ba \cdot ab, \tag{126}$$

$$ab \cdot bc = cb \cdot ba, \tag{129}$$

$$a(bc \cdot d) = (d \cdot cb)a, \tag{146}$$

$$a(b \cdot cd) = (dc \cdot b)a, \tag{148}$$

$$ab \cdot cd = db \cdot ca, \tag{149}$$

THEOREM 1. *In any IM-quasigroup $(Q, \cdot)$ the identities*

(1), (16), (17), (98), (98)', (101), (107), (113),
(120), (123), (124), (129), (146), (148), (149), (A)

and also the identities

(105), (115), (117) (B)

are mutually equivalent. Any of the identities (A) implies everyone of the identities (B) and each of the identities (94), (94)', (99), (126).

Proof. Obviously, in any quasigroup $(Q, \cdot)$ the identity (1) implies everyone of the mentioned identities except the identity (149), for the proof of which the mediality (III) is also necessary. Moreover, we have the implications

$(17)\,(c = b) \Rightarrow (1)$, $\quad (101)\,(c = a) \Rightarrow (1)$,

$(107)\,(c = a) \Rightarrow (1)$, $\quad (113)\,(c = b) \Rightarrow (1)$,

(123) $(c = a) \Rightarrow (1)$, (124) $(c = b) \Rightarrow (1)$,

(146) $(d = c = b) \Rightarrow (1)$, (148) $(d = c = b) \Rightarrow (1)$,

(149) $(b = a,\ d = c) \Rightarrow (1)$, (117) $(c = b) \Rightarrow (105)$,

(117) $(b = a) \Rightarrow (115)$, (129) $(b = a) \Rightarrow (16)$,

where e. g. (129) $(b = a) \Rightarrow (16)$ means that with $b = a$ from (129) in the IM-quasigroup $(Q, \cdot)$ it follows (16), eventually after a substitution of variables (as here: $c \to b$). It remains to prove some implications.

(16) $\Rightarrow$ (1): If we apply successively the identities (II), (16), (III), (IV) and again (II), we obtain

$$a \cdot ab = (a \cdot ab)(a \cdot ab) = (a \cdot ab)(ba \cdot a) = (a \cdot ba)(ab \cdot a) =$$
$$= (a \cdot ba)(a \cdot ba) = a \cdot ba,$$

wherefrom it follows $ab = ba$.

(98) $\Rightarrow$ (1): Because of (VI), (II), (98), (III), (V), (IV), (VI), (VI) and (V) we have successively

$$(a \cdot ab) \cdot ab = a(ab \cdot b) = a(ab \cdot b) \cdot a(ab \cdot b) = a(ab \cdot b) \cdot (b \cdot ab)a =$$
$$= a(b \cdot ab) \cdot (ab \cdot b)a = a(b \cdot ab) \cdot (ab \cdot a)(ba) = a(ba \cdot b) \cdot (a \cdot ba)(ba) =$$
$$= (a \cdot ba)(ab) \cdot (a \cdot ba)(ba) = (a \cdot ba)(ab \cdot ba) = (a \cdot ab) \cdot ba,$$

wherefrom it immediately follows $ab = ba$. Complementarily, it holds the implication $(98)' \Rightarrow (1)$.

(120) $\Rightarrow$ (1): By (II), (V), (V), (120), (III), (120), (VI), (III) and (II) we get

$$(ab \cdot ba)(ab \cdot ba) = ab \cdot ba = (a \cdot ba) \cdot (b \cdot ba) = a(b \cdot ba) \cdot (ba)(b \cdot ba) =$$
$$= (ab \cdot b)a \cdot (ba)(b \cdot ba) = (ab \cdot b)(ba) \cdot a(b \cdot ba) = (ab \cdot b)(ba) \cdot (ab \cdot b)a =$$
$$= (ab \cdot b)(ba \cdot a) = (ab \cdot ba) \cdot ba = (ab \cdot ba)(ba \cdot ba),$$

and so it holds $ab = ba$.

(107) $\Rightarrow$ (119): Let $a, b, c \in Q$ be any elements. There is an element $d \in Q$ such that it holds the equality

$$dc = b. \tag{XIV}$$

If we apply successively the equalities (XIV), (V), (VI), (IV), (107), (IV), (III), (107), (V), (VI), (III) and (XIV), then we obtain

$$a(b \cdot ac) = a(dc \cdot ac) = a \cdot (d \cdot ac)(c \cdot ac) = a(d \cdot ac) \cdot a(c \cdot ac) =$$
$$= a(d \cdot ac) \cdot a(ca \cdot c) = a(d \cdot ac) \cdot (c \cdot ac)a = a(d \cdot ac) \cdot (ca \cdot c)a =$$
$$= a(ca \cdot c) \cdot (d \cdot ac)a = (c \cdot ac)a \cdot (d \cdot ac)a = (c \cdot ac)(d \cdot ac) \cdot a =$$
$$= (cd \cdot ac)a = (ca \cdot dc)a = (ca \cdot b)a.$$

(117) $\Rightarrow$ (119): Let $a, b, c \in Q$ be any elements. There is $d \in Q$ such that it holds the equality

$$ad = b. \tag{XV}$$

Because of (XV), (V), (VI), (117), (VI), (XV), (III), (117), (VI) and (V) it follows successively

$$a(b \cdot ac) = a(ad \cdot ac) = a \cdot (a \cdot ac)(d \cdot ac) = a(a \cdot ac) \cdot a(d \cdot ac) =$$

$$= (ca \cdot a)a \cdot a(d \cdot ac) = (ca \cdot a)a \cdot (ad)(a \cdot ac) = (ca \cdot a)a \cdot b(a \cdot ac) =$$

$$= (ca \cdot a)b \cdot a(a \cdot ac) = (ca \cdot a)b \cdot (ca \cdot a)a = (ca \cdot a) \cdot ba = (ca \cdot b)a.$$

By Theorem 1 everyone of the identities (A) implies each of the identities (B) and identities (94), (94)′, (99). But, the converses of these implications do not hold, because the identities (B) hold also in the quasigroup $C\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)$, the identity (94) in the quasigroup $C(2)$, the identity (94)′ in the quasigroup $C(-1)$, and the identity (99) in the quasigroup $C\left(\frac{1}{2}(1 \pm \sqrt{5})\right)$, where the identities (A) do not hold. It remains an open question if a converse of the implication $(A) \Rightarrow$ $\Rightarrow (126)$ holds.

From Theorem 1 it follows that a special class of IM-quasigroups (for which $C\left(\frac{1}{2}\right)$ is an interpretation) can be characterized so that we demand the properties (II) and (III) and one of the identities (A). It is most natural to take the identity (1) and the corresponding class of quasigroups contains the so-called IMC-quasigroups (see [2], [3], [4]).

But, instead of three we can take only two identities, because it holds:

THEOREM 2. *In any idempotent quasigroup from the identity* (149) *it follows the mediality and every identity mentioned in Theorem* 1.

Proof. We already know that from (149) with $b = a$ and $d = c$, because of (II), it follows (1) and then from (1) it follows any of the mentioned identities. Moreover, because of (149) and (1) we obtain the mediality:

$$ab \cdot ca = db \cdot ca = ca \cdot db = ac \cdot bd.$$

2. $C\left(\frac{1}{2} \pm \frac{1}{2} i\right)$

$$ab \cdot ba = b, \tag{10}$$

$$ab \cdot a = ba \cdot b, \tag{11}$$

$$ab \cdot ca = bc \cdot b, \tag{46}$$

(69) $\quad (ab \cdot b)a = (b \cdot ba)b, \quad a(b \cdot ba) = b(ab \cdot b),$ (69)′

(81) $\quad (ab \cdot c)a = (c \cdot ba)c, \quad a(b \cdot ca) = b(ac \cdot b),$ (81)′

$$a(ba \cdot b) = (ba \cdot a)b, \quad (a \cdot ba)b = a(b \cdot ba), \tag{104, 104'}$$

$$a(b \cdot ca) = (ca \cdot b)c, \quad (ab \cdot c)a = b(c \cdot ab), \tag{125, 125'}$$

$$ab \cdot ba = cb \cdot bc, \tag{127}$$

$$ab \cdot ca = db \cdot cd. \tag{140}$$

THEOREM 3. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (10), (11), (46), (69), (69)', (81), (81)', (104), (104)', (125), (125)', (127) and (140) *are mutually equivalent.*

Proof. (10) ⇒ (11): By (VI), (II), (10) and (V) we have successively

$$(ab \cdot b)(ab \cdot a) = ab \cdot ba = (ab \cdot ba)(ab \cdot ba) = (ab \cdot ba)b = (ab \cdot b)(ba \cdot b),$$

wherefrom it follows (11).

(11) ⇒ (10): Because of (II), (VI), (11) and (V) it follows

$$(ab \cdot ba)(ab \cdot ba) = ab \cdot ba = (ab \cdot b)(ab \cdot a) = (ab \cdot b)(ba \cdot b) = (ab \cdot ba)b,$$

wherefrom we obtain the identity (10).

(10) & (11) ⇒ (46): Because of (VI), (11), (III), (10) and (IV) we get

$$ab \cdot ca = (ab \cdot c)(ab \cdot a) = (ab \cdot c)(ba \cdot b) = (ab \cdot ba) \cdot cb = b \cdot cb = bc \cdot b.$$

With the proven implications, the implications

(46) $(c = a)$ ⇒ (11), (127) $(c = b)$ ⇒ (10), (140) $(d = b)$ ⇒ (46), and the obvious implications (10) ⇒ (127) and (46) ⇒ (140) it follows the mutual equivalency of the identities (10), (11), (46), (127) and (140).

(11) ⇒ (81): By (V), (11), (III) and again (11) it follows

$$(ab \cdot c)a = (ab \cdot a) \cdot ca = (ba \cdot b) \cdot ca = (ba \cdot c) \cdot ba = (c \cdot ba)c.$$

(11) ⇒ (104): By triple application of (11) and by (V) we have

$$a(ba \cdot b) = a(ab \cdot a) = (ab \cdot a) \cdot ab = (ba \cdot b) \cdot ab = (ba \cdot a)b.$$

(104) ⇒ (11): Because of (IV), (VI), (104) and (V) we obtain

$$(ab \cdot a) \cdot ab = (a \cdot ba) \cdot ab = a(ba \cdot b) = (ba \cdot a)b = (ba \cdot b) \cdot ab,$$

wherefrom it follows (11).

(11) ⇒ (125): Owing to the identities (VI), (IV), (11), (III), (IV), (11), (V) it follows

$$a\,(b \cdot ca) = ab \cdot (a \cdot ca) = ab \cdot (ac \cdot a) = ab \cdot (ca \cdot c) =$$

$$= (a \cdot ca) \cdot bc = (ac \cdot a) \cdot bc = (ca \cdot c) \cdot bc = (ca \cdot b)\, c.$$

(69) ⇒ (10): We have successively, by (II), (VI), (VI), (69), (III), (IV), (III), (IV), (II), (V)

$$(ab \cdot ba)\,(ab \cdot ba) = ab \cdot ba = (ab \cdot b)\,(ab \cdot a) = (ab \cdot b)\,(ab) \cdot (ab \cdot b)\, a =$$

$$= (ab \cdot b)\,(ab) \cdot (b \cdot ba)\, b = (ab \cdot b)\,(b \cdot ba) \cdot (ab \cdot b) = (ab \cdot b) \cdot (b \cdot ba)(ab \cdot b) =$$

$$= (ab \cdot b) \cdot (b \cdot ab)\,(ba \cdot b) = (ab \cdot b) \cdot (ba \cdot b)\,(ba \cdot b) = (ab \cdot b)\,(ba \cdot b) =$$

$$= (ab \cdot ba)\, b,$$

wherefrom we obtain (10).

With the implications

$$(81)\,(b = a) \Rightarrow (11), \quad (125)\,(b = a) \Rightarrow (69)$$

we conclude that the identities (81), (104), (69) and (125) are equivalent with the identities (10), (11), (46), (127), and (140). The same is true for the complementary identities (81)′, (104)′, (69)′ and (125)′. Q. E. D.

Any of the identities mentioned in Theorem 3, together with (II) and (III), chraacterizes a class of IM-quasigroups. But, these quasigroups can be characterized by only one identity, because it holds:

THEOREM 4. *In any quasigroup* $(Q, \cdot)$ *from the identity* (46) *it follows that* $(Q, \cdot)$ *is an IM-quasigroup with all the above-mentioned identities.*

Proof. With $c = b = a$ from (46) it follows the idempotency (II). But, then with $b = a$ from (46) it follows (IV) and with $c = a$ from (46) it follows (11). Now, let $a, b, c \in Q$ be any elements. There is $d \in Q$ such that

$$da = c. \tag{XVI}$$

Now, we have by (XVI), (IV), (11), (XVI), (46), (11), (46) and (XVI) successively

$$ac \cdot bc = (a \cdot da) \cdot bc = (ad \cdot a) \cdot bc = (da \cdot d) \cdot bc =$$

$$= cd \cdot bc = db \cdot d = bd \cdot b = ab \cdot da = ab \cdot c,$$

i. e. the identity (V). Further, let $a, b, c, d \in Q$ be any elements. There is $e \in Q$ such that

$$ab = de. \tag{XVII}$$

By (V), (XVII) and (46) we have

$$bd \cdot e = be \cdot de = be \cdot ab = ea \cdot e,$$

wherefrom it follows

$$bd = ea. \tag{XVIII}$$

Because of (XVII), (46), (11), (46) and (XVIII) it follows successively

$$ab \cdot cd = de \cdot cd = ec \cdot e = ce \cdot c = ac \cdot ea = ac \cdot bd,$$

i.e. the mediality (III). The rest of our statement follows on the basis of Theorem 3.

3. $C\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)$

$$ab \cdot a = b, \tag{2}$$

$$ab \cdot a = cb \cdot c, \tag{12}$$

(25) $$(ab \cdot a)\, b = cb \cdot c, \quad a\,(b \cdot ab) = c \cdot ac,$$ (25)′

$$(ab \cdot a)\, a = (b \cdot ba)\, b, \tag{54}$$

(65) $$(ab \cdot a)\, c = (b \cdot bc)\, b, \quad a\,(b \cdot cb) = c\,(ac \cdot c),$$ (65)′

(89) $$(a \cdot ab)\, c = (a \cdot ac)\, b, \quad a\,(bc \cdot c) = b\,(ac \cdot c),$$ (89)′

$$a\,(ba \cdot b) = (b \cdot ab)\, a, \tag{105}$$

$$a\,(ba \cdot b) = (c \cdot ac)\, a, \tag{106}$$

$$a\,(bc \cdot b) = (b \cdot ab)\, c, \tag{108}$$

$$a\,(a \cdot ab) = (ba \cdot a)\, a, \tag{115}$$

$$a\,(b \cdot ac) = (ca \cdot b)\, a, \tag{117}$$

$$a\,(b \cdot ac) = (cb \cdot a)\,b, \tag{119}$$

(122) $$a\,(a \cdot bc) = (ca \cdot a)\,b,\ (ab \cdot c)\,c = b\,(c \cdot ca),$$ (122)′

(131) $$(ab \cdot a)\,c = (d \cdot bc)\,d,\ a\,(b \cdot cb) = d\,(ac \cdot d),$$ (131)′

$$a\,(bc \cdot b) = (d \cdot ad)\,c, \tag{137}$$

(144) $$(a \cdot bc)\,d = (a \cdot bd)\,c,\ a\,(bc \cdot d) = b\,(ac \cdot d),$$ (144)′

$$a\,(b \cdot cd) = (da \cdot b)\,c. \tag{147}$$

THEOREM 5. *In any IM-quasigroup* $(Q, \cdot)$ *the identities*

$$\begin{gathered}(2),\ (12),\ (25),\ (25)',\ (54),\ (65),\ (65)',\\ (89),\ (89)',\ (106),\ (108),\ (119),\ (122),\ (122)',\\ (131),\ (131)',\ (137),\ (144),\ (144)',\ (147)\end{gathered} \tag{C}$$

and also the identities

$$(105),\ (115),\ (117) \tag{B}$$

are mutually equivalent. Any of the identities (C) *implies every of the identities* (B).

Proof. First of all we have the implications

$$\begin{array}{lll}(12)\,(c = b) \Rightarrow (2), & (25)\,(c = b) \Rightarrow (2), & (65)\,(c = b) \Rightarrow (2),\\ (89)\,(c = a) \Rightarrow (2), & (106)\,(c = a) \Rightarrow (2), & (108)\,(c = b) \Rightarrow (2),\\ (119)\,(c = a) \Rightarrow (2), & (122)\,(c = a) \Rightarrow (2), & (131)\,(d = c) \Rightarrow (2),\\ (137)\,(d = a) \Rightarrow (2), & (144)\,(d = b) \Rightarrow (2), & (147)\,(d = a) \Rightarrow (2),\end{array}$$

and it is obvious that (2) implies everyone of the identities (12), (25), (54), (65), (106), (108), (131) and (137). As in Theorem 1 the equivalence of the identities (B) was proven, so it remains to prove the implication (54) $\Rightarrow$ (12) and that (2) implies each of the identities (89), (117), (119), (122), (144) and (147).

(54) $\Rightarrow$ (12): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = c. \tag{XIX}$$

By (II), (XIX), (VI), (VI), (V), (54), (III), (III), (VI), (VI), (XIX), (V) and (III) we have successively

$$\begin{aligned}&(b \cdot bc)\,b \cdot (b \cdot bc)\,b = (b \cdot bc)\,b = b\,(b \cdot ad) \cdot b = b\,(ba \cdot bd) \cdot b =\\ &= (b \cdot ba)\,(b \cdot bd) \cdot b = (b \cdot ba)\,b \cdot (b \cdot bd)\,b = (ab \cdot a)\,a \cdot (b \cdot bd)\,b =\end{aligned}$$

$$= (ab \cdot a)(b \cdot bd) \cdot ab = (ab \cdot b)(a \cdot bd) \cdot ab = (ab \cdot b)(ab \cdot ad) \cdot ab =$$

$$= (ab)(b \cdot ad) \cdot ab = (ab \cdot bc) \cdot ab = (a \cdot bc)(b \cdot bc) \cdot ab = (a \cdot bc)\, a \cdot (b \cdot bc) b,$$

wherefrom it follows the identity $(b \cdot bc)\, b = (a \cdot bc)\, a$. But, by substitution $bc \to c$ and then the substitution $b \leftrightarrow c$, this identity becomes (12).

(2) ⇒ (89): By (2), (III) and again (2) it follows

$$(a \cdot ab)\, c = (a \cdot ab)(ac \cdot a) = (a \cdot ac)(ab \cdot a) = (a \cdot ac)\, b.$$

(2) ⇒ (117): Because of (2), (IV), (V), (III), (VI) and (2) we obtain

$$a\,(b \cdot ac) = (ca \cdot c)(b \cdot ac) = (c \cdot ac)(b \cdot ac) = cb \cdot ac =$$

$$= ca \cdot bc = (ca \cdot b)(ca \cdot c) = (ca \cdot b)\, a.$$

(2) ⇒ (119): By (2), (IV), (V), (VI) and (2) we have

$$a\,(b \cdot ac) = (ca \cdot c)(b \cdot ac) = (c \cdot ac)(b \cdot ac) = cb \cdot ac = (cb \cdot a)(cb \cdot c) =$$

$$= (cb \cdot a)\, b.$$

(2) ⇒ (122): Owing to (2), (III), (IV) and (2) it follows

$$a\,(a \cdot bc) = (ca \cdot c)(a \cdot bc) = (ca \cdot a)(c \cdot bc) = (ca \cdot a)(cb \cdot c) = (ca \cdot a)\, b.$$

(2) ⇒ (144): By (2), (IV), (VI), (III), (VI), (IV) and (2) we get

$$(a \cdot bc)\, d = (a \cdot bc)(ad \cdot a) = (a \cdot bc)(a \cdot da) = a\,(bc \cdot da) =$$

$$= a\,(bd \cdot ca) = (a \cdot bd)(a \cdot ca) = (a \cdot bd)(ac \cdot a) = (a \cdot bd)\, c.$$

(2) ⇒ (147): Because of (2), (III), (IV) and (2) it follows

$$a\,(b \cdot cd) = (da \cdot d)(b \cdot cd) = (da \cdot b)(d \cdot cd) = (da \cdot b)(dc \cdot d) = (da \cdot b)\, c.$$

Q. E. D.

Any of the identities (C) implies everyone of the identities (B). But, the converses of these implications do not hold, because the identities (B) hold also in the quasigroup $C\left(\frac{1}{2}\right)$, where the identities (C) do not hold.

The identities (II) and (III) together with any of the identities (C) characterize a class of IM-quasigroups which was considered in [5]. But, instead of three it is enough to take only two identities, because we have:

THEOREM 6. *In any idempotent quasigroup* $(Q, \cdot)$ *the identities* (144), (144)′ *and* (147) *are mutually equivalent and any of these identities implies mediality and all identities* (B) *and* (C).

Proof. As the identities (II), (III) and (147) are complementary to itself and (144) and (144)′ are complementary identities, so it is sufficient to prove the implications (144) ⇒ (III), (144) ⇒ (144)′, (144) ⇒ (147), (147) ⇒ (144) and then to apply Theorem 5. If we put $d = b = a$ respectively $d = a$ in (144), then by (II) we obtain the identities $(a \cdot ac)\, a = ac$, $(a \cdot bc)\, a = (a \cdot ba)\, c$. By the substitution $ac \to b$ the first identity takes the form of the identity (2), and because of (2) the second identity takes the form $(a \cdot ba)\, c = bc$, wherefrom it follows

$$a \cdot ba = b. \qquad (2)'$$

By (144) and by four applications of (2) we have successively

$$(ab \cdot cd)(ac) \cdot b = (ab \cdot cd)(ab) \cdot c = cd \cdot c = d = bd \cdot b = (ac \cdot bd)(ac) \cdot b,$$

wherefrom it follows (III). By the four applications of (2) and then by (III) we obtain

$$[a(bc \cdot d) \cdot a] \cdot bc = (bc \cdot d) \cdot bc = d = (ac \cdot d) \cdot ac =$$
$$= [b(ac \cdot d) \cdot b] \cdot ac = [b(ac \cdot d) \cdot a] \cdot bc,$$

wherefrom it follows (144)'. Owing to (2)', (2), (2)', (2)' and (144) we get

$$(b \cdot cd) \cdot a(b \cdot cd) = a = da \cdot d = b(da \cdot b) \cdot d = b[c \cdot (da \cdot b)c] \cdot d =$$
$$= (b \cdot cd) \cdot (da \cdot b)c,$$

wherefrom it follows (147). On the other hand, from (147) with $d = c = b$ resp. $d = b = a$ it follows by (II) $ab = (ba \cdot b)b$, $a(a \cdot ca) = ac$, wherefrom we obtain $ba \cdot b = a$, $a \cdot ca = c$, i. e. the identities (2) and (2)'. Finally, by (147), (2)' and three applications of (2) we have

$$(a \cdot bc)d \cdot (a \cdot bd) = [d \cdot (a \cdot bc)d]a \cdot b = (a \cdot bc)a \cdot b = bc \cdot b = c =$$
$$= (a \cdot bd)c \cdot (a \cdot bd),$$

wherefrom it follows (144).

4. $C\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}i\right)$

$$ab \cdot ba = ba \cdot b, \tag{45}$$

$$a(b \cdot ab) = (ba \cdot a)b, \tag{121}$$

(130) $\quad ab \cdot ca = ba \cdot cb, \; ab \cdot ca = cb \cdot ac.$ (130)'

THEOREM 7. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (45), (121), (130) *and* (130)' *are mutually equivalent.*

Proof. (45) ⇒ (130): By (VI), (45), (III), (45), (IV), (V) and (III) we obtain

$$ab \cdot ca = (ab \cdot c)(ab \cdot a) = (ab \cdot c)(ba \cdot ab) = (ab \cdot ba)(c \cdot ab) =$$
$$= (ba \cdot b)(c \cdot ab) = (b \cdot ab)(c \cdot ab) = bc \cdot ab = ba \cdot cb.$$

(45) ⇒ (121): Because of (VI), (VI), (45), (III), (45) and (V) it follows

$$a(b \cdot ba) = ab \cdot (a \cdot ba) = (ab \cdot a)(ab \cdot ba) =$$
$$= (ab \cdot a)(ba \cdot b) = (ab \cdot ba) \cdot ab = (ba \cdot b) \cdot ab = (ba \cdot a)b.$$

(121) ⇒ (45): By (V), (V), (121), (III), (V), (III), (121), (V), (VI) and (II)

we have successively

$$ab \cdot ba = (a \cdot ba)(b \cdot ba) = a(b \cdot ba) \cdot (ba)(b \cdot ba) = (ba \cdot a) b \cdot (ba)(b \cdot ba) =$$
$$= (ba \cdot a)(ba) \cdot b(b \cdot ba) = (ba \cdot b) a \cdot b(b \cdot ba) = (ba \cdot b) b \cdot a(b \cdot ba) =$$
$$= (ba \cdot b) b \cdot (ba \cdot a) b = (ba \cdot b)(ba \cdot a) \cdot b = (ba \cdot ba) b = ba \cdot b.$$

Moreover, we have the implication (130) $(c = b) \Rightarrow$ (45) and the complementary implications (45) $\Rightarrow$ (130)′ and (130)′ $\Rightarrow$ (45). Q. E. D.

5. $C\left(\frac{1}{2}(1 \pm \sqrt{5})\right)$

(9) $$(a \cdot ab) b = a, \quad a(ab \cdot b) = b,$$ (9)′

(22) $$(ab \cdot a) a = ab \cdot b, \quad a(a \cdot ba) = b \cdot ba,$$ (22)′

(36) $$(a \cdot bc) c = ab \cdot b, \quad a(ab \cdot c) = b \cdot bc,$$ (36)′

(55) $$(ab \cdot a) a = (a \cdot bc) c, \quad a(a \cdot ba) = c(cb \cdot a),$$ (55)′

(86) $$(a \cdot ab) b = (a \cdot ac) c, \quad a(ab \cdot b) = c(cb \cdot b),$$ (86)′

$$a(ab \cdot b) = (b \cdot ba) a,$$ (99)

$$a(ab \cdot b) = (b \cdot bc) c,$$ (100)

(136) $$(a \cdot bc) c = (a \cdot bd) d, \quad a(ab \cdot c) = d(db \cdot c).$$ (136)′

THEOREM 8. *In any IM-quasigroup* $(Q, \cdot)$ *the identities*

(9), (9)′, (22), (22)′, (36), (36)′, (55),

(55)′, (86), (86)′, (100), (136), (136)′ $\qquad (D)$

are mutually equivalent. Any of the identities (D) *implies the identity* (99).

Proof. The implications

(86) $(c = a) \Rightarrow$ (9), (136) $(d = b) \Rightarrow$ (36), (136) $(d = a) \Rightarrow$ (55) and the obvious implications (9) $\Rightarrow$ (86), (36) $\Rightarrow$ (136) and (55) $\Rightarrow$ (136) prove the equivalence (9) $\Leftrightarrow$ (86) and the mutual equivalences of the identities (36), (55) and (136). As it holds the implication (136) $(b = a) \Leftrightarrow$ (86), so for the proof of the equivalence of all five considered identities it suffices to prove the implication (9) $\Rightarrow$ (36).

(9) $\Rightarrow$ (36): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$bd = a.$$ (XX)

Now, by (XX), (V), (V), (9), (VI), (VI), (XX), (III) and (9) we obtain

$$(a \cdot bc) c = (bd \cdot bc) c = (b \cdot bc)(d \cdot bc) \cdot c = (b \cdot bc) c \cdot (d \cdot bc) c = b \cdot (d \cdot bc) c =$$

$= b\,(d \cdot bc) \cdot bc = (bd)\,(b \cdot bc) \cdot bc = a\,(b \cdot bc) \cdot bc = ab \cdot (b \cdot bc)\,c = ab \cdot b.$

(9) $\Leftrightarrow$ (22): Because of (IV), (IV), (VI) and (VI) we have

$$a \cdot (ab \cdot a)\,a = a\,(ab \cdot a) \cdot a = (a \cdot ab)\,a \cdot a,$$

$$a\,(ab \cdot b) = (a \cdot ab) \cdot ab = (a \cdot ab)\,a \cdot (a \cdot ab)\,b,$$

and so the identities $a \cdot (ab \cdot a)\,a = a\,(ab \cdot b), (a \cdot ab)\,a \cdot (a \cdot ab)\,b = (a \cdot ab)\,a \cdot a$ are equivalent, i. e. the identities (22) and (9) are equivalent.

Therefore, the identities (9), (22), (36), (55), (86) and (136) are equivalent and the same is true for the complementary identities (9)′, (22)′, (36)′, (55)′, (86)′ and (136)′.

(9) $\Rightarrow$ (9)′: For any elements $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = b, \tag{XV}$$

and by (VI), (IV), (XV), (VI), (9), (V), (VI), (V), (XV) and (II) we obtain successively

$$ab \cdot a\,(ab \cdot b) = a \cdot b\,(ab \cdot b) = a \cdot (b \cdot ab)\,b = a \cdot (ad \cdot ab)\,b =$$

$$= a \cdot (a \cdot db)\,b = (a \cdot ab)\,b \cdot (a \cdot db)\,b = (a \cdot ab)\,(a \cdot db) \cdot b =$$

$$= a\,(ab \cdot db) \cdot b = a\,(ad \cdot b) \cdot b = (a \cdot bb)\,b = ab \cdot b,$$

wherefrom it follows (9)′.

Complementarily, it holds (9)′ $\Rightarrow$ (9). But, from (9) and (9)′ we obtain immediately

$$a\,(ab \cdot b) = b = (b \cdot bc)\,c,$$

i. e. the identity (100). Moreover, we have the implications

$$(100)\,(b = a) \Rightarrow (9), \quad (100)\,(c = a) \Rightarrow (99). \qquad \text{Q. E. D.}$$

Any of the identities (D) implies (99). The converse is not true, because the identity (99) holds also in the quasigroup $C\left(\frac{1}{2}\right)$, where the identities ($D$) do not hold.

For the characterization of the considered class of IM-quasigroups it is sufficient to take only two identities as we have:

THEOREM 9. *In any right distributive quasigroup* $(Q, \cdot)$ *the identity* (36) *implies idempotency, mediality and all identities considered in Theorem* 8.

Proof. With $c = b = a$ from (36) it follows idempotency and by (36), (V), (36) and again (36) we get successively

$$(ab \cdot cd)\,d = (ab \cdot c)\,c = (ac \cdot bc)\,c = (ac \cdot b)\,b = (ac \cdot bd)\,d,$$

wherefrom it follows mediality. The rest of the theorem follows now on the basis of Theorem 8.

6. $C(2)$

$$ab \cdot b = a, \tag{3}$$

$$ab \cdot b = ac \cdot c, \tag{13}$$

$$(ab \cdot b)\, a = ac \cdot c, \tag{26}$$

$$a\,(ab \cdot b) = ac \cdot c, \tag{37}$$

$$(ab \cdot a)\, a = (ab \cdot b)\, b, \tag{47}$$

$$(ab \cdot a)\, a = (ac \cdot c)\, b, \tag{48}$$

$$(ab \cdot a)\, c = (ac \cdot a)\, b, \tag{62}$$

$$(ab \cdot b)\, c = (ac \cdot b)\, b, \tag{76}$$

$$a\,(ab \cdot b) = (ab \cdot b)\, a, \tag{94}$$

$$a\,(ab \cdot b) = (ac \cdot c)\, a, \tag{96}$$

$$a\,(bc \cdot c) = (ab \cdot b)\, b, \tag{110}$$

$$a\,(bc \cdot c) = (ac \cdot c)\, b, \tag{111}$$

$$(ab \cdot b)\, c = (ac \cdot d)\, d, \tag{132}$$

$$a\,(bc \cdot c) = (ab \cdot d)\, d, \tag{138}$$

$$a\,(bc \cdot c) = (ad \cdot d)\, b, \tag{139}$$

$$(ab \cdot c)\, d = (ad \cdot c)\, b, \tag{141}$$

$$(a \cdot bc)\, d = (a \cdot dc)\, b. \tag{145}$$

THEOREM 10. *In any IM-quasigroup* $(Q, \cdot)$ *the identities*

$$\begin{array}{c}(3),\ (13),\ (26),\ (37),\ (47),\ (48),\ (62),\ (76),\ (96),\\ (110),\ (111),\ (132),\ (138),\ (139),\ (141),\ (145)\end{array} \tag{E}$$

are mutually equivalent and everyone of them implies the identity (94).

Proof. We have the implications

$$(13)\,(c = a) \Rightarrow (3), \qquad (26)\,(b = a) \Rightarrow (3),$$

$$(37)\,(b = a) \Rightarrow (3), \qquad (48)\,(b = a) \Rightarrow (3),$$

$$(76)\ (c = a) \Rightarrow (3), \qquad (96)\ (c = a) \Rightarrow (3),$$

$$(110)\ (c = b) \Rightarrow (3), \qquad (111)\ (c = b) \Rightarrow (3),$$

$$(132)\ (d = c = a) \Rightarrow (3), \quad (138)\ (d = b = a) \Rightarrow (3),$$

$$(139)\ (c = b = a) \Rightarrow (3), \quad (141)\ (d = a, c = b) \Rightarrow (3),$$

and it is obvious that the identity (3) implies each of the identities (13), (26), (37), (47), (76), (94), (96), (110), (111), (132), (138) and (139). Therefore, it suffices to prove some more implications.

(47) $\Rightarrow$ (3): Because of (II), (47), (III), (VI) and (II) we obtain

$$(ab \cdot b)\, b = (ab \cdot b)\, b \cdot (ab \cdot b)\, b = (ab \cdot a)\, a \cdot (ab \cdot b)\, b =$$
$$= (ab \cdot a)(ab \cdot b) \cdot ab = (ab \cdot ab) \cdot ab = ab,$$

wherefrom it follows (3).

(3) $\Rightarrow$ (62): By (V), (V), (III), and (3) it follows

$$(ab \cdot a)\, c = (ab \cdot c) \cdot ac = (ac \cdot bc) \cdot ac = (ac \cdot a)(bc \cdot c) = (ac \cdot a)\, b.$$

(62) $\Rightarrow$ (141): For any $a, b, c, d \in Q$ there is $e \in Q$ such that

$$ae = c. \tag{XXI}$$

Because of (II), (XXI), (VI), (V), (62), (III), (III), (XX), (V) and (III) we have successively

$$(ab \cdot c)\, d \cdot (ab \cdot c)\, d = (ab \cdot c)\, d = (ab \cdot ae)\, d = (ab \cdot a)(ab \cdot e) \cdot d =$$
$$= (ab \cdot a)\, d \cdot (ab \cdot e)\, d = (ad \cdot a)\, b \cdot (ab \cdot e)\, d = (ad \cdot a)(ab \cdot e) \cdot bd =$$
$$= (ad \cdot ab)(ae) \cdot bd = (ad \cdot ab)\, c \cdot bd = (ad \cdot c)(ab \cdot c) \cdot bd =$$
$$= (ad \cdot c)\, b \cdot (ab \cdot c) d,$$

wherefrom it follows the identity (141).

(141) $\Rightarrow$ (145): By (VI), (141) and (VI) we get

$$(a \cdot bc)\, d = (ab \cdot ac)\, d = (ad \cdot ac)\, b = (a \cdot dc)\, b.$$

(145) $\Rightarrow$ (141): Because of (V), (145) and (V) it follows

$$(ab \cdot c)\, d = (ac \cdot bc)\, d = (ac \cdot dc)\, b = (ad \cdot c)\, b. \qquad \text{Q. E. D.}$$

Any of the identities (E) implies the identity (94). The converse is not true, because the identity (94) holds also in the quasigroup $C\left(\frac{1}{2}\right)$, where the identities ($E$) do not hold.

The identities (II), (III) together with one of the identities (E) (e. g. the identity (3)) characterize a class of quasigroups, which was considered in [6].

But, instead of three we can take only two identities, because by the results of *D. Vakarelov* [6] it holds:

THEOREM 11. *In any idempotent quasigroup* $(Q, \cdot)$ *the identity* (141) *implies mediality and all identities mentioned in Theorem* 10.

Proof. We already know that with (II) it holds the implication (141) $\Rightarrow$ (3). For any $a, b, c \in Q$ there is $d \in Q$ such that

$$bc \cdot d = ab \cdot c. \tag{XXII}$$

By (3), (3), (XXII), (141), (141), (II), (3), (141), (II) and (XXII) it follows

$$ac \cdot bc = (ab \cdot b)\, c \cdot bc = [(ab \cdot c)\, c \cdot b]\, c \cdot bc = [(bc \cdot d)\, c \cdot b]\, c \cdot bc =$$
$$= [(bc \cdot d)\, c \cdot bc]\, c \cdot b = [(bc \cdot bc)\, c \cdot d]\, c \cdot b = [(bc \cdot c)\, d \cdot c]\, b =$$
$$= (bd \cdot c)\, b = (bb \cdot c)\, d = bc \cdot d = ab \cdot c,$$

which implies the identity (V). Now, because of (3), (V), (141), (V) and (3) we obtain

$$ab \cdot cd = (ab \cdot d)\, d \cdot cd = (ab \cdot d)\, c \cdot d = (ac \cdot d)\, b \cdot d = (ac \cdot d)\, d \cdot bd = ac \cdot bd,$$

i. e. mediality (III). Finally, it is sufficient to apply Theorem 10.

7. $C(-2)$

$$a\,(a \cdot ab) = ab \cdot b. \tag{40}$$

8. $C\left(\frac{2}{3}\right)$

$$ab \cdot ba = a \cdot ab, \tag{44$'$}$$

$$a\,(a \cdot ab) = (ab \cdot b)\, a, \tag{114}$$

$$ab \cdot bc = ac \cdot cb. \tag{128}$$

THEOREM 12. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (44)′, (114) *and* (128) *are mutually equivalent.*

Proof. (44)′ $\Rightarrow$ (114): By (44)′, (VI), (44)′, (V) and (V) we have

$$a\,(a \cdot ab) = a\,(ab \cdot ba) = (a \cdot ab)(a \cdot ba) = (ab \cdot ba)(a \cdot ba) =$$
$$= (ab \cdot a) \cdot ba = (ab \cdot b)\, a.$$

(44)′ $\Rightarrow$ (128): Because of (II), (V), (44)′, (VI) and (III) we obtain

$$(ab \cdot bc)(ab \cdot bc) = ab \cdot bc = (a \cdot bc)(b \cdot bc) =$$
$$= (a \cdot bc)(bc \cdot cb) = (ab \cdot ac)(bc \cdot cb) = (ab \cdot bc)(ac \cdot cb),$$

wherefrom it follows (128).

(114) $\Rightarrow$ (44)′: By (II), (VI), (IV), (VI), (114), (III), (VI) and (VI) we get

$$(ab \cdot ba)(ab \cdot ba) = ab \cdot ba = (ab \cdot b)(ab \cdot a) = (ab \cdot b)(a \cdot ba) =$$

$$= (ab \cdot b)\, a \cdot (ab \cdot b)(ba) = a\,(a \cdot ab) \cdot (ab \cdot b)(ba) = a\,(ab \cdot b) \cdot (a \cdot ab)(ba) =$$
$$= (a \cdot ab)(ab) \cdot (a \cdot ab)(ba) = (a \cdot ab)(ab \cdot ba),$$

wherefrom it follows (44)′.

Finally, we have also the implication (128) $(c = a) \Rightarrow$ (44)′.

9. $C\left(\frac{3}{2}\right)$

$$(ab \cdot a)\, a = (a \cdot ab)\, b, \tag{53}$$

$$(a \cdot bc)\, c = (a \cdot cb)\, b, \tag{93}$$

$$a\,(ab \cdot b) = (ab \cdot b)\, b. \tag{95}$$

THEOREM 13. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (53), (93) *and* (95) *are equivalent.*

Proof. We shall prove the equivalence of the identities (53), (93), (95) and the auxiliary identities

$$(ab \cdot ba)\, a = ab \cdot b, \tag{XXIII}$$

$$(ab \cdot ba)\, b = a \cdot ab. \tag{XXIV}$$

Besides the implication (93) $(c = a) \Rightarrow$ (53) we prove some more implications and equivalences.

(53) $\Rightarrow$ (93): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$cd = a. \tag{XXV}$$

By (XXV), (III), (III), (III), (VI), (III), (53), (V), (V), (VI), (II), (VI) and (XXV) we have successively

$$(a \cdot bc)\, c \cdot (c \cdot db)\, b = (cd \cdot bc)\, c \cdot (c \cdot db)\, b = (cb \cdot dc)\, c \cdot (c \cdot db)\, b =$$
$$= (cb \cdot dc)(c \cdot db) \cdot cb = (cb \cdot c)(dc \cdot db) \cdot cb = (cb \cdot c)(d \cdot cb) \cdot cb =$$
$$= (cb \cdot c)\, c \cdot (d \cdot cb)\, b = (c \cdot cb)\, b \cdot (d \cdot cb)\, b = (c \cdot cb)(d \cdot cb) \cdot b = (cd \cdot cb)\, b =$$
$$= (c \cdot db)\, b = (c \cdot db)\, b \cdot (c \cdot db)\, b = (cd \cdot cb)\, b \cdot (c \cdot db)\, b = (a \cdot cb)\, b\, (c \cdot db)\, b,$$

wherefrom it follows (93).

(53) $\Leftrightarrow$ (XXIII): Because of (II), (IV), (IV), (IV), (III), (VI) and (VI) we get

$$(ab \cdot a)\, a \cdot (ab \cdot a)\, a = (ab \cdot a)\, a = (a \cdot ba)a = a\,(ba \cdot a),$$
$$(ab \cdot a)\, a \cdot (a \cdot ab)\, b = (a \cdot ba)\, a \cdot (a \cdot ab)\, b =$$
$$= (a \cdot ba)(a \cdot ab) \cdot ab = a\,(ba \cdot ab) \cdot ab = a \cdot (ba \cdot ab)\, b$$

and it follows that the identities

$$(ab \cdot a)\, a \cdot (ab \cdot a)\, a = (ab \cdot a)\, a \cdot (a \cdot ab)\, b, \;\; a \cdot (ba \cdot ab)\, b = a\,(ba \cdot a)$$

are equivalent. So, the same is true for the identities (53) and (XXIII).

(95) $\Leftrightarrow$ (XXIV): By (V), (V), (IV), (V), (IV) and (V) we have

$$a(ab \cdot b) \cdot b(ab \cdot b) = ab \cdot (ab \cdot b) = (a \cdot ab)\, b,$$

$$(ab \cdot b)\, b \cdot b(ab \cdot b) = (ab \cdot b)\, b \cdot (b \cdot ab)\, b =$$

$$= (ab \cdot b)(b \cdot ab) \cdot b = (ab \cdot b)(ba \cdot b) \cdot b = (ab \cdot ba)\, b \cdot b$$

and hence the identities

$$a(ab \cdot b) \cdot b(ab \cdot b) = (ab \cdot b)\, b \cdot b(ab \cdot b), \quad (a \cdot ab)\, b = (ab \cdot ba)\, b \cdot b$$

are equivalent. Therefore, the same is true for the identities (95) and (XXIV).

(XXIII) $\Leftrightarrow$ (XXIV): According to (III), (IV), (III) and (VI) we get

$$(ab \cdot b)(a \cdot ab) = (ab \cdot a)(b \cdot ab) = (ab \cdot a)(ba \cdot b) =$$

$$= (ab \cdot ba) \cdot ab = (ab \cdot ba)\, a \cdot (ab \cdot ba)\, b$$

and the identities

$$(ab \cdot ba)\, a \cdot (a \cdot ab) = (ab \cdot b)(a \cdot ab),$$

$$(ab \cdot ba)\, a \cdot (a \cdot ab) = (ab \cdot b\, a)\, a \cdot (ab \cdot ba)\, b$$

are equivalent, i. e. the identities (XXIII) and (XXIV) are equivalent.

10. $C\left(\frac{1}{2}(2 \pm \sqrt{2})\right)$

$$ab \cdot b = ba \cdot a, \tag{14}$$

$$(ab \cdot b)\, a = (a \cdot ab)\, b, \tag{68}$$

$$(ab \cdot b)\, b = (b \cdot ba)\, a, \tag{74}$$

$$(ab \cdot c)\, c = (c \cdot ba)\, a. \tag{85}$$

THEOREM 14. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (14), (68), (74) *and* (85) *are equivalent.*

Proof. (14) $\Leftrightarrow$ (68): Because of (V), (IV), (III) and (V) we have

$$(ab \cdot b)\, a = (ab \cdot a) \cdot ba = (a \cdot ba) \cdot ba = ab \cdot (ba \cdot a),$$

$$(a \cdot ab)\, b = ab \cdot (ab \cdot b)$$

and it follows that the identity (68) is equivalent with the identity $ab \cdot (ba \cdot a) = ab \cdot (ab \cdot b)$, i. e. with the identity (14).

(14) $\Rightarrow$ (85): By (V), (V), (14), (III), (VI), (III), (14) and (V) we obtain successively

$$(ab \cdot c)\, c = (ac \cdot bc)\, c = (ac \cdot c)\,(bc \cdot c) = (ca \cdot a)\,(cb \cdot b) = (ca \cdot cb) \cdot ab =$$
$$= (c \cdot ab) \cdot ab = ca \cdot (ab \cdot b) = ca \cdot (ba \cdot a) = (c \cdot ba)\, a.$$

(74) $\Rightarrow$ (85): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$cd = b. \tag{XXVI}$$

By (III), (V), (III), (VI), (III), (XXVI), (VI), (74), (V), (VI), (V), (XXVI) and (II) we have

$$(ab \cdot c)\, c \cdot (c \cdot ba)\, a = (ab \cdot c)\,(c \cdot ba) \cdot ca = (ac \cdot bc)\,(c \cdot ba) \cdot ca =$$
$$= (ac \cdot c)\,(bc \cdot ba) \cdot ca = (ac \cdot c)\,(b \cdot ca) \cdot ca = (ac \cdot c)\, c \cdot (b \cdot ca)\, a =$$
$$= (ac \cdot c)\, c \cdot (cd \cdot ca)\, a = (ac \cdot c)\, c \cdot (c \cdot da)\, a = (c \cdot ca)\, a \cdot (c \cdot da)\, a =$$
$$= (c \cdot ca)\,(c \cdot da) \cdot a = c\,(ca \cdot da) \cdot a = c\,(cd \cdot a) \cdot a = (c \cdot ba)\, a =$$
$$= (c \cdot ba)\, a \cdot (c \cdot ba)\, a,$$

wherefrom it follows (85).

Finally, we have also the implications

$$(85)\,(b = a) \Rightarrow (14), \qquad (85)\,(c = b) \Rightarrow (74). \tag{XXVII}$$

But, it holds also:

THEOREM 15. *In any idempotent quasigroup* $(Q, \cdot)$ *the identity* (85) *implies the identities* (14), (68) *and* (74).

Proof. Besides the implications (XXVII), which were proved only by the idempotency of the quasigroup $(Q, \cdot)$, it suffices to prove the implication (14) & (74) $\Rightarrow$ (68):

$$(ab \cdot b)\, a = (ba \cdot a)\, a = (a \cdot ab)\, b.$$

11. $C(2 \pm \sqrt{3})$

$$(ab \cdot b)\, b = ba \cdot ab. \tag{75}$$

12. $C(-1 \pm \sqrt{2})$

$$a\,(b \cdot ba) = ba \cdot a, \tag{43}$$
$$(a \cdot ab)\, b = ba \cdot ab. \tag{88}$$

THEOREM 16. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (43) *and* (88) *are equivalent.*

Proof. (43) $\Rightarrow$ (88): Because of (V), (43), (VI), (43), (V) and (V) it follows

$$(a \cdot ab)\, b = ab \cdot (ab \cdot b) = ab \cdot b\,(a \cdot ab) = (ab \cdot b) \cdot (ab)\,(a \cdot ab) =$$
$$= b\,(a \cdot ab) \cdot (ab)\,(a \cdot ab) = (b \cdot ab)\,(a \cdot ab) = ba \cdot ab.$$

(88) $\Rightarrow$ (43): By (II), (VI), (VI), (IV), (88) and (III) we have

$$a(b \cdot ba) \cdot a(b \cdot ba) = a(b \cdot ba) = ab \cdot (a \cdot ba) = (ab \cdot a)(ab \cdot ba) =$$
$$= (a \cdot ba)(ab \cdot ba) = (a \cdot ba) \cdot (b \cdot ba) a = a(b \cdot ba) \cdot (ba \cdot a),$$

wherefrom it follows (43).

13. $C\left(\frac{1}{2}(3 \pm \sqrt{5})\right)$

$$ab \cdot b = ba, \tag{4}$$

$$ab \cdot c = cb \cdot a, \tag{15}$$

$$(ab \cdot b) a = ab, \tag{19}$$

$$(ab \cdot a) b = (b \cdot ba) a, \tag{61}$$

$$(ab \cdot b) c = (ac \cdot c) b, \tag{77}$$

$$a(ab \cdot b) = (ba \cdot a) a, \tag{97}$$

$$a(bc \cdot c) = (cb \cdot a) a, \tag{112}$$

$$(ab \cdot c) d = da \cdot cb. \tag{143}$$

THEOREM 17. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (4), (15), (19), (61), (77), (97), (112) *and* (143) *are mutually equivalent.*

Proof. (4) ⇒ (15): Because of (II), (V), (VI), (4), (V) and (III) we have

$$(ab \cdot c)(ab \cdot c) = ab \cdot c = ac \cdot bc = (ac \cdot b)(ac \cdot c) =$$
$$= (ac \cdot b) \cdot ca = (ab \cdot cb) \cdot ca = (ab \cdot c)(cb \cdot a),$$

wherefrom it follows (15).

(4) & (15) ⇒ (77): By (4), (15) and (4) it follows

$$(ab \cdot b) c = ba \cdot c = ca \cdot b = (ac \cdot c) b.$$

(19) ⇒ (4): According to (III), (IV), (V), (19) and (II) we get

$$ba \cdot (ab \cdot b) = (b \cdot ab) \cdot ab = (ba \cdot b) \cdot ab = (ba \cdot a) b = ba = ba \cdot ba,$$

and it follows (4).

Together with the implications

$$(15)(c = b) \Rightarrow (4), \quad (77)(c = a) \Rightarrow (19)$$

it follows that the identities (4), (15), (19) and (77) are equivalent.

(4) ⇒ (61): By (V), (4), (4) and (V) we obtain

$$(ab \cdot a) b = (ab \cdot b) \cdot ab = ba \cdot ab = ba \cdot (ba \cdot a) = (b \cdot ba) a.$$

(61) $\Rightarrow$ (4): Owing to (V), (II), (61), (III), (IV), (V) and (II) we have

$$(ab \cdot b) \cdot ab = (ab \cdot a)\, b = (ab \cdot a)\, b \cdot (ab \cdot a)\, b = (b \cdot ba)\, a \cdot (ab \cdot a)\, b =$$
$$= (b \cdot ba)(ab \cdot a) \cdot ab = (b \cdot ba)(a \cdot ba) \cdot ab = (ba \cdot ba) \cdot ab = ba \cdot ab,$$

wherefrom it follows (4).

(4) $\Rightarrow$ (97): By (4), (IV) and (4) it follows successively

$$a\,(ab \cdot b) = a \cdot ba = ab \cdot a = (ba \cdot a)\, a.$$

(97) $\Rightarrow$ (4): Because of (II), (97), (III), (IV), (V), (IV), (VI) and (II) we get

$$(ab \cdot b)\, b = (ab \cdot b)\, b \cdot (ab \cdot b)\, b = b\,(ba \cdot a) \cdot (ab \cdot b)\, b = b\,(ab \cdot b) \cdot (ba \cdot a)\, b =$$
$$= (b \cdot ab)\, b \cdot (ba \cdot a)\, b = (b \cdot ab)(ba \cdot a) \cdot b = (ba \cdot b)(ba \cdot a) \cdot b =$$
$$= (ba \cdot ba)\, b = ba \cdot b,$$

i. e. the identity (4).

(4) $\Rightarrow$ (112): By two applications of (4) it follows

$$a\,(bc \cdot c) = a \cdot cb = (cb \cdot a)\, a.$$

(15) $\Rightarrow$ (143): According to (15) and (III) we obtain

$$(ab \cdot c)\, d = dc \cdot ab = da \cdot cb.$$

Finally, we have also the implications

$$(112)\,(c = b) \Rightarrow (4), \qquad (143)\,(b = a) \Rightarrow (15).$$ Q. E. D.

THEOREM 18. *In any idempotent quasigroup* $(Q, \cdot)$ *the identity* (15) *implies mediality and all identities mentioned in Theorem* 17.

Proof. By (15) we get

$$ab \cdot cd = (cd \cdot b)\, a = (bd \cdot c)\, a = ac \cdot bd,$$

i. e. the mediality. Now, it suffices to apply Theorem 17.

14. $C\,(1 \pm i)$

$$(ab \cdot a)\, b = a, \tag{6}$$

$$(ab \cdot b)\, b = ab \cdot a, \tag{28}$$

$$(ab \cdot c)\, b = ac \cdot a, \tag{31}$$

$$(a \cdot ab)\, a = ab \cdot b, \tag{33}$$

$$(a \cdot bc)\, b = ac \cdot c, \tag{35}$$

$$(ab \cdot a)\, b = (ac \cdot a)\, c, \tag{57}$$

$$(ab \cdot c)\, b = (ad \cdot c)\, d, \tag{134}$$

$$(a \cdot bc)\, b = (a \cdot dc)\, d. \tag{135}$$

THEOREM 19. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (6), (28), (31), (33), (35), (57), (134) *and* (135) *are equivalent.*

Proof. (28) $\Rightarrow$ (31): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$bd = c. \tag{XXVIII}$$

By (II), (XXVIII), (VI), (V), (28), (III), (V), (III), (VI), (XXVIII), (III), (V) and (III) we have

$$\begin{aligned}(ab \cdot c)\, b \cdot (ab \cdot c)\, b &= (ab \cdot c)\, b = (ab \cdot bd)\, b = (ab \cdot b)(ab \cdot d) \cdot b = \\ &= (ab \cdot b)\, b \cdot (ab \cdot d)\, b = (ab \cdot a) \cdot (ab \cdot d)\, b = (ab)(ab \cdot d) \cdot ab = \\ &= (ab)(ad \cdot bd) \cdot ab = (ab)(ad \cdot c) \cdot ab = (ab)(ac \cdot dc) \cdot ab = \\ &= (a \cdot ac)(b \cdot dc) \cdot ab = (a \cdot ac)(bd \cdot bc) \cdot ab = (a \cdot ac)(c \cdot bc) \cdot ab = \\ &= (ac)(ac \cdot bc) \cdot ab = (ac)(ab \cdot c) \cdot ab = (ac \cdot a) \cdot (ab \cdot c)\, b,\end{aligned}$$

and it follows (31).

(134) $\Rightarrow$ (135): By (VI), (134) and (VI) we have

$$(a \cdot bc)\, b = (ab \cdot ac)\, b = (ad \cdot ac)\, d = (a \cdot dc)\, d.$$

(33) $\Rightarrow$ (28): Because of (II), (33) and (III) we obtain

$$(ab \cdot b)\, b \cdot (ab \cdot b)\, b = (ab \cdot b)\, b = (ab)(ab \cdot b) \cdot ab = (ab \cdot a) \cdot (ab \cdot b)\, b,$$

wherefrom it follows (28).

Together with the obvious implication (31) $\Rightarrow$ (134) and the implications

$$(135)\,(d = c) \Rightarrow (35), \quad (35)\,(b = a) \Rightarrow (33)$$

it follows the equivalence of the identities (31), (28), (33), (35), (134), and (135). Besides the obvious implication (6) $\Rightarrow$ (57) and the implications

$$(31)\,(c = b) \Rightarrow (6), \quad (57)\,(c = a) \Rightarrow (6),$$

it is sufficient to prove one more implication.

(6) $\Rightarrow$ (31): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = c. \tag{XIX}$$

By (II), (XIX), (VI), (V), (6), (VI), (VI), (XIX), (V) and (III) we get

$$\begin{aligned}(ab \cdot c)\, b \cdot (ab \cdot c)\, b &= (ab \cdot c)\, b = (ab \cdot ad)\, b = (ab \cdot a)(ab \cdot d) \cdot b = \\ &= (ab \cdot a)\, b \cdot (ab \cdot d)\, b = a \cdot (ab \cdot d)\, b = a\,(ab \cdot d) \cdot ab = (a \cdot ab)(ad) \cdot ab = \\ &= (a \cdot ab)\, c \cdot ab = (ac)(ab \cdot c) \cdot ab = (ac \cdot a) \cdot (ab \cdot c)\, b,\end{aligned}$$

Q. E. D.

wherefrom it follows (31).

Any of the identities mentioned in Theorem 19, together with (II) and (III), characterizes a class of IM-quasigroups. But these quasigroups can be characterized by only one identity, because it holds:

THEOREM 20. *In any quasigroup* $(Q, \cdot)$ *the identity* (35) *implies idempotency, mediality and all identities considered in Theorem* 19.

Proof. With $c = b$ from (35) it follows $bb = b$, i. e. the identity (II), and with $c = a$, because of (II), it follows the identity

$$(a \cdot ba)\, b = a. \tag{XXIX}$$

By (35), both sides of (135) are $ac \cdot c$, and so (135) holds. For any $a, b \in Q$ there is $c \in Q$ such that

$$cb = a. \tag{XXX}$$

By (II), (II), (135) and (XXX) we have

$$(ab \cdot a)(ab \cdot a) = ab \cdot a = (ab \cdot ab)\, a = (ab \cdot cb)\, c = (ab \cdot a)\, c,$$

wherefrom it follows the equality

$$ab \cdot a = c. \tag{XXXI}$$

Because of (XXXI), (XXX) and (XXIX) we obtain

$$(ab \cdot a)\, b = cb = a = (a \cdot ba)\, b$$

and it holds (IV). Owing to (35) and (II) it follows

$$(ab \cdot b)\, b = (ab \cdot ab)\, a = ab \cdot a,$$

i. e. the identity (28). By (28), (IV) and (XXIX) we have

$$(ab \cdot b)\, b \cdot b = (ab \cdot a)\, b = (a \cdot ba)\, b = a$$

and it follows the identity

$$(ab \cdot b)\, b \cdot b = a. \tag{XXXII}$$

For any $a, b, c \in Q$ there is $d \in Q$ such that

$$cd = b. \tag{XXVI}$$

Because of (XXVI), (35), (XXVI), (35) and (XXXII) if follows

$$(ab \cdot c)\, b \cdot c = (ab \cdot c)(cd) \cdot c = (ab \cdot c)\, d \cdot d = [(a \cdot cd)\, c \cdot d]\, d =$$
$$= (ad \cdot d)\, d \cdot d = a$$

and we have the identity

$$(ab \cdot c)\, b \cdot c = a. \tag{XXXIII}$$

By (XXXIII) and again (XXXIII) we get

$$(ab \cdot c)\, b \cdot c = a = (ad \cdot c)\, d \cdot c,$$

i. e. the identity (134). According to (134) and (135) we obtain

$$(ab \cdot cd)\, b = (ac \cdot cd)\, c = (ac \cdot bd)\, b,$$

wherefrom it follows mediality (III). The rest of our theorem is a consequence of Theorem 19.

The considered class of IM-quasigroups with the identity (35) is complementary to the class of IM-quasigroups with the identity

$$a(ba \cdot c) = b \cdot bc, \tag{35'}$$

for which $C(\pm i)$ is a model. These quasigroups are rot-quasigroups of *J. Duplák* [1].

15. $C\left(\frac{3}{2} \pm \frac{\sqrt{3}}{2} i\right)$

$$(ab \cdot b)b = a, \tag{8}$$

$$(ab \cdot b)b = (ac \cdot c)c. \tag{71}$$

THEOREM 21. *In any idempotent quasigroup* $(Q, \cdot)$ *the identities* (8) *and* (71) *are equivalent.*

Obvious.

16. $C\left(1 \pm \frac{\sqrt{2}}{2} i\right)$

$$(ab \cdot b)b = (a \cdot ab)a. \tag{73}$$

17. $C\left(\frac{3}{4} \pm \frac{\sqrt{7}}{4} i\right)$

$$(ab \cdot a)b = (a \cdot ab)a, \tag{60}$$

$$(a \cdot bc)b = (a \cdot cb)c, \tag{91}$$

$$(ab \cdot b)b = a(b \cdot ba), \tag{116'}$$

$$(ab \cdot c)b = a(c \cdot ba). \tag{118'}$$

THEOREM 22. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (60), (91), (116)′ *and* (118)′ *are equivalent.*

Proof. (60) $\Rightarrow$ (118)′: For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = c. \tag{XIX}$$

By (XIX), (VI), (V), (60), (V), (XIX), (IV), (III), (VI), (VI), (XIX), (V), (60), (VI), (III), (IV) and (VI) it follows

$$(ab \cdot c)b = (ab \cdot ad)b = (ab \cdot a)(ab \cdot d) \cdot b = (ab \cdot a)b \cdot (ab \cdot d)b =$$

$$= (a \cdot ab)\, a \cdot (ab \cdot d)\, b = (a \cdot ab)\, a \cdot (ad \cdot bd)\, b = (a \cdot ab)\, a \cdot (c \cdot bd)\, b =$$

$$= a\,(ab \cdot a) \cdot (c \cdot bd)\, b = a\,(c \cdot bd) \cdot (ab \cdot a)\, b = (ac)\,(a \cdot bd) \cdot (ab \cdot a)\, b =$$

$$= (ac)\,(ab \cdot ad) \cdot (ab \cdot a)\, b = (ac)\,(ab \cdot c) \cdot (ab \cdot a)\, b = (a \cdot ab)\, c \cdot (ab \cdot a)\, b =$$

$$= (a \cdot ab)\, c \cdot (a \cdot ab)\, a = (a \cdot ab) \cdot ca = ac \cdot (ab \cdot a) = ac \cdot (a \cdot ba) = a\,(c \cdot ba).$$

(116)′ ⇒ (118)′: For any $a, b, c \in Q$ there is $d \in Q$ such that

$$db = c. \qquad \text{(XXXIV)}$$

Because of (II), (XXXIV), (VI), (V), (116)′, (V), (XXXIV), (III), (VI), (III) and (V) we obtain

$$(ab \cdot c)\, b \cdot (ab \cdot c)\, b = (ab \cdot c)\, b = (ab \cdot db)\, b = (ab \cdot d)\,(ab \cdot b) \cdot b =$$

$$= (ab \cdot d)\, b \cdot (ab \cdot b)\, b = (ab \cdot d)\, b \cdot a\,(b \cdot ba) = (ab \cdot b)\,(db) \cdot a\,(b \cdot ba) =$$

$$= (ab \cdot b)\, c \cdot a\,(b \cdot ba) = (ab \cdot b)\, a \cdot c\,(b \cdot ba) = (ab \cdot b)\, a \cdot (cb)\,(c \cdot ba) =$$

$$= (ab \cdot b)\,(cb) \cdot a\,(c \cdot ba) = (ab \cdot c)\, b \cdot a\,(c \cdot ba)$$

and it follows (118)′.

(60) & (118)′ ⇒ (91): By (VI), (VI), (V), (60), (118)′, (IV), (VI), (IV), (V), (VI), (VI), (IV), (60), (III), (IV), (60), (V), (VI), and (VI) it holds

$$(a \cdot bc)\, b = (ab \cdot ac)\, b = (ab \cdot a)\,(ab \cdot c) \cdot b = (ab \cdot a)\, b \cdot (ab \cdot c)\, b =$$

$$= (a \cdot ab)\, a \cdot (ab \cdot c)\, b = (a \cdot ab)\, a \cdot a\,(c \cdot ba) = a\,(ab \cdot a) \cdot a\,(c \cdot ba) =$$

$$= a \cdot (ab \cdot a)\,(c \cdot ba) = a \cdot (a \cdot ba)\,(c \cdot ba) = a\,(ac \cdot ba) = a \cdot (ac \cdot b)\,(ac \cdot a) =$$

$$= a\,(ac \cdot b) \cdot a\,(ac \cdot a) = a\,(ac \cdot b) \cdot (a \cdot ac)\, a = a\,(ac \cdot b) \cdot (ac \cdot a)\, c =$$

$$= a\,(ac \cdot a) \cdot (ac \cdot b)\, c = (a \cdot ac)\, a \cdot (ac \cdot b)\, c = (ac \cdot a)\, c \cdot (ac \cdot b)\, c =$$

$$= (ac \cdot a)\,(ac \cdot b) \cdot c = (ac \cdot ab)\, c = (a \cdot cb)\, c.$$

Finally, we have the implications

(118)′ $(c = a)$ ⇒ (60), (118)′ $(c = b)$ ⇒ (116)′, (91) $(c = a)$ ⇒ (60).

18. $2q^3 - 1 = 0$

$$a\,(a \cdot ab) = b\,(b \cdot ba). \qquad (72)'$$

The equation **18** is obtained from the complementary equation $2q^3 - 6q^2 + 6q - 1 = 0$ by the substitution $q \to 1 - q$, which shall be used also in the future. The solutions of **18** are

$$q \approx 0{,}7937005, \; q \approx -0{,}3968503 \pm 0{,}6873648\, i.$$

$$\mathbf{19.}\ q^3 - 2q^2 + q - 1 = 0$$

$$(ab \cdot a)\, a = b, \tag{5}$$

$$(ab \cdot a)\, a = (cb \cdot c)\, c. \tag{52}$$

Obviously, it holds:

THEOREM 23. *In any idempotent quasigroup* $(Q, \cdot)$ *the identities* (5) *and* (52) *are equivalent.*

The solutions of the equation **19** are

$$q \approx 1{,}7548777,\ q \approx 0{,}1225612 \pm 0{,}7448618\, i.$$

$$\mathbf{20.}\ q^3 - 3q^2 + 2q - 1 = 0$$

$$(ab \cdot b)\, a = b, \tag{7}$$

$$(ab \cdot a)\, a = ba \cdot b, \tag{23}$$

$$(ab \cdot c)\, a = bc \cdot b, \tag{30}$$

$$(ab \cdot a)\, a = (cb \cdot a)\, c, \tag{51}$$

$$(ab \cdot b)\, a = (cb \cdot b)\, c, \tag{67}$$

$$(ab \cdot c)\, a = (db \cdot c)\, d. \tag{133}$$

THEOREM 24. *In any IM-quasigroup* $(Q. \cdot)$ *the identities* (7), (23), (30), (51), (67) *and* (133) *are mutually equivalent.*

Proof. (7) $\Rightarrow$ (30): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$bd = c. \tag{XXVIII}$$

By (II), (XXVIII), (VI), (V), (7), (VI), (VI), (XXVIII), (V) and (III) we have

$$(ab \cdot c)\, a \cdot (ab \cdot c)\, a = (ab \cdot c)\, a = (ab \cdot bd)\, a = (ab \cdot b)\,(ab \cdot d) \cdot a =$$
$$= (ab \cdot b)\, a \cdot (ab \cdot d)\, a = b \cdot (ab \cdot d)\, a = b\,(ab \cdot d) \cdot ba = (b \cdot ab)\,(bd) \cdot ba =$$
$$= (b \cdot ab)\, c \cdot ba = (bc)\,(ab \cdot c) \cdot ba = (bc \cdot b) \cdot (ab \cdot c)\, a,$$

wherefrom it follows (30).

(23) $\Rightarrow$ (7): Because of (II), (V), (VI), (23), (V) and (V) we get

$$(ab \cdot b)\, a \cdot (ab \cdot b)\, a = (ab \cdot b)\, a = (ab \cdot a) \cdot ba = (ab \cdot a)\, b \cdot (ab \cdot a)\, a =$$
$$= (ab \cdot a)\, b \cdot (ba \cdot b) = (ab \cdot a)\,(ba) \cdot b = (ab \cdot b)\, a \cdot b$$

and it holds (7).

Moreover, we have the implications

(133) $(c = a) \Rightarrow$ (51), (51) $(c = b) \Rightarrow$ (23), (67) $(c = b) \Rightarrow$ (7)

and the obvious implications (30) $\Rightarrow$ (133) and (7) $\Rightarrow$ (67). Therefore, we have finally

(7) $\Rightarrow$ (30) $\Rightarrow$ (133) $\Rightarrow$ (51) $\Rightarrow$ (23) $\Rightarrow$ (7) $\Leftrightarrow$ (67). Q. E. D.

The equation **20** has the solutions

$$q \approx 2{,}3247180,\ q \approx 0{,}3376410 \pm 0{,}5622795\, i.$$

21. $q^3 - 2q^2 + 3q - 1 = 0$

$$(ab \cdot a)\, b = ba, \tag{18}$$

$$a\,(ba \cdot b) = ba \cdot a, \tag{39}$$

$$(ab \cdot b)\, b = b \cdot ba, \tag{41'}$$

$$(ab \cdot c)\, b = b \cdot ca, \tag{42'}$$

$$(ab \cdot a)\, c = (cb \cdot c)\, a, \tag{64}$$

$$a\,(bc \cdot b) = (b \cdot ac)\, a. \tag{109}$$

THEOREM 25. *In any IM-quasigroups* $(Q, \cdot)$ *the identities* (18), (39), (41)', (42)', (64) *and* (109) *are equivalent.*

Proof. (18) $\Rightarrow$ (39): By (VI), (IV), (VI), (18), (V) and (18) it follows

$$a\,(ba \cdot b) = (a \cdot ba) \cdot ab = (ab \cdot a) \cdot ab = (ab \cdot a)\, a \cdot (ab \cdot a)\, b =$$
$$= (ab \cdot a)\, a \cdot ba = (ab \cdot a)\, b \cdot a = ba \cdot a.$$

(39) $\Rightarrow$ (109): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = c. \tag{XIX}$$

Because of (XIX), (VI), (V), (VI), (39), (VI), (VI), (XIX), (IV), (III), (III), (IV), (39), (VI) and (V) we obtain

$$a\,(bc \cdot b) = a \cdot (b \cdot ad)\, b = a \cdot (ba \cdot bd)\, b = a \cdot (ba \cdot b)\,(bd \cdot b) =$$
$$= a\,(ba \cdot b) \cdot a\,(bd \cdot b) = (ba \cdot a) \cdot a\,(bd \cdot b) = (ba \cdot a) \cdot (a \cdot bd)\,(ab) =$$
$$= (ba \cdot a) \cdot (ab \cdot ad)\,(ab) = (ba \cdot a) \cdot (ab \cdot c)\,(ab) = (ba \cdot a) \cdot (ab)\,(c \cdot ab) =$$
$$= (ba \cdot a) \cdot (ac)\,(b \cdot ab) = (ba \cdot ac) \cdot a\,(b \cdot ab) = (ba \cdot ac) \cdot a\,(ba \cdot b) =$$
$$= (ba \cdot ac)\,(ba \cdot a) = ba \cdot (ac \cdot a) = (b \cdot ac)\, a.$$

(109) $\Rightarrow$ (42)': For any $a, b, c \in Q$ there is $d \in Q$ such that it holds (XIX). According to (XIX), (VI), (109) and (XIX) we have

$$(ab \cdot c)\, b = (ab \cdot ad)\, b = (a \cdot bd)\, b = b\,(ad \cdot a) = b \cdot ca.$$

(42)' $\Rightarrow$ (109): By (42)' and (VI) it follows

$$a(bc \cdot b) = (ba \cdot bc)\, a = (b \cdot ac)\, a.$$

(41)′ ⇒ (42)′: For any $a, b, c \in Q$ there is $d \in Q$ such that

$$db = c. \tag{XXXIV}$$

Because of (XXXIV), (VI), (V), (41)′, (V), (XXXIV), (III), (41)′, (V) and (VI) it follows

$$(ab \cdot c)\, b = (ab \cdot db)\, b = (ab \cdot d)(ab \cdot b) \cdot b = (ab \cdot d)\, b \cdot (ab \cdot b)\, b =$$
$$= (ab \cdot d)\, b \cdot (b \cdot ba) = (ab \cdot b)(db) \cdot (b \cdot ba) = (ab \cdot b)\, c \cdot (b \cdot ba) =$$
$$= (ab \cdot b)\, b \cdot (c \cdot ba) = (b \cdot ba)(c \cdot ba) = bc \cdot ba = b \cdot ca.$$

As we have the implications

$$(109)\,(c = b) \Rightarrow (18),\ (42)'\,(c = b) \Rightarrow (41)',\ (64)\,(c = b) \Rightarrow (18),$$

so the identities (18), (39), (41)′, (42)′ and (109) are equivalent and it suffices to prove one more implication.

(18) & (42)′ ⇒ (64): By (IV), (V), (42)′, (III), (III), (18) and (V) we obtain

$$(ab \cdot a)\, c = (a \cdot ba)\, c = ac \cdot (ba \cdot c) = (c \cdot ac)(ba) \cdot ac =$$
$$= (cb)(ac \cdot a) \cdot ac = (cb \cdot a) \cdot (ac \cdot a)\, c = (cb \cdot a) \cdot ca = (cb \cdot c)\, a.$$ Q. E. D.

The equation **21** has the solutions

$$q \approx 0{,}4301597,\ q \approx 0{,}7849201 \pm 1{,}3071413\, i.$$

22. $q^3 - 3q^2 + 4q - 1 = 0$

$$(ab \cdot b)\, b = ba, \tag{20}$$

$$(ab \cdot a)\, b = ba \cdot a, \tag{24}$$

$$(ab \cdot c)\, b = bc \cdot a, \tag{32}$$

$$(ab \cdot a)\, c = (cb \cdot a)\, a, \tag{63}$$

$$(ab \cdot b)\, c = (cb \cdot b)\, a, \tag{78}$$

$$(ab \cdot c)\, d = (db \cdot c)\, a. \tag{142}$$

THEOREM 26. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (20), (24), (32) (63), (78), *and* (142) *are equivalent.*

Proof. (24) ⇒ (32): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = c. \tag{XIX}$$

Because of (II), (XIX), (VI), (V), (24), (III), (III), (XIX), (V) and (III) we have

$$(ab \cdot c)\, b \cdot (ab \cdot c)\, b = (ab \cdot c)\, b = (ab \cdot ad)\, b = (ab \cdot a)(ab \cdot d) \cdot b =$$

$$= (ab \cdot a)\, b \cdot (ab \cdot d)\, b = (ba \cdot a) \cdot (ab \cdot d)\, b = (ba)(ab \cdot d) \cdot ab =$$
$$= (b \cdot ab)(ad) \cdot ab = (b \cdot ab)\, c \cdot ab = (bc)(ab \cdot c) \cdot ab = (bc \cdot a) \cdot (ab \cdot c)\, b,$$

wherefrom it follows (32).

(20) $\Rightarrow$ (24): By (V), (VI), (20), (V) and (20) it follows

$$(ab \cdot a)\, b = (ab \cdot b) \cdot ab = (ab \cdot b)\, a \cdot (ab \cdot b)\, b = (ab \cdot b)\, a \cdot ba =$$
$$= (ab \cdot b)\, b \cdot a = ba \cdot a.$$

As we have the implication (32) $(c = b) \Rightarrow$ (20), so the identities (20), (24) and (32) are equivalent.

(20) & (32) $\Rightarrow$ (63): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$da = b. \qquad \text{(XXXV)}$$

By (V), (32), (XXXV), (V), (20), (V), (V), (VI) and (XXXV) it holds

$$(ab \cdot a)\, c = (ab \cdot c) \cdot ac = (ca \cdot b)\, a \cdot ac = (ca \cdot da)\, a \cdot ac = (cd \cdot a)\, a \cdot ac =$$
$$= (cd \cdot a)\, a \cdot (ca \cdot a)\, a = (cd \cdot a)(ca \cdot a) \cdot a = (cd \cdot ca)\, a \cdot a =$$
$$= (c \cdot da)\, a \cdot a = (cb \cdot a)\, a.$$

(63) $\Rightarrow$ (142): For any $a, b, c, d \in Q$ there is $e \in Q$ such that

$$ae = c. \qquad \text{(XXI)}$$

According to (II), (XXI), (VI), (V), (63), (III), (III), (XXI), (V) and (III) we have

$$(ab \cdot c)\, d \cdot (ab \cdot c)\, d = (ab \cdot c)\, d = (ab \cdot ae)\, d = (ab \cdot a)(ab \cdot e) \cdot d =$$
$$= (ab \cdot a)\, d \cdot (ab \cdot e)\, d = (db \cdot a)\, a \cdot (ab \cdot e)\, d = (db \cdot a)(ab \cdot e) \cdot ad =$$
$$= (db \cdot ab)(ae) \cdot ad = (db \cdot ab)\, c \cdot ad = (db \cdot c)(ab \cdot c) \cdot ad =$$
$$= (db \cdot c)\, a \cdot (ab \cdot c)\, d,$$

wherefrom it follows (142).

(78) $\Rightarrow$ (142): For any $a, b, c, d \in Q$ there is $e \in Q$ such that

$$be = c. \qquad \text{(XXXVI)}$$

By (II), (XXXVI), (VI), (V), (78), (III), (III), (XXXVI), (V) and (III) we get

$$(ab \cdot c)\, d \cdot (ab \cdot c)\, d = (ab \cdot c)\, d = (ab \cdot be)\, d = (ab \cdot b)(ab \cdot e) \cdot d =$$
$$= (ab \cdot b)\, d \cdot (ab \cdot e)\, d = (db \cdot b)\, a \cdot (ab \cdot e)\, d = (db \cdot b)(ab \cdot e) \cdot ad =$$
$$= (db \cdot ab)(be) \cdot ad = (db \cdot ab)\, c \cdot ad = (db \cdot c)(ab \cdot c) \cdot ad =$$
$$= (db \cdot c)\, a \cdot (ab \cdot c)\, d$$

and it holds (142).

Moreover, we have the implications

(63) $(b = a) \Rightarrow$ (20), (142) $(c = a) \Rightarrow$ (63), (142) $(c = b) \Rightarrow$ (78).

The theorem is proved.

The equation **22** has the solutions

$$q \approx 0{,}3176722, \; q \approx 1{,}3411639 \pm 1{,}1615414\, i.$$

23. $q^3 - q^2 - 2q + 1 = 0$

$$(a \cdot ab)\, b = ba, \tag{21}$$

$$a\,(ab \cdot b) = ba \cdot a, \tag{38}$$

$$(a \cdot ab)\, c = (c \cdot cb)\, a, \tag{90}$$

$$(a \cdot bc)\, c = c\,(ba \cdot a). \tag{102)'$$

THEOREM 27. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (21), (38), (90) *and* (102)' *are equivalent.*

Proof. (21) ⇒ (38): By (VI), (VI), (21), (V) and (21) it holds

$$a\,(ab \cdot b) = (a \cdot ab) \cdot ab = (a \cdot ab)\, a \cdot (a \cdot ab)\, b = (a \cdot ab)\, a \cdot ba =$$

$$= (a \cdot ab)\, b \cdot a = ba \cdot a.$$

(38) ⇒ (21): Because of (V), (V), (38), (III), (38), (VI) and (II) we have

$$(a \cdot ab)\, b = ab \cdot (ab \cdot b) = a\,(ab \cdot b) \cdot b\,(ab \cdot b) = (ba \cdot a) \cdot b\,(ab \cdot b) =$$

$$= (ba \cdot b) \cdot a\,(ab \cdot b) = (ba \cdot b)\,(ba \cdot a) = ba \cdot ba = ba.$$

(21) ⇒ (90): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$cd = b. \tag{XXVI}$$

By (II), (XXVI), (VI), (VI), (V), (21), (III), (VI), (VI), (XXVI), (V), (III) and (III) we obtain

$$(a \cdot ab)\, c \cdot (a \cdot ab)\, c = (a \cdot ab)\, c = a\,(a \cdot cd) \cdot c = a\,(ac \cdot ad) \cdot c =$$

$$= (a \cdot ac)\,(a \cdot ad) \cdot c = (a \cdot ac)\, c \cdot (a \cdot ad)\, c = ca \cdot (a \cdot ad)\, c =$$

$$= c\,(a \cdot ad) \cdot ac = (ca)\,(c \cdot ad) \cdot ac = (ca)\,(ca \cdot ca) \cdot ac = (ca)\,(ca \cdot b) \cdot ac =$$

$$= (ca)\,(cb \cdot ab) \cdot ac = (c \cdot cb)\,(a \cdot ab) \cdot ac = (c \cdot cb)\, a \cdot (a \cdot ab)c,$$

wherefrom it follows (90).

As we have the implications

$$(90)\,(c = b) \Rightarrow (21), \; (102)'\,(b = a) \Rightarrow (21),$$

so it suffices to prove one more implication.

(21) & (38) ⇒ (102)': For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = b. \tag{XV}$$

According to (XV), (V), (VI), (V), (21), (VI), (XV), (V), (III), (38), (III), (V), (V), and (VI) it follows

$$(a \cdot bc)\, c = a\,(ad \cdot c) \cdot c = a\,(ac \cdot dc) \cdot c = (a \cdot ac)\,(a \cdot dc) \cdot c =$$
$$= (a \cdot ac)\, c \cdot (a \cdot dc)\, c = ca \cdot (a \cdot dc)\, c = ca \cdot (ad \cdot ac)\, c = ca \cdot (b \cdot ac)\, c =$$
$$= ca \cdot (bc)\,(ac \cdot c) = (c \cdot bc) \cdot a\,(ac \cdot c) = (c \cdot bc)\,(ca \cdot a) =$$
$$= (c \cdot ca)\,(bc \cdot a) = (c \cdot ca)\,(ba \cdot ca) = (c \cdot ba) \cdot ca = c\,(ba \cdot a).$$

The equation **23** has the solutions

$$q \approx 1{,}8019377,\ q \approx -1{,}2469796,\ q \approx 0{,}4450419.$$

24. $q^3 - 4q^2 + 3q - 1 = 0$

$$(ab \cdot b)\, a = ba \cdot b, \tag{27}$$

$$(ab \cdot a)\, a = ba \cdot ab, \tag{56}$$

$$(ab \cdot c)\, a = (cb \cdot a)\, c, \tag{80}$$

$$(ab \cdot c)\, a = ba \cdot cb. \tag{82}$$

THEOREM 28. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (27), (56), (80) *and* (82) *are equivalent.*

Proof. (27) $\Rightarrow$ (56): By (27), (V), (IV), (III), (V), (27) and (VI) it holds

$$(ab \cdot a)\, a = (ba \cdot a)\, b \cdot a = (ba \cdot b)\,(ab) \cdot a = (b \cdot ab)\,(ab) \cdot a =$$
$$= (ba)\,(ab \cdot b) \cdot a = (ba \cdot a) \cdot (ab \cdot b)\, a = (ba \cdot a)\,(ba \cdot b) = ba \cdot ab.$$

(56) $\Rightarrow$ (27): Because of (II), (V), (VI), (56), (V), (V) and (III) we obtain

$$(ab \cdot b)\, a \cdot (ab \cdot b)\, a = (ab \cdot b)\, a = (ab \cdot a) \cdot ba = (ab \cdot a)\, b \cdot (ab \cdot a)\, a =$$
$$= (ab \cdot a)\, b \cdot (ba \cdot ab) = (ab \cdot b)\,(ab) \cdot (ba \cdot ab) = (ab \cdot b)\,(ba) \cdot ab =$$
$$= (ab \cdot b)\, a \cdot (ba \cdot b),$$

wherefrom it follows (27).

(27) & (56) $\Rightarrow$ (82): For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = c. \tag{XIX}$$

According to (XIX), (VI), (V), (56), (III), (III), (XIX), (27), (III), (IV), (VI) and (II) it follows

$$(ab \cdot c)\, a = (ab \cdot ad)\, a = (ab \cdot a)\,(ab \cdot d) \cdot a = (ab \cdot a)\, a \cdot (ab \cdot d)\, a =$$
$$= (ba \cdot ab) \cdot (ab \cdot d)\, a = (ba)\,(ab \cdot d) \cdot (ab \cdot a) = (b \cdot ab)\,(ad) \cdot (ab \cdot a) =$$
$$= (b \cdot ab)\, c \cdot (ab \cdot a) = (b \cdot ab)\, c \cdot (ba \cdot a)\, b = (b \cdot ab)\,(ba \cdot a) \cdot cb =$$
$$= (ba \cdot b)\,(ba \cdot a) \cdot cb = (ba \cdot ba) \cdot cb = ba \cdot cb.$$

(82) ⇒ (80): By (82), (III) and (82) we have

$$(ab \cdot c)\, a = ba \cdot cb = bc \cdot ab = (cb \cdot a)\, c.$$

Moreover, it holds the implication (80) $(c = b) \Rightarrow$ (27).

The equation **24** has the solutions

$$q \approx 3{,}1478990, \quad q \approx 0{,}4260505 \pm 0{,}3689894\, i.$$

25. $q^3 - 4q^2 + 5q - 1 = 0$

$$(ab \cdot b)\, b = ba \cdot a, \tag{29}$$

$$(ab \cdot c)\, c = (cb \cdot a)\, a. \tag{84}$$

THEOREM 29. *In any IM-quasigroup $(Q, \cdot)$ the identities* (29) *and* (84) *are equivalent.*

Proof. Besides the implication (84) $(c = b) \Rightarrow$ (29) we have by (V), (V), (29), (III), (III) and (29)

$$(ab \cdot c)\, c = (ac \cdot bc)\, c = (ac \cdot c)\,(bc \cdot c) = (ac \cdot c) \cdot (cb \cdot b)\, b =$$

$$= (ac)\,(cb \cdot b) \cdot cb = (a \cdot cb)\,(cb) \cdot cb = (cb \cdot a)\, a$$

and it holds (29) ⇒ (84).

The equation **25** has the solutions

$$q \approx 0{,}2451223, \quad q \approx 1{,}8774388 \pm 0{,}7448618\, i.$$

26. $q^3 - 3q + 1 = 0$

$$(a \cdot ab)\, b = ba \cdot a, \tag{34}$$

$$(a \cdot bc)\, c = (c \cdot ba)\, a. \tag{92}$$

THEOREM 30. *In any IM-quasigroup $(Q, \cdot)$ the identities* (34) *and* (92) *are equivalent.*

Proof. (34) ⇒ (92): By (VI), (V) (four times), (34), (III) (three times), (V), (34), (V) and (VI) it follows

$$(a \cdot bc)\, c = (ab \cdot ac)\, c = (ab \cdot c)\,(ac \cdot c) = (ac \cdot bc)\,(ac \cdot c) =$$

$$= (ac)\,(ac \cdot c) \cdot (bc)\,(ac \cdot c) = (a \cdot ac)\, c \cdot (bc)\,(ac \cdot c) = (ca \cdot a) \cdot (bc)\,(ac \cdot c) =$$

$$= (ca \cdot bc) \cdot a\,(ac \cdot c) = (cb \cdot ac) \cdot a\,(ac \cdot c) = (cb \cdot a) \cdot (ac)\,(ac \cdot c) =$$

$$= (cb \cdot a) \cdot (a \cdot ac)\, c = (cb \cdot a)\,(ca \cdot a) = (cb \cdot ca)\, a = (c \cdot ba)\, a.$$

Moreover, it holds (92) $(b = a) \Rightarrow$ (34).

The equation **26** has the solutions

$$q \approx 1{,}5320889, \quad q \approx -1{,}8793852, \quad q = 0{,}3472964.$$

$$\mathbf{27.}\ 2q^3 - 5q^2 + 3q - 1 = 0$$

$$(ab \cdot a)\, a = (ba \cdot a)\, b, \tag{49}$$

$$(ab \cdot c)\, a = (ba \cdot c)\, b. \tag{79}$$

THEOREM 31. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (49) *and* (79) *are equivalent.*

Proof. At first we have the implication (79) $(c = a) \Rightarrow$ (49). For any $a, b, c \in Q$ there is $d \in Q$ such that

$$db = c. \tag{XXXIV}$$

By (II), (XXXIV), (VI), (V), (49), (III), (III), (XXXIV), (V) and (III) we have

$$(ab \cdot c)\, a \cdot (ab \cdot c)\, a = (ab \cdot c)\, a = (ab \cdot db)\, a = (ab \cdot d)\,(ab \cdot b) \cdot a =$$

$$= (ab \cdot d)\, a \cdot (ab \cdot b)\, a = (ab \cdot d)\, a \cdot (ba \cdot b)\, b = (ab \cdot d)\,(ba \cdot b) \cdot ab =$$

$$= (ab \cdot ba)\,(db) \cdot ab = (ab \cdot ba)\, c \cdot ab = (ab \cdot c)\,(ba \cdot c) \cdot ab = (ab \cdot c)\, a \cdot (ba \cdot c) b,$$

wherefrom it follows (79). So it holds (49) $\Rightarrow$ (79).

The equation **27** has the solutions

$$q \approx 1{,}8294835, \ q \approx 0{,}3352582 \pm 0{,}4011273\, i.$$

$$\mathbf{28.}\ 2q^3 - 4q^2 + 2q - 1 = 0$$

$$(ab \cdot a)\, a = (ba \cdot b)\, b. \tag{50}$$

The solutions of the equation **28** are

$$q \approx 1{,}5651977, \ q \approx 0{,}2174011 \pm 0{,}5217137\, i.$$

$$\mathbf{29.}\ 2q^3 - 5q^2 + 5q - 1 = 0$$

$$(ab \cdot a)\, b = (ba \cdot a)\, a, \tag{58}$$

$$(ab \cdot c)\, b = (ba \cdot c)\, a. \tag{83}$$

THEOREM 32. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (58) *and* (83) *are equivalent.*

Proof. Besides the implication (83) $(c = a) \Rightarrow$ (58) we can prove also the implication (58) $\Rightarrow$ (83). For any $a, b, c \in Q$ there is $d \in Q$ such that

$$ad = c. \tag{XIX}$$

By (II), (XIX), (VI), (V), (58), (III), (III), (XIX), (V) and (III) we obtain

$$(ab \cdot c)\, b \cdot (ab \cdot c)\, b = (ab \cdot c)\, b = (ab \cdot ad)\, b = (ab \cdot a)\,(ab \cdot d) \cdot b =$$

$$= (ab \cdot a)\, b \cdot (ab \cdot d)\, b = (ba \cdot a)\, a \cdot (ab \cdot d)\, b = (ba \cdot a)(ab \cdot d) \cdot ab =$$
$$= (ba \cdot ab)(ad) \cdot ab = (ba \cdot ab)\, c \cdot ab = (ba \cdot c)(ab \cdot c) \cdot ab =$$
$$= (ba \cdot c)\, a \cdot (ab \cdot c)\, b,$$

wherefrom it follows (83),

The equation **29** has the solutions

$$q \approx 0{,}2610164, \; q \approx 1{,}1194918 \pm 0{,}8138346\, i.$$

30. $2q^3 - 4q^2 + 4q - 1 = 0$

$$(ab \cdot a)\, b = (ba \cdot b)\, a, \tag{59}$$
$$a\,(ba \cdot b) = (ba \cdot a)\, a. \tag{103}$$

THEOREM 33. *In any IM-quasigroup* $(Q, \cdot)$ *the identities* (59) *and* (103) *are equivalent.*

Proof. (59) ⇒ (103): By (VI), (IV), (VI), (59), (V), (VI), (VI), (59), (V), (V), (III), (59), (VI), (VI) and (II) it follows

$$a\,(ba \cdot b) = (a \cdot ba) \cdot ab = (ab \cdot a) \cdot ab = (ab \cdot a)\, a \cdot (ab \cdot a)\, b =$$
$$= (ab \cdot a)\, a \cdot (ba \cdot b)\, a = (ab \cdot a)(ba \cdot b) \cdot a = [(ab \cdot a)(ba) \cdot (ab \cdot a)\, b]\, a =$$
$$= [(ab \cdot a)\, b \cdot (ab \cdot a)\, a]\; [(ab \cdot a)\, b] \cdot a = [(ba \cdot b)\, a \cdot (ab \cdot a)\, a]\; [(ba \cdot b)\, a] \cdot a =$$
$$= [(ba \cdot b)(ab \cdot a) \cdot a]\; [(ba \cdot b)\, a] \cdot a = [(ba \cdot b)(ab \cdot a) \cdot (ba \cdot b)]\, a \cdot a =$$
$$= [(ba \cdot b)(ba) \cdot (ab \cdot a)\, b]\, a \cdot a = [(ba \cdot b)(ba) \cdot (ba \cdot b)\, a]\, a \cdot a =$$
$$= [(ba \cdot b)(ba \cdot a) \cdot a]\, a = (ba \cdot ba)\, a \cdot a = (ba \cdot a)\, a.$$

(103) ⇒ (59): Because of (IV), (V), (V), (103), (III), (103), (VI) and (V) it follows

$$(ab \cdot a)\, b = (a \cdot ba)\, b = ab \cdot (ba \cdot b) = a\,(ba \cdot b) \cdot b\,(ba \cdot b) = (ba \cdot a)\, a \cdot b\,(ba \cdot b) =$$
$$= (ba \cdot a)\, b \cdot a\,(ba \cdot b) = (ba \cdot a)\, b \cdot (ba \cdot a)\, a = (ba \cdot a) \cdot ba = (ba \cdot b)\, a.$$

The equation **30** has the solutions

$$q \approx 0{,}3522011, \; q \approx 0{,}8238994 \pm 0{,}8607166\, i.$$

31. $2q^3 - 6q^2 + 4q - 1 = 0$

$$(ab \cdot b)\, a = (ba \cdot a)\, b. \tag{66}$$

The equation **31** has the solutions

$$q \approx 2{,}1914879, \; q \approx 0{,}4042561 \pm 0{,}2544259\, i.$$

32. $q^3 - 5q^2 + 4q - 1 = 0$

$$(ab \cdot b)\, a = ba \cdot ab. \tag{70}$$

The solutions of **32** are

$$q \approx 4{,}0795956, \; q \approx 0{,}4602022 \pm 0{,}1825823\, i.$$

33. $2q^3 - 2q^2 - 2q + 1 = 0$

$$(a \cdot ab)\, b = (b \cdot ba)\, a. \tag{87}$$

The equation **33** has the solutions

$$q \approx 1{,}4516060, \; q \approx -0{,}8546377, \; q \approx 0{,}4030317.$$

The quasigroups from **1**—**5** are complementary to itself, and to any of quasigroups from **6**—**33** corresponds a complementary quasigroup.

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Dopustivi identiteti u kompleksnim IM-kvazigrupama. II.

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Sadržaj

U prvom dijelu rada nađeni su u kompleksnim IM-kvazigrupama svi primitivni dopustivi identiteti sa po najviše četiri faktora na svakoj strani. Ovdje se ispituju međusobni odnosi dobivenih identiteta u bilo kojoj IM-kvazigrupi i na taj način se karakteriziraju neke klase IM-kvazigrupa. Među tim klasama javljaju se već poznate kvazigrupe *D. Vakarelova* i rot-kvazigrupe *J. Dupláka*.

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