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ADMISSIBLE IDENTITIES IN COMPLEX IM-QUASIGROUPS. I.

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Abstract. By $a \circ b = (1 - q)\, a + qb$ (where q is a complex number different from 0 and 1) an idempotent medial quasigroup $(C, \circ)$ is defined in the field $(C, +, \cdot)$ of complex numbers. For some special value q in this quasigroup, it holds some additional identity besides the consequences of idempotency and mediality. All such identities with at most four factors on each side are found.

INTRODUCTION

Let $(C, +, \cdot)$ be the field of complex numbers and $\circ$ the operation on the set C defined by

$$a \circ b = (1 - q)\, a + qb, \tag{I}$$

where q is a complex number different from 0 and 1. It is easy to see that $(C, \circ)$ is a quasigroup with the properties of *idempotency* and *mediality*:

$$a \circ a = a, \tag{II}$$

$$(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d). \tag{III}$$

The quasigroup $(C, \circ)$ is said to be a *complex IM-quasigroup* and for a given number $q \in C \setminus \{0, 1\}$ this quasigroup is denoted by $C(q)$. Besides the identities (II) and (III) in the quasigroup $(C, \circ)$ it holds also all their consequences, e. g. the identities

$$(a \circ b) \circ a = a \circ (b \circ a), \tag{IV}$$

$$(a \circ b) \circ c = (a \circ c) \circ (b \circ c), \tag{V}$$

$$a \circ (b \circ c) = (a \circ b) \circ (a \circ c) \tag{VI}$$

of *elasticity* and *right* and *left distributivity*.

For some special values $q \in C \setminus \{0, 1\}$ besides the identities (II), (III) and their consequences it holds some additional identities, which are said to be the *admissible identities* in complex IM-quasigroups. So, for example

$$a \circ b = b \circ a \tag{1}$$

is an admissible identity, because it can be expressed by (I) in the form

$$(1 - 2q)\, a + (2q - 1)\, b = 0 \tag{VII}$$

and it holds iff $q = \frac{1}{2}$. Therefore the identity (1) characterizes the quasigroup $C\left(\frac{1}{2}\right)$ in the class of complex IM-quasigroups. But, the same quasigroup is characterized also by the identity

$$(a \circ b) \circ b = b \circ (b \circ a),$$

which is by (I) equivalent to (VII), too.

In general, any identity in the quasigroup $(C, \circ)$ is of the form

$$\varphi\,(a_1, \ldots, a_n) = \psi\,(a_1, \ldots, a_n), \tag{VIII}$$

where $a_1, \ldots, a_n$ are all variables which appear effectively on the left or right side of this identity. By (I) the identity (VIII) is equivalent to

$$p_1\,(q) \cdot a_1 + \ldots + p_n\,(q) \cdot a_n = 0,$$

where $p_1, \ldots, p_n$ are some polynomials in one variable over the set of integer real numbers. The identity (VIII) is an admissible identity iff the algebraic equations

$$p_1\,(q) = 0, \ldots, p_n\,(q) = 0$$

have a common solution $q \in C \setminus \{0, 1\}$. Herefrom it follows immediately:

PROPOSITION 1. If an admissible identity holds in the quasigroup $C\,(q)$, then it holds also in the quasigroup $C\,(\overline{q})$ (where $\overline{q}$ and q are the conjugate complex numbers), and moreover q is an algebraic number.

An identity in an IM-quasigroup $C\,(q)$ is said to be *deduced* iff it is equivalent to some other identity, which has a smaller total number of factors and not a greater number of variables and for the proof of equivalency, the property of left or right cancellation, the identities (II)—(VI) and the substitutions of variables can be used. For example, the identities

$$(a \circ b) \circ a = a \circ (a \circ b),$$

$$(a \circ b) \circ (b \circ a) = a \circ b,$$

$$(a \circ b) \circ (c \circ a) = a \circ (c \circ b)$$

are deduced, because they can be reduced to (1) by the substitution $a \circ b \to b$ resp. by the idempotency and cancellation resp. by the distributivity, mediality and cancellation.

The identities, which are not deduced, are said to be *primitive*. It is sufficient to restrict the study of admissible identities to the study of primitive admissible identities only.

Let n be a natural number. A »product« of n factors in the quasigroup $C(q)$ (with some arrangement of parentheses) is denoted by $\langle n \rangle$ and by $\langle n \rangle'$ is denoted a product $\langle n \rangle$ which has the factors ordered alphabetically, i. e. the first factor (from left to right) is a, the first factor different from a is b, etc.

In this paper we shall find all primitive admissible identities of the form $\langle m \rangle = \langle n \rangle$, where $m, n \in \{1, 2, 3, 4\}$ and the corresponding quasigroups $C(q)$ in which these identities hold. Obviously it is sufficient to take that $m \geq n$. Moreover, every identity of the form $\langle m \rangle = \langle n \rangle$ can be expressed in the form $\langle m \rangle' = \langle n \rangle$ by some substitution of the variables. So it holds:

PROPOSITION 2. It is sufficient to find all primitive admissible identities of the form $\langle m \rangle' = \langle n \rangle$, where $m \geq n$.

Further, we have:

PROPOSITION 3. Every variable which appears in the identity must appear at least twice.

Proof. If the variable x appears only once, then in the quasigroup $(C, \circ)$ for any choice of the values of other variables there is one and only one value of x which satisfies the observed identity conceived as an equation for x and therefore it is not an identity in the variable x.

In an identity $\langle m \rangle' = \langle n \rangle$ there are $m + n$ factors and by Proposition 3 it follows at once:

PROPOSITION 4. It is sufficient to find all primitive admissible identities of the form $\langle m \rangle' = \langle n \rangle$, where the variables are first $\left[\frac{m+n}{2}\right]$ letters of the alphabet.

From Proposition 4 it follows that in the case of identities of the form $\langle 2 \rangle' = \langle 1 \rangle$ it is sufficient to have only the variable a (and therefore besides the identity (II) there is no admissible identity of this form), in the case of identities of the forms $\langle 2 \rangle' = \langle 2 \rangle$, $\langle 3 \rangle' = \langle 1 \rangle$, $\langle 3 \rangle' = \langle 2 \rangle$ and $\langle 4 \rangle' = \langle 1 \rangle$ it is sufficient to have only two variables a and b, in the case of identities of the forms $\langle 3 \rangle' = \langle 3 \rangle$, $\langle 4 \rangle' = \langle 2 \rangle$ and $\langle 4 \rangle' = \langle 3 \rangle$, only three variables a, b and c, and in the case of identities of the form $\langle 4 \rangle' = \langle 4 \rangle$ only four variables a, b, c and d.

The quasigroups $C(q)$ and $C(1-q)$ are said to be *complementary* and two identities are said to be *complementary* iff one of them can be derived from the other by the mutual substitution of two factors in every »product«, i. e. by the substitutions of the form $x \circ y \to y \circ x$. By (I) it follows:

PROPOSITION 5. The identity complementary to an admissible identity in a quasigroup $C(q)$ is also admissible in the complementary quasigroup $C(1-q)$.

It is sufficient to find only one identity of any pair of complementary identities

ADMISSIBLE IDENTITIES OF THE FORMS $\langle 2 \rangle' = \langle 2 \rangle, \langle 3 \rangle' = \langle 1 \rangle, \langle 3 \rangle' = \langle 2 \rangle$ AND $\langle 4 \rangle' = \langle 1 \rangle$

The only one admissible identity of the form $\langle 2 \rangle' = \langle 2 \rangle$ is the identity (1) in the quasigroup $C\left(\frac{1}{2}\right)$.

With a view to find all primitive admissible identities of the forms $\langle 3\rangle' = \langle 1\rangle$ and $\langle 3\rangle' = \langle 2\rangle$ it is necessary to find first all products of the form $\langle 3\rangle'$ in variables a and b. These products are

$$(a \circ b) \circ a,\ (a \circ b) \circ b,\ a \circ (a \circ b),\ a \circ (b \circ a). \tag{IX}$$

But, it is sufficient to consider only the first two products (IX), because the last two products are complementary to them (the third product is complementary to the second after the substitution $a \leftrightarrow b$).

Now, it is sufficient to compare the first two products (IX) with the products

$$a,\ \ b,\ \ a \circ b,\ \ b \circ a$$

of the forms $\langle 1\rangle$ and $\langle 2\rangle$ in variables a and b. So, we have the following consideration:

$$(a \circ b) \circ a = a \Leftrightarrow a \circ b = a \Leftrightarrow q = 0,$$

$$(a \circ b) \circ a = b \Leftrightarrow q^2 - q + 1 = 0 \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{2}$$

$$(a \circ b) \circ a = a \circ b \Leftrightarrow a \circ b = a \Leftrightarrow q = 0,$$

$$(a \circ b) \circ a = b \circ a \Leftrightarrow a \circ b = b \Leftrightarrow q = 1,$$

$$(a \circ b) \circ b = a \Leftrightarrow q^2 - 2q = 0 \Leftrightarrow q = 2, \tag{3}$$

$$(a \circ b) \circ b = b \Leftrightarrow a \circ b = b \Leftrightarrow q = 1,$$

$$(a \circ b) \circ b = a \circ b \Leftrightarrow a \circ b = a \Leftrightarrow q = 0,$$

$$(a \circ b) \circ b = b \circ a \Leftrightarrow q^2 - 3q + 1 = 0 \Leftrightarrow q = \frac{1}{2}\left(3 \pm \sqrt{5}\right). \tag{4}$$

We obtained the admissible identities (2)—(4) in corresponding quasigroups $C(q)$, while the remaining identities are not admissible.

The products of the form $\langle 4\rangle$ can have one of the five forms

$$[(x \circ y) \circ u] \circ v,\quad [x \circ (y \circ u)] \circ v,\quad x \circ [(y \circ u) \circ v],$$

$$x \circ [y \circ (u \circ v)],\quad (x \circ y) \circ (u \circ v). \tag{X}$$

The third and fourth products (X) are complementary (by the substitutions $x \leftrightarrow v$, $y \leftrightarrow u$) to the second and first products (X). As we look for the admissible identities of the form $\langle 4\rangle' = \langle 1\rangle$, so in the first, second and fifth products (X) we take all possibilities $x, y, u, v \in \{a, b\}$ which give different products of the form $\langle 4\rangle'$. Because of (IV) and (II) the different products are:

$$[(a \circ b) \circ a] \circ a,\ [(a \circ b] \circ a] \circ b,\ [(a \circ b) \circ b] \circ a,\ [(a \circ b) \circ b] \circ b,$$

$$[a \circ (a \circ b)] \circ a,\ [a \circ (a \circ b)] \circ b,\ (a \circ b) \circ (b \circ a). \tag{XI}$$

By (IV) it holds

$$[a \circ (a \circ b)] \circ a = a \circ [(a \circ b) \circ a] = a \circ [a \circ (b \circ a)]$$

and the fifth product (XI) is complementary to the first product (XI). The remaining six products (XI) need to be compared with a or with b. So, we have:

$$[(a \circ b) \circ a] \circ a = a \Leftrightarrow (a \circ b) \circ a = a,$$

$$[(a \circ b) \circ a] \circ a = b \Leftrightarrow q^3 - 2q^2 + q - 1 = 0, \quad (5)$$

$$[(a \circ b) \circ a] \circ b = a \Leftrightarrow q^3 - 2q^2 + 2q = 0 \Leftrightarrow q = 1 \pm i, \quad (6)$$

$$[(a \circ b) \circ a] \circ b = b \Leftrightarrow (a \circ b) \circ a = b,$$

$$[(a \circ b) \circ b] \circ a = a \Leftrightarrow (a \circ b) \circ b = a,$$

$$[(a \circ b) \circ b] \circ a = b \Leftrightarrow q^3 - 3q^2 + 2q - 1 = 0, \quad (7)$$

$$[(a \circ b) \circ b] \circ b = a \Leftrightarrow q^3 - 3q^2 + 3q = 0 \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{3}\, i), \quad (8)$$

$$[(a \circ b) \circ b] \circ b = b \Leftrightarrow (a \circ b) \circ b = b,$$

$$[a \circ (a \circ b)] \circ b = a \Leftrightarrow q^3 - q^2 - q = 0 \Leftrightarrow q = \frac{1}{2}(1 \pm \sqrt{5}), \quad (9)$$

$$[a \circ (a \circ b)] \circ b = b \Leftrightarrow a \circ (a \circ b) = b,$$

$$(a \circ b) \circ (b \circ a) = a \Leftrightarrow 2q^2 - 2q = 0,$$

$$(a \circ b) \circ (b \circ a) = b \Leftrightarrow 2q^2 - 2q + 1 = 0 \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i. \quad (10)$$

Therefore, we have obtained further six admissible identities (5)—(10).

ADMISSIBLE IDENTITIES OF THE FORMS $\langle 3\rangle' = \langle 3\rangle$, $\langle 4\rangle' = \langle 2\rangle$ AND $\langle 4\rangle' = \langle 3\rangle$

In Table 1 are enumerated all products of the form $\langle 2\rangle$ in the variables a, b and c together with the coefficients of corresponding polynomials in the variable q, which stand by these variables after the representation of these products by means of (I) in the field $(C, +, \cdot)$.

Table 1.

$\langle 2\rangle$	a		b		c	
	1	q	1	q	1	q
$a \circ b$	1	-1		1		
$a \circ c$	1	-1				1
$b \circ a$		1	1	-1		
$b \circ c$			1	-1		1
$c \circ a$		1			1	-1
$c \circ b$				1	1	-1

By the multiplication of the products $\langle 2 \rangle$ from Table 1 by a, b or c from the left or right side we get 36 possible products of the form $\langle 3 \rangle$ in the variables a, b and c. It is sufficient to consider 30 products

$$\begin{array}{l}(a \bigcirc b) \bigcirc a, (a \bigcirc b) \bigcirc b, (a \bigcirc b) \bigcirc c, (a \bigcirc c) \bigcirc a, (a \bigcirc c) \bigcirc b, (a \bigcirc c) \bigcirc c, \\ (b \bigcirc a) \bigcirc a, (b \bigcirc a) \bigcirc b, (b \bigcirc a) \bigcirc c, (b \bigcirc c) \bigcirc a, (b \bigcirc c) \bigcirc b, (b \bigcirc c) \bigcirc c, \\ (c \bigcirc a) \bigcirc a, (c \bigcirc a) \bigcirc b, (c \bigcirc a) \bigcirc c, (c \bigcirc b) \bigcirc a, (c \bigcirc b) \bigcirc b, (c \bigcirc b) \bigcirc c,\end{array} \tag{XII}$$

$$\begin{array}{l}a \bigcirc (a \bigcirc b), c \bigcirc (a \bigcirc b), a \bigcirc (a \bigcirc c), b \bigcirc (a \bigcirc c), \\ b \bigcirc (b \bigcirc a), c \bigcirc (b \bigcirc a), a \bigcirc (b \bigcirc c), (b \bigcirc (b \bigcirc c), \\ b \bigcirc (c \bigcirc a), c \bigcirc (c \bigcirc a), a \bigcirc (c \bigcirc b), c \bigcirc (c \bigcirc b),\end{array} \tag{XIII}$$

because by (IV) the products $b \bigcirc (a \bigcirc b)$, $c \bigcirc (a \bigcirc c)$, $a \bigcirc (b \bigcirc a)$, $c \bigcirc (b \bigcirc c)$, $a \bigcirc (c \bigcirc a)$, $b \bigcirc (c \bigcirc b)$ are equal to the products $(b \bigcirc a) \bigcirc b$, $(c \bigcirc a) \bigcirc c$, $(a \bigcirc b) \bigcirc a$, $(c \bigcirc b) \bigcirc c$, $(a \bigcirc c) \bigcirc a$, $(b \bigcirc c) \bigcirc b$ respectively. But, on basis of Proposition 5 it is sufficient to find all admissible identities of the forms $\langle 3 \rangle' = \langle 3 \rangle$ and $\langle 4 \rangle' = \langle 3 \rangle$, where the products of the form $\langle 3 \rangle$ are taken from (XII). Therefore, in Table 2 are enumerated all these products with the corresponding coefficients.

Table 2.

$\langle 3 \rangle$	a			b			c		
	1	q	q^2	1	q	q^2	1	q	q^2
$(a \bigcirc b) \bigcirc a$	1	−1	1		1	−1			
$(a \bigcirc b) \bigcirc b$	1	−2	1		2	−1			
$(a \bigcirc b) \bigcirc c$	1	−2	1		1	−1		1	
$(a \bigcirc c) \bigcirc a$	1	−1	1					1	−1
$(a \bigcirc c) \bigcirc b$	1	−2	1		1			1	−1
$(a \bigcirc c) \bigcirc c$	1	−2	1					2	−1
$(b \bigcirc a) \bigcirc a$		2	−1	1	−2	1			
$(b \bigcirc a) \bigcirc b$		1	−1	1	−1	1			
$(b \bigcirc a) \bigcirc c$		1	−1	1	−2	1		1	
$(b \bigcirc c) \bigcirc a$		1		1	−2	1		1	−1
$(b \bigcirc c) \bigcirc b$				1	−1	1		1	−1
$(b \bigcirc c) \bigcirc c$				1	−2	1		2	−1
$(c \bigcirc a) \bigcirc a$		2	−1				1	−2	1
$(c \bigcirc a) \bigcirc b$		1	−1		1		1	−2	1
$(c \bigcirc a) \bigcirc c$		1	−1				1	−1	1
$(c \bigcirc b) \bigcirc a$		1			1	−1	1	−2	1
$(c \bigcirc b) \bigcirc b$					2	−1	1	−2	1
$(c \bigcirc b) \bigcirc c$					1	−1	1	−1	1

Among 30 products (XII) and (XIII) only five of them are of the form $\langle 3 \rangle'$ and they are enumerated in Table 3.

Table 3.

$\langle 3 \rangle'$	a			b			c		
	1	q	q^2	1	q	q^2	1	q	q^2
$(a \circ b) \circ a$	1	-1	1		1	-1			
$(a \circ b) \circ b$	1	-2	1		2	-1			
$(a \circ b) \circ c$	1	-2	1		1	-1		1	
$a \circ (a \circ b)$	1		-1			1			
$a \circ (b \circ c)$	1	-1			1	-1			1

By the multiplication of the products $\langle 3 \rangle$ from (XII) and (XIII) by a, b or c from the right or left side we get 180 products of the form $\langle 4 \rangle$ in the variables a, b and c. But, on the basis of Proposition 5 it is sufficient to consider only 90 products with the factors a, b or c on the right side with a view to get all admissible identities of the form $\langle 4 \rangle' = \langle 4 \rangle$. Moreover, by the mutual multiplication of any two products of the form $\langle 2 \rangle$ from Table 1 we further get 30 products of the form $\langle 4 \rangle$ in the variables a, b and c. But, it is not necessary to consider the products

$$(a \circ b) \circ (a \circ c),\ (a \circ b) \circ (c \circ b),\ (a \circ c) \circ (a \circ b),\ (a \circ c) \circ (b \circ a),$$
$$(a \circ c) \circ (b \circ c),\ (b \circ a) \circ (b \circ c),\ (b \circ a) \circ (c \circ a),\ (b \circ c) \circ (a \circ b),$$
$$(b \circ c) \circ (a \circ c),\ (b \circ c) \circ (b \circ a),\ (c \circ a) \circ (b \circ a),\ (c \circ a) \circ (c \circ b),$$
$$(c \circ b) \circ (a \circ b),\ (c \circ b) \circ (a \circ c),\ (c \circ b) \circ (c \circ a),$$

while by (III), (IV) and (VI) they are equal to the products

$$a \circ (b \circ c),\ (a \circ c) \circ b,\ a \circ (c \circ b),\ (a \circ b) \circ (c \circ a),\ (a \circ b) \circ c,$$
$$b \circ (a \circ c),\ (b \circ c) \circ a,\ (b \circ a) \circ (c \circ b),\ (b \circ a) \circ c,\ b \circ (c \circ a),$$
$$(c \circ b) \circ a,\ c \circ (a \circ b),\ (c \circ a) \circ b,\ (c \circ a) \circ (b \circ c),\ c \circ (b \circ a)$$

respectively. The remaining 105 products of the form $\langle 4 \rangle$ in the variables a, b and c are represented in Table 4.

Table 4.

$\langle 4 \rangle$	a: 1	a: q	a: q^2	a: q^3	b: 1	b: q	b: q^2	b: q^3	c: 1	c: q	c: q^2	c: q^3
$[(a \circ b) \circ a] \circ a$	1	−1	2	−1		1	−2	1				
$[(a \circ b) \circ a] \circ b$	1	−2	2	−1		2	−2	1				
$[(a \circ b) \circ a] \circ c$	1	−2	2	−1		1	−2	1		1		
$[(a \circ b) \circ b] \circ a$	1	−2	3	−1		2	−3	1				
$[(a \circ b) \circ b] \circ b$	1	−3	3	−1		3	−3	1				
$[(a \circ b) \circ b] \circ c$	1	−3	3	−1		2	−3	1		1		
$[(a \circ b) \circ c] \circ a$	1	−2	3	−1		1	−2	1		1	−1	
$[(a \circ b) \circ c] \circ b$	1	−3	3	−1		2	−2	1		1	−1	
$[(a \circ b) \circ c] \circ c$	1	−3	3	−1		1	−2	1		2	−1	
$[(a \circ c) \circ a] \circ a$	1	−1	2	−1						1	−2	1
$[(a \circ c) \circ a] \circ b$	1	−2	2	−1		1				1	−2	1
$[(a \circ c) \circ a] \circ c$	1	−2	2	−1						2	−2	1
$[(a \circ c) \circ b] \circ a$	1	−2	3	−1		1	−1			1	−2	1
$[(a \circ c) \circ b] \circ b$	1	−3	3	−1		2	−1			1	−2	1
$[(a \circ c) \circ b] \circ c$	1	−3	3	−1		1	−1			2	−2	1
$[(a \circ c) \circ c] \circ a$	1	−2	3	−1						2	−3	1
$[(a \circ c) \circ c] \circ b$	1	−3	3	−1		1				2	−3	1
$[(a \circ c) \circ c] \circ c$	1	−3	3	−1						3	−3	1
$[(b \circ a) \circ a] \circ a$		3	−3	1	1	−3	3	−1				
$[(b \circ a) \circ a] \circ b$		2	−3	1	1	−2	3	−1				
$[(b \circ a) \circ a] \circ c$		2	−3	1	1	−3	3	−1		1		
$[(b \circ a) \circ b] \circ a$		2	−2	1	1	−2	2	−1				
$[(b \circ a) \circ b] \circ b$		1	−2	1	1	−1	2	−1				
$[(b \circ a) \circ b] \circ c$		1	−2	1	1	−2	2	−1		1		
$[(b \circ a) \circ c] \circ a$		2	−2	1	1	−3	3	−1		1	−1	
$[(b \circ a) \circ c] \circ b$		1	−2	1	1	−2	3	−1		1	−1	
$[(b \circ a) \circ c] \circ c$		1	−2	1	1	−3	3	−1		2	−1	
$[(b \circ c) \circ a] \circ a$		2	−1		1	−3	3	−1		1	−2	1
$[(b \circ c) \circ a] \circ b$		1	−1		1	−2	3	−1		1	−2	1
$[(b \circ c) \circ a] \circ c$		1	−1		1	−3	3	−1		2	−2	1
$[(b \circ c) \circ b] \circ a$		1			1	−2	2	−1		1	−2	1
$[(b \circ c) \circ b] \circ b$					1	−1	2	−1		1	−2	1
$[(b \circ c) \circ b] \circ c$					1	−2	2	−1		2	−2	1
$[(b \circ c) \circ c] \circ a$		1			1	−3	3	−1		2	−3	1
$[(b \circ c) \circ c] \circ b$					1	−2	3	−1		2	−3	1
$[(b \circ c) \circ c] \circ c$					1	−3	3	−1		3	−3	1
$[(c \circ a) \circ a] \circ a$		3	−3	1					1	−3	3	−1
$[(c \circ a) \circ a] \circ b$		2	−3	1		1			1	−3	3	−1
$[(c \circ a) \circ a] \circ c$		2	−3	1					1	−2	3	−1
$[(c \circ a) \circ b] \circ a$		2	−2	1		1	−1		1	−3	3	−1
$[(c \circ a) \circ b] \circ b$		1	−2	1		2	−1		1	−3	3	−1
$[(c \circ a) \circ b] \circ c$		1	−2	1		1	−1		1	−2	3	−1
$[(c \circ a) \circ c] \circ a$		2	−2	1					1	−2	2	−1
$[(c \circ a) \circ c] \circ b$		1	−2	1		1			1	−2	2	−1
$[(c \circ a) \circ c] \circ c$		1	−2	1					1	−1	2	−1
$[(c \circ b) \circ a] \circ a$		2	−1			1	−2	1	1	−3	3	−1
$[(c \circ b) \circ a] \circ b$		1	−1			2	−2	1	1	−3	3	−1
$[(c \circ b) \circ a] \circ c$		1	−1			1	−2	1	1	−2	3	−1
$[(c \circ b) \circ b] \circ a$		1				2	−3	1	1	−3	3	−1
$[(c \circ b) \circ b] \circ b$						3	−3	1	1	−3	3	−1
$[(c \circ b) \circ b] \circ c$						2	−3	1	1	−2	3	−1
$[(c \circ b) \circ c] \circ a$		1				1	−2	1	1	−2	2	−1
$[(c \circ b) \circ c] \circ b$						2	−2	1	1	−2	2	−1
$[(c \circ b) \circ c] \circ c$						1	−2	1	1	−1	2	−1
$[a \circ (a \circ b)] \circ a$	1		−1	1			1	−1				
$[a \circ (a \circ b)] \circ b$	1	−1	−1	1		1	1	−1				

(4)′	a				b				c			
	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3
[a ○ (a ○ b)] ○ c	1	−1	−1	1			1	−1		1		
[c ○ (a ○ b)] ○ a		2	−2	1			1	−1	1	−2	1	
[c ○ (a ○ b)] ○ b		1	−2	1		1	1	−1	1	−2	1	
[c ○ (a ○ b)] ○ c		1	−2	1			1	−1	1	−1	1	
[a ○ (a ○ c)] ○ a	1		−1	1							1	−1
[a ○ (a ○ c)] ○ b	1	−1	−1	1		1					1	−1
[a ○ (a ○ c)] ○ c	1	−1	−1	1						1	1	−1
[b ○ (a ○ c)] ○ a		2	−2	1	1	−2	1				1	−1
[b ○ (a ○ c)] ○ b		1	−2	1	1	−1	1				1	−1
[b ○ (a ○ c)] ○ c		1	−2	1	1	−2	1			1	1	−1
[b ○ (b ○ a)] ○ a		1	1	−1	1	−1	−1	1				
[b ○ (b ○ a)] ○ b			1	−1	1		−1	1				
[b ○ (b ○ a)] ○ c			1	−1	1	−1	−1	1		1		
[c ○ (b ○ a)] ○ a		1	1	−1		1	−2	1	1	−2	1	
[c ○ (b ○ a)] ○ b			1	−1		2	−2	1	1	−2	1	
[c ○ (b ○ a)] ○ c			1	−1		1	−2	1	1	−1	1	
[a ○ (b ○ c)] ○ a	1	−1	1			1	−2	1			1	−1
[a ○ (b ○ c)] ○ b	1	−2	1			2	−2	1			1	−1
[a ○ (b ○ c)] ○ c	1	−2	1			1	−2	1		1	1	−1
[b ○ (b ○ c)] ○ a		1			1	−1	−1	1			1	−1
[b ○ (b ○ c)] ○ b					1		−1	1			1	−1
[b ○ (b ○ c)] ○ c					1	−1	−1	1		1	1	−1
[b ○ (c ○ a)] ○ a		1	1	−1	1	−2	1			1	−2	1
[b ○ (c ○ a)] ○ b			1	−1	1	−1	1			1	−2	1
[b ○ (c ○ a)] ○ c			1	−1	1	−2	1			2	−2	1
[c ○ (c ○ a)] ○ a		1	1	−1					1	−1	−1	1
[c ○ (c ○ a)] ○ b			1	−1		1			1	−1	−1	1
[c ○ (c ○ a)] ○ c			1	−1					1		−1	1
[a ○ (c ○ b)] ○ a	1	−1	1				1	−1		1	−2	1
[a ○ (c ○ b)] ○ b	1	−2	1			1	1	−1		1	−2	1
[a ○ (c ○ b)] ○ c	1	−2	1				1	−1		2	−2	1
[c ○ (c ○ b)] ○ a		1					1	−1	1	−1	−1	1
[c ○ (c ○ b)] ○ b						1	1	−1	1	−1	−1	1
[c ○ (c ○ b)] ○ c							1	−1	1		−1	1
(a ○ b) ○ (b ○ a)	1	−2	2			2	−2					
(a ○ b) ○ (b ○ c)	1	−2	1			2	−2				1	
(a ○ b) ○ (c ○ a)	1	−2	2			1	−1			1	−1	
(a ○ c) ○ (c ○ a)	1	−2	2							2	−2	
(a ○ c) ○ (c ○ b)	1	−2	1				1			2	−2	
(b ○ a) ○ (a ○ b)		2	−2		1	−2	2					
(b ○ a) ○ (a ○ c)		2	−2		1	−2	1				1	
(b ○ a) ○ (c ○ b)		1	−1		1	−2	2			1	−1	
(b ○ c) ○ (c ○ a)			1		1	−2	1			2	−2	
(b ○ c) ○ (c ○ b)					1	−2	2			2	−2	
(c ○ a) ○ (a ○ b)		2	−2				1		1	−2	1	
(c ○ a) ○ (a ○ c)		2	−2						1	−2	2	
(c ○ a) ○ (b ○ c)		1	−1			1	−1		1	−2	2	
(c ○ b) ○ (b ○ a)			1			2	−2		1	−2	1	
(c ○ b) ○ (b ○ c)						2	−2		1	−2	2	

Among 195 different products of the form $\langle 4 \rangle$ in the variables a, b and c only 30 of them are of the form $\langle 4 \rangle'$ and they are enumerated in Table 5.

Table 5.

	$\langle 4 \rangle'$	a				b				c			
		1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3
	$[(a \circ b) \circ a] \circ a$	1	−1	2	−1		1	−2	1				
	$[(a \circ b) \circ a] \circ b$	1	−2	2	−1		2	−2	1				
A	$[(a \circ b) \circ a] \circ c$	1	−2	2	−1		1	−2	1		1		
	$[(a \circ b) \circ b] \circ a$	1	−2	3	−1		2	−3	1				
	$[(a \circ b) \circ b] \circ b$	1	−3	3	−1		3	−3	1				
B	$[(a \circ b) \circ b] \circ c$	1	−3	3	−1		2	−3	1		1		
A	$[(a \circ b) \circ c] \circ a$	1	−2	3	−1		1	−2	1		1	−1	
B	$[(a \circ b) \circ c] \circ b$	1	−3	3	−1		2	−2	1		1	−1	
C	$[(a \circ b) \circ c] \circ c$	1	−3	3	−1		1	−2	1		2	−1	
	$[a \circ (a \circ b)] \circ a$	1		−1	1			1	−1				
	$[a \circ (a \circ b)] \circ b$	1	−1	−1	1		1	1	−1				
A	$[a \circ (a \circ b)] \circ c$	1	−1	−1	1			1	−1		1		
A	$[a \circ (b \circ c)] \circ a$	1	−1	1			1	−2	1			1	−1
B	$[a \circ (b \circ c)] \circ b$	1	−2	1			2	−2	1			1	−1
C	$[a \circ (b \circ c)] \circ c$	1	−2	1			1	−2	1		1	1	−1
	$a \circ [(a \circ b) \circ b]$	1		−2	1			2	−1				
A	$a \circ [(a \circ b) \circ c]$	1		−2	1			1	−1			1	
	$a \circ [(b \circ a) \circ b]$	1	−1	1	−1		1	−1	1				
A	$a \circ [(b \circ a) \circ c]$	1	−1	1	−1		1	−2	1			1	
B	$a \circ [(b \circ c) \circ b]$	1	−1				1	−1	1			1	−1
C	$a \circ [(b \circ c) \circ c]$	1	−1				1	−2	1			2	−1
	$a \circ [a \circ (a \circ b)]$	1			−1				1				
A	$a \circ [b \circ (a \circ c)]$	1	−1	1	−1		1	−1					1
	$a \circ [b \circ (b \circ a)]$	1	−1		1		1		−1				
A	$a \circ [a \circ (b \circ c)]$	1		−1				1	−1				1
B	$a \circ [b \circ (b \circ c)]$	1	−1				1		−1				1
A	$a \circ [b \circ (c \circ a)]$	1	−1		1		1	−1				1	−1
	$(a \circ b) \circ (b \circ a)$	1	−2	2			2	−2					
B	$(a \circ b) \circ (b \circ c)$	1	−2	1			2	−2				1	
A	$(a \circ b) \circ (c \circ a)$	1	−2	2			1	−1			1	−1	

By comparing anyone of the five products of the form $\langle 3 \rangle'$ from Table 3 with anyone of the 18 products of the form $\langle 3 \rangle$ from Table 2 we shall obtain all primitive admissible identities of the form $\langle 3 \rangle' = \langle 3 \rangle$ in the corresponding quasigroups $C(q)$. For the sake of simplicity we omit from the further lists all deduced primitive identities, all identities which are not admissible (i. e. any identity which reduces to a system of equations in the variable q without common solutions in the set $C \setminus \{0, 1\}$) and all identities which do not agree with Proposition 3. So, we have:

$$(a \circ b) \circ a = (b \circ a) \circ b \Leftrightarrow 2q^2 - 2q + 1 = 0 \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \qquad (11)$$

$$(a \circ b) \circ a = (c \circ b) \circ c \Leftrightarrow q^2 - q + 1 = 0 \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \qquad (12)$$

$$(a \circ b) \circ b = (a \circ c) \circ c \Leftrightarrow q^2 - 2q = 0 \Leftrightarrow q = 2, \qquad (13)$$

$$(a \circ b) \circ b = (b \circ a) \circ a \Leftrightarrow 2q^2 - 4q + 1 = 0 \Leftrightarrow q = \frac{1}{2}(2 \pm \sqrt{2}), \tag{14}$$

$$(a \circ b) \circ c = (c \circ b) \circ a \Leftrightarrow q^2 - 3q + 1 = 0 \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{15}$$

$$a \circ (a \circ b) = (b \circ a) \circ a \Leftrightarrow 2q - 1 = 0 \Leftrightarrow q = \frac{1}{2}, \tag{16}$$

$$a \circ (b \circ c) = (c \circ b) \circ a \Leftrightarrow 2q - 1 = 0 \Leftrightarrow q = \frac{1}{2}. \tag{17}$$

Analogously, we deduce all primitive admissible identities of the form $\langle 4 \rangle' = \langle 2 \rangle$ by the comparing of products from Tables 5 and 1, where by Proposition 5 it is sufficient to take only the first fifteen and the last three products from Table 5. As the result of this comparing we obtain:

$$[(a \circ b) \circ a] \circ b = b \circ a \Leftrightarrow q^3 - 2q^2 + 3q - 1 = 0, \tag{18}$$

$$[(a \circ b) \circ b] \circ a = a \circ b \Leftrightarrow q^3 - 3q^2 + q = 0 \qquad \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{19}$$

$$[(a \circ b) \circ b] \circ b = b \circ a \Leftrightarrow q^3 - 3q^2 + 4q - 1 = 0, \tag{20}$$

$$[a \circ (a \circ b)] \circ b = b \circ a \Leftrightarrow q^3 - q^2 - 2q + 1 = 0. \tag{21}$$

By the comparing of products from Tables 5 and 2 we get all primitive admissible identities of the form $\langle 4 \rangle' = \langle 3 \rangle$:

$$[(a \circ b) \circ a] \circ a = (a \circ b) \circ b \Leftrightarrow q^3 - q^2 - q = 0 \qquad \Leftrightarrow q = \frac{1}{2}(1 \pm \sqrt{5}), \tag{22}$$

$$[(a \circ b) \circ a] \circ a = (b \circ a) \circ b \Leftrightarrow q^3 - 3q^2 + 2q - 1 = 0, \tag{23}$$

$$[(a \circ b) \circ a] \circ b = (b \circ a) \circ a \Leftrightarrow q^3 - 3q^2 + 4q - 1 = 0, \tag{24}$$

$$[(a \circ b) \circ a] \circ b = (c \circ b) \circ c \Leftrightarrow q^3 - 2q^2 + 2q - 1 = 0,\ q^3 - q^2 + q = 0,\ q^2 - q + 1 = 0 \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{25}$$

$$[(a \circ b) \circ b] \circ a = (a \circ c) \circ c \Leftrightarrow q^3 - 2q^2 = 0,\ q^3 - 3q^2 + 2q = 0,\ q^2 - 2q = 0 \quad \Leftrightarrow q = 2, \tag{26}$$

$$[(a \circ b) \circ b] \circ a = (b \circ a) \circ b \Leftrightarrow q^3 - 4q^2 + 3q - 1 = 0, \tag{27}$$

$$[(a \circ b) \circ b] \circ b = (a \circ b) \circ a \Leftrightarrow q^3 - 2q^2 + 2q = 0 \qquad \Leftrightarrow q = 1 \pm i, \tag{28}$$

$$[(a \circ b) \circ b] \circ b = (b \circ a) \circ a \Leftrightarrow q^3 - 4q^2 + 5q - 1 = 0, \tag{29}$$

$$[(a \circ b) \circ c] \circ a = (b \circ c) \circ b \Leftrightarrow q^3 - 3q^2 + 2q - 1 = 0, \tag{30}$$

$$[(a \circ b) \circ c] \circ b = (a \circ c) \circ a \Leftrightarrow q^3 - 2q^2 + 2q = 0 \qquad \Leftrightarrow q = 1 \pm i, \tag{31}$$

$$[(a \circ b) \circ c] \circ b = (b \circ c) \circ a \Leftrightarrow q^3 - 3q^2 + 4q - 1 = 0, \tag{32}$$

$$[a \circ (a \circ b)] \circ a = (a \circ b) \circ b \Leftrightarrow q^3 - 2q^2 + 2q = 0 \qquad \Leftrightarrow q = 1 \pm i, \tag{33}$$

$$[a \circ (a \circ b)] \circ a = (b \circ a) \circ b \Leftrightarrow q^3 - q + 1 = 0, \tag{23$'$}$$

$$[a \circ (a \circ b)] \circ b = (b \circ a) \circ a \Leftrightarrow q^3 - 3q + 1 = 0, \tag{34}$$

$$[a \circ (b \circ c)] \circ b = (a \circ c) \circ c \Leftrightarrow q^3 - 2q^2 + 2q = 0 \qquad \Leftrightarrow q = 1 \pm i, \tag{35}$$

$$[a \circ (b \circ c)] \circ c = (a \circ b) \circ b \Leftrightarrow q^3 - q^2 - q = 0 \qquad \Leftrightarrow q = \frac{1}{2}(1 \pm \sqrt{5}), \tag{36}$$

$$a \circ [(a \circ b) \circ b] = (a \circ c) \circ c \Leftrightarrow q^3 - 3q^2 + 2q = 0,\ q^3 - 2q^2 = 0,\ q^2 - 2q = 0 \qquad \Leftrightarrow q = 2, \tag{37}$$

$$a \circ [(a \circ b) \circ b] = (b \circ a) \circ a \Leftrightarrow q^3 - q^2 - 2q + 1 = 0, \tag{38}$$

$$a \circ [(b \circ a) \circ b] = (b \circ a) \circ a \Leftrightarrow q^3 - 2q^2 + 3q - 1 = 0, \tag{39}$$

$$a \circ [(b \circ a) \circ b] = (c \circ a) \circ c \Leftrightarrow q^3 - 2q^2 + 2q - 1 = 0,\ q^3 - q^2 + q = 0,\ q^2 - q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{25'}$$

$$a \circ [a \circ (a \circ b)] = (a \circ b) \circ b \Leftrightarrow q^3 - q^2 - 2q = 0 \qquad \Leftrightarrow q = -2, \tag{40}$$

$$a \circ [a \circ (a \circ b)] = (b \circ a) \circ a \Leftrightarrow q^3 - q^2 + 2q - 1 = 0, \tag{41}$$

$$a \circ [a \circ (a \circ b)] = (b \circ a) \circ b \Leftrightarrow q^3 - q^2 + q - 1 = 0 \qquad \Leftrightarrow q = \pm i, \tag{28'}$$

$$a \circ [b \circ (a \circ c)] = (c \circ b) \circ a \Leftrightarrow q^3 - q^2 + 2q - 1 = 0, \tag{42}$$

$$a \circ [b \circ (a \circ c)] = (c \circ b) \circ c \Leftrightarrow q^3 - q^2 + q - 1 = 0 \qquad \Leftrightarrow q = \pm i, \tag{31'}$$

$$a \circ [b \circ (b \circ a)] = (b \circ a) \circ a \Leftrightarrow q^3 - q^2 - 3q + 1 = 0 \qquad \Leftrightarrow q = -1 \pm \sqrt{3}, \tag{43}$$

$$a \circ [b \circ (b \circ a)] = (b \circ a) \circ b \Leftrightarrow q^3 + q^2 - 2q + 1 = 0, \tag{27'}$$

$$a \circ [b \circ (c \circ a)] = (c \circ b) \circ c \Leftrightarrow q^3 - q + 1 = 0, \tag{30'}$$

$$(a \circ b) \circ (b \circ a) = (b \circ a) \circ a \Leftrightarrow 3q^2 - 4q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{3}, \tag{44}$$

$$(a \circ b) \circ (b \circ a) = (b \circ a) \circ b \Leftrightarrow 3q^2 - 3q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{6} i, \tag{45}$$

$$(a \circ b) \circ (c \circ a) = (b \circ c) \circ b \Leftrightarrow 2q^2 - 2q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i. \tag{46}$$

Here and from now on by $(k)'$ is denoted the identity complementary to the identity (k).

ADMISSIBLE IDENTITIES OF THE FORM $\langle 4 \rangle' = \langle 4 \rangle$

In the identities of the form $\langle 4 \rangle' = \langle 4 \rangle$ it is sufficient to consider the variables a, b, c and d. First of all we shall study the identities of this form in the variables a, b and c. For this purpose we compare the products from Tables 5 and 4. The results are:

$$[(a \circ b) \circ a] \circ a = [(a \circ b) \circ b] \circ b \Leftrightarrow q^2 - 2q = 0 \qquad \Leftrightarrow q = 2, \tag{47}$$

$$[(a \circ b) \circ a] \circ a = [(a \circ c) \circ c] \circ b \Leftrightarrow \begin{Bmatrix} q^2 - 2q = 0,\ q^3 - 2q^2 = 0, \\ q^3 - 3q^2 + 2q = 0 \end{Bmatrix} \Leftrightarrow q = 2, \tag{48}$$

$$[(a \circ b) \circ a] \circ a = [(b \circ a) \circ a] \circ b \Leftrightarrow 2q^3 - 5q^2 - 3q + 1 = 0, \tag{49}$$

$$[(a \circ b) \circ a] \circ a = [(b \circ a) \circ b] \circ b \Leftrightarrow 2q^3 - 4q^2 + 2q - 1 = 0, \tag{50}$$

$$[(a \circ b) \circ a] \circ a = [(c \circ b) \circ a] \circ c \Leftrightarrow q^3 - 3q^2 + 2q - 1 = 0, \tag{51}$$

$$[(a \circ b) \circ a] \circ a = [(c \circ b) \circ c] \circ c \Leftrightarrow q^3 - 2q^2 + q - 1 = 0, \tag{52}$$

$$[(a \circ b) \circ a] \circ a = [a \circ (a \circ b)] \circ b \Leftrightarrow 2q^3 - 3q^2 = 0, \quad \Leftrightarrow q = \frac{3}{2}, \tag{53}$$

$$[(a \circ b) \circ a] \circ a = [b \circ (b \circ a)] \circ b \Leftrightarrow q^2 - q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{54}$$

$$[(a \circ b) \circ a] \circ a = [a \circ (b \circ c)] \circ c \Leftrightarrow q^3 - q^2 - q = 0, \quad \Leftrightarrow q = \frac{1}{2}(1 \pm \sqrt{5}), \tag{55}$$

$$[(a \circ b) \circ a] \circ a = (b \circ a) \circ (a \circ b) \Leftrightarrow q^3 - 4q^2 + 3q - 1 = 0, \tag{56}$$

$$[(a \circ b) \circ a] \circ b = [(a \circ c) \circ a] \circ c \Leftrightarrow q^3 - 2q^2 + 2q = 0 \quad \Leftrightarrow q = 1 \pm i, \tag{57}$$

$$[(a \circ b) \circ a] \circ b = [(b \circ a) \circ a] \circ a \Leftrightarrow 2q^3 - 5q^2 + 5q - 1 = 0, \tag{58}$$

$$[(a \circ b) \circ a] \circ b = [(b \circ a) \circ b] \circ a \Leftrightarrow 2q^3 - 4q^2 + 4q - 1 = 0, \tag{59}$$

$$[(a \circ b) \circ a] \circ b = [a \circ (a \circ b)] \circ a \Leftrightarrow 2q^3 - 3q^2 + 2q = 0 \quad \Leftrightarrow q = \frac{3}{4} \pm \frac{\sqrt{7}}{4} i, \tag{60}$$

$$[(a \circ b) \circ a] \circ b = [b \circ (b \circ a]) \circ a \Leftrightarrow q^2 - 3q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{61}$$

$$[(a \circ b) \circ a] \circ c = [(a \circ c) \circ a] \circ b \Leftrightarrow q^3 - 2q^2 = 0 \quad \Leftrightarrow q = 2, \tag{62}$$

$$[(a \circ b) \circ a] \circ c = [(c \circ b) \circ a] \circ a \Leftrightarrow q^3 - 3q^2 + 4q - 1 = 0, \tag{63}$$

$$[(a \circ b) \circ a] \circ c = [(c \circ b) \circ c] \circ a \Leftrightarrow q^3 - 2q^2 + 3q - 1 = 0, \tag{64}$$

$$[(a \circ b) \circ a] \circ c = [(b \circ (b \circ c)] \circ b \Leftrightarrow \left\{ \begin{matrix} q^3 - 2q^2 + 2q - 1 = 0 \\ q^2 - q + 1 = 0, \; q^3 - q^2 + q = 0 \end{matrix} \right\} \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{65}$$

$$[(a \circ b) \circ b] \circ a = [(b \circ a) \circ a] \circ b \Leftrightarrow 2q^3 - 6q^2 + 4q - 1 = 0, \tag{66}$$

$$[(a \circ b) \circ b] \circ a = [(b \circ a) \circ b] \circ b \Leftrightarrow 2q^3 - 5q^2 + 3q - 1 = 0, \tag{49''}$$

$$[(a \circ b) \circ b] \circ a = [(c \circ b) \circ b] \circ c \Leftrightarrow q^3 - 3q^2 + 2q - 1 = 0, \tag{67}$$

$$[(a \circ b) \circ b] \circ a = [a \circ (a \circ b)] \circ b \Leftrightarrow 2q^3 - 4q^2 + q = 0 \quad \Leftrightarrow q = \frac{1}{2}(2 \pm \sqrt{2}), \tag{68}$$

$$[(a \circ b) \circ b] \circ a = [b \circ (b \circ a)] \circ b \Leftrightarrow 2q^2 - 2q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \tag{69}$$

$$[(a \circ b) \circ b] \circ a = (b \circ a) \circ (a \circ b) \Leftrightarrow q^3 - 5q^2 + 4q - 1 = 0, \tag{70}$$

$$[(a \circ b) \circ b] \circ b = [(a \circ c) \circ c] \circ c \Leftrightarrow q^3 - 3q^2 + 3q = 0 \quad \Leftrightarrow q = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i, \tag{71}$$

$$[(a \circ b) \circ b] \circ b = [(b \circ a) \circ a] \circ a \Leftrightarrow 2q^3 - 6q^2 + 6q - 1 = 0, \tag{72}$$

$$[(a \circ b) \circ b] \circ b = [(b \circ a) \circ b] \circ a \Leftrightarrow 2q^3 - 5q^2 + 5q - 1 = 0, \tag{58''}$$

$$[(a \circ b) \circ b] \circ b = [a \circ (a \circ b)] \circ a \Leftrightarrow 2q^3 - 4q^2 + 3q = 0 \quad \Leftrightarrow q = 1 \pm \frac{\sqrt{2}}{2} i, \tag{73}$$

$$[(a \circ b) \circ b] \circ b = [b \circ (b \circ a)] \circ a \Leftrightarrow 2q^2 - 4q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2}(2 \pm \sqrt{2}), \tag{74}$$

$$[(a \circ b) \circ b] \circ b = (b \circ a) \circ (a \circ b) \Leftrightarrow q^3 - 5q^2 + 5q - 1 = 0 \quad \Leftrightarrow q = 2 \pm \sqrt{3}, \tag{75}$$

$$[(a \circ b) \circ b] \circ c = [(a \circ c) \circ a] \circ a \Leftrightarrow \left\{ \begin{matrix} q^2 - 2q = 0, \; q^3 - 3q^2 + 2q = 0, \\ q^3 - 2q^2 = 0 \end{matrix} \right\} \quad \Leftrightarrow q = 2, \tag{48''}$$

$$[(a \circ b) \circ b] \circ c = [(a \circ c) \circ b] \circ b \Leftrightarrow q^3 - 2q^2 = 0 \quad \Leftrightarrow q = 2, \tag{76}$$

$$[(a \circ b) \circ b] \circ c = [(a \circ c) \circ c] \circ b \Leftrightarrow q^3 - 3q^2 + q = 0 \qquad \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{77}$$

$$[(a \circ b) \circ b] \circ c = [(c \circ b) \circ b] \circ a \Leftrightarrow q^3 - 3q^2 + 4q - 1 = 0, \tag{78}$$

$$[(a \circ b) \circ c] \circ a = [(b \circ a) \circ c] \circ b \Leftrightarrow 2q^3 - 5q^2 + 3q - 1 = 0, \tag{79}$$

$$[(a \circ b) \circ c] \circ a = [(c \circ b) \circ a] \circ c \Leftrightarrow q^3 - 4q^2 + 3q - 1 = 0, \tag{80}$$

$$[(a \circ b) \circ c] \circ a = [c \circ (b \circ a)] \circ c \Leftrightarrow 2q^2 - 2q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \tag{81}$$

$$[(a \circ b) \circ c] \circ a = (b \circ a) \circ (c \circ b) \Leftrightarrow q^3 - 4q^2 + 3q - 1 = 0, \tag{82}$$

$$[(a \circ b) \circ c] \circ b = [(b \circ a) \circ c] \circ a \Leftrightarrow 2q^3 - 5q^2 + 5q - 1 = 0, \tag{83}$$

$$[(a \circ b) \circ c] \circ c = [(c \circ b) \circ a] \circ a \Leftrightarrow q^3 - 4q^2 + 5q - 1 = 0, \tag{84}$$

$$[(a \circ b) \circ c] \circ c = [(c \circ (b \circ a)] \circ a \Leftrightarrow 2q^2 - 4q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}(2 \pm \sqrt{2}), \tag{85}$$

$$[a \circ (a \circ b)] \circ a = [(b \circ a) \circ a] \circ b \Leftrightarrow 2q^2 - 2q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \tag{69$''$}$$

$$[a \circ (a \circ b)] \circ a = [(b \circ a) \circ b] \circ b \Leftrightarrow q^2 - q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{54$'$}$$

$$[a \circ (a \circ b)] \circ a = [(c \circ a) \circ c] \circ b \Leftrightarrow \left\{ \begin{matrix} q^2 - q + 1 = 0, \ q^3 - q^2 + q = 0 \\ q^3 - 2q^2 + 2q - 1 = 0 \end{matrix} \right\} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{65$'$}$$

$$[a \circ (a \circ b)] \circ a = [b \circ (b \circ a)] \circ b \Leftrightarrow 2q^3 - 2q^2 + 1 = 0, \tag{50$'$}$$

$$[a \circ (a \circ b)] \circ a = [c \circ (c \circ b)] \circ c \Leftrightarrow q^3 - q^2 + 1 = 0, \tag{52$'$}$$

$$[a \circ (a \circ b)] \circ a = (b \circ a) \circ (a \circ b) \Leftrightarrow q^3 - q^2 - 2q + 1 = 0, \tag{56$'$}$$

$$[a \circ (a \circ b)] \circ b = [(b \circ a) \circ a] \circ a \Leftrightarrow 2q^2 - 4q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}(2 \pm \sqrt{2}), \tag{74$''$}$$

$$[a \circ (a \circ b)] \circ b = [(b \circ a) \circ b] \circ a \Leftrightarrow q^3 - 3q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{61$''$}$$

$$[a \circ (a \circ b)] \circ b = [a \circ (a \circ c)] \circ c \Leftrightarrow q^3 - q^2 - q = 0 \qquad \Leftrightarrow q = \frac{1}{2}(1 \pm \sqrt{5}), \tag{86}$$

$$[a \circ (a \circ b)] \circ b = [b \circ (b \circ a)] \circ a \Leftrightarrow 2q^3 - 2q^2 - 2q + 1 = 0, \tag{87}$$

$$[a \circ (a \circ b)] \circ b = (b \circ a) \circ (a \circ b) \Leftrightarrow q^3 + q^2 - 3q + 1 = 0 \qquad \Leftrightarrow q = -1 \pm \sqrt{2}, \tag{88}$$

$$[a \circ (a \circ b)] \circ c = [a \circ (a \circ c)] \circ b \Leftrightarrow q^3 - q^2 + q = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{89}$$

$$[a \circ (a \circ b)] \circ c = [c \circ (c \circ b)] \circ a \Leftrightarrow q^3 - q^2 - 2q + 1 = 0, \tag{90}$$

$$[a \circ (b \circ c)] \circ a = [(c \circ b) \circ a] \circ c \Leftrightarrow 2q^2 - 2q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \tag{81$''$}$$

$$[a \circ (b \circ c)] \circ b = [a \circ (c \circ b)] \circ c \Leftrightarrow 2q^3 - 3q^2 + 2q = 0 \qquad \Leftrightarrow q = \frac{3}{4} \pm \frac{\sqrt{7}}{4} i, \tag{91}$$

$$[a \circ (b \circ c)] \circ c = [(c \circ b) \circ a] \circ a \Leftrightarrow 2q^2 - 4q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2}(2 \pm \sqrt{2}), \tag{85)''}$$

$$[a \circ (b \circ c)] \circ c = [c \circ (b \circ a)] \circ a \Leftrightarrow q^3 - 3q + 1 = 0, \tag{92}$$

$$[a \circ (b \circ c)] \circ c = [a \circ (c \circ b)] \circ b \Leftrightarrow 2q^3 - 3q^2 = 0 \quad \Leftrightarrow q = \frac{3}{2}, \tag{93}$$

$$a \circ [(a \circ b) \circ b] = [(a \circ b) \circ b] \circ a \Leftrightarrow 2q^3 - 5q^2 + 2q = 0 \quad \Leftrightarrow q = \frac{1}{2},\ q = 2, \tag{94}$$

$$a \circ [(a \circ b) \circ b] = [(a \circ b) \circ b] \circ b \Leftrightarrow 2q^3 - 5q^2 + 3q = 0 \quad \Leftrightarrow q = \frac{3}{2}, \tag{95}$$

$$a \circ [(a \circ b) \circ b] = [(a \circ c) \circ c] \circ a \Leftrightarrow \left\{ \begin{matrix} 2q^3 - 5q^2 + 2q = 0 \\ q^3 - 2q^2 = 0,\ q^3 - 3q^2 + 2q = 0 \end{matrix} \right\} \quad \Leftrightarrow q = 2, \tag{96(}$$

$$a \circ [(a \circ b) \circ b] = [(b \circ a) \circ a] \circ a \Leftrightarrow q^2 - 3q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{97}$$

$$a \circ [(a \circ b) \circ b] = [(b \circ a) \circ b] \circ a \Leftrightarrow 2q - 1 = 0 \quad \Leftrightarrow q = \frac{1}{2}, \tag{98}$$

$$a \circ [(a \circ b) \circ b] = [b \circ (b \circ a)] \circ a \Leftrightarrow 2q^3 - 3q^2 - q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2},\ q = \frac{1}{2}(1 \pm \sqrt{5}), \tag{99}$$

$$a \circ [(a \circ b) \circ b] = [b \circ (b \circ a)] \circ b \Leftrightarrow 2q^3 - 3q^2 + 1 = 0 \quad \Leftrightarrow q = -\frac{1}{2}, \tag{53)'}$$

$$a \circ [(a \circ b) \circ b] = [b \circ (b \circ c)] \circ c \Leftrightarrow \left\{ \begin{matrix} q^3 - 2q^2 + 1 = 0, \\ 2q^3 - 3q^2 - q + 1 = 0, \quad q^3 - q^2 - q = 0 \end{matrix} \right\} \Leftrightarrow q = \frac{1}{2}(1 \pm \sqrt{5}), \tag{100}$$

$$a \circ [(a \circ b) \circ b] = (a \circ b) \circ (b \circ a) \Leftrightarrow q^3 - 4q^2 + 2q = 0 \quad \Leftrightarrow q = 2 \pm \sqrt{2}, \tag{88)'}$$

$$a \circ [(a \circ b) \circ c] = [c \circ (b \circ a)] \circ a \Leftrightarrow \left\{ \begin{matrix} 2q^3 - 3q^2 - q + 1 = 0, \\ 2q^3 - 3q^2 + q = 0, \quad 2q - 1 = 0 \end{matrix} \right\} \quad \Leftrightarrow q = \frac{1}{2}, \tag{101}$$

$$a \circ [(a \circ b) \circ c] = [c \circ (c \circ b)] \circ a \Leftrightarrow q^3 - 2q^2 - q + 1 = 0, \tag{102}$$

$$a \circ [(b \circ a) \circ b] = [(b \circ a) \circ a] \circ a \Leftrightarrow 2q^3 - 4q^2 + 4q - 1 = 0, \tag{103}$$

$$a \circ [(b \circ a) \circ b] = [(b \circ a) \circ a] \circ b \Leftrightarrow 2q^3 - 4q^2 + 3q - 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \tag{104}$$

$$a \circ [(b \circ a) \circ b] = [(b \circ a) \circ b] \circ a \Leftrightarrow 2q^3 - 3q^2 + 3q - 1 = 0 \quad \Leftrightarrow q = \frac{1}{2},\ q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{105}$$

$$a \circ [(b \circ a) \circ b] = [(b \circ a) \circ b] \circ b \Leftrightarrow 2q^3 - 3q^2 + 2q - 1 = 0 \quad \Leftrightarrow q = \frac{1}{4} \pm \frac{\sqrt{7}}{4} i, \tag{60)'}$$

$$a \circ [(b \circ a) \circ b] = [(c \circ a) \circ c] \circ a \Leftrightarrow \left\{ \begin{matrix} 2q^3 - 3q^2 + 3q - 1 = 0, \\ q^3 - q^2 + q = 0,\ q^3 - 2q^2 + 2q - 1 = 0 \end{matrix} \right\} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{106}$$

$$a \circ [(b \circ a) \circ b] = [b \circ (b \circ a)] \circ a \Leftrightarrow 2q - 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}, \tag{98)'$$

$$a \circ [(b \circ a) \circ c] = [c \circ (a \circ b)] \circ a \Leftrightarrow \left\{\begin{matrix} 2q^3 - 3q^2 + 3q - 1 = 0, \\ 2q^3 - 3q^2 + q = 0, \quad 2q - 1 = 0 \end{matrix}\right\} \Leftrightarrow q = \frac{1}{2}, \tag{107}$$

$$a \circ [(b \circ c) \circ b] = [(b \circ a) \circ b] \circ c \Leftrightarrow \left\{\begin{matrix} q^3 - 2q^2 + 2q - 1 = 0, \\ 2q^3 - 3q^2 + 3q - 1 = 0, \ q^3 - q^2 + q = 0 \end{matrix}\right\} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{108}$$

$$a \circ [(b \circ c) \circ b] = [(c \circ a) \circ c] \circ c \Leftrightarrow \left\{\begin{matrix} q^3 - 2q^2 + 2q - 1 = 0, \\ q^3 - q^2 + q = 0, \quad q^2 - q + 1 = 0 \end{matrix}\right\} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{65)'$$

$$a \circ [(b \circ c) \circ b] = [b \circ (a \circ c)] \circ a \Leftrightarrow q^3 - 2q^2 + 3q - 1 = 0, \tag{109}$$

$$a \circ [(b \circ c) \circ c] = [(a \circ b) \circ b] \circ b \Leftrightarrow \left\{\begin{matrix} q^3 - 3q^2 + 2q = 0, \\ q^2 - 2q = 0, \quad q^3 - 2q^2 = 0 \end{matrix}\right\} \Leftrightarrow q = 2, \tag{110}$$

$$a \circ [(b \circ c) \circ c] = [(a \circ c) \circ c] \circ b \Leftrightarrow \left\{\begin{matrix} q^3 - 3q^2 + 2q = 0, \ q^3 - 2q^2 = 0 \\ 2q^3 - 5q^2 + 2q = 0 \end{matrix}\right\} \Leftrightarrow q = 2, \tag{111}$$

$$a \circ [(b \circ c) \circ c] = [(c \circ b) \circ a] \circ a \Leftrightarrow q^2 - 3q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{112}$$

$$a \circ [(b \circ c) \circ c] = [c \circ (b \circ a)] \circ a \Leftrightarrow q^3 - q^2 - 2q + 1 = 0, \tag{102)'$$

$$a \circ [(b \circ c) \circ c] = [c \circ (c \circ b)] \circ a \Leftrightarrow \left\{\begin{matrix} 2q - 1 = 0, \quad 2q^3 - 3q^2 + q = 0, \\ 2q^3 - 3q^2 - q + 1 = 0 \end{matrix}\right\} \Leftrightarrow q = \frac{1}{2}, \tag{113}$$

$$a \circ [a \circ (a \circ b)] = [(a \circ b) \circ b] \circ a \Leftrightarrow 3q^2 - 2q = 0 \qquad \Leftrightarrow q = \frac{2}{3}, \tag{114}$$

$$a \circ [a \circ (a \circ b)] = [(b \circ a) \circ a] \circ a \Leftrightarrow 2q^3 - 3q^2 + 3q - 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}, q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{115}$$

$$a \circ [a \circ (a \circ b)] = [(b \circ a) \circ a] \circ b \Leftrightarrow 2q^3 - 3q^2 + 2q - 1 = 0 \qquad \Leftrightarrow q = \frac{1}{4} \pm \frac{\sqrt{7}}{4} i, \tag{116}$$

$$a \circ [a \circ (a \circ b)] = [(b \circ a) \circ b] \circ a \Leftrightarrow 2q^3 - 2q^2 + 2q - 1 = 0, \tag{103)'$$

$$a \circ [a \circ (a \circ b)] = [(b \circ a) \circ b] \circ b \Leftrightarrow 2q^3 - 2q^2 + q - 1 = 0 \qquad \Leftrightarrow q = \pm \frac{\sqrt{2}}{2} i, \tag{73)'$$

$$a \circ [a \circ (a \circ b)] = [a \circ (a \circ b)] \circ b \Leftrightarrow 2q^3 - q^2 - q = 0 \qquad \Leftrightarrow q = -\frac{1}{2}, \tag{95)'$$

$$a \circ [a \circ (a \circ b)] = [b \circ (b \circ a)] \circ a \Leftrightarrow q^2 + q - 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}(-1 \pm \sqrt{5}), \tag{97)'$$

$$a \circ [a \circ (a \circ b)] = [c \circ (c \circ a)] \circ b \Leftrightarrow \left\{\begin{matrix} q^2 - 1 = 0, \ q^3 - q = 0 \\ q^3 - q^2 - q + 1 = 0 \end{matrix}\right\} \Leftrightarrow q = -1, \tag{110)'$$

$$a \circ [a \circ (a \circ b)] = (a \circ b) \circ (b \circ a) \Leftrightarrow q^3 + 2q^2 - 2q = 0 \qquad \Leftrightarrow q = -1 \pm \sqrt{3}, \tag{75)'$$

$$a\circ[b\circ(a\circ c)]=[(c\circ a)\circ b]\circ a\Leftrightarrow 2q^3-3q^2+3q-1=0 \quad \Leftrightarrow q=\frac{1}{2},\ q=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i, \qquad (117)$$

$$a\circ[b\circ(a\circ c)]=[(c\circ a)\circ b]\circ c\Leftrightarrow 2q^3-3q^2+2q-1=0 \quad \Leftrightarrow q=\frac{1}{4}\pm\frac{\sqrt{7}}{4}i, \qquad (118)$$

$$a\circ[b\circ(a\circ c)]=[(c\circ b)\circ a]\circ b\Leftrightarrow\begin{Bmatrix} q^3-2q^2+2q-1=0,\ q^3-q^2+q=0, \\ 2q^3-3q^2+3q-1=0 \end{Bmatrix}\Leftrightarrow q=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i, \qquad (119)$$

$$a\circ[b\circ(b\circ a)]=[(a\circ b)\circ a]\circ b\Leftrightarrow 2q^3-2q^2+q=0 \quad \Leftrightarrow q=\frac{1}{2}\pm\frac{1}{2}i, \qquad (104)'$$

$$a\circ[b\circ(b\circ a)]=[(a\circ b)\circ b]\circ a\Leftrightarrow 2q^3-3q^2+q=0 \quad \Leftrightarrow q=\frac{1}{2}, \qquad (120)$$

$$a\circ[b\circ(b\circ a)]=[(a\circ b)\circ b]\circ b\Leftrightarrow 2q^3-3q^2+2q=0 \quad \Leftrightarrow q=\frac{3}{4}\pm\frac{\sqrt{7}}{4}i, \qquad (116)'$$

$$a\circ[b\circ(b\circ a)]=[(b\circ a)\circ a]\circ a\Leftrightarrow 3q^2-4q+1=0 \quad \Leftrightarrow q=\frac{1}{3}, \qquad (114)'$$

$$a\circ[b\circ(b\circ a)]=[(b\circ a)\circ a]\circ b\Leftrightarrow 3q^2-3q+1=0 \quad \Leftrightarrow q=\frac{1}{2}\pm\frac{\sqrt{3}}{6}i, \qquad (121)$$

$$a\circ[b\circ(b\circ a)]=[(b\circ a)\circ b]\circ b\Leftrightarrow 2q^2-2q+1=0 \quad \Leftrightarrow q=\frac{1}{2}\pm\frac{1}{2}i, \qquad (69)'$$

$$a\circ[b\circ(b\circ a)]=[b\circ(b\circ a)]\circ a\Leftrightarrow 2q^3-q^2-2q+1=0 \quad \Leftrightarrow q=\frac{1}{2},\ q=-1, \qquad (94)'$$

$$a\circ[b\circ(b\circ a)]=[b\circ(b\circ a)]\circ b\Leftrightarrow 2q^3-q^2-q-q+1=0, \qquad (49)'$$

$$a\circ[b\circ(b\circ a)]=[c\circ(c\circ a)]\circ a\Leftrightarrow\begin{Bmatrix} 2q^3-q^2-2q+1=0, \\ q^3-q=0,\ q^3-q^2-q+1=0 \end{Bmatrix}\Leftrightarrow q=-1, \qquad (96)'$$

$$a\circ[b\circ(b\circ a)]=(b\circ a)\circ(a\circ b)\Leftrightarrow q^3+2q^2-3q+1=0, \qquad (70)'$$

$$a\circ[a\circ(b\circ c)]=[(c\circ a)\circ a]\circ b\Leftrightarrow\begin{Bmatrix} q^3-2q^2+2q-1=0,\ q^2-q+1=0, \\ 2q^3-3q^2+3q-1=0 \end{Bmatrix}\Leftrightarrow q=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i, \qquad (122)$$

$$a\circ[a\circ(b\circ c)]=[(c\circ b)\circ a]\circ a\Leftrightarrow\begin{Bmatrix} 2q-1=0,\ 2q^2-3q+1=0, \\ 2q^3-3q^2+3q-1=0 \end{Bmatrix}\Leftrightarrow q=\frac{1}{2}, \qquad (123)$$

$$a\circ[a\circ(b\circ c)]=[c\circ(c\circ b)]\circ a\Leftrightarrow q^2+q-1=0 \quad \Leftrightarrow q=\frac{1}{2}(-1\pm\sqrt{5}), \qquad (112)'$$

$$a\circ[b\circ(b\circ c)]=[(c\circ a)\circ b]\circ b\Leftrightarrow\begin{Bmatrix} q^3-2q^2+2q-1=0,\ q^3-q^2+q=0, \\ 2q^3-3q^2+3q-1=0 \end{Bmatrix}\Leftrightarrow q=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i, \qquad (122)'$$

$$a\circ[b\circ(b\circ c)]=[(c\circ b)\circ b]\circ a\Leftrightarrow\begin{Bmatrix} 2q-1=0,\ 2q^3-3q^2+q=0, \\ 2q^3-3q^2+3q-1=0 \end{Bmatrix}\Leftrightarrow q=\frac{1}{2}, \qquad (124)$$

$$a\circ[b\circ(b\circ c)]=[b\circ(b\circ a)]\circ c\Leftrightarrow\begin{Bmatrix} q^3-q^2-q+1=0, \\ 2q^3-q^2-2q+1=0,\ q^3-q=0 \end{Bmatrix}\Leftrightarrow q=-1, \qquad (111)'$$

$$a \circ [b \circ (b \circ c)] = [c \circ (c \circ a)] \circ c \Leftrightarrow \begin{cases} q^3 - q^2 - q + 1 = 0, \\ q^3 - q = 0,\ q^2 - 1 = 0 \end{cases} \Leftrightarrow q = -1, \quad (48)'$$

$$a \circ [b \circ (c \circ a)] = [(a \circ c) \circ b] \circ c \Leftrightarrow 2q^3 - 3q^2 + 2q = 0 \quad \Leftrightarrow q = \frac{3}{4} \pm \frac{\sqrt{7}}{4} i, \quad (118)'$$

$$a \circ [b \circ (c \circ a)] = [(c \circ a) \circ b] \circ c \Leftrightarrow 2q^2 - 2q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \quad (125)$$

$$a \circ [b \circ (c \circ a)] = [b \circ (c \circ a)] \circ b \Leftrightarrow 2q^2 - 2q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \quad (81)'$$

$$a \circ [b \circ (c \circ a)] = (c \circ a) \circ (b \circ c) \Leftrightarrow q^3 + q^2 - 2q + 1 = 0, \quad (82)'$$

$$(a \circ b) \circ (b \circ a) = [(b \circ a) \circ a] \circ a \Leftrightarrow q^3 - 5q^2 + 5q - 1 = 0 \quad \Leftrightarrow q = 2 \pm \sqrt{3}, \quad (75)''$$

$$(a \circ b) \circ (b \circ a) = [(b \circ a) \circ a] \circ b \Leftrightarrow q^3 - 5q^2 + 4q - 1 = 0, \quad (70)''$$

$$(a \circ b) \circ (b \circ a) = [(b \circ a) \circ b] \circ b \Leftrightarrow q^3 - 4q^2 + 3q - 1 = 0, \quad (56)''$$

$$(a \circ b) \circ (b \circ a) = [b \circ (b \circ a)] \circ a \Leftrightarrow q^3 + q^2 - 3q + 1 = 0 \quad \Leftrightarrow q = -1 \pm \sqrt{2}, \quad (88)''$$

$$(a \circ b) \circ (b \circ a) = [b \circ (b \circ a)] \circ b \Leftrightarrow q^3 + q^2 - 2q + 1 = 0, \quad (56)'$$

$$(a \circ b) \circ (b \circ a) = (b \circ a) \circ (a \circ b) \Leftrightarrow 4q^2 - 4q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2}, \quad (126)$$

$$(a \circ b) \circ (b \circ a) = (c \circ b) \circ (b \circ c) \Leftrightarrow 2q^2 - 2q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i, \quad (127)$$

$$(a \circ b) \circ (b \circ c) = (a \circ c) \circ (c \circ b) \Leftrightarrow 3q^2 - 2q = 0 \quad \Leftrightarrow q = \frac{2}{3}, \quad (128)$$

$$(a \circ b) \circ (b \circ c) = (b \circ a) \circ (a \circ c) \Leftrightarrow 3q^2 - 4q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{3}, \quad (128)'$$

$$(a \circ b) \circ (b \circ c) = (c \circ b) \circ (b \circ a) \Leftrightarrow 2q - 1 = 0 \quad \Leftrightarrow q = \frac{1}{2}, \quad (129)$$

$$(a \circ b) \circ (c \circ a) = [(b \circ a) \circ c] \circ b \Leftrightarrow q^3 - 4q^2 + 3q - 1 = 0, \quad (82)''$$

$$(a \circ b) \circ (c \circ a) = [(c \circ a) \circ b] \circ c \Leftrightarrow q^3 - 4q^2 + 3q - 1 = 0, \quad (82)''$$

$$(a \circ b) \circ (c \circ a) = (b \circ a) \circ (c \circ b) \Leftrightarrow 3q^2 - 3q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{6} i, \quad (130)$$

$$(a \circ b) \circ (c \circ a) = (c \circ a) \circ (b \circ c) \Leftrightarrow 3q^2 - 3q + 1 = 0 \quad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{6} i. \quad (130)''$$

Here and from now on by $(k)''$ is denoted an identity which by a suitable substitution of variables transforms into the identity (k). We wish to note that the deduced admissible identities are not mentioned in the previous list. One such identity is the identity $a \circ [(b \circ c) \circ c] = [(a \circ b) \circ c] \circ a$, which by (V) transforms into the identity $a \circ [(b \circ c) \circ c] = [(a \circ c) \circ (b \circ c)] \circ a$ and then by the successive substitutions $b \bigcirc c \to b$ and $c \leftrightarrow b$ in a simpler identity $[(a \bigcirc b) \bigcirc c] \bigcirc a = a \bigcirc (c \bigcirc b)$ of the form $\langle 4 \rangle' = \langle 3 \rangle$. The last identity is not mentioned among the identities (22)—(46), while it can be further simplified. Because of (VI) we shall write $[(a \bigcirc b) \bigcirc c] \bigcirc a = (a \bigcirc c) \bigcirc (a \bigcirc b)$, i. e. after the succesive substitutions $a \bigcirc b \to b$ and $b \leftrightarrow c$ finally $(a \bigcirc b) \bigcirc c = (c \bigcirc b) \bigcirc a$ and this is the identity (15) of the form $\langle 3 \rangle' = \langle 3 \rangle$.

Now, let us consider the identities of the form $\langle 4 \rangle' = \langle 4 \rangle$, where in the product of the form $\langle 4 \rangle'$ there are only the variables a, b, c, but in the product of the form $\langle 4 \rangle$ there is also the variable d. Because of Proposition 3 the variable d must appear at least twice in this product $\langle 4 \rangle$. If the product $\langle 4 \rangle'$ has two factors a and two factors b, then the product $\langle 4 \rangle$ must have a factor c, because in the contrary case the substitution $d \to c$ gives the previously considered identities. But, then in our case the product $\langle 4 \rangle$ has two factors c and two factors d, and these factors do not appear on the left side of the identity $\langle 4 \rangle' = \langle 4 \rangle$. After the application of the identity (I) we obtain the polynomials $\gamma(q)$ and $\delta(q)$ as the coefficients of the variables c and d and it holds obviously the identity $\gamma(q) + \delta(q) = 1$. (More generally, in any product of the form $\langle n \rangle$ the sum of polynomials, which are the coefficients of the variables, is equal to 1.) Therefore, these polynomials $\gamma(q)$ and $\delta(q)$ are relatively prime and the equations $\gamma(q) = 0$ and $\delta(q) = 0$ do not have a common solution, i. e. there is no one admissible identity of this form. If the product $\langle 4 \rangle'$ has three factors a and one factor b, then in the product $\langle 4 \rangle$ must appear a factor b. But this product has already at least two factors d and so it can not have any factor c. Thus, by the substitution $d \to c$ our identity transforms into a previous identity.

From the preceding consideration it follows that it is unneccessary to study the identities of the form $\langle 4 \rangle' = \langle 4 \rangle$, where the product $\langle 4 \rangle'$ has only two of the variables a, b, c and the product $\langle 4 \rangle$ has at least two factors d. Therefore, it is sufficient to study the products of the form $\langle 4 \rangle'$ of one of three types (which is already denoted in Table 5):

A) the product $\langle 4 \rangle'$ has two factors a, one factor b and one factor c;

B) the product $\langle 4 \rangle'$ has two factors b, one factor a and one factor c;

C) the product $\langle 4 \rangle'$ has two factors c, one factor a and one factor b.

Because of Proposition 3 in the case *A*) the product $\langle 4 \rangle$ has two factors d, one factor b and one factor c and we have analogous conclusions in the cases *B*) and *C*).

In order to obtain all products of the form $\langle 4 \rangle$ in the case *A*) in the first, second and fifth products (X) for two of the four factors x, y, u, v we must take the variable d, and for the other two factors the variables b, c. In the cases of the first or second product (X) the two factors d can be taken in five different ways (in the first product it is unneccessary to consider the case $x = y = d$ and in the second product the case $y = v = d$), while in the case of the product $(x \bigcirc y) \bigcirc (u \bigcirc v)$ it is unneccessary to consider the cases $x = y = d$ and $u = v = d$ and because of (V) and (VI) the cases $y = v = d$ and $x = u = d$. Moreover, by (IV) and (III) it holds $[d \bigcirc (b \bigcirc d)] \bigcirc c = [(d \bigcirc b) \bigcirc d] \bigcirc c$, $[d \bigcirc (c \bigcirc d)] \bigcirc b = [(d \bigcirc c) \bigcirc d] \bigcirc b$, $(d \bigcirc c) \bigcirc (b \bigcirc d) = (d \bigcirc b) \bigcirc (c \bigcirc d)$. Therefore, there remain all together 21 products of the form $\langle 4 \rangle$ in the case *A*). These products are enumerated in Table 6. *A*) and analogously in the cases *B*) and *C*) we have Tables 6. *B*) and 6. *C*).

Table 6.*A*)

⟨4⟩	*b*				*c*				*d*			
	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3
$[(b \circ c) \circ d] \circ d$	1	−3	3	−1		1	−2	1		2	−1	
$[(c \circ b) \circ d] \circ d$		1	−2	1	1	−3	3	−1		2	−1	
$[(b \circ d) \circ c] \circ d$	1	−3	3	−1		1	−1			2	−2	1
$[(c \circ d) \circ b] \circ d$		1	−1		1	−3	3	−1		2	−2	1
$[(b \circ d) \circ d] \circ c$	1	−3	3	−1		1				2	−3	1
$[(c \circ d) \circ d] \circ b$		1			1	−3	3	−1		2	−3	1
$[(d \circ b) \circ c] \circ d$		1	−2	1		1	−1		1	−2	3	−1
$[(d \circ c) \circ b] \circ d$		1	−1			1	−2	1	1	−2	3	−1
$[(d \circ b) \circ d] \circ c$		1	−2	1		1			1	−2	2	−1
$[(d \circ c) \circ d] \circ b$		1				1	−2	1	1	−2	2	−1
$[b \circ (c \circ d)] \circ d$	1	−2	1			1	−2	1		1	1	−1
$[c \circ (b \circ d)] \circ d$		1	−2	1	1	−2	1			1	1	−1
$[b \circ (d \circ c)] \circ d$	1	−2	1				1	−1		2	−2	1
$[c \circ (d \circ b)] \circ d$			1	−1	1	−2	1			2	−2	1
$[d \circ (b \circ c)] \circ d$		1	−2	1			1	−1	1	−1	1	
$[d \circ (c \circ b)] \circ d$			1	−1		1	−2	1	1	−1	1	
$[d \circ (d \circ b)] \circ c$			1	−1		1			1	−1	−1	1
$[d \circ (d \circ c)] \circ b$		1					1	−1	1	−1	−1	1
$(b \circ d) \circ (d \circ c)$	1	−2	1	1			1			2	−2	
$(c \circ d) \circ (d \circ b)$			1		1	−2	1			2	−2	
$(d \circ b) \circ (c \circ d)$		1	−1			1	−1		1	−2	2	

Table 6.*B*)

⟨4⟩	*a*				*c*				*d*			
	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3
$[(a \circ c) \circ d] \circ d$	1	−3	3	−1		1	−2	1		2	−1	
$[(c \circ a) \circ d] \circ d$		1	−2	1	1	−3	3	−1		2	−1	
$[(a \circ d) \circ c] \circ d$	1	−3	3	−1		1	−1			2	−2	1
$[(c \circ d) \circ a] \circ d$		1	−1		1	−3	3	−1		2	−2	1
$[(a \circ d) \circ d] \circ c$	1	−3	3	−1		1				2	−3	1
$[(c \circ d) \circ d] \circ a$		1			1	−3	3	−1		2	−3	1
$[(d \circ a) \circ c] \circ d$		1	−2	1		1	−1		1	−2	3	−1
$[(d \circ c) \circ a] \circ d$		1	−1			1	−2	1	1	−2	3	−1
$[(d \circ a) \circ d] \circ c$		1	−2	1		1			1	−2	2	−1
$[(d \circ c) \circ d] \circ a$		1				1	−2	1	1	−2	2	−1
$[a \circ (c \circ d)] \circ d$	1	−2	1			1	−2	1		1	1	−1
$[c \circ (a \circ d)] \circ d$		1	−2	1	1	−2	1			1	1	−1
$[a \circ (d \circ c)] \circ d$	1	−2	1				1	−1		2	−2	1
$[c \circ (d \circ a)] \circ d$			1	−1	1	−2	1			2	−2	1
$[d \circ (a \circ c)] \circ d$		1	−2	1			1	−1	1	−1	1	
$[d \circ (c \circ a)] \circ d$			1	−1		1	−2	1	1	−1	1	
$[d \circ (d \circ a)] \circ c$			1	−1		1			1	−1	−1	1
$[d \circ (d \circ c)] \circ a$		1					1	−1	1	−1	−1	1
$(a \circ d) \circ (d \circ c)$	1	−2	1				1			2	−2	
$(c \circ d) \circ (d \circ a)$			1		1	−2	1			2	−2	
$(d \circ a) \circ (c \circ d)$		1	−1			1	−1		1	−2	2	

Table 6.*C*)

$\langle 4\rangle$	*a*				*b*				*d*			
	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3
$[(a \circ b) \circ d] \circ d$	1	−3	3	−1		1	−2	1		2	−1	
$[(b \circ a) \circ d] \circ d$		1	−2	1	1	−3	3	−1		2	−1	
$[(a \circ d) \circ b] \circ d$	1	−3	3	−1		1	−1			2	−2	1
$[(b \circ d) \circ a] \circ d$		1	−1		1	−3	3	−1		2	−2	1
$[(a \circ d) \circ d] \circ b$	1	−3	3	−1		1				2	−3	1
$[(b \circ d) \circ d] \circ a$		1			1	−3	3	−1		2	−3	1
$[(d \circ a) \circ b] \circ d$		1	−2	1		1	−1		1	−2	3	−1
$[(d \circ b) \circ a] \circ d$		1	−1			1	−2	1	1	−2	3	−1
$[(d \circ a) \circ d] \circ b$		1	−2	1		1			1	−2	2	−1
$[(d \circ b) \circ d] \circ a$		1				1	−2	1	1	−2	2	−1
$[a \circ (b \circ d)] \circ d$	1	−2	1			1	−2	1		1	1	−1
$[b \circ (a \circ d)] \circ d$		1	−2	1	1	−2	1			1	1	−1
$[a \circ (d \circ b)] \circ d$	1	−2	1				1	−1		2	−2	1
$[b \circ (d \circ a)] \circ d$			1	−1	1	−2	1			2	−2	1
$[d \circ (a \circ b)] \circ d$		1	−2	1			1	−1	1	−1	1	
$[d \circ (b \circ a)] \circ d$			1	−1		1	−2	1	1	−1	1	
$[d \circ (d \circ a)] \circ b$			1	−1		1			1	−1	−1	1
$[d \circ (d \circ b)] \circ a$		1					1	−1	1	−1	−1	1
$(a \circ d) \circ (d \circ b)$	1	−2	1				1			2	−2	
$(b \circ d) \circ (d \circ a)$			1		1	−2	1			2	−2	
$(d \circ a) \circ (b \circ d)$		1	−1			1	−1		1	−2	2	

Now, by comparing the corresponding products of the form $\langle 4\rangle'$ from Table 5 with the products of the form $\langle 4\rangle$ from one of Tables 6, we obtain the new primitive admissible identities:

$$[(a \circ b) \circ a] \circ c = [d \circ (b \circ c)] \circ d \Leftrightarrow \left\{\begin{matrix} q^3-2q^2+2q-1=0, \\ q^3-q^2+q=0,\ q^2-q+1=0 \end{matrix}\right\} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i, \tag{131}$$

$$[(a \circ b) \circ c] \circ a = [(a \circ c) \circ d] \circ d \Leftrightarrow \left\{\begin{matrix} q^3 - 3q^2 + 2q = 0, \\ q^3 - 2q^2 = 0,\ q^2 - 2q = 0 \end{matrix}\right\} \Leftrightarrow q = 2, \tag{132}$$

$$[(a \circ b) \circ c] \circ a = [(d \circ b) \circ c] \circ d \Leftrightarrow q^3 - 3q^2 + 2q - 1 = 0, \tag{133}$$

$$[(a \circ b) \circ c] \circ b = [(a \circ d) \circ c] \circ d \Leftrightarrow q^3 - 2q^2 + 2q = 0 \qquad \Leftrightarrow q = 1 \pm i, \tag{134}$$

$$[(a \circ b) \circ c] \circ c = [(a \circ d) \circ d] \circ b \Leftrightarrow \left\{\begin{matrix} q^3 - 2q^2 = 0,\ q^2 - 2q = 0 \\ q^3 - 3q^2 + 2q = 0 \end{matrix}\right\} \Leftrightarrow q = 2, \tag{132$''$}$$

$$[a \circ (b \circ c)] \circ a = [(d \circ b) \circ d] \circ c \Leftrightarrow \left\{\begin{matrix} q^2-q+1=0,\ q^3-q^2+q=0, \\ q^3-2q^2+2q-1=0 \end{matrix}\right\} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i, \tag{131$''$}$$

$$[a \circ (b \circ c)] \circ b = [a \circ (d \circ c)] \circ d \Leftrightarrow q^3 - 2q^2 + 2q = 0 \qquad \Leftrightarrow q = 1 \pm i, \tag{135}$$

$$[a \circ (b \circ c)] \circ c = [a \circ (b \circ d)] \circ d \Leftrightarrow q^3 - q^2 - q = 0 \qquad \Leftrightarrow q = \frac{1}{2}(1 \pm \sqrt{5}), \tag{136}$$

$$a \circ [(b \circ c) \circ b] = [(d \circ a) \circ d] \circ c \Leftrightarrow q^3 - 2q^2 + 2q - 1 = 0,\ q^3 - q^2 + q = 0 \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{137}$$

$$a \circ [(b \circ c) \circ b] = [d \circ (a \circ c)] \circ d \Leftrightarrow \begin{Bmatrix} q^3 - 2q^2 + 2q - 1 = 0, \\ q^3 - q^2 + q = 0,\ q^2 - q + 1 = 0 \end{Bmatrix} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{131$'$}$$

$$a \circ [(b \circ c) \circ c] = [(a \circ b) \circ d] \circ d \Leftrightarrow \begin{Bmatrix} q^3 - 3q^2 + 2q = 0, \\ q^3 - 2q^2 = 0,\ q^2 - 2q = 0 \end{Bmatrix} \Leftrightarrow q = 2, \tag{138}$$

$$a \circ [(b \circ c) \circ c] = [(a \circ d) \circ d] \circ b \Leftrightarrow q^3 - 3q^2 + 2q = 0,\ q^3 - 2q^2 = 0 \qquad \Leftrightarrow q = 2, \tag{139}$$

$$a \circ [a \circ (b \circ c)] = [d \circ (d \circ b)] \circ c \Leftrightarrow q^2 - 1 = 0,\ q^3 - q = 0,\ q^3 - q^2 - q + 1 = 0 \Leftrightarrow q = -1, \tag{138$'$}$$

$$a \circ [b \circ (b \circ c)] = [d \circ (d \circ a)] \circ c \Leftrightarrow q^3 - q^2 - q + 1 = 0,\ q^3 - q = 0 \qquad \Leftrightarrow q = -1, \tag{139$'$}$$

$$(a \circ b) \circ (c \circ a) = (d \circ b) \circ (c \circ d) \Leftrightarrow 2q^2 - 2q + 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{1}{2} i. \tag{140}$$

Finally, we must find all admissible identities of the form $\langle 4 \rangle' = \langle 4 \rangle$ in the variables a, b, c, d, where the product $\langle 4 \rangle'$ has all these variables and so any of these variables appears once. By Proposition 3 the same happens in the product $\langle 4 \rangle$. The needed products $\langle 4 \rangle'$ can be obtained from the products (X) by the substitution $x = a$, $y = b$, $u = c$, $v = d$. These products are enumerated in Table 7.

Table 7.

$\langle 4' \rangle$	a				b				c				d			
	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3
$[(a \circ b) \circ c] \circ d$	1	−3	3	−1		1	−2	1		1	−1			1		
$[a \circ (b \circ c)] \circ d$	1	−2	1			1	−2	1			1	−1		1		
$a \circ [(b \circ c) \circ d]$	1	−1				1	−2	1			1	−1			1	
$a \circ [b \circ (c \circ d)]$	1	−1				1	−1				1	−1				1
$(a \circ b) \circ (c \circ d)$	1	−2	1			1	−1			1	−1				1	

In the case of the product of the form $\langle 4 \rangle$ it is sufficient to consider again only the first, second and fifth products (X) and then for (x, y, u, v) to take all permutations of the set $\{a, b, c, d\}$. Because of (III) only 12 of 24 products of the form $(x \circ \circ y) \circ (u \circ v)$ are mutually different. Therefore, it is sufficient to consider the 60 products in Table 8.

Table 8.

⟨4⟩	a				b				c				d			
	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3	1	q	q^2	q^3
$[(a \circ b) \circ c] \circ d$	1	−3	3	−1		1	−2	1		1	−1			1		
$[(a \circ b) \circ d] \circ c$	1	−3	3	−1		1	−2	1		1				1	−1	
$[(a \circ c) \circ b] \circ d$	1	−3	3	−1		1	−1			1	−2	1		1		
$[(a \circ c) \circ d] \circ b$	1	−3	3	−1		1				1	−2	1		1	−1	
$[(a \circ d) \circ b] \circ c$	1	−3	3	−1		1	−1			1				1	−2	1
$[(a \circ d) \circ c] \circ b$	1	−3	3	−1		1				1	−1			1	−2	1
$[(b \circ a) \circ c] \circ d$		1	−2	1	1	−3	3	−1		1	−1			1		
$[(b \circ a) \circ d] \circ c$		1	−2	1	1	−3	3	−1		1				1	−1	
$[(b \circ c) \circ a] \circ d$		1	−1		1	−3	3	−1		1	−2	1		1		
$[(b \circ c) \circ d] \circ a$		1			1	−3	3	−1		1	−2	1		1	−1	
$[(b \circ d) \circ a] \circ c$		1	−1		1	−3	3	−1		1				1	−2	1
$[(b \circ d) \circ c] \circ a$		1			1	−3	3	−1		1	−1			1	−2	1
$[(c \circ a) \circ b] \circ d$		1	−2	1		1	−1		1	−3	3	−1		1		
$[(c \circ a) \circ d] \circ b$		1	−2	1		1			1	−3	3	−1		1	−1	
$[(c \circ b) \circ a] \circ d$		1	−1			1	−2	1	1	−3	3	−1		1		
$[(c \circ b) \circ d] \circ a$		1				1	−2	1	1	−3	3	−1		1	−1	
$[(c \circ d) \circ a] \circ b$		1	−1			1			1	−3	3	−1		1	−2	1
$[(c \circ d) \circ b] \circ a$		1				1	−1		1	−3	3	−1		1	−2	1
$[(d \circ a) \circ b] \circ c$		1	−2	1		1	−1			1			1	−3	3	−1
$[(d \circ a) \circ c] \circ b$		1	−2	1		1				1	−1		1	−3	3	−1
$[(d \circ b) \circ a] \circ c$		1	−1			1	−2	1		1			1	−3	3	−1
$[(d \circ b) \circ c] \circ a$		1				1	−2	1		1	−1		1	−3	3	−1
$[(d \circ c) \circ a] \circ b$		1	−1			1				1	−2	1	1	−3	3	−1
$[(d \circ c) \circ b] \circ a$		1				1	−1			1	−2	1	1	−3	3	−1
$[a \circ (b \circ c)] \circ d$	1	−2	1			1	−2	1			1	−1		1		
$[a \circ (b \circ d)] \circ c$	1	−2	1			1	−2	1		1					1	−1
$[a \circ (c \circ b)] \circ d$	1	−2	1				1	−1		1	−2	1		1		
$[a \circ (c \circ d)] \circ b$	1	−2	1			1				1	−2	1			1	−1
$[a \circ (d \circ b)] \circ c$	1	−2	1				1	−1		1				1	−2	1
$[a \circ (d \circ c)] \circ b$	1	−2	1			1					1	−1		1	−2	1
$[b \circ (a \circ c)] \circ d$		1	−2	1	1	−2	1				1	−1		1		
$[b \circ (a \circ d)] \circ c$		1	−2	1	1	−2	1			1					1	−1
$[b \circ (c \circ a)] \circ d$			1	−1	1	−2	1			1	−2	1		1		
$[b \circ (c \circ d)] \circ a$		1			1	−2	1			1	−2	1			1	−1
$[b \circ (d \circ a)] \circ c$			1	−1	1	−2	1			1				1	−2	1
$[b \circ (d \circ c)] \circ a$		1			1	−2	1				1	−1		1	−2	1
$[c \circ (a \circ b)] \circ d$		1	−2	1			1	−1	1	−2	1			1		
$[c \circ (a \circ d)] \circ b$		1	−2	1		1			1	−2	1				1	−1
$[c \circ (b \circ a)] \circ d$			1	−1		1	−2	1	1	−2	1			1		
$[c \circ (b \circ d)] \circ a$		1				1	−2	1	1	−2	1				1	−1
$[c \circ (d \circ a)] \circ b$			1	−1		1			1	−2	1			1	−2	1
$[c \circ (d \circ b)] \circ a$		1					1	−1	1	−2	1			1	−2	1
$[d \circ (a \circ b)] \circ c$		1	−2	1			1	−1		1			1	−2	1	
$[d \circ (a \circ c)] \circ b$		1	−2	1		1					1	−1	1	−2	1	
$[d \circ (b \circ a)] \circ c$			1	−1		1	−2	1		1			1	−2	1	
$[d \circ (b \circ c)] \circ a$		1				1	−2	1			1	−1	1	−2	1	
$[d \circ (c \circ a)] \circ b$			1	−1		1				1	−2	1	1	−2	1	
$[d \circ (c \circ b)] \circ a$		1					1	−1		1	−2	1	1	−2	1	
$(a \circ b) \circ (c \circ d)$	1	−2	1			1	−1			1	−1				1	
$(a \circ b) \circ (d \circ c)$	1	−2	1			1	−1				1			1	−1	
$(a \circ c) \circ (d \circ b)$	1	−2	1				1			1	−1			1	−1	
$(b \circ a) \circ (c \circ d)$		1	−1		1	−2	1			1	−1				1	
$(b \circ a) \circ (d \circ c)$		1	−1		1	−2	1				1			1	−1	
$(b \circ c) \circ (d \circ a)$			1		1	−2	1			1	−1			1	−1	
$(c \circ a) \circ (b \circ d)$		1	−1			1	−1		1	−2	1				1	
$(c \circ a) \circ (d \circ b)$		1	−1				1		1	−2	1			1	−1	
$(c \circ b) \circ (d \circ a)$			1			1	−1		1	−2	1			1	−1	
$(d \circ a) \circ (b \circ c)$		1	−1			1	−1				1		1	−2	1	
$(d \circ a) \circ (c \circ b)$		1	−1				1			1	−1		1	−2	1	
$(d \circ b) \circ (c \circ a)$			1			1	−1			1	−1		1	−2	1	

By comparing products from Tables 7 and 8 we obtain the remaining primitive admissible identities:

$$[(a \circ b) \circ c] \circ d = [(a \circ d) \circ c] \circ b \Leftrightarrow q^3 - 2q^2 = 0 \qquad \Leftrightarrow q = 2, \tag{141}$$

$$[(a \circ b) \circ c] \circ d = [(d \circ b) \circ c] \circ a \Leftrightarrow q^3 - 3q^2 + 4q - 1 = 0, \tag{142}$$

$$[(a \circ b) \circ c] \circ d = (d \circ a) \circ (c \circ b) \Leftrightarrow \begin{Bmatrix} q^3-4q^2+4q-1=0, \\ q^3-3q^2+q=0, \quad q^2-3q+1=0 \end{Bmatrix} \Leftrightarrow q = \frac{1}{2}(3 \pm \sqrt{5}), \tag{143}$$

$$[a \circ (b \circ c)] \circ d = [a \circ (b \circ d)] \circ c \Leftrightarrow q^3 - q^2 + q = 0 \qquad \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{144}$$

$$[a \circ (b \circ c)] \circ d = [a \circ (d \circ c)] \circ b \Leftrightarrow q^3 - 2q^2 = 0 \qquad \Leftrightarrow q = 2, \tag{145}$$

$$a \circ [(b \circ c) \circ d] = [d \circ (c \circ b)] \circ a \Leftrightarrow 2q - 1 = 0, 2q^3 - 3q^2 + q = 0 \Leftrightarrow q = \frac{1}{2}, \tag{146}$$

$$a \circ [b \circ (c \circ d)] = [(d \circ a) \circ b] \circ c \Leftrightarrow \begin{Bmatrix} q^3-2q^2+2q-1=0, \quad q^3-q^2+q=0, \\ 2q^3-3q^2+3q-1=0 \end{Bmatrix} \Leftrightarrow q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \tag{147}$$

$$a \circ [b \circ (c \circ d)] = [(d \circ c) \circ b] \circ a \Leftrightarrow \begin{Bmatrix} 2q-1=0, \quad 2q^3-3q^2+q=0, \\ 2q^3-3q^2+3q-1=0 \end{Bmatrix} \Leftrightarrow q = \frac{1}{2}, \tag{148}$$

$$a \circ [b \circ (c \circ d)] = (c \circ b) \circ (d \circ a) \Leftrightarrow \begin{Bmatrix} q^2+q-1=0, \quad q^3-2q+1=0, \\ q^3+q^2-q=0 \end{Bmatrix} \Leftrightarrow q = \frac{1}{2}(-1 \pm \sqrt{5}), \tag{143'}$$

$$(a \circ b) \circ (c \circ d) = (d \circ b) \circ (c \circ a) \Leftrightarrow 2q - 1 = 0 \qquad \Leftrightarrow q = \frac{1}{2}. \tag{149}$$

The identities (1)—(149) and their complementary identities are all primitive admissible identities of the form $\langle m \rangle' = \langle n \rangle$, where $m, n \in \{1, 2, 3, 4\}$ and $m \geq n$.

All results deduced here in the field of complex numbers can be brought over into any of its subfields. It may be interesting to do an analogous consideration in any field of a finite characteristic.

Proposition 1 affirms the statement: if an admissible identity holds in a quasigroup $C(q)$, then q is an algebraic number. It would be interesting to investigate if the converse statement is true: for every algebraic number q there is an admissible identity which holds in the quasigroup $C(q)$.

Naturally, an analogous consideration can be applied for the study of all primitive admissible identities of the form $\langle m \rangle' = \langle n \rangle$, where $m \geq n$ and $m > 4$. But, for this purpose some more simple methods must be found, while the number of all necessary comparisons increases very rapidly in the dependence of the number m.

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Dopustivi identiteti u kompleksnim IM-kvazigrupama. I.

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Sadržaj

Ako je $(C, +, \cdot)$ polje kompleksnih brojeva i $q \in C \setminus \{0, 1\}$, tada je formulom $a \circ b = (1 - q)\, a + qb$ definirana idempotentna medijalna kvazigrupa $(C, \circ)$. Za neku posebnu vrijednost q u kvazigrupi $(C, \circ)$ može osim posljedica idempotentnosti i medijalnosti vrijediti još neki dodatni identitet, tzv. dopustivi identitet. Nađeni su svi primitivni dopustivi identiteti s ne više od četiri faktora na svakoj strani.

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