Geometric Simplicity Theory

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Abstract

We prove the group configuration theorem in simple theories, a very abstract result reconstructing a group (action) from a certain independence-theoretic configuration of points, and argue that such a result gives rise to 'geometric simplicity theory' (i.e. analogues of methods and results of geometric stability theory).

The proof involves studying the behaviour of multivalued algebraic structures like polygroups and polyspaces, a development of the theory of independence for almost hyperimaginaries, and a sophisticated blowup procedure.

Some of the corollaries of the group configuration theorem we obtain include finding the group associated to a polygroup in a simple theory, interpreting a vector space over a finite field inside a one-based ω-categorical theory of SU-rank 1, and showing how pseudolinearity implies one-basedness under the assumption of ω-categoricity.

Declaration of authenticity

This thesis has been composed by myself.

Important parts of the thesis come from my collaboration with Itay Ben-Yaacov and Frank O. Wagner on [BTW] and some from my collaboration with just Frank O. Wagner ([TW]). My contribution has been clearly indicated throughout the chapters. Moreover, since [BTW] will also form part of Ben-Yaacov's forthcoming PhD thesis at Université Paris 7, I made an effort to emphasize his substantial contribution.

Ivan Tomašić

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Introduction

In the beginning, there was Morley's theorem from the 1960's, stating that if a complete countable first order theory T is categorical in an uncountable cardinality, it must in fact be categorical in all uncountable cardinalities.

After that came Shelah's monumental programme ([S]), concerned with classifying theories in terms of structure/non-structure dichotomies, where a structure theorem states that a theory has relatively few models and thus it makes sense to try and classify its models (Morley's theorem being the first example of such a theorem), while a non-structure theorem asserts that a theory has a maximal number of models in a given cardinality, their classification thus being hopeless.

On the other hand, Zil'ber developed in the 1970's a variety of methods and results which became known as *geometric stability theory*, oriented more towards the qualitative classification of theories. After showing many results concerning ω_1-categorical theories, he conjectured that if M is a model of such a theory, then M must essentially be either a vector space, or an algebraically closed field, or a degenerate structure (see [Z1]–[Z6]). We will refer to this conjecture as *Zil'ber's trichotomy.* The methods consisted of studying the combinatorial pregeometries arising on strongly minimal sets inside a given structure, and the relationship between the properties (e.g. (non) local modularity, triviality, etc.) of those pregeometries and the structure as a whole. Interpretability of nice algebraic structures (e.g. groups or fields) inside a structure had significant consequences.

In 1980's, Hrushovski brought geometric stability theory to its peak ([H0]–[H8]). He showed in particular that Zil'ber's conjecture does not hold in full generality ([H5]), and then later with Zil'ber that it does hold in Zariski geometries [HZ]. Also, his work on the Mordell-Lang and Manin-Mumford conjectures ([H6], [H8]) clearly demonstrated that 'excursions' outside the classical model theory of stable structures can answer fundamental questions about the classical structures themselves. Some of the theories he considered we now know to be *simple.*

Simple theories were defined by Shelah in [S93], but only became a subject of intense investigation after Kim managed to prove symmetry and transitivity of forking in [Ki] and especially after Kim and Pillay's proof of the Independence Theorem. The program was apparent: to generalise well-understood methods of stability in this new and mysterious context. With more or less difficulty, the usual techniques of forking calculus, canonical bases, orthogonality, regularity, coordinatisation, type-definable groups and Hrushovski's amalgamation construction were translated into simplicity. There were extremely difficult problems intrinsic to simple theories which were also resolved, e.g. elimination of hyperimaginaries for supersimple theories ([BPW]).

However, for a long time, there was no results reminiscent of geometric stability theory. In particular, there was no group existence theorems. And yet, it was apparent there should be, since at least in all the known examples, speaking loosely, we either have a stable 'reduct' controlling forking (e.g. random graph, vector space over a finite field with a bilinear form, algebraically closed fields with an automorphism), or the structure can be 'embedded' into a stable one (pseudofinite fields), so the 'geometry' should come from these stable structures.

The first attempts on the *group configuration theorem* (for stable theories first recognised by Zil'ber and then in full generality by Hrushovski) in simple theories by Ben-Yaacov [BY] and myself [To] yielded some partial results and indicated the way to proceed. The collaboration of Ben-Yaacov, Wagner and myself gave a satisfactory solution to the problem in [BTW]. Furthermore, there were several different lines of attack on the binding group theorem by Hart, Shami and Wagner. It seemed for a while that it was just a matter of guessing the right definition of the group of automorphisms, but then Wagner reduced the problem to the group configuration ([W2]), showing that it is indeed the only group existence theorem around. Since then there were other developments in the direction of geometric simplicity, see [Va] and [dPK]. My purpose in this thesis is to present the proof of the group configuration theorem for simple theories, as well as its application in developing *geometric simplicity theory*, with a view to Zil'ber-type trichotomy for simple theories.

In Chapter 1, we deal with prerequisites needed to develop further chapters, to make the exposition as self-contained as possible. Thus, on one hand, we discuss simple theories, independence relations and combinatorial pregeometries arising within, and on the other, we define and study basic properties of polygroups, because even though they appear quite naturally, we cannot really consider them to be standard mathematical objects.

In Chapter 2, we develop the theory of germs of type-definable generic partial multiactions needed for the proof of the group configuration theorem. In the stable case, germs can be developed quickly and efficiently, as shown in the introduction to the chapter. In the simple case, however, the situation is very intricate so the whole chapter is dedicated to it, based on [BY]. It is shown how a polygroup chunk arises from the usual algebraic quadrangle (group configuration), thus completing the first step towards constructing a group. In the stable case, at this stage a group chunk would be derived, and a direct application of the Hrushovski-Weil style group chunk theorem would yield a group. A polygroup chunk proves to be much more troublesome.

To overcome the difficulties with the polygroup chunk, in Chapter 4, we present a variety of blowup procedures which give a group chunk, where the group chunk theorem will apply and provide us with the sought-after group. The construction is of algebraic-geometric nature (hence the name) and has a surprising similarity to the reconstruction of the division ring from a projective geometry (maybe not so surprising if we remind ourselves that the group configuration is a highly abstract analogue of the very same classical construction). Three variants of the construction due to myself are given (it should be noted here that the idea by Ben-Yaacov of using the core relation in a 'blowup-like' context resulted in the first successful blowup construction): the first is as hyperdefinable as possible, the second is written to look like von Neumann's proof from [vN], and the third to be as universal as possible–in fact, to have the usual universal property of blowing-up from algebraic geometry.

Unfortunately, the construction of Chapter 4 works only up to a certain invariant relation, thus giving what we call a *gradedly almost hyperdefinable*, and not necessarily hyperdefinable group chunk, so in Chapter 3 we develop the machinery (our presentation modifies slightly Ben-Yaacov's notes for the purpose of [BTW]) needed to handle almost hyperimaginary elements, gradedly almost hyperdefinable polygroups and polygroup chunks. In particular, gradedly almost hyperdefinable polygroups are defined, as well as their generic elements, whose existence and basic properties are shown (for hyperdefinable polygroups in supersimple theories this was originally done in [To1]). The group (space) chunk theorem is proved in this category as well, based on ideas of [To], Ben-Yaacov noting it preserves almost hyperdefinability.

In Chapter 5 we give a few applications of the group configuration framework. Firstly, we show how each gradedly almost hyperdefinable polygroup has a group closely related to it, solving a fundamental problem of classical hypergroup theory. There are two approaches to the problem, the first by Wagner and myself, the second by Ben-Yaacov. We then demonstrate that no problems with almost definability arise in an ω-categorical theory where the group configuration gives an interpretable group (action). This result is due to myself. Then, we find a vector space interpreted in an one-based SU-rank 1 simple theory and argue that this partially solves the well-known stable forking hypothesis, and also leads towards a Zil'ber trichotomy-type result for simple theories. And, finally, we reconstruct the original group action and prove that pseudolinearity implies one-basedness under the assumption of ω-categoricity. These last three applications are due to Wagner and myself.

CHAPTER 1

Preliminaries

This chapter is devoted to fixing notation and stating the facts necessary for the rest of the thesis. Even though it is impossible to give a brief outline of the theory of simplicity without repeating e.g. the entire thesis of Kim ([Ki]) and quite a few research papers, or the first several chapters of Wagner's book ([W]), in Section 1.2 we attempt to at least set out the facts we will need, assuming the reader is somewhat familiar with basic first order model theory and stability. In Section 1.3 we study combinatorial (pre)geometries which are important for developing geometric stability (and later simplicity) theory. In Section 1.4 we expound the definitions and basic notions of the classical theory of polygroups, since most mathematicians are not likely to have encountered it.

1.1. Conventions and notation

Let T be a first order theory with infinite models in a language L, $\mathfrak{C}$ the corresponding monster model (highly saturated and strongly homogeneous model of the theory) and $\mathfrak{C}^{\mathrm{eq}}$ the associated enrichment by imaginaries.

Arbitrary small (we do not wish to be precise about the meaning of 'small' which is after all a common practice in stability theory, mostly it will mean 'of cardinality less than $|\mathfrak{C}|$') tuples of elements of $\mathfrak{C}^{\mathrm{eq}}$ will usually be denoted by a, $b\dots$, and 'small' subsets of $\mathfrak{C}^{\mathrm{eq}}$ will be denoted by A, $B\dots$ The concatenation of tuples a and b is usually written as ab (or occasionally $a\hat{\ }b$), and the union of sets A and B is sometimes written simply as AB. The group of automorphisms of $\mathfrak{C}$ fixing a set A is denoted by $\mathrm{Aut}_A(\mathfrak{C})$. We write $a \equiv_A b$ to express that $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. Our assumptions on $\mathfrak{C}$ allow us to use principles of the kind: $a \equiv_A b$ if and only if there is an automorphism of $\mathfrak{C}^{\mathrm{eq}}$ fixing A and taking a to b. By $\mathrm{dcl}(A)$ we denote the set of all elements of $\mathfrak{C}^{\mathrm{eq}}$ which are fixed by any automorphism fixing A, and $\mathrm{acl}(A)$ is used for the set of elements of $\mathfrak{C}^{\mathrm{eq}}$ which have only finitely many conjugates by automorphisms fixing A.

Furthermore, we write $A \approx B$ for $A \cap B \neq \emptyset$, which becomes extremely useful when dealing with multivalued algebraic objects. If R is an equivalence relation, we write $A \approx_R B$ if $A/R \cap B/R \neq \emptyset$, and $A =_R B$ if $A/R = B/R$ as sets. The partitive set of A is denoted by $\mathbb{P}(A)$, and the nonempty subsets of A by $\mathbb{P}^*(A)$.

1.2. Simplicity

I leave to several futures (not to all) my garden of forking paths.
JORGE LUIS BORGES, *The Garden of Forking Paths*

The concept of forking was introduced by Shelah in early 1970s, and a smooth theory was developed in a stable setting in [S], showing that it gives a good notion of independence in the sense that if $tp(a/B)$ does not fork over A ($\subseteq B$), then a satisfies no more dependency relations with elements of B than with elements of A. In 1977, Lascar and Poizat gave an alternative approach to forking [LPo] which replaced Shelah's 'combinatorial' definition, and resulted in popularization of forking, followed by books by Pillay [P1], Lascar [L4], and papers by Harnik and Harrington [HH] and Makkai [Mk].

Then, in [S93], Shelah introduced a new class of theories generalizing stable theories called simple, and noticed that the theory of forking might behave well in this context, but he didn't prove all the good properties of forking and was forced to introduce the concept of weak dividing which allowed him to solve a combinatorial problem he considered in that paper. However, work of Hrushovski in certain algebraic examples of simple theories [H3], Hrushovski and Pillay [HP1], [HP2] and Hrushovski and Chatzidakis [ChH], showed that there is indeed a good notion of independence in all cases, and an important result called the Independence Property was isolated.

The greatest contribution to the theory of forking in simple context came with work of Kim and Pillay, when first Kim proved the symmetry of dividing in presence of simplicity in [Ki], and then Kim and Pillay were able to prove all the relevant good properties, including the above mentioned Independence Theorem, see [KP], [KP1], [Ki1]. Some of the important problems that have been investigated are the equivalence of the notions of Lascar strong type and strong type, connected to the problem of eliminating hyperimaginaries ([PP], [BPW], [LP], [P5]), and the existence of canonical bases ([HKP]) in $\mathfrak{C}^{eq}$.

A natural question, motivated by the lack of examples, posed by Hart and others, was whether forking in simple theories (under certain reasonable assumptions like elimination of hyperimaginaries) is in some way represented or determined by stable formulae; this became known as the *stable forking hypothesis*. Needless to say, all the known simple theories satisfy this in one way or another. Kim an Pillay offer a solution to an interesting approximation to the problem, as well as a better understanding of canonical bases in the supersimple case, see [KP2]. Also, we shall prove in Chapter 5 that certain structures have stable reducts 'preserving' forking, thus showing a very strong form of stable forking.

We start with Shelah's original definition of forking in terms of dividing in an arbitrary theory.

Definition 1.2.1.
- A formula $\varphi(x,b)$ *divides over* A, if there exist $\{b_i|i<\omega\}$ and $k<\omega$ such that for every $i<\omega$, $\text{tp}(b_i/A)=\text{tp}(b/A)$ and the set $\{\varphi(x,b_i)|i<\omega\}$ is k-contradictory.
- A (partial) type p *divides over* A if there is a formula $\varphi(x,b)$ with $p\vdash\varphi(x,b)$ and $\varphi(x,b)$ divides over A.
- A formula $\varphi(x,b)$ *forks over* A if there are $n<\omega$ and $\{\varphi_i(x,b^i)|i<n\}$ such that $\varphi(x,b)\vdash\bigvee_{i<n}\varphi_i(x,b^i)$ and for every $i<n$, $\varphi_i(x,b^i)$ divides over A.
- A (partial) type p *forks over* A if there exists a formula $\varphi(x,b)$ such that $p\vdash\varphi(x,b)$ and $\varphi(x,b)$ forks over A.

Definition 1.2.2. A theory T has the *tree property*, if there is a formula $\varphi(x,y)$ with the *tree property*, i. e., there are $k<\omega$ and $\{a_\eta|\eta\in{}^{\omega>}\omega\}$ such that

- for every $\eta\in{}^{\omega}\omega$, the set $\{\varphi(x,a_{\eta\restriction l})|l<\omega\}$ is consistent;
- for every $\eta\in{}^{\omega>}\omega$, the set $\{\varphi(x,a_{\eta\frown n})|n<\omega\}$ is k-contradictory.

Shelah originally defined a theory to be simple, if it did not have the tree property.

Let us write $A\mathop{\smile\!\!\!\!\!\!|}^{\text{f}}_C B$ (to be read 'A is (forking-)independent from B over C') if for every finite $a\in A$, $\text{tp}(a/BC)$ does not fork over C. Kim has shown that in theories without the tree property, forking coincides with dividing and it is possible to prove properties listed in the definition below for this notion of independence. There are many beautiful accounts of this, see [Ki], [KP], [KP1], [P4], or [W]. We shall, however, adopt an axiomatic approach to simple theories.

Definition 1.2.3. A theory is *simple* if it has an *independence relation*, i.e. a ternary relation $\mathop{\smile\!\!\!\!\!\!\!\mid}\;$ on subsets of the monster model satisfying:

[Invariance]	$\mathop{\smile\!\!\!\!\!\!\!\mid}\;$ is invariant under automorphisms of $\mathfrak{C}$;
[Finite Character]	$A \mathop{\smile\!\!\!\!\!\!\!\mid}_E B$ if and only if for all finite $a \in A$, $b \in B$, $a \mathop{\smile\!\!\!\!\!\!\!\mid}_E b$;
[Transitivity]	$A \mathop{\smile\!\!\!\!\!\!\!\mid}_E BC$ if and only if $A \mathop{\smile\!\!\!\!\!\!\!\mid}_E B$ and $A \mathop{\smile\!\!\!\!\!\!\!\mid}_{EB} C$;
[Symmetry]	$A \mathop{\smile\!\!\!\!\!\!\!\mid}_E B$ if and only if $B \mathop{\smile\!\!\!\!\!\!\!\mid}_E A$;
[Extension]	for all A, B, E, there is A' with $A' \equiv_E A$ and $A' \mathop{\smile\!\!\!\!\!\!\!\mid}_E B$;
[Local Character]	for every finite a and all B, there is $E \subseteq B$ with $\lvert E\rvert \leq \lvert T\rvert$ such that $a \mathop{\smile\!\!\!\!\!\!\!\mid}_E B$.
[Independence Theorem]	if M is a model, $a_0 \equiv_M a_1$, $a_i \mathop{\smile\!\!\!\!\!\!\!\mid}_M b_i$ for $i < 2$ and $b_0 \mathop{\smile\!\!\!\!\!\!\!\mid}_M b_1$, there is $a \mathop{\smile\!\!\!\!\!\!\!\mid}_M b_1 b_2$ with $a \equiv_{Mb_0} a_0$ and $a \equiv_{Mb_1} a_1$.

A theory is *supersimple*, if in 'Local Character' above, E can in fact be found finite. The property replacing the Independence Theorem in the characterization of *stable* theories is Stationarity: for a model M and a set B, if $a_0 \equiv_M a_1$ and $a_0 \mathop{\smile\!\!\!\!\!\!\!\mid}_M B$, $a_1 \mathop{\smile\!\!\!\!\!\!\!\mid}_M B$, then $a_0 \equiv_{MB} a_1$.

A justification for such a definition can be found in [KP] (or [W], 2.6.1):

Theorem 1.2.4. *Let T be simple in the sense of 1.2.3. Then T does not have the tree property and* $\mathop{\smile\!\!\!\!\!\!\!\mid}\; = \mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{f}}$.

One of the particularly useful properties of independence which can be derived from the others, is invariance under algebraic closure, i.e. $A \mathop{\smile\!\!\!\!\!\!\!\mid}_C B$ if and only if $\mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$.

Definition 1.2.5. An infinite sequence $\langle a_i : i < \alpha\rangle$ is *indiscernible* over a set A (sometimes called A-indiscernible), if for every $n < \omega$, for every $i_0 < \cdots < i_{n-1} < \alpha$ and $j_0 < \cdots < j_{n-1} < \alpha$, $\mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/A) = \mathrm{tp}(a_{j_0}, \ldots, a_{j_{n-1}}/A)$.

Indiscernible sequences of arbitrary length can be found (and the existing ones can be made longer) inside our model by the well-known combinatorial argument using Ramsey's theorem, giving this subject a combinatorial/set theoretic flavour. The following lemma, however, a complete proof of which can be found in [GIL], can be used in producing indiscernible sequences with special properties:

Lemma 1.2.6. *For every set A and sequence $\langle a_i : i < \beth_{(2^{(\lvert T\rvert+\lvert A\rvert+\lambda)})^+}\rangle$ of tuples of length λ, there exists an A-indiscernible sequence $\langle b_n : n < \omega\rangle$ such that for every $n < \omega$ there are $i_0 < \cdots < i_{n-1}$ with* $\mathrm{tp}(b_0, \ldots, b_{n-1}/A) = \mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/A)$.

Definition 1.2.7. We say that a set X is independent over A, if for every $x \in X$, $x \mathop{\smile\!\!\!\!\!\!\!\mid}_A X - \{x\}$.

Using the properties of independence, it is easy to see that a sequence $\langle a_i : i < \alpha\rangle$ is A-independent if and only if for each $i < \alpha$, $a_i \mathop{\smile\!\!\!\!\!\!\!\mid}_A \langle a_j : j < i\rangle$.

Definition 1.2.8. A sequence $\langle a_i : i < \alpha\rangle$ is a *Morley sequence* over A if it is both A-indiscernible and A-independent.

The existence of Morley sequences is shown by refining a long independent sequence (which exists by Extension) using 1.2.6. These sequences play a crucial role in Kim's arguments.

Definition 1.2.9. Let A be a set. The group of *Lascar strong* automorphisms of $\mathfrak{C}$ over A is the subgroup of $\mathrm{Aut}_A(\mathfrak{C})$ generated by all automorphisms fixing some model $M \supseteq A$, usually denoted $\mathrm{LAut}_A(\mathfrak{C})$. Two tuples a and b have the same *Lascar strong* type over A, denoted $a \equiv^L_A b$, if they are conjugate by a Lascar strong automorphism over A. The equality of Lascar strong types over A is clearly an equivalence relation, the equivalence class of a denoted by $\text{lstp}(a/A)$.

It can be shown that $\text{LAut}_A(\mathfrak{C})$ is a normal subgroup of $\text{Aut}_A(\mathfrak{C})$, and that equality of Lascar strong types over A is the finest bounded A-invariant equivalence relation, see [W]. The notion of Lascar strong type was studied in [L3]. Lascar points out that in stable theories, $\text{lstp}(a/A) = \text{lstp}(b/A)$ if and only if $\text{stp}(a/A) = \text{stp}(b/B)$. In any case, it is clear that the role of strong types from stability is taken over by Lascar strong types in simplicity since it is possible to prove the Independence Theorem for Lascar strong types ([KP], see also [Sha] and [P4] for more elegant proofs):

Theorem 1.2.10. *Let $a_0 \equiv^L_A a_1$, $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A b_i$ for $i < 2$ and $b_0 \mathop{\smile\!\!\!\!\!\!\mid}_A b_1$, there is $a \mathop{\smile\!\!\!\!\!\!\mid}_A b_1 b_2$ with $a \equiv_{Ab_0} a_0$ and $a \equiv_{Ab_1} a_1$.*

While canonical bases in stable theories exist in $\mathfrak{C}^{\text{eq}}$, in simple theories it became apparent ([HKP]) that we need to expand somewhat our universe.

Definition 1.2.11. Let x and y be a tuples of (possibly infinite, but small) order type α, and let $E(x, y)$ be a type-definable equivalence relation. We call E *countable* if the partial type defining E is, and *finitary* if α is finite. A *hyperimaginary* element of type E is just an equivalence class a_E for some a of length α. A hyperimaginary is countable or finitary if the corresponding relation is.

Lemma 1.2.12. *In any complete theory, every type-definable equivalence relation is the intersection of countable equivalence relations.*

It should be noted that it is no longer possible to treat hyperimaginaries as elements of the structure, as it was for imaginaries, see [W], 3.1.6. In spite of that fact, we denote by $\mathfrak{C}^{\text{heq}}$ the collection of all countable hyperimaginaries, and it is still possible to make sense of (Lascar strong) types of hyperimaginaries and develop the theory of independence satisfying the same axioms as the ones mentioned above; we refer the reader to [W].

The closure operators from $\mathfrak{C}^{\text{eq}}$ extend naturally, since every automorphism of $\mathfrak{C}$ extends uniquely to $\mathfrak{C}^{\text{heq}}$.

Definition 1.2.13.
(1) We say that $a \in \text{dcl}(A)$ for hyperimaginary a and A, if a is fixed by all the automorphisms fixing A. As an object, we define $\text{dcl}(A)$ to be the set of all countable hyperimaginaries which are fixed under all A-automorphisms (the reason for allowing only countable hyperimaginaries being our desire to keep $\text{dcl}(A)$ small, which is not a serious restriction by 1.2.12).
(2) We say that $a \in \text{bdd}(A)$ if a has only boundedly many conjugates under A-automorphisms. We let $\text{bdd}(A)$ be the set of all countable hyperimaginaries which are bounded over A.

The bounded closure takes on the role of algebraic closure in the hyperimaginary universe; in particular, $a \mathop{\smile\!\!\!\!\!\!\mid}_c b$ if and only if $\text{bdd}(ac) \mathop{\smile\!\!\!\!\!\!\mid}_{\text{bdd}(c)} \text{bdd}(bc)$. Furthermore, in a simple theory, $a \equiv^L_A b$ if and only if $a \equiv_{\text{bdd}(A)} b$.

Definition 1.2.14. The SU-rank, sometimes called the Lascar rank, is the least function from the collection of all types (over parameters in the monster model) to $\text{On} \cup \{\infty\}$, such that for every α, $\text{SU}(p) \geq \alpha + 1$ if there is a forking extension q of p with $\text{SU}(q) \geq \alpha$.

The properties of SU-rank can be summarised in ([W], Section 5.1):

Proposition 1.2.15. *SU-rank in a simple theory T has the following properties (where $\oplus$ denotes the Cantor commutative sum of ordinals):*

(1) SU *is automorphism invariant;*
(2) $q \vdash p$ *implies* $\mathrm{SU}(q) \leq \mathrm{SU}(p)$*;*
(3) $\mathrm{SU}(p) = 0$ *if and only if p is bounded;*
(4) *if* $\mathrm{SU}(p) < \infty$ *and* $\alpha \leq \mathrm{SU}(p)$*, then p has an extension q with* $\mathrm{SU}(q) = \alpha$*;*
(5) *if* $q \vdash p$ *and* $\mathrm{SU}(q) < \infty$*, then q is a nonforking extension of p if and only if* $\mathrm{SU}(q) = \mathrm{SU}(p)$*;*
(6) *T is supersimple if and only if $SU(p) < \infty$ for every (real) type p;*
(7) $\mathrm{SU}(a/bA) + \mathrm{SU}(b/A) \leq \mathrm{SU}(ab/A) \leq \mathrm{SU}(a/ba) \oplus \mathrm{SU}(b/A)$*;*
(8) *if* $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$*, then* $\mathrm{SU}(ab/A) = \mathrm{SU}(a/A) \oplus \mathrm{SU}(b/A)$.

Let us consider now the definability of various objects discussed above. Regarding definability of independence (see [W], 2.3.15, 3.2.9), we will use the following without explicit mention. Sometimes it is possible to show definability even if p is not complete ([W], proof of 4.7.1).

Lemma 1.2.16. *The condition $\exists x[x \models p \wedge x \mathop{\smile\!\!\!\!\!\!\mid}_A b \wedge \Phi(x,b)]$, for a complete (hyperimaginary) type p over A and any partial type $\Phi(x,y)$, is type-definable.*

For the definability of equality of Lascar strong types for hyperimaginaries, see e.g. [W], 3.2.12:

Lemma 1.2.17. *The relation $\mathrm{LS}(y,x;y',x')$, true if $x = x'$ and $y \equiv^L_x y'$ is a hyperdefinable equivalence relation.*

We say that a type $p(x,a)$ (where a is a hyperimaginary element) is an *amalgamation base* if (the hyperimaginary version of) the Independence Theorem holds over p, i.e. any two nonforking extensions of p over a-independent sets can be amalgamated. In particular, all Lascar strong types are amalgamation bases.

Definition 1.2.18. Let p be an amalgamation base.

- The *amalgamation class* of p is $\mathcal{P}_p = \{r | r$ is an amalgamation base and there are amalgamation bases $p_0, \ldots, p_n$ such that $p = p_0,\ r = p_n,$ and for every $i < n,\ p_i$ and p_{i+1} have a common nonforking extension$\}$.
- We call a set B of (hyper)imaginaries a *canonical base for p*, if every automorphism fixes $\mathcal{P}_p$ setwise if and only if it fixes B pointwise.

It is clear that any two canonical bases for a fixed amalgamation base p are interdefinable, so we sometimes abuse the language and speak about 'the' canonical base, denoted by $\mathrm{Cb}(p)$. The following very important theorem is from [HKP].

Theorem 1.2.19. *In a simple theory, every (hyperimaginary) amalgamation base p over A has a hyperimaginary canonical base A_0, i.e., A_0 has the following properties:*

(1) *p does not fork over A_0.*
(2) *$p \restriction_{A_0}$ is an amalgamation base.*
(3) *If q is an amalgamation base over B such that p and q have a common nonforking extension, then $A_0 \subseteq \mathrm{dcl}(B)$.*
(4) *If q is over B and p and q have a common nonforking extension, then $A_0 \subseteq \mathrm{bdd}(B)$.*

The proof goes roughly by defining a *generically transitive* relation on the conjugates of parameters of A, identifying A and A' if $p(x,A)$ and $p(x,A')$ have a common nonforking extension. Then, the canonical base will be the class of A and it will be a hyperimaginary by the following lemma ([W], 3.3.1).

Lemma 1.2.20. *Suppose R_0 is type-definable reflexive symmetric relation on a partial type π,* generically transitive *in the sense that whenever $a, a', a'' \models \pi$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}_a a''$ and $R_0(a,a')$ and $R_0(a,a'')$ hold, then $R_0(a',a'')$ holds. Then, the transitive closure R of R_0 equals the 2-step iteration of R_0 and is thus a type-definable equivalence relation. Furthermore, $R(a,b)$ holds for some $a, b \models \pi$ if and only if there is some $c \models \pi$ with $c \mathop{\smile\!\!\!\!\!\!\mid}_a b$ and $c \mathop{\smile\!\!\!\!\!\!\mid}_b a$ such that $R_0(a,c)$ and $R_0(c,b)$ hold.*

Definition 1.2.21. A theory T is *one-based,* if $a \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{bdd}(a)\cap\mathrm{bdd}(b)} b$ for any a, b.

A characterisation of one-based theories is given by ([W], 3.5.18):

Proposition 1.2.22. *A theory is one-based if and only if every real type* $\mathrm{tp}(a/A)$ *is based on* $\mathrm{bdd}(a)$*, i.e.* $\mathrm{Cb}(a/A) \in \mathrm{bdd}(a)$.

We might ask ourselves whether hyperimaginaries are really neccessary in simple theories, motivating the following definition. We say that a theory admits *elimination of hyperimaginaries* if every hyperimaginary is interdefinable with a sequence of imaginary element. Now we can state all the known results connecting these concepts. In [BPW], using a quite involved analysability argument, the authors managed to prove:

Theorem 1.2.23. *Any supersimple theory admits elimination of hyperimaginaries.*

Elimination of hyperimaginaries, being equivalent to the statement that on the set of realizations of a fixed complete type, every type-definable equivalence relation is an intersection of definable equivalence relations, was proved for stable theories already in [PP]. Also, the equivalence of Lascar strong types and strong types was proved by Buechler in [Bu] for a certain class of simple theories he called *low.*

Now, elimination of hyperimaginaries clearly implies the existence of canonical bases in $\mathfrak{C}^{eq}$, and the converse is also true for simple T, as shown in [LP]. Also, elimination of hyperimaginaries easily implies the equivalence of Lascar strong types and strong types (since, of course, $\mathrm{acl} = \mathrm{bdd}$ becomes true).

The definition of canonical bases as given above is not completely satisfactory, especially when compared to the stable case, where canonical base of a stationary type $p(x) \in S(A)$ in $\mathfrak{C}^{eq}$ is given by $\mathrm{Cb}(p) = \mathrm{dcl}(\bigcup\{\ulcorner d_p x\varphi(x,y)\urcorner | \varphi(x,y) \in L(T)\})$, where by $d_p x\varphi(x,y)$ we denote the φ-definition of p and by $\ulcorner X \urcorner$ we denote the canonical parameter or name of a definable set X.

In their efforts to understand the stable forking hypothesis, one form of which could say that, in a simple theory, canonical bases of amalgamation bases can be obtained as unions of names of definitions corresponding to stable formulae (similarly as above in stable theories), Kim and Pillay proved a very close approximation to it in [KP2].

Definition 1.2.24.
(1) A complete type p over A is *foreign* to an A-invariant family of partial types Σ, if for all $a \models p$, $B \mathop{\smile\!\!\!\!\!\!\mid}_A a$, and realisations $\bar{c}$ of possibly forking extensions of types in Σ over B, we always have $a \mathop{\smile\!\!\!\!\!\!\mid}_{AB} \bar{c}$.
(2) A type is *regular* if it is unbounded and foreign to all its forking extensions.

Clearly, any type p with $\mathrm{SU}(p) = 1$ is regular, since all its forking extensions are bounded types.

Having fixed the terminology above, let us proceed by giving examples of simple theories. Firstly, all the stable theories are simple, including e.g. the theory of the trivial structure (with just equality), theory of vector spaces over division rings, algebraically closed fields, separably closed fields, etc. Below are some examples of simple unstable theories.

Example 1.2.25. The theory of the *random graph*, i. e., the theory of an irreflexive symmetric binary operation such that for all n, for all distinct $x_1, \ldots, x_n, y_1, \ldots, y_n$ there is z such that $R(z, x_i)$ for $i \in \{1, \ldots, n\}$ and $\neg R(z, y_i)$ for all $i \in \{1, \ldots, n\}$. This theory is ω-categorical, with quantifier elimination, and it can be shown that any complete 1-type does not divide over a finite set (in fact over the empty set if p is not algebraic). Thus, by 1.2.3, this theory is simple (in fact supersimple of SU-rank 1). It is not stable because the formula $R(x, y)$ has the order property (it is even more obvious that it has the *independence* property).

Example 1.2.26. Let V be an infinite vector space over a finite field, and $\langle\,,\rangle$ a non-degenerate bilinear form on V. Then, $(V, \langle\,,\rangle)$ is unstable, because using a variant of the Gramm-Schmidt procedure we can find an orthogonal basis and then easily a formula having the independence property. It is supersimple of SU-rank 1 and ω-categorical.

Example 1.2.27. A field F is *pseudofinite* if F is perfect, for every $n > 1$, F has a unique algebraic extension of degree n and F is pseudo-algebraically closed (PAC), i. e., every (absolutely irreducible) variety over F has an F-rational point. It can be shown that these properties are first-order axiomatizable; call the resulting theory Psf. Psf was shown to be decidable by Ax, Duret [Du] proved it unstable, the structure of definable sets was then studied in detail by Chatzidakis, Macintyre and van den Dries in [ChMv]. In pseudofinite fields, dimension corresponds to SU-rank and is actually the algebraic-geometrical dimension of the Zariski closure, and so forking corresponds to algebraic independence. This was essentially known to Hrushovski in [H3], where he proved the Independence Theorem for types over relatively algebraically closed subfields (after adding a fixed suitable set of constants). From this, simplicity is clear and it also follows that $\mathrm{lstp} = \mathrm{stp}$, canonical bases exist in the real world, which also implies elimination of imaginaries.

Example 1.2.28. ACFA is the model companion of the theory of fields with a distinguished automorphism σ (i.e. the theory of existentially closed fields with an automorphism). The existence of the model companion is due to van den Dries, Macintyre and Wood ([Ma]), and a deep model-theoretic analysis of ACFA can be found in [ChH].

Let $(F, \sigma) \models$ ACFA be saturated. For $A \subseteq F$, let $\langle A \rangle$ be the smallest subfield of F closed under σ. Chatzidakis and Hrushovski introduce the following relation of independence: $B \perp_A C$ if $\langle AB \rangle$ is algebraically independent from $\langle AC \rangle$ over $\langle A \rangle$, and prove the Independence Theorem for types $p(x) \in S(A)$ where A equals the field theoretic algebraic closure of $\langle A \rangle$. It follows that ACFA is simple and this notion of independence coincides with nonforking. It can also be seen that canonical bases exist in the real world, which implies elimination of imaginaries. In fact, ACFA is supersimple.

Example 1.2.29. Theories with 'generic' predicates and/or automorphisms, as studied in [ChP]. We begin with a complete theory T with quantifier elimination in a language L. Let P (resp. σ) be a new predicate (resp. function) symbol. If T has a model companion in $L(P)$ (resp. $L(\sigma)$), we call it T_P (resp. T_σ). It can be shown that if T is simple, T_P is simple and if T is stable, T_σ is simple.

Thus, the ACFA example above is a special case of this construction.

Example 1.2.30. A *smoothly approximable* structure, as introduced by Lachlan, is a countable relational ω-categorical structure M which is the union of an increasing chain of finite *homogeneous substructures* of M (such a substructure is a subset A of M such that for every finite tuples $a, b \in A$, $\mathrm{tp}_M(a) = \mathrm{tp}_M(b)$ if and only if there is an automorphism of M fixing A setwise and taking a to b). By [CHL], ω-categorical,

ω-stable structures are smoothly approximable. A monumental treatise of smoothly approximable structures is currently in preparation by Hrushovski and Cherlin [CH].

A rank notion giving rise to a notion of independence can be defined for which the Independence Theorem over algebraically closed sets can be proved and thus we get simplicity.

It is a remarkable fact that in spite of all the work being done in simple theories, the above list of examples remains exhaustive; there are (virtually) no more examples of simple unstable theories, apart from Hrushovski's amalgamation construction modified for simple theories, see [H7], [E], [Pou] (where stable forking for structures obtained from the amalgamation construction is shown), for general framework compare with [W1]; there's also an extensive work of Baldwin on the topic, for bibliography see `http://www.math.uic.edu/~jbaldwin/`. In particular, we haven't got any 'bad' examples where lstp = stp, elimination of hyperimaginaries or stable forking property would fail.

1.3. Combinatorial geometries

In this section we recall some definitions regarding combinatorial geometries needed for understanding the concepts of geometric stability (and simplicity) theory. For more details, we refer the reader to [P3].

Definition 1.3.1. A (combinatorial) *pregeometry* (or a *matroid*) is a pair (S, cl) consisting of a set S and a closure operation $\mathrm{cl} : \mathbb{P}(S) \to \mathbb{P}(S)$ such that:

(1) $X \subseteq \mathrm{cl}(X)$;
(2) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$;
(3) if $a \in \mathrm{cl}(Xb) - \mathrm{cl}(X)$ then $b \in \mathrm{cl}(Xa)$ (Steinitz exchange);
(4) if $a \in \mathrm{cl}(X)$ then $a \in \mathrm{cl}(Y)$ for some finite $Y \subseteq X$.

The pregeometry (S, cl) is a *geometry*, if $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(\{a\}) = \{a\}$ for every $a \in S$. It is *homogeneous* if for every closed subset X of S and $a, b \in S - X$, there is an automorphism of S (a cl-preserving permutation of S) fixing X and taking a to b.

Remark 1.3.2.
(1) To any pregeometry (S, cl) we can associate a *canonical geometry* (S', cl'), where $S' = \{cl(\{a\}) : a \in S - \mathrm{cl}(\emptyset)\}$, and for $X \subseteq S$, $\mathrm{cl}'(\{\mathrm{cl}(\{a\}) : a \in X\}) = \{\mathrm{cl}(\{b\}) : b \in \mathrm{cl}(X)\}$.
(2) If (S, cl) is a pregeometry and $A \subseteq S$, we can *localize* at A to obtain (S, cl_A), where $\mathrm{cl}_A(X) := \mathrm{cl}(AX)$ for $X \subseteq S$.
(3) We invoke the reader's knowledge of linear algebra in order to remark that Steinitz exchange axiom gives rise to notions of independence and dimension. Thus, if we are in a pregeometry (S, cl), we say that a set A is *independent* over a set B, if for every $a \in A$, $a \notin \mathrm{cl}((A - \{a\}) \cup B)$. We say that $A_0 \subseteq A$ is a *basis* for A over B, if A_0 is independent over B and $A \subseteq \mathrm{cl}(A_0B)$. All bases for A over B will have the same cardinality denoted $\dim(A/B)$. We say that A is *independent* from B over C if $\dim(A'/CB) = \dim(A'/C)$ for all finite $A' \subseteq A$. This is easily shown to be a symmetric relation.

Definition 1.3.3. Let (S, cl) be a pregeometry. It is said to be:

- *trivial* or *degenerate*, if for every $X \subseteq S$, $\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(\{a\}) : a \in X\}$;
- *modular*, if for any closed sets $X, Y \subseteq S$, X is independent from Y over $X \cap Y$, or, equivalently, if for any finite-dimensional closed sets X and Y, $\dim(X) + \dim(Y) = \dim(XY) + \dim(X \cap Y)$;
- *locally modular*, if some localization at a point is modular;
- *projective*, if it is non-trivial and modular;

· *locally finite*, if $\mathrm{cl}(A)$ is finite for any finite A.

Example 1.3.4. Let S be a nonempty set and let $\mathrm{cl}(A) = A$ for any $A \subseteq S$. Then (S, cl) is a trivial homogeneous (pre)geometry.

Example 1.3.5. Let F be a division ring and V be a κ-dimensional vector space over F. For $A \subseteq V$, let $\mathrm{cl}(A) := \mathrm{span}_F(A)$. Then (V, cl) is a homogeneous modular pregeometry and its associated geometry is called $(\kappa-1)$-dimensional projective geometry over F. If we let $\mathrm{affcl}(A)$ be the smallest F-affine subspace of V containing A, we get a geometry (V, affcl) which is locally modular (localisation at 0 takes us into the situation above) but not modular.

Example 1.3.6. If K is an algebraically closed field of infinite transcendence degree over its prime subfield, and if we let $\mathrm{cl}(A)$ be the field-theoretic algebraic closure of A in K, (K, cl) is a homogeneous geometry which is not locally modular.

The above are the classical examples of pregeometries which are stable. We can add some random structure to make them simple unstable and less well-behaved, in view of the examples given in the previous section; see also [Va] and [dPK]. Moreover, we have the following.

Lemma 1.3.7. *Let p be a (complete) regular type over A in a simple theory and let D be the set of realisations of p. For $B \subseteq D$, we let $\mathrm{cl}(B) := \{b \in D : b \not\perp_A B\}$. Then, (D, cl) is a pregeometry.*

Proof. The only difficult part is showing that $\mathrm{cl}(\mathrm{cl}(B)) \subseteq \mathrm{cl}(B)$. If $b \in \mathrm{cl}(\mathrm{cl}(B))$, then $b \not\perp_A \mathrm{cl}(B)$, so there is $\bar{c} \in \mathrm{cl}(B)$ such that $b \not\perp_A B\bar{c}$ and for every i, $c_i \not\perp_A B$. If $b \perp_A B$, it follows by regularity that $b \perp_{AB} \bar{c}$ and $b \perp_A B\bar{c}$, which is a contradiction. □

Remark 1.3.8. In particular, if $\mathrm{SU}(p) = 1$ in the above, the closure operation becomes just $\mathrm{cl}(B) := D \cap \mathrm{acl}(BA)$. Furthermore, if we denote by $G(D)$ the set of all SU-rank 1 elements (over A) in $D^{\mathrm{eq}} := \mathrm{dcl}(D \cup A) \subseteq \mathfrak{C}^{\mathrm{eq}}$, $G(D)$ is a pregeometry with full $\mathrm{acl}(\cdot \cup A)$ (in $\mathfrak{C}^{\mathrm{eq}}$).

Proof. The first part is trivial. To see that $G(D)$ is a pregeometry with $\mathrm{cl}(X) := \mathrm{acl}(XA)$, the property $\mathrm{cl}(\mathrm{cl}(B)) \subseteq \mathrm{cl}(B)$ for every B follows from the fact that acl is a closure operator, and Steinitz exchange follows from symmetry of forking since for $a \in G(D)$, as $\mathrm{SU}(a/A) = 1$, $a \in \mathrm{cl}(B)$ if and only if $a \not\perp_A B$. □

Definition 1.3.9. Let D be a set of realisations of some p over A with $\mathrm{SU}(p) = 1$. A *plane curve* in D is a Lascar strong type $q = \mathrm{lstp}(ab/A)$ with $\mathrm{SU}(q) = 1$, and $\{a, b, A\}$ pairwise independent. We say that D is *k-linear* if there is a plane curve q with $\mathrm{SU}(\mathrm{Cb}(q)) = k$ and for every plane curve q', $\mathrm{SU}(\mathrm{Cb}(q')) \leq k$. It is *pseudolinear* if it is k-linear for some k. If the set is 1-linear, we just call it linear.

For quite some time it was unclear which is the 'right' definition of 'local modularity' in simple theories. The following result from [dPK] resolves the ambiguity.

Theorem 1.3.10. *The following statements are equivalent for a solution set D of an SU-rank 1 Lascar strong type:*

(1) *D (D^{eq}) is one-based;*
(2) *D is linear;*
(3) *$G(D)$ is linear;*
(4) *$G(D)$ is modular;*
(5) *$G(D)_A$ is linear, for any (some) small A;*
(6) *$G(D)_A$ is modular, for any (some) small A.*

The study of hypergroups and multivalued mathematical structures was initiated in 1934 by Marty [Mar]. It was noticed around that time that certain structures like double coset spaces maintain some group theoretic behaviour. For more information we refer the reader to [Co].

Definition 1.4.1. A *hypergroup* is a pair $(H, *)$ consisting of a set H and a hyperoperation $* : H \times H \to \mathbb{P}^*(H)$ (with each pair of elements (a, b) we associate a nonempty set $a * b$), such that

- for all $a \in H$, $a * H = H * a = H$;
- for all $a, b, c \in H$, $a * (b * c) = (a * b) * c$ (as sets).

A hypergroup $(H, *)$ is a *polygroup* if additionally

- there is a *scalar* identity $e \in H$, i. e. for every $a \in H$, $a * e = e * a = \{a\}$;
- for each $b \in H$ there is a unique $b^{-1} \in H$ such that for each $a \in H$, $a * b^{-1} = \{x \in H \mid a \in x * b\}$ and $b^{-1} * a = \{x \in H \mid a \in b * x\}$.

The way to interpret '$(a * b) * c$' above, since $a * b$ is a set, is to take $\bigcup_{d \in a*b} d * c$. We do not wish to consider hypergroups any further, because the requirement that something be a hypergroup carries too little algebraic information: just consider the trivial example of $(A, *)$, where $A \neq \emptyset$ and $a * b := \{a, b\}$ (although there are some interesting hypergroups which are not polygroups).

Definition 1.4.2. A *polyspace* $(P, X, *)$ consists of a polygroup $(P, *)$, a set X, and a multivalued map $* : P \times X \to \mathbb{P}^*(X)$, such that:

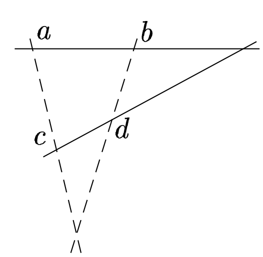
(1) for every $a, b \in P$, $x \in X$, $(a * b) * x = a * (b * x)$;
(2) for every $x \in X$, $e * x = \{x\}$;
(3) for every $a \in P$, $x \in X$, $y \in a * x$ if and only if $x \in a^{-1} * y$.

The next two examples are the principal examples of polygroups and have played an important role in the development of the blowup construction from Chapter 4.

Example 1.4.3. Let G be a group, and H a (not necessarily normal) subgroup. The double coset space $G \,//\, H$ is a polygroup with the multioperation $HaH * HbH := \{HahbH : h \in H\}$.

Example 1.4.4. A projective geometry is an incidence system (P, L, I) consisting of a set of points P, a set of lines L and an incidence relation $I \subseteq P \times L$ satisfying the following axioms:

(1) any line contains at least three points;
(2) two distinct points a, b are contained in a unique line denoted by $L(a, b)$;
(3) if a, b, c, d are distinct points and $L(a, b)$ intersects $L(c, d)$, then $L(a, c)$ must intersect $L(b, d)$ (Pasch axiom), as shown in the figure.



Let $P' := P \cup \{e\}$, where e is not in P, and define:

- for $a \neq b \in P$, $a \circ b := L(a, b) \setminus \{a, b\}$;
- for $a \in P$, if any line contains exactly three points, put $a \circ a := \{e\}$, otherwise $a \circ a := \{a, e\}$;
- for $a \in P'$, $e \circ a = a \circ e := \{a\}$.

Then it is easily verified that $(P', \circ)$ is a polygroup; compare to 1.3.5 and 5.4.6.

Lemma 1.4.5. *Let $(P, *)$ be a polygroup. Then the following properties are equivalent to associativity:*

(1) *(Second form of associativity). For every $a, b, c \in P$ and $a' \in a * b$, $c' \in b * c$, $a' * c \approx a * c'$.*
(2) *(Transposition). $a * d \approx b * c$ if and only if $b^{-1} * a \approx c * d^{-1}$ (notice the similarity to the Pasch axiom in 1.4.4).*

Proof.
(1) Assume the usual associativity, and let $a' \in a * b$, $c' \in b * c$. Then, $a \in a' * b^{-1} \subseteq a' * (c * c'^{-1}) = (a' * c) * c'^{-1}$, so we can find $d \in a' * c$ such that $a \in d * c'^{-1}$.

Assume now the second form of associativity, and let $d \in (a * b) * c$. There will be $a' \in a * b$ with $d \in a' * c$. We have $b \in a^{-1} * a'$, so there will be $d' \in b * c \cap a^{-1} * d$, i.e. $d \in a * (b * c)$.
(2) is similar. □

Notice that the second form of associativity (for every $a, b \in P$, $x \in X$ and every $a' \in a * b$ and $x' \in b * x$, $a' * x \approx a * x'$) is equivalent to the associativity of the multiaction in polyspaces as well.

Definition 1.4.6. An equivalence relation R on a hypergroup H is *regular* on the right if for every $x, y \in H$, xRy implies that for every $a \in H$, $x * a =_R y * a$. It is *strongly regular* on the right if xRy implies that for all a, $x' \in x * a$ and $y' \in y * a$, $x'Ry'$. Analogously we define (strong) regularity on the left. An equivalence is (strongly) regular if it is (strongly) regular both on the left and right.

It is an immediate consequence of the definition that if an equivalence relation R is regular, then $a_R * b_R \subseteq (a * b)_R$ for all $a, b \in H$.

Lemma 1.4.7. *If $(P, *)$ is a polygroup and R is regular, then $(P/R, \circ)$ is a polygroup, where $a_R \circ b_R := (a * b)/R$. If R happens to be strongly regular, the quotient will be a group.*

The product is clearly well-defined and all the properties of polygroups descend to quotients.

Definition 1.4.8. If $(P, *)$ is a polygroup, we call $Q \subseteq P$ a *subpolygroup* if Q is itself a polygroup with the same multioperation $*$.

Remark 1.4.9. It should be remarked here that in model-theoretic language, the above definition is a substructure in the 'functional' sense, since $*$ on Q is the same as on P, which is a much stronger condition than e.g. requiring that $(Q, *\restriction_Q)$ be a polygroup, which would yield a substructure in a 'relational' sense.

This strong definition allows us to talk about the *index* of Q in P, as the number of (left) cosets of Q covering P, since it is clear that different cosets are disjoint.

Definition 1.4.10. Let $(P, *)$ be a polygroup. A subpolygroup $N < P$ is called *normal*, if for every $a \in P$, $a * N = N * a$.

Lemma 1.4.11. *Let us say that a and b are equivalent modulo N, if $a * N \approx N * b$. This is a regular equivalence relation.*

Proof. It is clearly reflexive and symmetric, by normality of N. For transitivity, let $a * N \approx N * b$ and $b * N \approx N * c$. Then $b \in N^{-1} * (a * N) \cap (N * c) * N^{-1}$, so by the transposition property, $a * N * N \approx N * N * c$, i.e. $a * N \approx N * c$. If we denote the coset $a * N$ by a_N, it is clear that $a_N * b_N \subseteq (a * b)_N$, showing regularity. □

Thus, we can quotient by normal subpolygroups.

Definition 1.4.12. We shall say that a map $\varphi : P \to P'$ between two polygroups is a *homomorphism*, if $\varphi(a * b) \subseteq \varphi(a) * \varphi(b)$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a, b \in P$. Denoting $a_\varphi := \varphi^{-1}(\varphi(a))$, we say that a homomorphism φ is of

- *type 1,* if $\varphi^{-1}(\varphi(a) * \varphi(b)) = (a_\varphi * b_\varphi)_\varphi$;
- *type 2,* if $(a*b)_\varphi = \varphi^{-1}(\varphi(a)*\varphi(b))$ (which in turn implies $a_\varphi * b_\varphi \subseteq (a*b)_\varphi$);
- *type 3,* if $\varphi^{-1}(\varphi(a) * \varphi(b)) = a_\varphi * b_\varphi$;
- *type 4,* if it is of type 2 and type 3.

It is an *isomorphism*, if it is bijective and $\varphi(a * b) = \varphi(a) * \varphi(b)$.

Clearly, type j implies type i for $i < j$, and type 1 is equivalent to the induced structure $(P/\varphi, *_\varphi)$ being a polygroup, where $a_\varphi *_\varphi b_\varphi := \{c_\varphi : c \in a_\varphi * b_\varphi\}$. Also, for type 2, the corresponding equivalence relation (whose classes are a_φ) is regular. We shall not need any further results about polygroups in chapters to follow, but let us just mention a few results from [Ja], showing that the theory is reasonably well-behaved.

Proposition 1.4.13.
1. *(First isomorphism theorem). Let $\varphi : P \to P'$ be a surjective homomorphism of type 2 between polygroups P and P'. Then $P/\ker(\varphi) \cong P'$.*
2. *(Second isomorphism theorem). In a polygroup, suppose M is a subpolygroup, and N normal subpolygroup. Then $\langle M, N\rangle/N \cong M/M \cap N$, where $\langle M, N\rangle$ is the subpolygroup generated by $M \cup N$.*
3. *(Third isomorphism theorem). Suppose $M < N$ are normal subpolygroups of a polygroup P. Then $P/N \cong (P/M)/(M/N)$.*
4. *(Jordan-Hölder theorem). In a polygroup, let $M < N$ be subpolygroups. Suppose $M = K_0 \subseteq \cdots \subseteq K_k = N$ and $M = L_0 \subseteq \cdots \subseteq L_l = N$, where each K_i is maximal proper normal subpolygroup in K_{i+1} for $i < k$, and similarly for L_i. Then there is a bijective correspondence between $\{K_{i+1}/K_i : i < k\}$ and $\{L_{j+1}/L_j : j < l\}$ such that the correspondents are isomorphic.*

CHAPTER 2

Germs and group configuration

The main tool for proving the group configuration theorem in the stable case is the machinery of germs of definable functions. They can be developed as follows:

Definition 2.1. Let T be a stable theory.

(1) Let p and q be stationary types over $\emptyset$. We say that a strong type $r(x,y)$ over some f defines a function from p to q if:
  (a) $r(x,y) \to p(x) \wedge q(y)$;
  (b) if $r(a,b)$, then $a \downarrow f$, $b \downarrow f$, $b \in \mathrm{dcl}(fa)$ and in such a case we may write $b = f(a)$; if also $a \in \mathrm{dcl}(fb)$, we say the function is invertible.

(2) Two functions $r_1, r_2 : p \to q$ (defined over f_1, f_2) are equivalent if there is $a \downarrow f_1 f_2$ such that $f_1(a) = f_2(a)$, i.e. if r_1 and r_2 have a common nonforking extension, which is a definable equivalence relation, and clearly the class of r, called the *germ* of r is in fact $\sigma := \mathrm{Cb}(r)$. Furthermore, σ again defines a function $p \to q$ since *definability descends*: if $b \in \mathrm{dcl}(af)$ and $\sigma = \mathrm{Cb}(ab/f)$, it is still true that $b \in \mathrm{dcl}(a\sigma)$.

(3) To see that germs can be composed, let $\sigma : p \to q$ and $\tau : q \to s$ be germs. If we define $\tau \cdot \sigma := \mathrm{Cb}(a\tau(\sigma(a))/\sigma\tau)$ for some $a \downarrow \tau\sigma$, this is a germ $p \to s$ and the fact that $\mathrm{tp}(a\tau(\sigma(a))/\sigma\tau)$ is stationary implies $\tau \cdot \sigma \in \mathrm{dcl}(\tau\sigma)$.

What are the obstacles in the simple case? Firstly, definability does not descend (the most we can conclude from $b \in \mathrm{dcl}(af)$, if $\sigma = \mathrm{Cb}(ab/f)$, is that $b \in \mathrm{acl}(a\sigma)$):

Example 2.2. (Pillay) Suppose we have three sorts, called P, Q and R and let Q be a 2-cover of P, i.e. there is a surjection $\pi : Q \to P$ with each fibre of size 2. Write $\pi^{-1}(a)$ as $\{a_1, a_2\}$. Let S be a random bipartite graph S between P and R and let $f, g : P \times R \to Q$ be functions such that if $S(b,a)$, then $f(b,a) = a_1$ and $g(b,a) = a_2$; otherwise, $f(b,a) = a_2$ and $g(b,a) = a_1$. It can be shown that $\langle P, Q, R, f, g \rangle$ is a simple structure (notice we have 'forgotten' S). Consider now $\mathrm{tp}(aa_1/b)$, where e.g. $S(b,a)$; clearly $f(b,a) = a_1$ and $a_1 \in \mathrm{dcl}(ab)$. But $\mathrm{Cb}(aa_1/b) = \emptyset$, and obviously $a_1 \notin \mathrm{dcl}(a)$.

Secondly, in the simple case, there is no hope for a property similar to (3) above, see 2.3.4 which expresses this in a more specific language.

To overcome the first (in fact nonessential) difficulty, we consider multifunctions instead of functions, and to deal with the second, we need to *complete* after each composition.

Material in the first three sections is almost all from [BY], with a few modifications and simplifications (e.g. we consider completions and reductions in any of the variables, thus replacing Ben-Yaacov's 'strongness on the left and right'). For Section 2.4, I had a proof scheme in the simple case (in particular I had recognised the importance of what in the present language is called 'π with $\pi^{-1} \circ \pi$ generic') but was lacking the right definition of germs. However, upon seeing Ben-Yaacov's definition, I managed to reprove the results of that section independently.

2.1. Partial generic multiactions

From now onwards, we shall be working in a simple theory.

Definition 2.1.1. Let $\pi_i(x_i)$, $i < \alpha$ be partial types (in hyperimaginary sorts) over a hyperimaginary parameter e. Their product, $\bigotimes_{i<\alpha} \pi_i(\langle x_i : i < \alpha\rangle)$ is the partial type, if one exists, which is true of $\langle a_i : i < \alpha\rangle$ if and only if for every i, $\pi(a_i)$, and $\langle a_i : i < \alpha\rangle$ is independent over e.

Definition 2.1.2. A partial type π over e has *definable independence* if for every π' over e, $\pi \otimes_e \pi'$ exists.

The following is straightforward from the definition.

Proposition 2.1.3.
(1) *Every complete type has definable independence over its domain.*
(2) *If π_i has definable independence for all $i < \alpha$, then $\bigotimes_{i<\alpha} \pi_i$ exists and has definable independence.*
(3) *If π has definable independence and $\pi' \vdash \pi$, then π' has it too.*
(4) *If π has definable independence over e, $\sim$ is an e-hyperdefinable equivalence relation, π is $\sim$-invariant, then $\pi/\sim$ has definable independence over e.*

Definition 2.1.4. Let $\pi(x, y, z)$ be a partial type in three hyperimaginary sorts over $\emptyset$. We say that π defines a *partial generic multiaction* if:

(1) $\pi \restriction_x$, $\pi \restriction_y$ and $\pi \restriction_z$ have definable independence;
(2) $\pi(x, y, z)$ implies that x, y, z are pairwise independent;
(3) for any f, a, there are at most boundedly many b such that $\pi(f, a, b)$, and in that case we write $b \in f(a)$ or $b \approx f(a)$.

We shall use the following notation: $\mathrm{Fun}(\pi) = \pi \restriction_x$, $\mathrm{Arg}(\pi) = \pi \restriction_y$, $\mathrm{Val}(\pi) = \pi \restriction_z$, and if $f \in \mathrm{Fun}(\pi)$, $\Gamma(f)(y, z) := \pi(f, y, z)$ and usually we will identify f with its graph, writing '$xy \models f$' in place of '$xy \models \Gamma(f)$'.

In the above, it is enough to require that just $\mathrm{Fun}(\pi)$ and $\mathrm{Arg}(\pi)$ have definable independence, since for $b \in \mathrm{Val}(\pi)$ and any e, $b \mathop{\smile\!\!\!\!\!\!|}\, e$ if and only if there are $f \in \mathrm{Fun}(\pi)$ and $a \in \mathrm{Arg}(\pi)$ with $fa \mathop{\smile\!\!\!\!\!\!|}\, e$.

Definition 2.1.5. Let π be a partial generic multiaction.

(1) π is *invertible*, if for every b there is at most boundedly many a's with $\pi(f, a, b)$, i.e. if π^{-1} is a generic action, where $\pi^{-1}(f, b, a)$ if and only if $\pi(f, a, b)$;
(2) π is *complete* in x if for any $f \in \mathrm{Fun}(\pi)$, $\Gamma(f)$ is a Lascar strong type (or an amalgamation base) over f; similarly we define completeness in y or z; if we just say 'complete' without mentioning a variable, it will mean 'complete in x';
(3) π is *reduced* (in x) if it is complete (in x) and whenever there is $a \mathop{\smile\!\!\!\!\!\!|}\, fg$ with $f(a) \approx g(a)$, then $f = g$; it is reduced in y, if it is complete in y and whenever there is $f \mathop{\smile\!\!\!\!\!\!|}\, aa'$ with $f(a) \approx f(a')$, then $a = a'$; if π is invertible and complete in z, we say it is reduced in z if whenever there are $f \mathop{\smile\!\!\!\!\!\!|}\, b, b'$ and a such that $b, b' \in f(a)$, then $b = b'$.
(4) π is trivial, if $\pi(a, b, c)$ implies that $\{a, b, c\}$ is independent.

Definition 2.1.6. Two partial generic actions $\pi(x, y, z)$ and $\pi'(x', y, z)$ are *isomorphic* if there is a hyperdefinable bijection $\varphi : \mathrm{Fun}(\pi) \to \mathrm{Fun}(\pi')$ such that for every $f \in \mathrm{Fun}(\pi)$, $\Gamma(f) = \Gamma(\varphi(f))$.

Two constructions, yielding examples of complete and reduced multiactions, are discussed in the next section.

Definition 2.2.1. Let $\pi(x,y,z)$ be a partial generic multiaction. Let $\underline{x}$ be the sort $(yz,x)/\mathrm{LS}$, where $\mathrm{LS}(yz,x,y'z',x')$ is the hyperdefinable equivalence relation from 1.2.17 saying that $x=x'$ and $yz \equiv^L_x y'z'$. An element of this sort can be identified with a pair f_p, where f is of the x sort, and p is a Lascar strong type over f in the yz sort. Let $\underline{\pi}((y'z',x)/\mathrm{LS},y,z) = \pi(x,y,z) \wedge \mathrm{LS}(yz,x,y'z',x)$, i.e. $\underline{\pi}(f_p,a,b)$ if and only if $b \in f(a)$ and $\mathrm{lstp}(ab/f) = p$. This $\underline{\pi}$ we shall call the *completion* of π (in x).

For $f \in \mathrm{Fun}(\pi)$, we define $\underline{f} := \{(ab,f)_{\mathrm{LS}} : b \in f(a)\}$, which is obviously a bounded hyperdefinable set. In the same way, we can complete with respect to y and z.

Proposition 2.2.2. (1) *The completion of a partial generic multiaction (in any of the variables) is a complete partial generic multiaction (in the respective variable).*

(2) *If a multiaction $\pi(x,y,z)$ is complete in some variable, (e.g. x), the completion with respect to some other variable (say y) will still be complete in x.*

Proof. Part (1) is easy. For (2), let $\pi(x,y,z)$ be complete in x, and let $\underline{\pi}_y$ be the completion with respect to y. We need to show that $\underline{\pi}_y$ is still complete in x. Suppose $\underline{\pi}_y(f,a_p,b)$ and $\underline{\pi}_y(f,a'_{p'},b')$; then $\pi(f,a,b)$, $\mathrm{lstp}(fb/a) = p$, $\pi(f,a',b')$ and $\mathrm{lstp}(fb'/a') = p'$. By assumption of completeness in x, let $\varphi : ab \equiv^L_f a'b'$. But then, by invariance of Lascar strong types, $\varphi(p) = p'$, so in fact $\varphi : a_p b \equiv^L_f a'_{p'}b'$. □

Definition 2.2.3. Let $\pi(x,y,z)$ be a complete multiaction, $f,g \in \mathrm{Fun}(\pi)$. Let $f \sim_1 g$ (or sometimes we write $\sim_1^x$) if there is $a \mathop{\smile\!\!\!\!\!\mid}\, fg$ such that $f(a) \approx g(a)$, and let $\sim$ be the transitive closure of $\sim_1$. It is clear by the definable independence of $\mathrm{Fun}(\pi)$ that $\sim_1$ is hyperdefinable, as well as reflexive and symmetric, and we will show below that $\sim$ is also hyperdefinable. The class of a function f, denoted $\bar{f}$, we call the *germ* of f, and we define the *reduction* of π, $\bar{\pi}(\bar{x},y,z)$ (where $\bar{x} = x/\sim$, the sort of germs) as the partial type such that $\bar{\pi}(\bar{f},a,b)$ if and only if there is $f \in \bar{f}$ with $\pi(f,a,b)$. The set of all germs of π is denoted $\mathrm{Germ}(\pi)$. In a similar way, we can reduce with respect to y and z, by dividing by the corresponding relations $\sim^y$, $\sim^z$ ($a \sim^y_1 a'$ if there is $f \mathop{\smile\!\!\!\!\!\mid}\, aa'$, $f(a) \approx f(a')$, $b \sim^z_1 b'$ if there is $f \mathop{\smile\!\!\!\!\!\mid}\, bb'$ and $a \mathop{\smile\!\!\!\!\!\mid}\, f$ such that $b,b' \in f(a)$).

Lemma 2.2.4. *Relations $\sim^x_1$, and $\sim^y_1$ are generically transitive. If π is invertible, so is $\sim^z_1$ and we have that $\sim^y$ and $\sim^z$ are such that any two elements which are related are in fact interbounded. Thus, $\sim^x$ and $\sim^y$ are hyperdefinable, and so is $\sim^z$ when π is invertible.*

Proof. First of all, let us show that $f \sim^x_1 g$ if and only if $\Gamma(f)$ and $\Gamma(g)$ have a common nonforking extension. If $a \mathop{\smile\!\!\!\!\!\mid}\, fg$ with $b \in f(a) \cap g(a)$, then obviously $ab \mathop{\smile\!\!\!\!\!\mid}_f g$ and $ab \mathop{\smile\!\!\!\!\!\mid}_g f$ so ab realises a common nonforking extension of $\Gamma(f)$ and $\Gamma(g)$. The converse is also straightforward.

Assume now $f \sim_1 g$, $g \sim_1 h$ and $f \mathop{\smile\!\!\!\!\!\mid}_g h$. In other words, $\Gamma(f)$ and $\Gamma(g)$ have a common nonforking extension, the same as $\Gamma(g)$ and $\Gamma(h)$. By completeness, $\Gamma(g)$ is a Lascar strong type. Thus, since $f \mathop{\smile\!\!\!\!\!\mid}_g h$, we can apply the Independence Theorem and get that $\Gamma(f)$ and $\Gamma(h)$ have a common nonforking extension and thus $\sim^x_1$ is generically transitive.

To see the same for e.g. $\sim^y_1$ when π is complete in y, let $a \sim^y_1 a'$, $a \sim^y_1 a''$, $a' \mathop{\smile\!\!\!\!\!\mid}_a a''$. There will be some $f \mathop{\smile\!\!\!\!\!\mid}\, aa'$ and b, $f' \mathop{\smile\!\!\!\!\!\mid}\, aa''$ and b' such that $b \in f(a) \cap f(a')$,

$b' \in f'(a) \cap f'(a'')$. Then $fb \equiv^L_a f'b'$, $fb \mathop{\smile\!\!\!\!\!\!\mid}_a a'$, $f'b' \mathop{\smile\!\!\!\!\!\!\mid}_a a''$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_a a''$, so by the Independence Theorem we may assume $fb = f'b' \mathop{\smile\!\!\!\!\!\!\mid}_a a'a''$ and thus $a' \sim^y_1 a''$.

When π is invertible, if e.g. $a \sim^y_1 a'$, let $f \mathop{\smile\!\!\!\!\!\!\mid} aa'$ and b such that $b \in f(a) \cap f(a')$. Then in particular $f \mathop{\smile\!\!\!\!\!\!\mid}_a a'$, and, since $a' \in \mathrm{bdd}(fa)$ ($a' \in f^{-1}(b)$, $b \in f(a)$), we get $a' \mathop{\smile\!\!\!\!\!\!\mid}_a a'$ so $a' \in \mathrm{bdd}(a)$.

Finally, use 1.2.20, to get that $\sim^?$ is hyperdefinable, for $? \in \{x, y, z\}$. □

Definition 2.2.5. In view of the above considerations, if π is not necessarily complete, for $f \in \mathrm{Fun}(\pi)$, we let $\hat{f}$ be the set of germs of all completions of $\Gamma(f)$ to a Lascar strong type.

Remark 2.2.6. It is clear that for complete π, $f \in \mathrm{Fun}(\pi)$, $\bar{f} = \mathrm{Cb}(\Gamma(f))$ by [HKP].

Proposition 2.2.7. *If π is a complete partial generic multiaction (in x), then $\bar{\pi}$ is a reduced partial generic multiaction (and similarly in y and z). Moreover, if e.g. π was complete in x and y, and we reduce with respect to x, the result is still complete in y. If π is nontrivial, so is the completion.*

Proof. $\mathrm{Fun}(\bar{\pi})$ has definable independence as a quotient of $\mathrm{Fun}(\pi)$. If $\bar{\pi}(\bar{f}, a, b)$, there is $f \in \bar{f}$ with $\pi(f, a, b)$, so $\{f, a, b\}$ is pairwise independent and therefore $\{\bar{f}, a, b\}$ as well. By the previous remark, $ab \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{f}} f$, so $b \mathop{\smile\!\!\!\!\!\!\mid}_{a\bar{f}} af$, and, as $b \in \mathrm{bdd}(af)$, $b \mathop{\smile\!\!\!\!\!\!\mid}_{a\bar{f}} b$, which implies $b \in \mathrm{bdd}(a\bar{f})$.

The 'moreover' part is easy. □

Definition 2.2.8. Two partial generic multiactions π and π' are *equivalent*, written $\pi \approx \pi'$, if their reductions are isomorphic.

2.3. Composing germs

Definition 2.3.1. Let $\pi(u, x, y)$ and $\pi'(v, y, z)$ be partial generic multiactions. Define $\pi' \circ \pi((v, u), x, z)$ to be the partial type such that $\pi' \circ \pi((g, f), a, c)$ if:

(1) $\{g, f, a\}$ is independent;
(2) $c \in g \circ f(a)$, i.e. there is $b \in f(a)$ such that $c \in g(b)$.

Proposition 2.3.2. (1) *$\pi' \circ \pi$ always exists and it is a generic action;*
(2) *$(\pi' \circ \pi)^{-1} = \pi^{-1} \circ \pi'^{-1}$.*

Proof.
(1) Since $\mathrm{Fun}(\pi)$ and $\mathrm{Fun}(\pi')$ have definable independence, so does $\mathrm{Fun}(\pi) \times \mathrm{Fun}(\pi')$, and $\pi' \circ \pi$ exists. As $\mathrm{Fun}(\pi' \circ \pi) \vdash \mathrm{Fun}(\pi) \times \mathrm{Fun}(\pi')$, it also has definable independence. It is easy to check the definable independence of $\mathrm{Arg}(\pi' \circ \pi)$ and the boundedness property. Suppose $\pi' \circ \pi(f, g, a, c)$, so $b \in f(a)$, $c \in g(b)$, and $\{f, g, a\}$ is independent. In particular, $a \mathop{\smile\!\!\!\!\!\!\mid} fg$, so $a \mathop{\smile\!\!\!\!\!\!\mid}_f g$, and, as $b \in \mathrm{bdd}(fa)$, $b \mathop{\smile\!\!\!\!\!\!\mid}_f g$. Since $b \mathop{\smile\!\!\!\!\!\!\mid} f$, we get that $b \mathop{\smile\!\!\!\!\!\!\mid} fg$. Continuing, $b \mathop{\smile\!\!\!\!\!\!\mid}_g f$, so, by $c \in \mathrm{bdd}(gb)$ and $c \mathop{\smile\!\!\!\!\!\!\mid} g$, we conclude $c \mathop{\smile\!\!\!\!\!\!\mid} fg$, as required.
(2) is trivial. □

Proposition 2.3.3. *Let π and π' be partial generic multiactions with $\mathrm{Arg}(\pi') = \mathrm{Val}(\pi)$ being Lascar strong types. Then, for any $f \in \mathrm{Fun}(\pi)$, $g \in \mathrm{Fun}(\pi')$ with $f \mathop{\smile\!\!\!\!\!\!\mid} g$, $g \circ f \in \mathrm{Fun}(\pi' \circ \pi)$. If π' is nontrivial, so is the composition.*

Proof. Let $f \mathop{\smile\!\!\!\!\!\!\mid} g$ and let $ab \models f$, $b'c \models g$. By assumption, $b \equiv^L b'$, $b \mathop{\smile\!\!\!\!\!\!\mid} f$, $b' \mathop{\smile\!\!\!\!\!\!\mid} g$, $f \mathop{\smile\!\!\!\!\!\!\mid} g$, so by the Independence Theorem, we may assume $b = b' \mathop{\smile\!\!\!\!\!\!\mid} fg$. We may also assume $a \mathop{\smile\!\!\!\!\!\!\mid}_{bf} g$, and so $g \mathop{\smile\!\!\!\!\!\!\mid} abf$ and $a \mathop{\smile\!\!\!\!\!\!\mid} fg$, so $g \circ f$ is defined on a. □

Remark 2.3.4. It is clear that in a general simple theory, a composition of two complete multiactions need not be complete any more. Thus, if we want that the composition of two germs lives on germs again, we need to observe all the possible completions of the composition and their germs. This is how multivaluedness appears in simple theories, in some sense a perfectly natural consequence of the crucial difference between simplicity and stability, simple theories not having stationarity which helps to determine objects uniquely.

The following obvious but important lemmas allow us to formalise these considerations.

Lemma 2.3.5. *Let* $f \in \mathrm{Fun}(\pi)$, $g \in \mathrm{Fun}(\pi')$, $f \perp g$ *and* $h \in \mathrm{Germ}(\pi' \circ \pi)$. *We say that* abc witness *that* $h \in \widehat{g \circ f}$ *and write* $abc \models h \in \widehat{g \circ f}$ *if* $a \perp fgh$ *and* $ab \models f$, $bc \models g$, $ac \models h$. *With this definition,* $h \in \widehat{g \circ f}$ *if and only if there are witnesses for it.*

Lemma 2.3.6. *Let* $\pi' \circ \pi$ *be defined,* $f \in \mathrm{Germ}(\pi)$, $g \in \mathrm{Germ}(\pi')$, $h \in \mathrm{Germ}(\pi' \circ \pi)$, $f \perp g$.

(1) *If* π *is invertible and* $f \perp h$, *then* $abc \models h \in \widehat{g \circ f}$ *if and only if* $bac \models g \in \widehat{h \circ f^{-1}}$.
(2) *If* π' *is invertible and* $g \perp h$, *then* $abc \models h \in \widehat{g \circ f}$ *if and only if* $acb \models f \in \widehat{g^{-1} \circ h}$.

A desirable property of germs is that reduction and composition commute. Indeed, we have:

Lemma 2.3.7. *Let* $f \in \mathrm{Fun}(\pi)$, $g \in \mathrm{Fun}(\pi')$, $f \perp g$. *Then* $\widehat{g \circ f} = \bigcup_{\bar{f} \in \hat{f}, \bar{g} \in \hat{g}} \widehat{\bar{g} \circ \bar{f}}$.

Proof. Let $abc \models h \in \widehat{g \circ f}$, let $\bar{f} := \mathrm{Cb}(ab/f)$, $\bar{g} := \mathrm{Cb}(bc/g)$. Then clearly $h \in \widehat{\bar{g} \circ \bar{f}}$.

On the other hand, let $abc \models h \in \widehat{\bar{g} \circ \bar{f}}$. We can rechoose them such that $abc \perp_{\bar{f}\bar{g}h} f$, or even $abc \perp_{\bar{f}\bar{g}} f$, as $h \in \mathrm{bdd}(\bar{f}\bar{g})$. Then, as $f \perp \bar{g}$, $f \perp_{\bar{f}} abc\bar{g}h$. If $a'b' \models f$ with $\bar{f} = \mathrm{Cb}(a'b'/f)$, since $a'b' \perp_{\bar{f}} f$, $ab \perp_{\bar{f}} \bar{g}h$ and $f \perp_{\bar{f}} \bar{g}h$, by the Independence Theorem we may assume that $ab \models f$ and $ab \perp_{\bar{f}} f\bar{g}h$. Moreover, we may assume that $c \perp_{ab\bar{f}\bar{g}} f$ and also $abc \perp_{f\bar{g}h} g$, yielding $g \perp_{\bar{g}} abcfh$. As above for f, using the Independence Theorem, we may assume that $bc \models g$ and $bc \perp_{\bar{g}} fgh$, but then $abc \models h \in \widehat{g \circ f}$. □

Corollary 2.3.8. $\pi' \circ \pi \approx \bar{\pi}' \circ \bar{\pi}$. *Moreover, if* $\pi_i \approx \pi'_i$ *then* $\pi_1 \circ \pi_0 \approx \pi'_1 \circ \pi'_0$.

Definition 2.3.9. We say that the composition $\pi' \circ \pi$ is *generic* if for any $f \in \mathrm{Germ}(\pi)$, $g \in \mathrm{Germ}(\pi')$ with $f \perp g$ and $h \in \widehat{g \circ f}$, we have $h \perp f$ and $h \perp g$.

There is another potential obstacle to successfully defining a multioperation: a composition of two, three or more germs might not be of the same sort. Luckily, the following solves the problem.

Theorem 2.3.10. *Let* $\pi(x, y, z)$ *be an invertible partial generic multiaction, which is also complete in* y, $\mathrm{Arg}(\pi)$ *be a Lascar strong type and the composition* $\pi^{-1} \circ \pi$ *generic.*

(0) *If we denote* $\hat{\pi} = \overline{\pi^{-1} \circ \pi}$, *we have that* $\hat{\pi} \circ \hat{\pi} \approx \hat{\pi}$.

This yields a multioperation (with boundedly many values) $* : P \otimes P \to \mathbb{P}^{\mathrm{bdd}}(P)$, *for* $P = \mathrm{Germ}\,\hat{\pi}$, *given by* $g * f := \widehat{g \circ f}$, *for* $f \perp g$ *such that* $(P, *)$ *is in fact a hyperdefinable polygroup chunk, i.e.*

(1) *(Generic independence) for every* $f \perp g$, *if* $h \in f * g$, *then* $h \perp f$ *and* $h \perp g$;
(2) *(Generic associativity) for* $\{f, g, h\}$ *independent,* $(f * g) * h = f * (g * h)$ *(as sets);*

(3) *(Generic surjectivity) for any $f \perp g$, there is h such that $g \in f * h$; in fact, $g \in f * h$ if and only if $h \in f^{-1} * g$.*

Proof.
(0) Let $abc \models h \in \widehat{h_1 \circ h_0}$, for $h_0 \perp h_1 \in \mathrm{Fun}(\hat{\pi})$. There are $f_0, g_0 \in \mathrm{Fun}(\pi)$ and d_0 such that $ad_0b \models h_0 \in \widehat{f_0^{-1} \circ g_0}$ and $f_1, g_1 \in \mathrm{Fun}(\pi)$ such that $bd_1c \models h_1 \in \widehat{g_1^{-1} \circ f_1}$. By completeness in y, $f_0d_0 \equiv_b^L f_1d_1$. Also, $f_0d_0 \perp_b ah_0$ (by genericity of $\pi^{-1} \circ \pi$, $h_0 \perp f_0$, so $abd_0 \models g_0 \in \widehat{f_0 \circ h_0}$, so in particular $b \perp f_0h_0$, and thus $f_0 \perp_b h_0$, so $f_0d_0 \perp_b ah_0$) and similarly $f_1d_1 \perp_b ch_1$, and $ah_0 \perp_b ch_1$. Thus, by the Independence Theorem, we can assume that $f_1d_1 = f_0d_0 =: fd$ and $ah_0 \perp_{bdf} ch_1$ and also $ag_0h_0 \perp_{bdf} cg_1h_1$. In particular, $ag_0 \perp_{bdf} g_1$, and by assumptions above, $ag_0 \perp_d bf$, so $ag_0 \perp_d fg_0$, $ag_0 \perp fg_1$, $a \perp fg_0g_1$ and finally $a \perp fg_0g_1h_0h_1h$, so $adc \models h \in \widehat{g_1^{-1} \circ g_0}$.

Conversely, let $adc \models h \in \widehat{g_1^{-1} \circ g_0}$, $g_0, g_1 \in \mathrm{Fun}(\pi)$, $g_0 \perp g_1$. Since $d \in \mathrm{Val}(\pi)$, we can find $f \in \mathrm{Fun}(\pi)$ and b with $bd \models f$ and we may assume that $bf \perp_d acg_0g_1h$, so $ag_0 \perp_{bdf} cg_1$. Now, $f \perp g_0g_1$, $a \perp fg_0$, $b \perp fg_1$, so we may take $h_0 := \mathrm{Cb}(ab/fg_0) \in \widehat{f^{-1} \circ g_0}$ and $h_1 := \mathrm{Cb}(bc/fg_1) \in \widehat{g_1^{-1} \circ f}$, and we have that $abc \models h \in \widehat{h_1 \circ h_0}$.
(1) follows from genericity of $\pi^{-1} \circ \pi$.
(2) Let $l \in (f * g) * h$; so there are k, $bcd \models k \in f * g$ and $ab'd' \models l \in k * h$. Therefore, we have $bd \equiv_k^L b'd'$, $bd \perp_k fg$, $b'd' \perp_k lh$, $fg \perp_k lh$, so by the Independence Theorem we may assume that $b'd' = bd \perp_k fglh$. Then, $abc \models m := \mathrm{Cb}(ac/gh) \in g * h$ and $abd \models l \in f * m$.
(3) is easy by 2.3.6. ☐

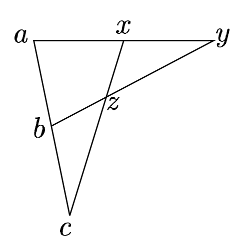
Remark 2.3.11. In the previous theorem, requiring that $\pi(x, y, z)$ be complete in y on top of the assumption of genericity of $\pi^{-1} \circ \pi$ is not too strong, since if, for example, we had π such that $\pi^{-1} \circ \pi$ is generic, and if π_y is the completion in y, then $\pi_y^{-1} \circ \pi_y$ is also generic.

Proof. Let $a_pbc_q \models h \in \widehat{g^{-1} \circ f} \in \mathrm{Germ}(\pi_x^{-1} \circ \pi_x)$. In particular, $xyz \models h' := \mathrm{Cb}(xz/fg) \in \widehat{g^{-1} \circ f} \in \mathrm{Germ}(\pi^{-1} \circ \pi)$, so by the assumption, $h' \perp f$ and $h' \perp g$. However, as $x_pz_q \in \mathrm{bdd}(xz)$, $xz \perp_{h'} fg$ implies that $x_pz_q \perp_{h'} fg$, so $h \in \mathrm{bdd}(h')$ and thus $h \perp f$, $h \perp g$, as required. ☐

2.4. Group configuration

Definition 2.4.1. An *algebraic quadrangle* is a diagram (a, b, c, x, y, z), where:

- any pair and any non-collinear triple is independent;
- $\mathrm{bdd}(ab) = \mathrm{bdd}(ac) = \mathrm{bdd}(bc)$;
- $\mathrm{bdd}(xa) = \mathrm{bdd}(ya)$,
- $\mathrm{bdd}(zc) = \mathrm{bdd}(xc)$,
- $\mathrm{bdd}(yb) = \mathrm{bdd}(zb)$;
- b is interbounded with $\mathrm{Cb}(yz/b)$,
- a is interbounded with $\mathrm{Cb}(xy/a)$,
- c is interbounded with $\mathrm{Cb}(xz/c)$.



Clearly, replacing any element in an algebraic quadrangle by an interbounded element gives another algebraic quadrangle, and we call such quadrangles *algebraically equivalent*.

Although it is possible, we do not attempt to find, given an algebraic quadrangle, an equivalent one where each point is actually definable over the other two on the same line, since this property is not preserved by reduction anyway, in view of 2.2.

Theorem 2.4.2. *Let (a, b, c, x, y, z) be an algebraic quadrangle. We may assume (by considering an equivalent quadrangle with y and z replaced by* $\mathrm{bdd}(y)$ *and* $\mathrm{bdd}(z)$*) that y and z are boundedly closed. If we let $\pi(\xi, \eta, \zeta) = \mathrm{lstp}(byz)$, then π is invertible partial generic multiaction, complete in η and ζ, with $\pi^{-1} \circ \pi$ generic.*

Proof. Let $b_1 \mathop{\smile\!\!\!\!\!\!|}\, b_2 \models \mathrm{tp}(b)$ and let $tut_1 \models h \in \widehat{b_1^{-1} \circ b_2}$. We need to show that $h \mathop{\smile\!\!\!\!\!\!|}\, b_1$ and $h \mathop{\smile\!\!\!\!\!\!|}\, b_2$. Without loss of generality, $b_2 = b$. We have $btu \models \pi$, $b_1 t_1 u \models \pi$, so let $\varphi \in LAut_b$ such that $tu \mapsto yz$ and $b_1 t_1 u \mapsto b_1' y_1 z$. Then $\varphi(\mathrm{lstp}(tt_1/bb_1)) = \mathrm{lstp}(yy_1/bb_1')$ and it's enough to show $\varphi(h) \mathop{\smile\!\!\!\!\!\!|}\, b$ and $\varphi(h) \mathop{\smile\!\!\!\!\!\!|}\, b_1'$.

So, it boils down to: starting with $h = \mathrm{Cb}(yy_1/bb_1)$ for some $\{b, b_1, y\}$ independent with $b_1 y_1 z \equiv byz$, we need to show $h \mathop{\smile\!\!\!\!\!\!|}\, b$ and $h \mathop{\smile\!\!\!\!\!\!|}\, b_1$.

Since $b_1 y_1 \equiv_z by$, $b_1 y_1 \mathop{\smile\!\!\!\!\!\!|}_z by$ and $by \mathop{\smile\!\!\!\!\!\!|}_z cx$, the Independence Theorem implies (as $z = bdd(z)$ so in fact $b_1 y_1 \equiv_z^L by$) there is $b_2 y_2 \mathop{\smile\!\!\!\!\!\!|}_z bcxy$ with $b_2 y_2 \equiv_{byz} b_1 y_1$ (*) and $b_2 y_2 \equiv_{cxz} by$ (**). Let a_2 such that $a_2 b_2 c x y_2 z \equiv abcxyz$. Let $h_2 := \mathrm{Cb}(yy_2/bb_2)$; then, if we manage to show $h_2 \mathop{\smile\!\!\!\!\!\!|}\, b$ and $h_2 \mathop{\smile\!\!\!\!\!\!|}\, b_2$ we'll be done by (*). We may also assume $b_2 y_2 \mathop{\smile\!\!\!\!\!\!|}_z abcxy$ (***). From here on, we proceed as in [P3], 5.4.7:

(i). $aa_2bb_2 \mathop{\smile\!\!\!\!\!\!|}\, y$. From (***), (**) and $b \mathop{\smile\!\!\!\!\!\!|}\, z$, we get $b_2 \mathop{\smile\!\!\!\!\!\!|}\, abcy$, which, together with $y \mathop{\smile\!\!\!\!\!\!|}\, abc$ gives $y \mathop{\smile\!\!\!\!\!\!|}\, abb_2c$, but $a_2 \in \mathrm{bdd}(b_2 c)$ so $y \mathop{\smile\!\!\!\!\!\!|}\, aa_2bb_2$.

(ii). $y_2 \in \mathrm{bdd}(aa_2y)$. This is easy, and so is the following claim:

(iii). $y_2 \in \mathrm{bdd}(bb_2y)$.

From the above, forking calculus yields:

(iv). $aa_2 \mathop{\smile\!\!\!\!\!\!|}\, b$. (v). $aa_2 \mathop{\smile\!\!\!\!\!\!|}\, b_2$.

Now, by (i), (ii), (iii), $\mathrm{lstp}(yy_2/aa_2bb_2)$ does not fork over both aa_2, bb_2 and thus

$$h_2 = \mathrm{Cb}(yy_2/bb_2) = \mathrm{Cb}(yy_2/aa_2) \in \mathrm{bdd}(aa_2).$$

By Claims (iv) and (v), it is independent from each b and b_2, as required.

□

CHAPTER 3

Almost hyperimaginaries

Classes of finite tuples for definable equivalence relations in a monster model $\mathfrak{C}$ of a first order theory, called *imaginaries*, were organised into the multi-sorted first order structure called $\mathfrak{C}^{eq}$ by Shelah, to allow a smoother development of stability theory. This turned out particularly important in geometric stability theory, since canonical bases of types exist in $\mathfrak{C}^{eq}$.

On the other hand, the construction of canonical bases in simple theories [HKP] seems to require quotienting by type-definable equivalence relations, which lead to the introduction of *hyperimaginaries* as classes of (possibly infinite, but small) tuples with respect to type-definable equivalence relations.

Alas, not even the hyperimaginary universe is general enough, since in our construction of the group from the polygroup chunk, certain relations appear, which, at the moment, we cannot guarantee to be type-definable. They belong, however, to a very well behaved class of invariant relations and it is possible to develop some model theory for quotients over such relations, called *graded almost hyperimaginaries*, as shown in this chapter.

In sections 3.1 and 3.2 we define *ultraimaginaries* and *graded almost hyperimaginaries* and develop a satisfactory 'forking calculus', or a theory of independence for them, while in Section 3.3, we study the *gradedly almost hyperdefinable polygroups* and *polyspaces* in simple theories. The *core relation* needed to identify elements that happen to be in the intersection of independent enough products is discussed in Section 3.4, and we prove the Hrushovski-Weil style group (and space) chunk theorem for the gradedly almost hyperdefinable case in Section 3.5. One might pose a question whether it is necessary to consider graded almost hyperimaginaries, since if we observe the hyperdefinable polygroup chunk obtained in 2.3.10, the core relation on it has bounded classes (any two related elements are interbounded), and it would be much easier to develop the theory for quotients over such equivalence relations. The reason for choosing such generality is that in order to apply the group chunk theorem, we might be forced to leave the realm of relations with bounded classes. An additional bonus for choosing to work in the category of gradedly almost hyperdefinable groups in simple theories is that our subsequent results will show it 'closed under group configuration'.

The fact that ultraimaginaries might be of importance in the construction of a group from the polygroup chunk and that the proof of the group chunk theorem will go through was already noted in [To], where the group I obtained lived on ultraimaginaries, but the relation I was forced to use was not as well-behaved as the present core relation and I lacked control over the dimension of such objects. A possibility of studying hyperdefinable polygroups and properties of generic elements in the supersimple case was first recognised in [To1], then improved by Ben-Yaacov to the framework of gradedly almost hyperdefinable polygroups in general simple theories. For the presentation here we adopt a different definition of stratified ranks and correct, by distinguishing 'full' structures, some inaccuracies present in a working version of Ben-Yaacov's notes. The group chunk theorem in the simple (hyperdefinable) case is by Wagner, see [W], 4.7.1. For the gradedly almost hyperdefinable case, it is given in [To] for ultraimaginaries, and Ben-Yaacov noted that that it preserves almost hyperdefinability. The full details presented here were not written down before. I have not seen the space chunk theorem 3.5.3 appear anywhere in the literature.

It should be remarked here that the framework of graded almost hyperimaginaries might turn out to be unnecessary generality because it is indeed our hope, since most of the construction can be done hyperdefinably, that the relation in question can be made type-definable as well. Nevertheless, at least for now, it is required.

3.1. Ultraimaginaries and almost-hyperimaginaries

To clarify some of the notation, if R is an equivalence relation, the R-class of a is denoted by a_R or a/R. When R is just reflexive and symmetric, $a_R := \{x : xRa\}$, '$a \in_R A$' means there is $a' \in A$ with aRa', '$A \approx_R B$' means there are $a \in A$, $b \in B$ with aRb, and '$A =_R B$' means that for every $a \in A$ there is $b \in B$ with aRb and vice versa. If R_i comes from some grading, then $\in_i$, $\approx_i$ and $=_i$ will stand for $\in_{R_i}$, $\approx_{R_i}$ and $=_{R_i}$, respectively.

Definition 3.1.1. Let $(I, \leq)$ be a directed partial order.

(1) An *I-graded equivalence relation* R is a direct limit of reflexive symmetric type-definable relations R_i such that:
   (a) if $i \leq j$, then $R_i \subseteq R_j$;
   (b) for every i, j, there is k ($i \leq k$, $j \leq k$) such that xR_iyR_jz implies xR_kz;
(2) An invariant equivalence relation R is *almost type-definable* if there is a reflexive symmetric type-definable relation R' finer than R such that each R-class can be covered by boundedly many R'-classes. It is *I-gradedly almost type-definable* if it is I-graded and almost type-definable.
(3) A class of an invariant equivalence relation is called an *ultraimaginary*. A class of a (gradedly) almost type-definable relation is called a *(graded) almost hyperimaginary*.

In the end, the objects we will be dealing with will live on graded almost hyperimaginaries. Our approach is partly inspired by the development of hyperimaginaries in [HKP].

Convention 3.1.2. From this point onwards, in order to save precious letters of the alphabet, we abuse the notation somewhat, and write a_R and b_R even when a and b are not even of the same sort, since it will always be clear from the context which relation is involved in a_R and which in b_R.

Furthermore, we might have defined *(gradedly) almost hyperdefinable* equivalence relations on hyperimaginaries as the ones which can be covered by boundedly many classes of *hyperdefinable* reflexive and symmetric relations, but this approach gives the same objects as quotienting tuples of real elements by (gradedly) almost type-definable equivalence relations from 3.1.1, so we use these terms interchangeably.

Definition 3.1.3. Two ultraimaginaries a_R and b_R have the same type over a hyperimaginary c, denoted $a_R \equiv_c b_R$, if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv_c b'$. They have the same Lascar strong type, denoted $a_R \equiv^L_c b_R$ if there are $a' \in a_R$ and $b' \in b_R$ with $a' \equiv^L_c b'$.

Lemma 3.1.4. *For ultraimaginaries a_R and b_R and a hyperimaginary c, the following statements are equivalent:*

(1) $a_R \equiv_c b_R$;
(2) *there is an automorphism fixing c and taking a_R to b_R;*

(3) *for every* $a' \in a_R$ *there is* $b' \in b_R$ *with* $a' \equiv_c b'$.

Similarly, for Lascar strong types, the following are equivalent:

(1) $a_R \equiv^L_c b_R$;

(2) a_R *and* b_R *are equivalent modulo any bounded c-invariant equivalence relation;*

(3) *for every* $a' \in a_R$ *there is* $b' \in b_R$ *with* $a' \equiv^L_c b'$.

Definition 3.1.5. An infinite sequence $\langle a_{iR} : i \in I \rangle$ of ultraimaginaries is *indiscernible* over a hyperimaginary set A if there is an A-indiscernible sequence $\langle c_i : i \in I \rangle$ such that $c_i \in a_{iR}$.

3.2. Dividing

Definition 3.2.1. We say that two ultraimaginaries a_R and b_Q are independent over a hyperimaginary c, if there are $a' \in a_R$ and $b' \in b_Q$ such that $a' \downarrow_c b'$.

We exercise particular care not to consider independence *over* an ultraimaginary, or between infinite tuples of ultraimaginaries, due to complications caused by the lack of compactness. Naturally, a finite set of ultraimaginaries can be considered as a single ultraimaginary by juxtaposing the equivalence relations.

Proposition 3.2.2. *Independence (defined between finite tuples of ultraimaginaries over a hyperimaginary set) has the following properties:*

(1) *(Bounded elements) If* $a_R \downarrow_c b_R$*, and* $a'_R \in \mathrm{bdd}(a_R c)$*, then* $a'_R \downarrow_c b_R$*. Conversely, if R is graded and* $a_R \downarrow_{b'} a_R$ *for every* $b' \in b_R$*, then* $a_R \in \mathrm{bdd}(b_R)$*.*

(2) *(Extension) for any* a_R*,* A*,* b_R *there is an* $a'_R \equiv_A a_R$ *such that* $a'_R \downarrow_A b_R$*;*

(3) *(Symmetry)* $a_R \downarrow_A b_R$ *if and only if* $b_R \downarrow_A a_R$*;*

(4) *(Transitivity) If* $a_R \downarrow_A b_R c_R$ *and* $b_R \downarrow_A c_R$ *then* $a_R b_R \downarrow_A c_R$*.* $a_R \downarrow_A bc_R$ *if and only if* $a_R \downarrow_A b$ *and* $a_R \downarrow_{Ab} c_R$*.*

(5) *(Local Character). For every* a_R *and* A*, there is* $A_0 \subseteq A$ *with* $|A_0| \leq |T|$ *such that* $a_R \downarrow_{A_0} A$*.*

(6) *(Independence Theorem) If* $a_{0R} \equiv^L_A a_{1R}$*,* $a_{iR} \downarrow_A b_{iR}$ *for* $i < 2$*,* $b_{0R} \downarrow_A b_{1R}$*, then there is* a_R *with* $a_R b_{iR} \equiv_A a_{iR} b_{iR}$ *and* $a_R \downarrow_A b_{0R} b_{1R}$*.*

Proof.

(1) We may assume $a \downarrow_c b$ and let $\bar{a}'$ be any enumeration of representatives of all $a_R c$-conjugates of a'_R. We may assume $\bar{a}' \downarrow_{ac} b$, which implies $a\bar{a}' \downarrow_c b$. Of course, there will be i such that $a' R a'_i$ and then $a'_i \downarrow_c b$ implies that $a'_R \downarrow_c b_R$.

For the converse, suppose a_R is not bounded over b_R; there will be a b-indiscernible sequence $a_\alpha b_\alpha$ such that $\alpha \neq \beta$ implies $a_{\alpha R}$ and $a_{\beta R}$ are distinct, and $b_\alpha R b$. Moreover, by an automorphism, we may assume it is b'-indiscernible and $a_0 b_0 = ab$, for some $b' R b$. The existence of such a sequence implies $a'' \not\downarrow_{b'} a'$ for any $a' R a$ and $a'' R a$: suppose $a' R_i a$, $a'' R_i a$, hence $a'' R_j a'$ for some j and let a'_α be such that $a'_\alpha a_\alpha \equiv_{b'} a' a$; it is still true that $\alpha \neq \beta$ implies a'_α is not R-related to a'_β. We may in fact assume that a'_α is b'-indiscernible with $a'_0 = a'$. Let k such that $R_j(x, y) \wedge R_j(x, z) \to R_k(y, z)$; by indiscernibility of a'_α, we can find $\psi(y, z) \in R_k$ such that $\alpha \neq \beta$ implies $\neg\psi(a'_\alpha, a'_\beta)$. By compactness, find a formula $\varphi(x, y) \in R_j$ such that $\varphi(x, y) \wedge \varphi(x, z) \to \psi(y, z)$. But then $\varphi(x, a') \in \mathrm{tp}(a''/b'a')$, and $\varphi(x, a')$ 2-divides over b' since if e.g. $\varphi(c, a'_\alpha) \wedge \varphi(c, a'_\beta)$, then $\psi(a'_\alpha, a'_\beta)$, which is a contradiction.

(2), (3) and (5) are easy.

(4) If $a_R \downarrow_A b_R c_R$ and $b_R \downarrow_A c_R$, we may assume that $a \downarrow_A bc$, and let $b' R b$, $c' R c$ with $b' \downarrow_A c'$. We may assume $b'c' \downarrow_{Abc} a$, so $a \downarrow_A bb'cc'$ and we are done. The rest is similar.

(6) Let $a_{0R} \equiv^L_A a_{1R}$, $a_{iR} \mathop{\smile\!\!\!\!\!\mid}_A b_{iR}$, $b_{0R} \mathop{\smile\!\!\!\!\!\mid}_A b_{1R}$. We may assume $a_0 \equiv^L_A a_1$ and $b_0 \mathop{\smile\!\!\!\!\!\mid}_A b_1$. Let $a'_i R a_i$ and $b'_i R b_i$ such that $a'_i \mathop{\smile\!\!\!\!\!\mid}_A b'_i$. In fact, it is possible to choose b'_i such that $b'_i \equiv^L_A b_i$ (let b''_i realise the nonforking extension of $\mathrm{lstp}(b_i/Ab'_i)$ to $Ab'_i a'_i$; then $b''_i \mathop{\smile\!\!\!\!\!\mid}_{Ab'_i} a'_i$ and by above $a'_i \mathop{\smile\!\!\!\!\!\mid}_A b'_i b''_i$). So, let a''_i be such that $a'_i b'_i \equiv^L_A a''_i b_i$. Now, since $a''_i \equiv^L_A a'_i$, $a'_i R a_i$, and $a_0 \equiv^L_A a_1$, we have $a''_{0R} \equiv^L_A a''_{1R}$. Thus, there will be some $a'''_i R a''_i$ such that $a'''_0 \equiv^L_A a'''_1$ and we may assume $a'''_i \mathop{\smile\!\!\!\!\!\mid}_A b_i$ (let a''''_i realise the nonforking extension of $\mathrm{lstp}(a'''_i/Aa''_i)$ to $Aa''_i b_i$; then $a''''_i \mathop{\smile\!\!\!\!\!\mid}_{Aa''_i} b_i$, but $b_i \mathop{\smile\!\!\!\!\!\mid}_A a''_i$ implies $b_i \mathop{\smile\!\!\!\!\!\mid}_A a''''_i$). Now just apply the (hyperimaginary) Independence Theorem to this situation. □

Let us show now the special properties of independence for almost hyperimaginaries.

Lemma 3.2.3. *Let R be almost type-definable, witnessed by R', $a, b, c \in \mathfrak{C}$. Then, if $a_R \mathop{\smile\!\!\!\!\!\mid}_c b$, there is $a'R'a$ such that $a' \mathop{\smile\!\!\!\!\!\mid}_c b$.*

Proof. Let $a''Ra$ such that $a'' \mathop{\smile\!\!\!\!\!\mid}_c b$, and let $\bar{a}$ be such that $a/R = \bigcup a_i/R'$. We may assume $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{a''} cb$, which implies $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_c b$. Now, there is i such that $a \in a_i/R'$, so we may take $a' := a_i$. □

As a corollary, we get the first order characterisation of independence for almost hyperimaginaries:

Corollary 3.2.4. *Let R be almost type-definable, witnessed by R'. Then $a_R \mathop{\smile\!\!\!\!\!\mid}_c b_R$ if and only if there are $a'R'a$, $b'R'b$ such that $a' \mathop{\smile\!\!\!\!\!\mid}_c b'$.*

Proof. If $a_R \mathop{\smile\!\!\!\!\!\mid}_c b_R$, let $b_0 R b$ such that $a_R \mathop{\smile\!\!\!\!\!\mid}_c b_0$. Then, by the lemma above, there is $a'R'a$ with $a' \mathop{\smile\!\!\!\!\!\mid}_c b_0$, and, in particular, $a' \mathop{\smile\!\!\!\!\!\mid}_c b_R$. By the lemma again, there is $b'R'b$ with $a' \mathop{\smile\!\!\!\!\!\mid}_c b'$. The converse is trivial. □

Remark 3.2.5. It is clear by the above corollary that we can make sense of independence between infinite (small) tuples of almost hyperimaginaries and in that case we will have the Finite Character of independence as well.

3.3. Almost hyperdefinable polygroups

Let us describe the category of gradedly almost hyperdefinable (multi)structures and maps we shall be working in.

Definition 3.3.1. Assume $S = S_0/R$, $S' = S'_0/R'$, where S_0 and S'_0 are type-definable sets, R, R' are I and J-graded almost type-definable relations and $f(x, y)$ a type-definable relation on $S_0 \times S'_0$. Then f defines a gradedly almost type-definable partial multimap $S_0/R \to S'_0/R'$ if:

(1) there exists $j \in J$ such that for every $a \in S$, $f(a)$ can be covered by a bounded set of R'_j-classes (i.e. the fact that $f(a)/R$ is bounded is witnessed by some R'_j);
(2) for every R_i, there is R'_j such that for every $a \in S$, $f(a/R_i) \subseteq f(a)/R'_j$.

If in (1), at most a single R'_j-class is needed, then f induces a partial map. If $f(a) \neq \emptyset$ for every $a \in S$, we say the (multi)map is total.

An equational multistructure is a structure in a language consisting of symbols for multivalued functions (operations), '$\in$' and '$=$', where axioms are universal closures of formulas of the form $\bigwedge_n x_n \in \tau_n \to \bigvee_m y_m \in \sigma_m$, where τ_n and σ_m are terms.

A gradedly almost type definable equational (multi)structure S in some theory is given as $S = S_0/R$, where S_0 is a type-definable set, R is a gradedly almost hyperdefinable equivalence relation and each n-ary (multi)operation f on S is a gradedly definable (multi)map $f : S_0^n/R^n \to S_0/R$ such that, for every axiom of the form

$$\bigwedge_n x_{k_n} \in \tau_n(x_0, \ldots, x_{N-1}) \to \bigvee_m x_{l_m} \in \sigma_m(x_0, \ldots, x_{N-1}),$$

for every $i \in I$ there is $j \in I$ such that for every a_k/R, $k < N$, if the premise holds up to R_i (for every n, there are $a'_k R_i a_k$, $k < N$ with $a'_{k_n} \in \tau_n(a'_0, \ldots, a'_{N-1})$), one of the conclusions holds up to R_j.

Convention 3.3.2. In view of 3.1.2, even if our structure is multi-sorted, with sorts $S_i = S_i^0/R_i$, we shall denote all R_i's by the same letter, e.g. R, since knowing that $x \in S_i^0$ clearly indicates that x_R should be read as x_{R_i}. For example, a gradedly almost hyperdefinable polyspace $(P_0/R, X_0/Q)$ will be denoted as $(P_0/R, X_0/R)$.

Let us explore the extent of the above definition on an example of a gradedly almost definable polygroup. It is possible to formulate a similar statement at a 'meta-level' for an arbitrary equational multistructure, but we wish to be extra precise here for reasons that will emerge in 3.3.4.

Lemma 3.3.3. *Let $(P = P_0/R, *)$ be a gradedly almost hyperdefinable polygroup. Then, for every i and j, there is k such that:*

(1) *$a \in_i b' * c'$, $b' R_j b$, $c' R_j c$ implies $a \in_k b * c$.*
(2) *$d \in_i (a' * b') * c'$, $a' R_j a$, $b' R_j b$, $c' R_j c$ implies $d \in_k a * (b * c)$.*
(3) *$c \in_i a' * b'$, $a' R_j a$, $b' R_j b$ implies $b \in_k a^{-1} * c$, and symmetrically for inverses from the right.*

Proof.
(1) If $a R_i a' \in b' * c'$, $b' R_j b$, $c' R_j c$, by the fact that $*$ induces a graded multimap, $a' \in_{j'} b * c$ for some j', and then $a \in_k b * c$ for some k.
(2) Assume the situation is as in the statement. Let $d R_i d' \in (a' * b') * c'$, so the premise of the axiom $t \in (x * y) * z \to t \in x * (y * z)$ is satisfied up to $R_{\max\{i,j\}}$, so there is l such that the conclusion holds up to l, i.e. there are $d'' R_l d$, $a'' R_l a$, $b'' R_l b$, $c'' R_l c$ with $d'' \in a'' * (b'' * c'')$, so let $e'' \in b'' * c''$ such that $d'' \in a'' * e''$. By (1), there will be l' such that $e'' R_{l'} e$ for some $e \in b * c$, and again by (1), there will be l'' such that $d'' \in_{l''} a * e$, or, in other words, $d R_l d'' R_{l''} a * (b * c)$.
(3) Follows in a similar fashion to (2), using that $^{-1}$ induces a graded map. □

Remark 3.3.4. There is a complication which is not apparent when dealing with hyperdefinable objects, but presents itself here. It concerns the 'right' definition of a graded map. The above definition is minimal such that the stratified ranks for gradedly almost hyperdefinable polygroups, to be defined below, have sensible properties like e.g. translation invariance. However, when trying to do the same for gradedly almost hyperdefinable polyspaces, this definition is not rich enough. Namely, the following lemma may not hold for polyspaces, and in case we want a uniform treatment for polygroups and polyspaces, we have to require the conclusion of the lemma in the definition of graded maps and call such maps and corresponding structures *full*.

In this exposition, we choose the minimal definition, as we have reservations about the naturality of the alternative one, but then we have to prove the existence of generics for polygroups using stratified ranks first, and then obtain the existence of generics for polyspaces as a corollary.

Lemma 3.3.5. *Let $(P_0/R, *)$ be a gradedly almost hyperdefinable polygroup and let R_0 be such that $z \in x * y$ implies $y \in_0 x^{-1} * z$ and the same from the right. Then, for every i there is j such that if $c \in_i a * b$, there is $a' R_j a$ with $c \in_0 a' * b$.*

Proof. Let $c R_i c' \in a * b$. Then $a \in_0 c' * b^{-1}$, and by (1) of the previous lemma, $a \in_j c * b^{-1}$, so let $a R_j a' \in c * b^{-1}$, but then $c \in_0 a' * b$. □

Remark 3.3.6. I believe it should be possible to handle even more general objects, where e.g. S is a direct limit of S_i/R_i for $i \in I$, and a map f from S to $S' = \lim_{j \in J} S'_j/R'_j$ consists of an increasing index map $\iota : I \to J$ and type-definable maps $f_i : S_i \to S'_{\iota(i)}$, where we would have a similar requirement for uniformity of satisfaction of axioms. This actually seems like a more natural setup from a classical mathematical perspective, but we shall not need such generality in the applications that follow.

Definition 3.3.7. If $(P, X) = (P_0/R, X_0/R)$ is a gradedly almost hyperdefinable polyspace, $g \in P_0$ and π is a partial type extending X_0, we denote by $g * \pi(x)$ the partial type which is true of x if there is $y \models \pi$ such that $x \in g * y$. If Q is some type-definable reflexive and symmetric relation, by $(\pi/Q)(x)$ we denote the partial type satisfied by x if there is $y \models \pi$ with xQy.

For the rest of this chapter we shall assume we are in a simple theory, even though some of the definitions would make sense in an arbitrary theory as well.

Definition 3.3.8. Let $(P, X) = (P_0/R, X_0/R)$ be a gradedly almost hyperdefinable polyspace in a simple theory. We say that x_R is a *generic* element of X over a set A if for every $g \mathop{\smile\!\!\!\!\!\!\!\;|}_A x_R$, for every $y \in g \cdot x$, $y_R \mathop{\smile\!\!\!\!\!\!\!\;|} Ag$. Since every polygroup acts on itself by left and right translation, we can define both left and right generics in a polygroup.

In polygroups, the usual properties of generics follow. In particular, left and right generics coincide and we just call them generics:

Lemma 3.3.9. *Let $P = P_0/R$ be a gradedly almost hyperdefinable polygroup.*

(1) *If a_R is left generic in P, $g_R \mathop{\smile\!\!\!\!\!\!\!\;|} a_R$, $b_R \in g_R * a_R$, then b_R is left generic;*
(2) *if g_R is left generic, then g_R^{-1} is left generic;*
(3) *if g_R is left generic, it is right generic, too.*

Proof.
(1) is a particular case of 3.3.13.
(2) Let g_R be left generic, and let $g' \mathop{\smile\!\!\!\!\!\!\!\;|} g$ realise $\mathrm{tp}(g)$. Let $h_R \in g'_R * g_R$ and we may assume $h \mathop{\smile\!\!\!\!\!\!\!\;|} g'$. Then $g_R^{-1} \in h_R^{-1} * g'_R$ is generic by the previous claim.
(3) follows easily from (2). □

From now on, let R_0 witness the almost type-definability of R, and let it be high enough in the grading such that $y \in x * z^{-1}$ implies $x \in_0 y * z$, and let R_1 be coarser than R_0^2 and such that $x \in_0 y * z$ and yR_0y' imply $x \in_1 y' * z$.

Definition 3.3.10. Let $\mathrm{D}(\cdot, \varphi, \epsilon, \lambda, k) \geq \alpha$, for a formula $\varphi(x, z)$, $\epsilon(x, y) \in R_1$, cardinal λ and $k \in \omega$ be the least ordinal-valued function on partial types π satisfying:

- $\mathrm{D}(\pi, \varphi, \epsilon, \lambda, k) \geq 0$ if π is consistent;
- $\mathrm{D}(\pi, \varphi, \epsilon, \lambda, k) \geq \alpha$ for α limit, if $\mathrm{D}(\pi, \varphi, \epsilon, \lambda, k) \geq \beta$ for all $\beta < \alpha$;
- $\mathrm{D}(\pi, \varphi, \epsilon, \lambda, k) \geq \alpha + 1$ if there is $f \in P$, and $\langle c_i : i < \lambda \rangle$ such that
  (1) $\{\varphi(x, c_i) : i < \lambda\}$ is ϵ-k-inconsistent, i.e. $\bigwedge_{i \in I} \exists y[\epsilon(x, y) \wedge \varphi(y, b_i)]$ is not realised for any $I \subset \omega$, $|I| = k$;
  (2) for every $i < \lambda$, $\mathrm{D}((f * \pi)/R_1(x) \cup \varphi(x, c_i), \varphi, \epsilon, \lambda, k) \geq \alpha$;

Remark 3.3.11. For any φ, ϵ and k, the rank $\mathrm{D}(\cdot, \varphi, \epsilon, \omega, k)$ is closed and continuous: $\mathrm{D}(\pi, \varphi, \epsilon, \omega, k) = \min\{\mathrm{D}(\psi, \varphi, \epsilon, \omega, k) : \pi \vdash \psi\}$. Furthermore, if $\pi(x) = \pi(x, A)$ is a partial type with parameters from A, then for every $n < \omega$ there is a partial type $\nu_n^\pi(X)$ (equivalent to the consistency of a certain tree) such that $\models \nu_n^\pi(A')$ if and only if $\mathrm{D}(\pi(x, A'), \varphi, \epsilon, \omega, k) \geq n$. By compactness then, it is clear that if e.g. f and $c_i, i < \omega$ witness that $\mathrm{D}(\pi, \varphi, \epsilon, \omega, k) \geq n + 1$, we can in fact find $d_i, i < \lambda$ for any λ, with the same property as c_i, and thus it follows that $\mathrm{D}(\pi, \varphi, \epsilon, \lambda, k) = \mathrm{D}(\pi, \varphi, \epsilon, \omega, k)$

for any $\lambda \geq \omega$, so from now on we may omit λ. Also, if φ, ϵ and k are clear from the context, we shall write $\mathrm{D}(\pi)$ for shorthand.

Proof. By induction on n we show that $\mathrm{D}(\psi, \varphi, \epsilon, k) \geq n$ for all ψ provable from π implies $\mathrm{D}(\pi, \varphi, \epsilon, k) \geq n$, and there is a partial type ν_n^π such that $\models \nu_n^\pi(A')$ if and only if $\mathrm{D}(\pi(x, A')) \geq n$. Clearly this holds for $n = 0$, where $\nu_0^\pi(X)$ is the partial type $\exists x\, \pi(x, X)$. Assume the statement holds for n and $\mathrm{D}(\psi) \geq n+1$ for all $\psi(x, A)$ implied by $\pi(x, A)$. Thus, for every such ψ there are b^ψ and $\langle b_i^\psi : i < \omega\rangle$ such that for all $i < \omega$, $\mathrm{D}((b^\psi * \psi(x, A))/R_1 \cup \varphi(x, b_i^\psi)) \geq n$ and $\{\varphi(x, b_i^\psi) : i < \omega\}$ is ϵ-k-inconsistent. By inductive assumption there is $\nu_n^\psi(X, x, y)$ true of (A, b^ψ, b_i^ψ) for every $i < \omega$ such that $\mathrm{D}((b' * \psi(x, A'))/R_1 \cup \varphi(x, b'')) \geq n$ if and only if $\models \nu_n^\psi(A', b', b'')$. Clearly, if $\psi \vdash \psi'$, then $\nu_n^\psi \vdash \nu_n^{\psi'}$. By compactness, there are $\langle b', b_i' : i < \omega\rangle$ such that $\nu_n^\psi(A, b', b_i')$ for all ψ implied by π and all $i < \omega$, and $\{\varphi(x, b_i') : i < \omega\}$ is ϵ-k-inconsistent. By induction hypothesis $\mathrm{D}((b' * \pi)/R_1 \cup \varphi(x, b_i')) \geq n$ for all $i < \omega$ so $\mathrm{D}(\pi) \geq n+1$ by definition. We may define $\nu_{n+1}^\pi(X)$ as

$$\exists x \exists \langle y_i : i < \omega\rangle \left[\bigwedge_{i<\omega, \pi\vdash\psi} \nu_n^\psi(X, x, y_i) \wedge \bigwedge_{I\subset\omega, |I|=k} \neg\exists z \bigwedge_{s=1}^{k} \exists y\, [\epsilon(z, y) \wedge \varphi(y, y_s)] \right].$$

□

Even though the definition of stratified ranks makes sense for arbitrary polyspaces, the lemma below needs fullness.

Lemma 3.3.12. *Stratified ranks on graded almost definable polygroups have the following properties:*

(1) Ultrametric Property. *If $\pi \vdash \bigvee_{i<\beta} \pi_i$, for π, π_i type-definable, then $\mathrm{D}(\pi) \leq \sup\{\mathrm{D}(\pi_i) : i < \beta\}$.*
(2) Translation Invariance. $\mathrm{D}(\pi) = \mathrm{D}((g * \pi)/R_j)$ *for every $g \in P$ and R_j.*
(3) Finiteness. *For every π, φ, ϵ, k, $\mathrm{D}(\pi, \varphi, \epsilon, k) < \omega$.*
(4) Witnessing Dividing. *Suppose $q \supseteq p$ are complete types over $B \supseteq A$. Then q/R_1 doesn't divide over A if and only if $\mathrm{D}(p, \varphi, \epsilon, k) = \mathrm{D}(q, \varphi, \epsilon, k)$, for every φ, ϵ and k.*

Proof.
(1) Let $|\beta| < \lambda$. We proceed by induction. Assume $\mathrm{D}(\pi) \geq \alpha + 1$; let g and $c_i, i < \lambda$ be such that for every i, $\mathrm{D}((g * \pi)/R_1(x) \cup \varphi(x, c_i)) \geq \alpha$ and $\{\varphi(x, c_i) : i < \lambda\}$ is ϵ-k-contradictory. Then, since $(g * \pi)/R_1(x) \wedge \varphi(x, c_i) \vdash \bigvee_{j<\beta}(g * \pi_j)/R_1 \wedge \varphi(x, c_i)$, by induction hypothesis, there is $j = j(i)$ such that $\mathrm{D}((g * \pi_j)/R_1 \cup \varphi(x, c_i)) \geq \alpha$. By the pigeonhole principle applied to $i \mapsto j(i)$, there is $j_0 < \beta$ such that for λ-many c_i's, $\mathrm{D}((g * \pi_{j_0})/R_1 \cup \varphi(x, c_i)) \geq \alpha$ and thus $\mathrm{D}(\pi_{j_0}) \geq \alpha + 1$.
(2) Suppose $\mathrm{D}((g * \pi)/R_j) \geq \alpha + 1$; there are $f \in P$ and $c_i : i < \lambda$ such that for every $i < \lambda$, $\mathrm{D}((f*(g*\pi)/R_j)/R_1 \cup \varphi(x, c_i)) \geq \alpha$ and $\varphi(x, c_i)$ is ϵ-k-inconsistent. Consider an x realising the above type; there are $z \models \pi$, $yR_jy' \in g * z$, $x \in_1 f * y$. This means that $x \in_{j'} f * y' \subseteq f * (g * z)$ for some higher j', and in fact $x \in_k (f * g) * z$ for some higher k. By 3.3.5, there is $h'R_l f * g$ such that $x \in_0 h' * z$. Let $\{h_\beta : \beta < \mu\}$ be boundedly many elements such that $(f*g)/R_l \subseteq \bigcup_{\beta<\mu} h_\beta/R_0$. Thus, there must be a β such that hR_0h_β. By assumption on R_1, as $x \in_0 h * z$, we get that $x \in_1 h_\beta * z$. In other words, $(f * (g * \pi)/R_j)/R_1 \vdash \bigvee_{\beta<\mu}(h_\beta * \pi)/R_1$, so, by the ultrametric property, there is h_β such that $\mathrm{D}((h_\beta * \pi)/R_1 \cup \varphi(x, c_i)) \geq \alpha$. However, we might have started with $\lambda > \mu$, so even though the choice of h_β depends on c_i, we can apply the pigeonhole principle to get a fixed h such that there are still λ many i's with $\mathrm{D}((h * \pi)/R_1 \cup \varphi(x, c_i)) \geq \alpha$, witnessing $\mathrm{D}(\pi) \geq \alpha + 1$.

(3) If some $D(\pi, \varphi, \epsilon, k) \geq \omega$, the formula φ would have the tree property, contradicting simplicity.
(4) Suppose q/R_1 divides over A; let $\varphi_0(x, b_0) \in q/R_1$ be a dividing formula, and let $\langle b_i : i < \omega \rangle$ be an indiscernible sequence in $\mathrm{tp}(b_0/A)$ such that $\{\varphi_0(x, b_i) : i < \omega\}$ is inconsistent. But then, there will be an $\epsilon \in R_1$ and $\varphi \in q$ such that $\exists y\ \epsilon(x, y) \wedge \varphi(y, b_0) \vdash \varphi_0(x, b_0)$. Then, as $q \vdash p \cup \varphi(x, b_0)$, we have that for every $i < \omega$, $D(p \cup \varphi(x, b_i)) \geq D(q)$, and as $\{\varphi(x, b_i) : i < \omega\}$ are ϵ-k-inconsistent, we have that $D(p) \geq D(q) + 1$.

For the converse, let us show by induction on n that if $a_R \mathop{\smile\!\!\!\!\!\!|}_A B$ and $D(a/A) \geq n$, then $D(a/B) \geq n$. This is clear for $n = 0$, so suppose $D(a/A) \geq n + 1$. There are b and $\langle b_i : i < \omega \rangle$ such that for every $i < \omega$, $D((b * \mathrm{tp}(a/A))/R_1 \wedge \varphi(x, b_i)) \geq n$ and $\{\varphi(x, b_i) : i < \omega\}$ is ϵ-k-contradictory. By continuity of ranks, we can extend $(b * \mathrm{tp}(a/A))/R_1 \wedge \varphi(x, b_i)$ to a complete type p over Abb_0 of the same rank and we may assume that some $c \in_R b * a$ realises it. Let $a' R_0 a$ with $a' \mathop{\smile\!\!\!\!\!\!|}_A B$ and we may assume $bb_0 \mathop{\smile\!\!\!\!\!\!|}_{Aa'} B$, implying $B \mathop{\smile\!\!\!\!\!\!|}_A a'bb_0$ and $c_R \mathop{\smile\!\!\!\!\!\!|}_{Abb_0} B$. By induction hypothesis, $D(c/Abb_0) \geq n$ implies $D(c/Bbb_0) \geq n$. As $b_0 \mathop{\smile\!\!\!\!\!\!|}_{Ab} B$, we may assume that $\langle b_i : i < \omega \rangle$ is indiscernible over Bb, so $\langle b_i : i < \omega \rangle$ witness that $D(c/Bb) \geq n + 1$. Thus, by translation invariance, $D(a/B) \geq D(a/Bb) = D(c/Bb) \geq n + 1$. □

Proposition 3.3.13. (1) *There exist generic elements in gradedly almost hyperdefinable polygroups.*
(2) *If $(P, X) = (P_0/R, X_0/R)$ is a gradedly almost hyperdefinable polyspace $x_R \in X$ is generic if and only if there is a generic $g_R \in P$ and $x_{0R} \in X$ with $g_R \mathop{\smile\!\!\!\!\!\!|} x_{0R}$ such that $x_R \in g_R * x_{0R}$.*
(3) *Generic elements exist in every orbit of a gradedly almost hyperdefinable polyspace.*

Proof.
(1) Let $(P_0/R, *)$ be the polygroup in question. Enumerate all the possible triplets (φ, ϵ, k) and choose a type $p \vdash x \in P_0$ with maximal $D(\cdot, \varphi, \epsilon, k)$-rank in the lexicographic order with respect to the above enumeration. Let $a \models p$ and let $g \mathop{\smile\!\!\!\!\!\!|} a_R$, $b \in g * a$. Then, we know that, since $a_R \mathop{\smile\!\!\!\!\!\!|} g$ (implying $\mathrm{tp}(a/g)/R_1$ does not divide over $\emptyset$), $D(a, \varphi, \epsilon, k) = D(a/g, \varphi, \epsilon, k) = D(b/g, \varphi, \epsilon, k) \leq D(b, \varphi, \epsilon, k)$. By the choice of a, we have equality and $b_R \mathop{\smile\!\!\!\!\!\!|} g$ (the equality $D(a/g) = D(b/g)$ is obtained as follows: from $b_R \in g_R * a_R$ we can deduce that $b \in_j g * a$ for some R_j, so $\mathrm{tp}(b/g) \vdash (g * \mathrm{tp}(a/g))/R_j$ and by translation invariance we get that $D(b/g) \leq D(a/g)$; similarly, from $a_R \in g_R^{-1} * b_R$, we get $D(a/g) \leq D(b/g)$).
(2) Let $x_{0R} \in X$, $g_R \in P$ generic with $g_R \mathop{\smile\!\!\!\!\!\!|} x_{0R}$ and let $x_R \in g_R * x_{0R}$. Without loss of generality, $g \mathop{\smile\!\!\!\!\!\!|} x_0$. Take any $f \mathop{\smile\!\!\!\!\!\!|} x_R$. We may assume that $f \mathop{\smile\!\!\!\!\!\!|} x$ and furthermore that $gx_0 \mathop{\smile\!\!\!\!\!\!|}_x f$, implying $gx_0x \mathop{\smile\!\!\!\!\!\!|} f$. Let $y_R \in f_R * x_R \subseteq f_R * (g_R * x_{0R}) = (f_R * g_R) * x_{0R}$, so there is $h_R \in f_R * g_R$ with $y_R \in h_R * x_{0R}$. As g_R is generic, $h_R \mathop{\smile\!\!\!\!\!\!|} f$. Now, we have $f \mathop{\smile\!\!\!\!\!\!|} gx_0$, which implies $x_0 \mathop{\smile\!\!\!\!\!\!|}_f g$ and $x_0 \mathop{\smile\!\!\!\!\!\!|}_f h_R$ (since $h_R \in \mathrm{bdd}(fg)$), so $x_0 f \mathop{\smile\!\!\!\!\!\!|} h_R$, $f \mathop{\smile\!\!\!\!\!\!|} x_0 h_R$ and subsequently, as $y_R \in \mathrm{bdd}(x_0 h_R)$, we get the required $f \mathop{\smile\!\!\!\!\!\!|} y_R$.

On the other hand, if $x_R \in X$ is generic, find a generic $g_R \in G$ such that $x \mathop{\smile\!\!\!\!\!\!|} g$ and let $x_0 \in g^{-1} * x$. Then $g_R \mathop{\smile\!\!\!\!\!\!|} x_{0R}$ by genericity of x_R.
(3) is a trivial corollary of (1) and (2). □

Definition 3.3.14. Let S_0 be a type-definable set, R a gradedly almost type-definable equivalence relation on S_0, $* : S_0/R \otimes S_0/R \to S_0/R$ a gradedly type-definable multimap, and a gradedly defined map ${}^{-1} : S_0/R \to S_0/R$ such that whenever $a_R \mathop{\smile\!\!\!\!\!\!|} b_R$, $a * b$ is defined (in particular, we require that for every i, there is j such that if

$a_R \mathop{\smile\!\!\!\!\!\!\,|}\, b_R$, $a/R_i * b/R_i \subseteq (a*b)/R_j$). We say that $S = (S_0/R, *)$ is a gradedly almost hyperdefinable polygroup chunk, if there is some R_0 in the grading of R such that:

(1) *Generic independence:* If $a \mathop{\smile\!\!\!\!\!\!\,|}\, b$, $c \in a*b$, then $c_R \mathop{\smile\!\!\!\!\!\!\,|}\, a$ and $c_R \mathop{\smile\!\!\!\!\!\!\,|}\, b$.
(2) *Generic associativity:* If $\{a, b, c\}$ is independent, $(a*b)*c =_0 a*(b*c)$.
(3) *Generic inverse:* If $b_R \mathop{\smile\!\!\!\!\!\!\,|}\, c_R$ and $a \in b*c$, then $b \in_0 a*c^{-1}$, and, if $b \in a*c^{-1}$, then $a \in_0 b*c$; similarly from the left.

If furthermore X_0/R is a gradedly almost hyperdefinable set, and $* : S_0/R \otimes X_0/R \to X_0/R$ a gradedly type-definable multimap (i.e. for every $a_R \mathop{\smile\!\!\!\!\!\!\,|}\, x_R$, $a*x$ is defined), we say that $(S, X, *) = (S_0/R, X_0/R, *)$ forms a *polyspace chunk*, if the R_0 from above can be coarsened such that:

(1') If $a \mathop{\smile\!\!\!\!\!\!\,|}\, x$, $y \in a*b$, then $y_R \mathop{\smile\!\!\!\!\!\!\,|}\, a$.
(2') If $\{a, b, x\}$ is independent, $(a*b)*x =_0 a*(b*x)$.
(3') If $a_R \mathop{\smile\!\!\!\!\!\!\,|}\, x_R$ and $y \in a*x$, then $x \in_0 a^{-1}*y$.

Proposition 3.3.15. *Let $P = P_0/R$ be a gradedly almost hyperdefinable (poly)group. Then the generic elements of P form a gradedly almost hyperdefinable (poly)group chunk.*

Proof. By continuity of ranks, it follows that an element a_R is generic if and only if $D(a, \varphi, \epsilon, k) = D(P_0, \varphi, \epsilon, k)$ for all φ, ϵ and k, which is a type-definable condition S_0, and S_0/R is the required almost hyperdefinable (poly)group chunk. □

A dual problem is whether from a (poly)group chunk we can reconstruct a (poly)group. An answer in the case of a group chunk is provided by 3.5.1. As for polygroup chunks, in Chapter 4 we will present a construction of a gradedly almost hyperdefinable group chunk, given a gradedly almost hyperdefinable polygroup chunk, *coreless* in the sense below.

3.4. The core relation

Definition 3.4.1. Let $S = S_0/R$ be an I-gradedly almost hyperdefinable polygroup chunk.

(1) For $a, a' \in S_0$ we say that $a \sim_{i1} a'$ if there is $x_R \mathop{\smile\!\!\!\!\!\!\,|}\, a_R a'_R$ such that $a*x \approx_i a'*x$. Let $\sim_{in}$ be the n-th iterate of $\sim_{i1}$, and let $\sim$ be the $I \times \omega$-graded direct limit of those, called the *core relation.*
(2) S is *coreless* if $\sim$ is the same as R, i.e. for every $(i, n) \in I \times \omega$, there is j such that R_j is coarser than $\sim_{in}$.

Lemma 3.4.2. (1) *For every i there is j such that if $a \sim_{i1} a'$, there is $x \mathop{\smile\!\!\!\!\!\!\,|}\, aa'$ with $a*x \approx_j a'*x$.*
(2) *$\sim$ is an $I \times \omega$-graded almost type-definable equivalence relation on P such that every $\sim$-class contains boundedly many R-classes (i.e. if $a_R \sim a'_R$, then $\mathrm{bdd}(a_R) = \mathrm{bdd}(a'_R)$ as almost-hyperimaginaries).*
(3) *For every i there is j such that $a \sim_{i1} a'$ if and only if there are c, b, $c_R \mathop{\smile\!\!\!\!\!\!\,|}\, a_R a'_R$ with $a, a' \in_j b*c$ (and the other way around).*
(4) *$\sim$ is (gradedly and generically) regular: for every $(i, n) \in I \times \omega$ there is $(j, m) \in I \times \omega$ such that whenever $a \sim_{in} a'$ and $b_R \mathop{\smile\!\!\!\!\!\!\,|}\, a_R$ (and thus also $b_R \mathop{\smile\!\!\!\!\!\!\,|}\, a'_R$), for every $c \in a*b$ there is $c' \in a'*b$ with $c \sim_{jm} c'$.*
(5) *By regularity, $P/\sim$ is again a graded almost hyperdefinable polygroup chunk and it is coreless.*

Proof.

(1) Let $x_R \perp a_R a'_R$ such that $a * x \approx_i a' * x$. There are $x_0 R_0 x$, $a_0 R_0 a$, $a'_0 R_0 a'$ such that $x_0 \perp a_0 a'_0$. By gradedness of $*$, $a_0 * x_0 \approx_k a'_0 * x_0$ for some k and we may assume that $x_0 \perp_{a_0 a'_0} aa'$. Thus, $x_0 \perp aa'$ and by gradedness of $*$ again, $a * x_0 \approx_j a' * x_0$ for some j.
(2) If $a \sim_{i1} a'$, then there is j and $x \perp aa'$ with $a * x \approx_j a' * x$. Then, $x \perp_a a'$, and since $a'_R \in \mathrm{bdd}(xa)$ ($a'_R \in (a_R * x_R) * x_R^{-1}$), we get that $a'_R \perp_a a'_R$. Of course, we could have done the same for any $a_0 R a$, since if e.g. $a_0 R_i a \sim_{i1} a'$, then clearly $a_0 \sim_{k1} a'$ for some k. Therefore, $a'_R \in \mathrm{bdd}(a_R)$.

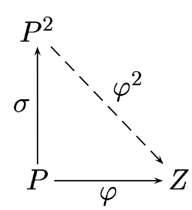
Thus, if R_0 witnesses the almost type-definability of R, $\sim_{01}$ will witness the almost type-definability of $\sim$.
(3) Let $x_R \perp a_R a'_R$ with $a * x \approx_i a' * x$, and let y be in this intersection (up to R_i). Then, there is j such that $a, a' \in_j y * x^{-1}$, and these are as required.
(4) Suppose $a \sim_{i1} a'$, and let (by previous claims) x, y and i' be such that $x \perp aa'$ and $a, a' \in_{i'} x * y$. We may additionally choose $xy \perp_{aa'} b$. If $c \in a * b$, there is i'' such that $c \in_{i''} (x * y) * b =_0 x * (y * b)$, so there will be a $z \in y * b$ with $c \in_{i''} x * z$. On the other hand, by associativity (from $x * (y * b) =_0 (x * y) * b$), we get that $x * z \approx_{j'} a' * b$, so pick c' inside up to $R_{j'}$. Thus, we may assume (by coarsening $R_{j'}$) that $c, c' \in_{j'} x * z$, and since $x \perp aa'b$, we get that $c \sim_{j1} c'$ for some higher j.
(5) To see corelessness, let $g_\sim \perp a_\sim b_\sim$ such that $a, b \in_{jn} g * h$. Let $a' \sim_{jn} a$ and $b' \sim_{jn} b$ with $a', b' \in g * h$. Then, since each $\sim$-class is covered by boundedly many R-classes, $g_R \perp a'_R b'_R$, so we have $a \sim_{jn} a' \sim_{01} b' \sim_{jn} b$ and $a \sim_{j,2n+1} b$. □

It is possible to define a similar relation on a gradedly almost hyperdefinable polygroup, and it will have all the analogous properties to the above, and even more: the class of the scalar identity element will form a subpolygroup, and the quotient by the relation will be the quotient by the subpolygroup. We skip the details as we will not need it later. The definition is as follows.

Definition 3.4.3. Let $P = P_0/R$ be an I-graded almost hyperdefinable polygroup. We will say $a \sim_{i1} a'$ if there is a generic element x_R with $x_R \perp a_R a'_R$ such that $a * x \approx_i a' * x$. Let $\sim_{in}$ be the n-closure of $\sim_{i1}$, and let $\sim$ be the direct limit of all of those.

3.5. Group chunk theorem

Theorem 3.5.1. *Let $P = (P_0/R, *)$ be a graded almost hyperdefinable group chunk in a simple theory. There is a gradedly almost hyperdefinable group G and a graded bijection σ between P and the set of generics of G, such that generically $*$ is mapped to the group operation. Moreover, the construction of $G = P^2$ has the following universal property: for any polygroup Z and a graded homomorphism $\varphi : P \to Z$ into generics of Z (i.e. $\varphi(a * b) \in \varphi(a) * \varphi(b)$ for $a \perp b$, and $\varphi(a^{-1}) = \varphi(a)^{-1}$), there is a unique graded $\varphi^2 : G \to Z$ such that $\varphi = \varphi^2 \circ \sigma$. In particular, such G is unique up to a gradedly almost type-definable isomorphism.*



Proof. It was already noted in [To] that the proof of the group chunk theorem by Wagner ([W], 4.7.1) is quite robust and can be done at the level of ultraimaginaries however bad the relation we are quotienting is. The point is that the independence conditions can be restricted to representatives.

In what follows, we assume that R_0 is coarse enough to witness the graded almost hyperdefinability of R and to 'absorb' all the algebraic operations (e.g. if $z \in x * y$, then $x \in_0 z * y^{-1}$). In particular, if $x \mathop{\smile\!\!\!\!\!\!\vert}\ yA$, we can find $z \in_0 x * y$ with $z \mathop{\smile\!\!\!\!\!\!\vert}\ yA$ ($x \mathop{\smile\!\!\!\!\!\!\vert}\ yA$ implies $x \mathop{\smile\!\!\!\!\!\!\vert}_y A$, so for $z' \in x * y$, $z'_R \in \mathrm{bdd}(xy)$ and $z'_R \mathop{\smile\!\!\!\!\!\!\vert}\ y$ yield $z'_R \mathop{\smile\!\!\!\!\!\!\vert}\ yA$ and there is zR_0z' with $z \mathop{\smile\!\!\!\!\!\!\vert}\ ya$ and $z \in_0 x * y$).

Let us write $(a,b)Q_i(a',b')$ for pairs from P_0 if there are $x, y \in P_0$ such that $x \mathop{\smile\!\!\!\!\!\!\vert}\ aba'b'$, $y \mathop{\smile\!\!\!\!\!\!\vert}\ aba'b'$ and $a * x \approx_i a' * y$, $b * x \approx_i b' * y$. Let Q be the direct limit of Q_i and let $[a,b] := (a,b)/Q$.

Claim: Q is gradedly almost type-definable equivalence relation.

Clearly Q_i's are reflexive and symmetric type-definable relations. For almost type-definability, let R_0 witness the almost type-definability of R. Fix some (a,b) and suppose e.g. $(a',b') \in (a,b)/Q_i$; there are x and y, $x \mathop{\smile\!\!\!\!\!\!\vert}\ aba'b'$, $y \mathop{\smile\!\!\!\!\!\!\vert}\ aba'b'$ with $a * x \approx_i a' * y$ and $b * x \approx_i b' * y$. Let $a/R \subseteq \bigcup_{\alpha<\mu} a_\alpha/R_0$ and $b/R \subseteq \bigcup_{\alpha<\mu} b_\alpha/R_0$ (for small μ) and without loss of generality $xy \mathop{\smile\!\!\!\!\!\!\vert}_{aba'b'} \bar{a}\bar{b}$.

By 3.3.5, we can find $a''R_ja$ and $b''R_jb$ with $a'' * x \approx_1 a' * y$ and $b'' * x \approx_1 b' * y$ for some fixed R_1. There will be a_α with $a''R_0a_\alpha$ and b_β with $b''R_0b_\beta$, so for a fixed $2 \in I$, we will have $a' * y \approx_2 a_\alpha * x$, $b' * y \approx_2 b_\beta * x$, and thus $(a',b')Q_2(a_l, b_k)$, meaning that $(a,b)/Q$ can be covered by the bounded union $\bigcup_{\alpha<\mu,\beta<\mu}(a_\alpha, b_\beta)/Q_2$.

Let us show the graded transitivity. Choose x, y witnessing $(a,b)Q_i(a',b')$ with $xy \mathop{\smile\!\!\!\!\!\!\vert}_{aba'b'} a''b''$, and x', y' witnessing $(a,b)Q_j(a'',b'')$ with $x'y' \mathop{\smile\!\!\!\!\!\!\vert}_{aba''b''} a'b'xy$. In particular, this implies that each of x, y, x', y' is independent of $aba'b'a''b''$. Take $u \in P_0$ independent of everything. Then, for some $t \in_0 x * u$, $t \mathop{\smile\!\!\!\!\!\!\vert}\ aba'b'a''b''xx'yy'$ by generic independence. By generic surjectivity, let $u' \in_0 x'^{-1} * t$ with $u' \mathop{\smile\!\!\!\!\!\!\vert}\ aba'b'a''b''x'y'$. Then, by generic associativity (as all the triples below are independent):

$$\begin{aligned} a' * (y * u) &\approx_0 (a' * y) * u \approx_{i'} (a' * x) * u \approx_0 a * (x * u) \\ &\approx_1 a * (x' * u') \approx_0 (a * x') * u' \approx_{j'} (a'' * y') * u' \approx_0 a'' * (y' * u'), \end{aligned}$$

for some constant R_1, i' depending only on i and j' depending on j, and thus $a' * (y * u) \approx_k a'' * (y' * u')$ for some k depending only on i and j. Similarly we get $b' * (y * u) \approx_k b'' * (y' * u')$. Now, since $*$ defines a (single-valued) product, we may assume that $y * u$ is covered by a single R_0-class, so we only may need to coarsen k a little to choose $v \in_0 y * u$ and $v' \in_0 y' * u'$ with $a' * v \approx_k a'' * v'$ and $b' * v \approx_k b'' * v'$, showing $(a',b')Q_k(a'',b'')$.

Claim: if $a * x \approx_i b * x$ for some x with $x \mathop{\smile\!\!\!\!\!\!\vert}\ a$, $x \mathop{\smile\!\!\!\!\!\!\vert}\ b$, there is j such that for every y with $y \mathop{\smile\!\!\!\!\!\!\vert}\ a$, $y \mathop{\smile\!\!\!\!\!\!\vert}\ b$, $a * y \approx_j b * y$.

Suppose we have z with $z \mathop{\smile\!\!\!\!\!\!\vert}\ ax$ and $z \mathop{\smile\!\!\!\!\!\!\vert}\ bx$. Let $c \in_0 z^{-1} * x$, with $c \mathop{\smile\!\!\!\!\!\!\vert}\ x$. We may assume $c \mathop{\smile\!\!\!\!\!\!\vert}_{xz} ab$, which will imply $a \mathop{\smile\!\!\!\!\!\!\vert}\ cxz$ and $b \mathop{\smile\!\!\!\!\!\!\vert}\ cxz$ and $\{a, c, x\}$, $\{b, c, x\}$ will be independent triples. Thus,

$$a * z \approx_1 a * (x * c) \approx_0 (a * x) * x \approx_{i'} (b * x) * c \approx_0 b * (x * c) \approx_1 b * z,$$

for some constant R_1, and i' depending only on i, so there will be k depending only on i such that $a * z \approx_k b * z$. Now, if we let $z \mathop{\smile\!\!\!\!\!\!\vert}\ abxy$, by above we will have that $a * z \approx_k b * z$, but, since also $y \mathop{\smile\!\!\!\!\!\!\vert}\ az$ and $y \mathop{\smile\!\!\!\!\!\!\vert}\ bz$, by the proof above, there will be j such that $a * y \approx_j b * y$.

Claim: If x is such that $x \mathop{\smile\!\!\!\!\!\!\vert}\ ab$, then for every i, there is j such that if $a' \in_i a * x$, $b' \in_i b * x$, $(a,b)Q_j(a',b')$. If $c \mathop{\smile\!\!\!\!\!\!\vert}\ ab$, there is k and there are d, d' with $d \mathop{\smile\!\!\!\!\!\!\vert}\ ab$ and $d' \mathop{\smile\!\!\!\!\!\!\vert}\ ab$ such that $(a,b)Q_k(c,d)$ and $(a,b)Q_k(d',c)$.

Let $y \mathop{\smile\!\!\!\!\!\!\vert}\ abx$. As $a'_R \in \mathrm{bdd}(ax)$ and $b'_R \in \mathrm{bdd}(bx)$, there will be $a''R_0a'$ and $b''R_0b'$ such that $y \mathop{\smile\!\!\!\!\!\!\vert}\ aba''b''x$. Let $z \in_0 x * y$ such that $z \mathop{\smile\!\!\!\!\!\!\vert}\ aba''b''$. Then, $a * z \approx_1 a * (x * y) \approx_0 (a * x) * y \approx_{i'} a'' * y$ and $b * z \approx_1 b * (x * y) \approx_0 (b * x) * y \approx_{i'} b'' * y$ for some R_1 and

i' depending only on i. We may assume $yz \mathop{\smile\!\!\!\!\!\!\mid}_{aba''b''} a'b'$, so $y \mathop{\smile\!\!\!\!\!\!\mid} aba'b'$, $z \mathop{\smile\!\!\!\!\!\!\mid} aba'b'$ and $a * z \approx_{i''} a'' * y \approx_1 a' * y$ by gradedness, as $a'R_0a''$, so $a * z \approx_j b' * y$ and similarly $a * z \approx_j b' * y$ for some j depending only on i, showing $(a, b)Q_j(a', b')$.

If $c \mathop{\smile\!\!\!\!\!\!\mid} ab$, let $x \in_0 a^{-1} * c$ with $x \mathop{\smile\!\!\!\!\!\!\mid} ab$. Then, $c \in_1 a * x$ for some R_1, pick any $d \in_1 b * x$ and by the previous paragraph there is R_k such that $(c, d)Q_k(a, b)$.

Now, for (a_0, b_0) and (b_1, c_0), we can pick $b \mathop{\smile\!\!\!\!\!\!\mid} a_0b_0b_1c_0$ so by the previous claim there will be $(a, b)Q_k(a_0, b_0)$ and $(b, c)Q_k(b_1, c_0)$. Thus, it makes sense to define $(a_0, b_0) \circ (b_1, c_0) = \{(a, c) : \exists b, (a, b)Q_k(a_0, b_0) \wedge (b, c)Q_k(b_1, c_0)\}$, this relation clearly being type-definable.

Claim: The relation $\circ$ gradedly defines a single-valued operation on R-classes.

To see this, it is enough to show that for every i, there is j such that if $(a', b')Q_i(a, b)$ and $(b', c')Q_i(b, c)$, then $(a', c')Q_j(a, c)$. Let $x \mathop{\smile\!\!\!\!\!\!\mid} aba'b'$ and $y \mathop{\smile\!\!\!\!\!\!\mid} aba'b'$ with $a * x \approx_i a' * y$, $b * x \approx_i b' * y$, $x' \mathop{\smile\!\!\!\!\!\!\mid} bcb'c'$ and $y' \mathop{\smile\!\!\!\!\!\!\mid} bcb'c'$ with $b * x \approx_i b' * y$, $c * x \approx_i c' * y$. We may choose $x'y' \mathop{\smile\!\!\!\!\!\!\mid}_{bcb'c'} aa'xy$, so in particular $x' \mathop{\smile\!\!\!\!\!\!\mid} x$. By generic surjectivity, let $z \in_0 x^{-1} * x'$ with $z \mathop{\smile\!\!\!\!\!\!\mid} aba'b'xy$ (since $x' \mathop{\smile\!\!\!\!\!\!\mid} abca'b'c'xy$), and the line below makes sense (where $1 \in I$ depends only on 0):

$$b' * y' \approx_i b * x' \approx_1 b * (x * z) \approx_0 (b * x) * z \approx_{i'} (b' * y) * z \approx_0 b' * (y * z).$$

Let $t \in_0 y * z$ with $t \mathop{\smile\!\!\!\!\!\!\mid} y$ (implying $t \mathop{\smile\!\!\!\!\!\!\mid} b'$) such that $b' * y' \approx_k b' * t$, k depending only on i. As $b' \mathop{\smile\!\!\!\!\!\!\mid} y'$ and $b' \mathop{\smile\!\!\!\!\!\!\mid} t$, by one of the previous claims we get that there is k' such that for every u with $u \mathop{\smile\!\!\!\!\!\!\mid} y'$, $u \mathop{\smile\!\!\!\!\!\!\mid} t$, $u * y' \approx_{k'} u * t$. In particular,

$$a' * y' \approx_{k'} a' * t \approx_1 a' * (y * z) \approx_0 (a' * y) * z \approx_{i'} (a * x) * z \approx_0 a * (x * z) \approx_1 a * x'.$$

Thus, for some j depending only on i, $a' * y' \approx_j a * x'$, and similarly $c' * y \approx_j c * x'$, so $(a', c')Q_j(a, c)$.

Claim: There is j such that for every a, b, $(a, a)Q_j(b, b)$.

Let $c \mathop{\smile\!\!\!\!\!\!\mid} ab$, $x \in_0 a^{-1} * c$ and $y \in_0 b^{-1} * c$ such that $x \mathop{\smile\!\!\!\!\!\!\mid} ab$ and $y \mathop{\smile\!\!\!\!\!\!\mid} ab$. By the previous claim, there is some i with $(a, a)Q_i(c, c)Q_i(b, b)$ so $(a, a)Q_j(b, b)$ for some fixed j.

Clearly, $[a, a]$ will act as a unit for multiplication, and any class $[a, b]$ has an inverse $[b, a]$.

Claim: Multiplication is (gradedly) associative, i.e. there is k such that $((a, b) \circ (a', b')) \circ (a'', b'') =_k (a, b) \circ ((a', b') \circ (a'', b''))$.

Let $x \mathop{\smile\!\!\!\!\!\!\mid} aba'b'a''b''$ and find $a_0 \in_0 a' * x$ and $b_0 \in_0 b' * x$ with $a_0 \mathop{\smile\!\!\!\!\!\!\mid} aba''b''$ and $b_0 \mathop{\smile\!\!\!\!\!\!\mid} aba''b''$. By the previous claims, there is i such that we can find c and c' with $(a, b)Q_i(c, a_0)$ and $(a'', b'')Q_i(b_0, c')$. By gradedness of $\circ$, there will be j with $((a, b) \circ (a', b')) \circ (a'', b'') =_j ((c, a_0) \circ (a_0, b_0)) \circ (b_0, c') =_j (c, c')$, but the same result is obtained (up to Q_j) from $(a, b) \circ ((a', b') \circ (a'', b''))$, so the existence of k is clear. Thus, the set of Q-classes forms an almost hyperdefinable group $G = (P_0 \times P_0)/Q$ with $\circ$ inducing the operation.

Claim: For every i there is k such that for any $x \mathop{\smile\!\!\!\!\!\!\mid} a$ and $y \mathop{\smile\!\!\!\!\!\!\mid} a$, any $a' \in_i a * x$, $a'' \in_i a * y$, $(a', x)Q_k(a'', y)$.

Let $u \mathop{\smile\!\!\!\!\!\!\mid} axy$ and $z \in_0 x^{-1} * u$ with $z \mathop{\smile\!\!\!\!\!\!\mid} axy$, $z' \in_0 y^{-1} * u$ with $z' \mathop{\smile\!\!\!\!\!\!\mid} axy$. Since $a' * z =_{i'} (a * x) * z =_0 a * (x * z) =_1 a * (y * z') =_0 (a * y) * z' =_{i'} a'' * z'$, pick $t \in_j a' * z$ and $t \in_j a'' * z'$ and by previous claims $(a', x)Q_{j'}(t, u)Q_{j'}(a'', y)$, so $(a', x)Q_k(a'', y)$ for some k.

The above claim gives rise to a gradedly almost type-definable map $\sigma : P \to G$.

Since the reader should be accustomed to extending the classical arguments to accommodate almost-hyperimaginaries by now, we shall only run through the remaining intermediate claims needed for the proof at the ultraimaginary level.

Claim: $G = \sigma(P)^2$.

If $a, b \in P_0$, pick $x \downarrow ab$. Let $y \in_0 a * x^{-1}$ with $y \downarrow ab$ and let $y' \in_0 x * b^{-1}$ with $y' \downarrow ab$. Then $[a,b] = [a,x] \circ [x,b] = [y * x, x] \circ [y' * b, b] \in \sigma(P)^2$.

Claim: σ is injective.

Suppose $\sigma(a) = \sigma(b)$ for $a, b \in P_0$. If $x \downarrow ab$, $[a * x, x] = [b * x, x]$, so $[a,b] = [a * x, b * x] = [a * x, x] \circ [b * x, x]^{-1} = 1_G = [a,a]$. Thus, there are $x \downarrow ab$ and $y \downarrow ab$ with $a * x =_R a * y =_R b * x$; therefore $a * z =_R b * z$ for all z with $z \downarrow a$, $z \downarrow b$. Take $c \downarrow ab$ and let $a_1 =_R a * c^{-1}$ with $a_1 \downarrow ab$, and $b_1 =_R b * c^{-1}$ with $b_1 \downarrow ab$. For any $y \downarrow a a_1 b b_1 c$ the triples $\{a_1, c, y\}$ and $\{b_1, c, y\}$ are independent, so:

$$a_1 * (c * y) =_R (a_1 * c) * y =_R a * y =_R b * y =_R (b_1 * c) * y =_R b_1 * (c * y).$$

Therefore $a_1 * x =_R b_a * x$ for all x with $x \downarrow a_1$ and $x \downarrow b_1$. Thus $a =_R a_1 * x =_R b_1 * x =_R b$.

Claim: σ generically preserves multiplication and maps P onto the generics of G.

Let $a \downarrow b$ and take $x \downarrow ab$. Then

$$\sigma(a * b) = [a * b * x, x] = [a * b * x, b * x] \circ [b * x, x] = \sigma(a) \circ \sigma(b).$$

Let $c \in P_0$ and consider a and b with $ab \downarrow c$ and let $c' R c * a$ with $c' \downarrow ab$. Then $\sigma(c) = [c', a]$ and $\sigma(c) \circ [a,b] = [c', b]$. As $c' \downarrow ab$, let $d R c' * b^{-1}$ with $d \downarrow ab$. So $\sigma(c) \circ [a,b] = [d * b, b] = \sigma(d)$. Hence $\sigma(c) \circ [a,b] \downarrow [a,b]$ and $\sigma(c)$ is generic.

Suppose now we have a graded homomorphism $\varphi : P \to Z$. We may define $\varphi^2([a,b]) := \varphi(b) * \varphi(a)^{-1} \cap \varphi(b') * \varphi(a')^{-1}$, for some $[a', b'] = [a,b]$ with $a' \downarrow ab$ (such (a', b') can be found by translating by $x \downarrow ab$). Clearly such a map is well-defined and single-valued, seen as follows. If $[a,b] = [a', b']$, let $t \downarrow a b a' b'$ such that $b * (a^{-1} * t) = b * (a'^{-1} * t)$, so $\varphi(b) * \varphi(a^{-1}) * \varphi(t) \approx \varphi(b') * \varphi(a'^{-1}) * \varphi(t)$; pick z in the intersection. Then $\varphi(b) * \varphi(a)^{-1} \approx z * \varphi(t)^{-1} \approx \varphi(b') * \varphi(a')^{-1}$, and choose h in the first and k in the second intersection. Since $\varphi(t) \downarrow hk$, they are core-related and the map has nonempty value. It is also single-valued as each two potential values are core-related. The reader can check that this can be done gradedly.

The uniqueness part follows from the universal property. □

Remark 3.5.2. If in the previous theorem the map φ happens to be (generically) onto of type 3 (i.e. if $\varphi(C) \in a * b$ for $a \downarrow b$, there are A and B with $\varphi(A) = a$, $\varphi(B) = b$ and $C = A * B$), the induced map φ^2 is onto of type 3.

Proof. Let us show that φ^2 is of type 3. Assume $\varphi^2([E,F]) = z \in u * v$. We may assume $E \downarrow uv$ and $F \downarrow uv$ by translation, and let $X \downarrow EFuv$ and $E' := E * X$, $F' := F * X$, so by definition of φ^2, $z = f * e^{-1} \cap f' * e'^{-1}$, where $\varphi(E) = e$, $\varphi(F) = f$, $\varphi(E') = e'$ and $\varphi(F') = f'$. Since φ is a homomorphism, $\varphi(X) =: x \in e^{-1} * e' \cap f^{-1} * f'$. Now, since $z \in f' * e'^{-1} \cap u * v$, by transposition we can find $b' \in u^{-1} * f' \cap v * e'$. By the second form of associativity, from $u^{-1} * (f' * x^{-1}) = (u^{-1} * f') * x^{-1}$ we get that $u^{-1} * f \approx b' * x^{-1}$ and similarly $v * e \approx b' * x^{-1}$. Since $x \downarrow efuv$, any two elements from the two intersections are core-equivalent and we can choose $b \in u^{-1} * f \cap v * e \cap b' * x^{-1}$. By the fact that x, b and b' are generic and φ is generically onto of type 3, let B and B' be such that $B' = B * X$, $\varphi(B) = b$ and $\varphi(B') = b'$. Now, it is easily checked that $[E,B] = [E', B']$, $[B,F] = [B', F']$ and they witness that $u = \varphi^2([E,B])$, $v = \varphi^2([B,F])$ and $[E,F] = [E,B] \circ [B,F]$. □

Corollary 3.5.3. *If $(P, Y, *)$ ($*$ standing for both the group chunk operation and the generic action) is a gradedly almost hyperdefinable space chunk in a simple theory, then there is a gradedly almost hyperdefinable space $(G, X, \circ)$ and a graded bijection σ between (P, Y) and the generics of (G, X) such that generically $*$ is mapped onto the group operation and action $\circ$.*

Proof. We shall not worry about the graded almost hyperdefinability of objects we construct in this proof, assuming the reader can rewrite the material in the style of the previous proof. Alternatively, the reader can imagine we are working in the hyperdefinable category.

We need to consider the dual equivalence relation between the pairs from P to the one given in the proof above, since we wish to deal here with the generic action of P on X from the left: we say that $(a,b)L(a'b')$ if there are $u \downarrow aba'b'$, $v \downarrow aba'b'$ with $u * a = v * a'$ and $u * b = v * b'$. Then it is clear that $G := (P \times P)/L$ is a group with the usual $\circ$. Note that $[a,b] = [a',b']$ if and only if there is $t \downarrow aba'b'$ with $(t * a^{-1}) * b = (t * a'^{-1}) * b'$.

Let $X := (P \times P \times Y)/L'$, where $(a,b,x)L'(a',b',x')$ if there is an $u \downarrow abxa'b'x'$ such that $((u * a^{-1}) * b) * x = ((u * a'^{-1}) * b') * x'$. It is easily shown that this is an equivalence relation and we denote the class of (a,b,x) by $[a,b,x]$.

Notice, if $[a,b] = [a',b']$, then trivially $[a,b,x] = [a',b',x]$. Thus, for any $[a',b'] \in G$ and $[b'',c',x]$, by results for L-classes, we can find $[a,b] = [a',b']$ and $[b,c,x] = [b'',c',x']$, and we can define the action of G on X by $[a,b] \circ [b,c,x] := [a,c,x]$. To see that it is well-defined, let $[a',b'] = [a,b]$ and $[b',c',x'] = [b,c,x]$. Let $u \downarrow aba'b'$ such that $(u * a^{-1}) * b = (u * a'^{-1}) * b'$ and we can pick $v \downarrow bcxb'c'x'u$ such that $((v * b^{-1}) * c) * x = ((v * b'^{-1}) * c) * x$. Choose any $z \downarrow abcxa'b'c'x'uv$ and let z' be such that $z^{-1} = z' * v$ (in particular, z' is again independent of everything). Now, since $\{z, u * a^{-1}, b\}$ is an independent triple (and similarly with dashes), by generic associativity we get that $(u * a^{-1}) * (b * z) = (u * a'^{-1}) * (b' * z) =: t^{-1}$ (where t is again independent of everything). Also, since $\{z', (v * b^{-1}) * c, x\}$, $\{z', v * b^{-1}, c\}$ and $\{z', v, b\}$ are independent triples,

$$\begin{aligned} z' * (((v * b^{-1}) * c) * x) &= (z' * ((v * b^{-1}) * c)) * x = ((z' * (v * b^{-1})) * c) * x \\ &= (((z' * v) * b^{-1}) * c) * x = ((z^{-1} * b^{-1}) * c) * x. \end{aligned}$$

Similarly we get that $z' * (((v * b'^{-1}) * c') * x') = ((z^{-1} * b'^{-1}) * c') * x'$. Now, since $(b * z)^{-1} = t * (u * a^{-1}) = (t * u) * a^{-1}$ and $(b' * z)^{-1} = (t * u) * a'^{-1}$, we get that $(((t*u)*a^{-1})*c)*x = (((t*u)*a'^{-1})*c')*x'$ so $t*u$ witnesses that $[a,c,x] = [a',c',x']$.

It is clear that (P,Y) embeds into (G,X) via $x \mapsto [a,a,x]$ for $x \in X$ and any $a \in G$ and, $a \mapsto [b, b * a]$ for any $a \in G$ and $b \downarrow a$. To see e.g. that for $a \downarrow x$, $\sigma(a * x) = \sigma(a) \circ \sigma(x)$, it is enough to notice that if whenever $a \downarrow b$ and $a \downarrow x$, $[b,b,a * x] = [b, b * a, x]$. By the previous theorem, it is clear that P is in bijection with generics of G. Suppose now that x_0 is generic in X. Then, by 3.3.13, there are $g \in G$ generic and $y \in X$ such that $g \downarrow y$ and $x_0 = g \circ y$. Let $y = [b,c,x]$ such that $g \downarrow bcx$ and $g = [a, a * g]$ for $a \downarrow gbcx$. Now, since $a * g * b^{-1} \downarrow bc$, $g * b^{-1} * c \downarrow a$ and $g * b^{-1} * c \downarrow x$,

$$\begin{aligned} g \circ y =& [a, a * g] \circ [b,c,x] = [a, a * g] \circ [a * g, (a * g * b^{-1}) * c, x] \\ =& [a, a * (g * b^{-1} * c), x] = [a, a, (g * b^{-1} * c) * x], \end{aligned}$$

showing that x_0 is in the image of σ. □

3.6. Stabilizers

In this section, we work in the hyperdefinable category, assuming the interested reader can extend the results to gradedly almost hyperdefinable objects. If π is a partial type over a set A, we say it is *generic* if it is contained in a generic type.

Lemma 3.6.1. *The following are equivalent in a hyperdefinable polygroup P:*

(1) *π is generic over A;*
(2) *for every $b \in P$, $b * \pi$ does not fork over $\emptyset$;*
(3) *for every $b \in P$, $b * \pi$ does not fork over A.*

Proof.
(1) $\Rightarrow$ (2). Since π is generic, let $a \models \pi$ be generic over A with $a \downarrow_A b$. By genericity, if $c \in b * a$, $c \downarrow Ab$, but $c \models b * \pi$ so $b * \pi$ does not fork over $\emptyset$.
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1). Let b be generic over A. Then $b * \pi$ does not fork over A so there are $a \models \pi$ and $c \in b * a$ with $c \downarrow_A b$. Therefore b and b^{-1} are generic over Ac, and so is $a \in b^{-1} * c$. □

Definition 3.6.2. Let P be a (hyperdefinable) polygroup, and let p be a type in P over A. Let $S(p) := \{g \in P : g * p \cup p \text{ doesn't fork over } A\}$, and let $\mathrm{stab}(p)$ be the subpolygroup of P generated by $S(p)$.

Lemma 3.6.3. (1) *$g \in S(p)$ if and only if there are $u, v \models p$, $u \downarrow_A g$, $v \downarrow_A g$, $u \in g * v$;*
(2) *$g \in S(p)$ if and only if $g^{-1} \in S(p)$;*
(3) *if $g \downarrow_A g' \in S(p)$ and p is an amalgamation base, then $g * g' \approx S(p)$.*

Proof.
(1) By definition, if $g \in S(p)$, there are $u, v \models p$ such that $u \in g * v$ and $u \downarrow_A g$. Then, for every $\mathrm{D}(\cdot) = \mathrm{D}(\cdot, \varphi, \epsilon, k)$,

$$\mathrm{D}(p) = \mathrm{D}(v/A) \geq \mathrm{D}(v/Ag) = D(u/Ag) = D(u/A) = D(p).$$

Thus we have equality all the way through and $v \downarrow_A g$.
(2) is a trivial consequence of (1).
(3) Let $u, v \models p$, $u \downarrow_A g$, $v \downarrow_A g$, $u \in g*v$ and $v', w \models p$, $v' \downarrow_A g'$, $w \downarrow_A g'$, $u \in g'*v$. Now, by the Independence Theorem, we may assume that $v' = v \downarrow_A gg'$. Thus, we have $u \in g * (g' * w) = (g * g') * w$, so there is $g'' \in g * g'$ with $u \in g'' * w$, and obviously $u \downarrow_A g''$, so $g'' \in S(p)$. □

Remark 3.6.4. Clearly $S(p)$ is a hyperdefinable set (over A). Unfortunately, the condition proved above, that $g \downarrow_A g' \in S(p)$ implies $g * g' \approx S(P)$ is too weak to show (without invoking the later techniques used in blowing-up) that $\mathrm{stab}(p)$ is generated in finitely many steps from $S(P)$ (if we had $g * g' \subseteq S(p)$, then it would be generated in two steps like in the case of groups). Nevertheless, it will be an invariant (over A) object.

Lemma 3.6.5. *If X is a generic hyperdefinable (over A) set, the subpolygroup H generated by X has a bounded index in P.*

Proof. Suppose H has an unbounded index in P. Then, we can find an A-indiscernible sequence $\langle g_i : i < \omega \rangle$, such that $\{g_i * H : i < \omega\}$ are distinct cosets. Then, $\{g_i * X : i < \omega\}$ is inconsistent, implying that $g_0 * X$ forks over A, so, by 3.6.1, X is not generic. □

Lemma 3.6.6. *A type p (over $\emptyset$, say) in a hyperdefinable polygroup is generic if and only if $S(p)$ is generic.*

Proof. If p is generic, let $u \downarrow v \models p$ and let $g \in u * v^{-1}$. By genericity of u and v, $u \downarrow g$, $v \downarrow g$, and $u \in g * v$, so $g \in S(p)$ is generic.
The converse if similarly straightforward. □

CHAPTER 4

Blowup

Our original aim was to obtain a group from a group configuration in a simple theory. In Chapter 2, we managed to extract a polygroup chunk $(P, *)$ from this information, and it is perfectly natural to hope that some variant of Hrushovski-Weil group chunk theorem (see 3.5.1) will yield a group, or at least a polygroup. The technical difficulties imposed by multivaluedness, however, are overwhelming, one of the reasons being that this polygroup chunk, in spite of some good properties like *generalised associativity* to be discussed below, has some serious shortcomings. For example, even if $y, y' \in f(x)$, for $f \in P$, $x \downarrow f$, it can a priori happen that $y \downarrow y'$. Also, if $h, h' \in f * g$ for $f \downarrow g \in P$, we might have $h \downarrow h'$.

Generalised associativity properties allow us to *blow up* the original polygroup chunk and obtain an improved polygroup chunk which actually becomes a group chunk modulo the core relation from 3.4, where 3.5.1 is going to be applicable. The construction has a familiar algebraic-geometric flavour which, together with the universal property it satisfies, should hopefully provide a good enough justification for its name.

However, there is an even more important historical reference to be made here. Model theoretic group configuration originates in the classical reconstruction of a division ring from a projective geometry by von Staudt, Hilbert, Veblen-Young, Artin and von Neumann. A blowup procedure extracting a group (resp. space) chunk out of the polygroup (resp. polyspace) chunk is written in Section 4.2 in a way which is, at the notational level, completely parallel to some parts of von Neumann's proof from [vN]. Also, the analogy to linear algebra (e-numbers being matrices, e-tuples vectors, multiplication of numbers corresponding to matrix multiplication, action of a number on a tuple corresponding to action of a matrix on a vector) we find rather amusing. This is not at all surprising given that, at a very abstract level, the axioms of a polygroup correspond to the axioms of a modular lattice.

Of course, it is desirable to keep the construction hyperdefinable as long as we can, because our hope is that the resulting group can be obtained hyperdefinable. That is why in Section 4.1 we present a blowup construction which produces a (hyperdefinable) partial generic multiaction which in fact is the improved polygroup chunk. One drawback of this construction is that it is even more localised than the previous one.

In order to obtain a construction of a global character, in Section 4.3 we discuss a sheaf-like (or manifold-like, depending on the reader's mathematical taste) blowup, constructed by pasting together the local blowups. It satisfies the universal property normally associated with blowing-up in algebraic geometry.

The first rather naive attempt to obtain a group from the polygroup chunk just by quotienting described in [To], in the particular example of a double coset space $G /\!/ H$ corresponds to descending to the group G/N, where N is the normal closure of H. This was unsatisfactory since we couldn't guarantee the nontriviality of the resulting group. Wagner suggested that, since we require $G /\!/ H$ to be a type-definable polygroup, one of the sufficient conditions for boundedness of products is that the family of conjugates of H be uniformly commensurable. But then, Schlichting's theorem ([W], 4.2.4) implies the existence of a normal subgroup N uniformly commensurable with the family of conjugates of H and G/N is thus a nontrivial group. This suggested that it would be better to look for the group 'interpreted' in the polygroup chunk, which resulted in a variety of blowup procedures by Ben-Yaacov and myself. I present three of mine here. It is interesting to notice a posteriori that a construction of a character similar to our blowup was used by Pillay and Kowalski in [KoP] to obtain the group configuration in the special case of ACFA, where the 'blowup map' is roughly $a \mapsto \langle \sigma^i(a) : i \in \mathbb{Z} \rangle$.

4.1. Hyperdefinable blowup

Let $(P, *)$ be the polygroup chunk of germs obtained from a suitable partial generic multiaction π, as in 2.3.10. We present here a construction of an improved polygroup chunk where whenever $y, y' \in f(x)$, $\mathrm{bdd}(y) = \mathrm{bdd}(y')$ and if $h, h' \in f * g$, $\mathrm{bdd}(h) = \mathrm{bdd}(h')$ and thus quotienting by the core relation gives an almost hyperdefinable group chunk. Intuitively, we force each point into an intersection of 'independent enough' products, which suffices to determine it better (e.g. in an algebraic quadrangle, each point is determined up to interboundedness, or in the example 1.4.4, a point is uniquely determined in the intersection of two lines).

Lemma 4.1.1. *There are e and f in P and x_0, x, y_0, y such that $x \mathop{\smile\!\!\!\!\!\vert}\, ef$, $x_0 x \models e$, $y_0 y \models e$, $xy \models f$.*

Proof. Start with some $x'_0 x \models e'$, $xy' \models f'$. Now, $x \equiv^L y'$, $x \mathop{\smile\!\!\!\!\!\vert}\, e'$, $y' \mathop{\smile\!\!\!\!\!\vert}\, x$, $e' \mathop{\smile\!\!\!\!\!\vert}\, x$, so by the Independence Theorem, there is $y \mathop{\smile\!\!\!\!\!\vert}\, xe'$, $\varphi : x \equiv_{e'} y$, $\psi : y' \equiv_x y$, and now $x'_0 x \models e'$, $\varphi(x'_0) y \models e'$, $xy \models \psi(f')$. Now $y \mathop{\smile\!\!\!\!\!\vert}\, xe'$ implies $e' \mathop{\smile\!\!\!\!\!\vert}\, xy$, so let e realise the nonforking extension of $\mathrm{tp}(e'/xy)$ to xyf, where $f = \psi(f')$. Thus, for some x_0 and y_0, $x_0 x \models e$, $y_0 y \models e$, $xy \models f$ and $x \mathop{\smile\!\!\!\!\!\vert}\, ef$. □

From now on, fix an e as in the lemma.

Definition 4.1.2. Let $\tilde{x}$ stand for $x_0 x$, $\tilde{f}$ for $\langle f, {}_0 f, f_0 \rangle$. We define a partial generic multiaction π^2 as follows: $\pi^2(\tilde{f}, \tilde{x}, \tilde{y})$ if $f \mathop{\smile\!\!\!\!\!\vert}\, e$, $x_0 x y \models f_0 \in f * e$, $x y y_0 \models {}_0 f \in e^{-1} * f$, and in such a case we shall write $\tilde{x}\tilde{y} \models \tilde{f}$.

We can complete and reduce π^2 to get $\bar{\pi}^2$, although the reduction doesn't change the completion $\underline{\pi}^2$: if $\tilde{x} \mathop{\smile\!\!\!\!\!\vert}_e \tilde{f}_p \tilde{g}_q$ with $\tilde{y} \in \tilde{f}_p(\tilde{x}) \cap \tilde{g}_q(\tilde{x})$, then $x \mathop{\smile\!\!\!\!\!\vert}\, fg$, $y \in f(x) \cap g(x)$, so $f = g$ since they are both germs. Similarly, $x_0 y$ witness $f_0 = g_0$ and $x y_0$ witness ${}_0 f = {}_0 g$. Now, we claim that, in the language from Chapter 2 (we abuse the notation a little in what follows, as we omit '$_p$' from $\tilde{f}_p \in \mathrm{Fun}(\bar{\pi}^2)$, but we bear in mind that $\mathrm{lstp}(\tilde{x}\tilde{y}/\tilde{f})$ for $\tilde{x}\tilde{y} \models \tilde{f}$ is fixed):

Theorem 4.1.3. *The pair $(P^2, *)$, where $P^2 = \mathrm{Fun}(\bar{\pi}^2)$ and multioperation $* : P^2 \otimes_e P^2 \to \mathbb{P}^{\mathrm{bdd}}(P^2)$ is given by $\tilde{g} * \tilde{f} := \widehat{\tilde{g} \circ \tilde{f}}$ (for $\tilde{f} \mathop{\smile\!\!\!\!\!\vert}_e \tilde{g}$), forms a (hyperdefinable) polygroup chunk with:*

- *for all $\tilde{y}, \tilde{y}' \in \tilde{f}(\tilde{x})$, $\mathrm{bdd}(\tilde{y}) = \mathrm{bdd}(\tilde{y}')$;*
- *for all $\tilde{h}, \tilde{h}' \in \tilde{f} * \tilde{g}$, $\mathrm{bdd}(\tilde{h}) = \mathrm{bdd}(\tilde{h}')$ (in fact, $\tilde{h}$ and $\tilde{h}'$ are core-related).*

Proof. For the reader who refuses to believe the notation as a 'black box', here is a detailed proof that for any $\tilde{f} \mathop{\smile\!\!\!\!\!\vert}_e \tilde{g}$, $\tilde{f} * \tilde{g} \neq \emptyset$ (the reason essentially being that $\mathrm{Arg}(\pi^2) = \Gamma(e)$ is an amalgamation base).

Let $\tilde{x}\tilde{y} \models \tilde{f}$, $\tilde{y}'\tilde{z} \models \tilde{g}$. As $\tilde{y} \equiv_e^L \tilde{y}'$, $\tilde{y} \mathop{\smile\!\!\!\!\!\vert}_e \tilde{f}$, $\tilde{y}' \mathop{\smile\!\!\!\!\!\vert}_e \tilde{g}$, by the Independence Theorem, we may assume that $\tilde{y}' = \tilde{y} \mathop{\smile\!\!\!\!\!\vert}_e \tilde{f}\tilde{g}$ and we can take $h := \mathrm{Cb}(xz/fg) \in f * g \cap g_0 * {}_0 f$,

$h_0 := \mathrm{Cb}(x_0z/gf_0) \in h * e \cap g * f_0$, ${}_0h := \mathrm{Cb}(xz_0/{}_0gf) \in e^{-1} * h \cap {}_0g * f$. In the case as above we write $\tilde{x}\tilde{y}\tilde{z} \models \tilde{h} \in \tilde{g} * \tilde{f}$.

It is easy to see this product is e.g. associative: let $\{\tilde{f}, \tilde{g}, \tilde{h}\}$ be e-independent, let $\tilde{x}\tilde{y}\tilde{z} \models \tilde{k} \in \tilde{f} * \tilde{g}$ and $\tilde{u}\tilde{x}'\tilde{z}' \models \tilde{l} \in \tilde{k} * \tilde{h}$. Since we are in a complete situation, $\tilde{x}\tilde{z} \equiv^L_{\tilde{k}} \tilde{x}'\tilde{z}'$, $\tilde{x}\tilde{z} \downarrow_{\tilde{k}} \tilde{f}\tilde{g}$, $\tilde{x}'\tilde{z}' \downarrow_{\tilde{k}} \tilde{l}\tilde{h}$, and $\tilde{f}\tilde{g} \downarrow_{\tilde{k}} \tilde{l}\tilde{h}$, so by the Independence Theorem, we may assume that $\tilde{x}'\tilde{z}' = \tilde{x}\tilde{z} \downarrow_{\tilde{k}} \tilde{f}\tilde{g}\tilde{h}\tilde{l}$, and if we let $m := \mathrm{Cb}(uy/gh)$, $m_0 := \mathrm{Cb}(u_0y/gh_0)$, ${}_0m := \mathrm{Cb}(uy_0/{}_0gh)$, we have $\tilde{l} \in \tilde{f} * \tilde{m}$ and $\tilde{m} \in \tilde{g} * \tilde{h}$.

If $\tilde{y}, \tilde{y}' \in \tilde{f}(\tilde{x})$, then $y, y' \in f(x) \cap f_0(x_0)$, but as $fx \downarrow_y f_0x_0$ ($x_0xy \models f_0 \in f * e$, so $x_0 \downarrow ff_0$ and as $f_0 \downarrow f$ we get $f \downarrow f_0x_0$ and the required independency follows from $y \in \mathrm{bdd}(f_0x_0)$ and $x \in \mathrm{bdd}(fy)$), this implies $y' \downarrow_y y'$ so $y' \in \mathrm{bdd}(y)$. Similarly one sees the interboundedness of y_0 and y_0'.

If $\tilde{h}, \tilde{h}' \in \tilde{f} * \tilde{g}$, then $h, h' \in f * g \cap g_0 * {}_0f$, and $f \downarrow {}_0gf_0$ ($ef \downarrow g$ implies $ff_0 \downarrow g_0$ which gives the required) so h and h' are core-related and thus interbounded. The interboundedness of h_0 and h_0', as well as ${}_0h$ and ${}_0h'$ follows in a similar way. □

The following proposition states that our polygroup chunk has 'generalised associativity' for each α.

Proposition 4.1.4. *Let $\{g_i : i < \alpha\}$ be independent germs. Then, for any choice of $\{{}_0g_i \in g_0^{-1} * g_i : 0 < i < \alpha\}$, there are $\{{}_ig_j \in g_i^{-1} * g_j : i \neq j < \alpha\}$ such that for all $\{i, j, k\}$, ${}_ig_j \in {}_ig_k * {}_kg_j$, and ${}_ig_j{}^{-1} = {}_jg_i$. Moreover, for any $f \downarrow \{g_i, {}_ig_j : i \neq j < \alpha\}$, we can find $f_i \in f * g_i$ with $f_j \in f_i * {}_ig_j$ for every $i \neq j$.*

We call any such $\langle g_i, {}_ig_j : i \neq j < \alpha\rangle$ an α-frame.

Proof. Let $a_ib_ic_i$ witness ${}_0g_i \in g_0^{-1} * g_i$, i.e. $a_ib_i \models \Gamma(g_i)$, $c_ib_i \models \Gamma(g_0)$, $a_i \downarrow g_0g_{i0}g_i$.

By induction on k, it is possible to assume: (‡) there are bc with $bc \downarrow_{g_0} \{g_i, {}_0g_i : 0 < i \leq k\}$ such that $a_ibc \models {}_0g_i \in g_0^{-1} * g_i$, for $0 < i \leq k$. For $k = 1$ there's nothing to prove. For the successor case, suppose (‡) holds for k and some bc. By hypotheses of the proposition, $c_{k+1}b_{k+1} \equiv^L_{g_0} cb$, $cb \downarrow_{g_0} \{g_i, {}_0g_i : i \leq k\}$, $c_{k+1}b_{k+1} \downarrow_{g_0} g_{k+1}$ and $g_{k+1} \downarrow_{g_0} \{g_i, {}_0g_i : i \leq k\}$ so by the Independence Theorem we may assume $c_{k+1}b_{k+1} = cb \downarrow_{g_0} \{g_i, {}_0g_i : i \leq k+1\}$. If l is a limit and (‡) holds for all $k < l$, then by compactness we can find bc which are 'good' for all $k < l$ and we continue by the Independence Theorem as in the successor case. This induction finishes at α; rename cb as a_0b.

Now, for $i \neq j > 0$, let ${}_ig_j := \mathrm{Cb}(a_ja_i/g_ig_j)$. From the above independencies, it is clear that a_jba_i witness ${}_ig_j \in g_i^{-1} * g_j$, and $a_ja_ka_i$ witness ${}_ig_j \in {}_ig_k * {}_kg_j$.

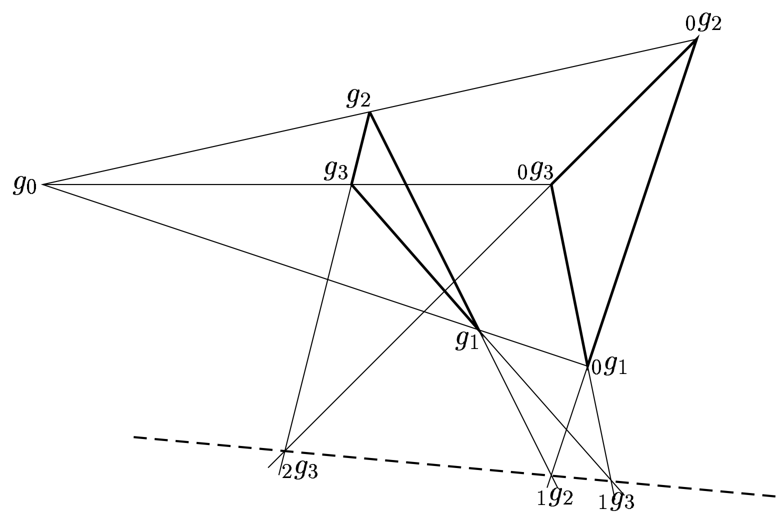
Now let $f \downarrow \{g_i, {}_ig_j : i \neq j < \alpha\}$, and let e.g. $ed \models \Gamma(f)$. By above, we have $a_ib \models \Gamma(g_i)$ for all i. Since $Arg(\pi)$ is a Lascar strong type, $b \equiv^L d$. Also, $b \downarrow \{g_i, {}_ig_j : i \neq j < \alpha\}$ and $d \downarrow f$, so by the Independence Theorem we may assume that $b = d \downarrow \{f\} \cup \{g_i, {}_ig_j : i \neq j < \alpha\}$ and for f_i we may take $\mathrm{Cb}(a_ie/g_if)$. □

Remark 4.1.5. The polygroup chunk from 4.1.3 will obviously give a gradedly almost hyperdefinable group chunk after quotienting by the core relation. Moreover, it is possible to generalise the construction as follows, with the idea of getting the group chunk in a *hyperdefinable* way.

Denote by $C(f)$ the intersection of all products containing f. If there is a frame $e = \langle e_i, {}_ie_j : i \neq j < \alpha\rangle$ such that for every g, h with $\{g, h, e\}$ independent we have that if $f \in \bigcap_{i<\alpha} g_i * {}_ih$ for $g_i \in g * e_i$ and $g_j \in g_i * {}_ie_j$, ${}_ih \in e_i^{-1} * h$ and ${}_ih \in {}_ie_j * {}_jh$, then $C(f) = \bigcap_{i<\alpha} g_i * h_i$ (i.e. the intersection on the right hand side stabilises), then $f \sim^\alpha f'$ if $C(f) = C(f')$ will become a type-definable equivalence relation, and we can do the blowup construction in such a way that whenever $\tilde{h}, \tilde{h}' \in \tilde{f} * \tilde{g}$ we have $\tilde{h} \sim^\alpha \tilde{h}'$, thus yielding a hyperdefinable group chunk. This property is not so unreasonable to ask for, since in the motivating example of reconstructing a group from a double coset space mentioned in the introduction to this chapter, it is possible to find an n-frame such that the intersection of any n products as above is a singleton (after some initial normalizations). Also, as mentioned in the introduction to this section, in the example of reconstructing a group from a projective space, a 2-frame suffices.

The construction of π^α goes by finding a suitable frame $\bar{e} = \langle e_i, {}_ie_j : i \neq j < \alpha\rangle$ with $x_ix \models e_i$, $i < \alpha$, $x \downarrow \bar{e}$ and $x_ixx_j \models {}_je_i \in e_j^{-1} * e_i$. Denote by $\tilde{x} := \langle x_i : i < \alpha\rangle\hat{\ }x$, and we fix $p := \mathrm{lstp}(\tilde{x}/\bar{e})$. Then, if $\tilde{f} := f\hat{\ }\langle {}_if, f_i, {}_if_j : i \neq j < \alpha\rangle$, we let $\pi^\alpha(\tilde{f}, \tilde{x}, \tilde{y})$ if $f \downarrow \bar{e}$, $\tilde{x}, \tilde{y} \models p$, $x_ixy \models f_i \in f * e_i$, $xyy_i \models {}_if \in e_i^{-1} * f$, $x_iyy_j \models {}_jf_i \in e_j^{-1} * f_i$, $x_ixy_j \models {}_jf_i \in {}_jf * e_i$ and in such a case we write $\tilde{x}\tilde{y} \models \tilde{f}$. The construction continues by completing, with the product of $\tilde{f}_q \downarrow_{\bar{e}} \tilde{g}_r$ given as (now forgetting the types q and r and witnesses) $\tilde{h}$ such that $h \in f * g \cap \bigcap_i f_i * {}_ig$, ${}_ih \in e_i^{-1} * h \cap {}_if * g \cap \bigcap_j {}_if_j * {}_jg$, $h_i \in h * e_i \cap f * g_i \cap \bigcap_j f_j * {}_jg_i$, ${}_ih_j \in {}_ih * e_j \cap e_i^{-1} * h_j \cap {}_if * g_j \cap \bigcap_k {}_if_k * {}_kg_j$, thus determining the product up to $\sim^\alpha$.

Remark 4.1.6. If we consider what the generalised associativity means for the polygroup associated to a projective geometry from 1.4.4, the Pasch axiom is equivalent to associativity (i.e. generalised associativity for $\alpha = 3$), and e.g. the Desargues axiom corresponds to generalised associativity for $\alpha = 4$, as shown in the figure below. Hence, both in our and the classical case, we can 'coordinatise' only in the presence of a 'higher associativity' condition.



Remark 4.1.7. A negative feature of this construction is that for an arbitrary $f \downarrow e$, it may be impossible to find witnesses x_0xy_0y as in 4.1.1, so not every $f \downarrow e$ can be blown up. This will be rectified in the next section at the expense of losing hyperdefinability. Nevertheless, the rank situation is still good, i.e. $\mathrm{SU}(P^2) = \mathrm{SU}(P)$, since if $f \downarrow e$ are from 4.1.1, at least every $f' \models \mathrm{tp}(f/e)$ can be blown up.

4.2. The 'classical' blowup construction

Let $(S, X) = (S_0/R, X_0/Q)$ be a coreless gradedly almost hyperdefinable polyspace chunk, where R is I-graded and Q is J-graded. Assume R_0 is like in 3.3.14. In what follows, we often claim without a proof the existence of certain R_ι's such that some algebraic manipulations give a result up to R_ι, and in each such case it will follow from almost hyperdefinability and gradedness of the polyspace chunk, e.g. if $\{a_R, b_R, c_R\}$ is independent, we can find $a' R_0 a$, $b' R_0 b$, $c' R_0 c$ such that $\{a', b', c'\}$ is independent, so $(a' * b') * c' =_0 a' * (b' * c')$, and, by gradedness of $*$, $(a * b) * c =_\iota a * (b * c)$ for some ι.

Definition 4.2.1. A tuple $e = \langle e_i, e_{ij} : i \neq j < n \rangle$ from S_0, we shall call an *n-frame* (up to R_ϵ), or sometimes just a *frame*, if:

(1) $\{e_{i R} : i < n\}$ is independent;
(2) for every $i \neq j < n$, $e_{ij} \in_\epsilon e_i^{-1} * e_j$;
(3) for every $i \neq j < n$, $e_{ij} =_\epsilon e_{ji}^{-1}$;
(4) for every $i, j, k < n$, $e_{ij} \in_\epsilon e_{ik} * e_{kj}$.

Given an n-frame e, a tuple (a matrix?) $a = \langle a_{ij} : i \neq j < n \rangle$ from S_0 is called an *e-number* (up to R_ι), if there exists an $a \in S_0$, $a_R \downarrow e$ such that:

(1) for every $i \neq j < n$, $a_{ij} \in_\iota e_i^{-1} * a * e_j$;
(2) for every i, j, k, $a_{ij} \in_\iota a_{ik} * e_{kj}$ and $a_{ij} \in_\iota e_{ik} * a_{kj}$.

A tuple $\langle {}_i x : i < n \rangle$ in X_0 is called a *left e-tuple* (up to some Q_ξ), provided there exists an $x \in X_0$ such that:

(1) for every $i < n$, ${}_i x \in e_i^{-1} * x$;
(2) for every $i \neq j < n$, ${}_i x \in_\xi e_{ij} * {}_j x$.

We define *right e-tuples* dually.

Lemma 4.2.2. *There is an R_ϵ such that, given $\{e_{i R} : i < n\}$ independent, for any choice of $e_{0i} \in e_0^{-1} * e_i$, $0 < i < n$, we can find e_{ij}, $0 < i < n$, $j < n$ such that $\langle e_i, e_{ij} : i \neq j < n \rangle$ is a frame (up to R_ϵ).*

Proof. Let $e_{0i} \in e_0^{-1} * e_i$ be given, and put $e_{i0} := e_{0i}^{-1}$. Now, $e_0^{-1} \in e_{0i} * e_i^{-1} \cap e_{0j} * e_j^{-1}$ (up to some R_ϵ), and therefore $e_{i0} * e_{0j} \approx_\epsilon e_i^{-1} * e_j$, (for some coarser R_ϵ), and we can pick an element e_{ij} in both.

We claim that this will form a frame. By the second form of associativity, since e.g. $(e_{13} * e_3^{-1}) * e_2 =_R e_{13} * (e_3^{-1} * e_2)$, we get that $e_1^{-1} * e_2 \approx_\epsilon e_{13} * e_{32}$ (for some R_ϵ), so pick h in the intersection.

So, we have obtained $e_{12}, h \in_\epsilon e_1^{-1} * e_2$, and $e_{1R} \downarrow e_{12R} h_R$ as it is easy to check that $e_{1R} \downarrow e_{01R} e_{02R} e_{03R}$, which implies $e_{1R} \downarrow e_{01R} e_{02R} e_{13R} e_{32R}$. Therefore, by corelessness, $e_{12} \in_\epsilon e_{13} * e_{32}$ for some higher R_ϵ. We can check the other relations in a similar fashion. □

From now on, fix an n-frame e up to some R_ϵ, and let $X_e^0 := \{x \in X_0 : x_Q \downarrow e\}$ and $S_e^0 := \{a \in S_0 : a_R \downarrow e\}$ and denote by $(S_e, X_e) = (S_e^0/R, X_e^0/Q)$ the resulting almost hyperdefinable polyspace chunk over e.

Lemma 4.2.3. *There is Q_ξ such that, given $x \in X_e^0$, for any choice of ${}_0 x \in e_0^{-1} * x$, we can find (uniquely up to Q) ${}_i x$, $0 < i < n$ such that $\langle {}_i x : i < n \rangle$ is an e-tuple (up to Q_ξ).*

Proof. Let ${}_0 x \in e_0^{-1} * x$ be given. Since $(e_{10} * e_0^{-1}) * x =_Q e_{10} * (e_0^{-1} * x)$, by the second form of associativity, we get that $e_1^{-1} * x \approx_\xi e_{10} * {}_0 x$ (for some Q_ξ), so we can pick ${}_1 x$ inside.

Similarly, we get that $e_{20} * {}_0 x \approx_\xi e_2^{-1} * x \approx_\xi e_{21} * {}_1 x$; pick h in the first, k in the second intersection. Thus, since $h, k \in_\xi e_2^{-1} * x$, and $e_{2R} \downarrow h_Q k_Q$ (the last independency follows from $e_{2R} \downarrow e_{01R} e_{02R0} x_Q$, which is easily verified), h and k are core-related and thus, by corelessness, we can define ${}_2 x$ up to some higher Q_ξ. We proceed analogously. □

Lemma 4.2.4. *There is a certain R_ι higher in the grading than R_ϵ such that, given any $a \in S_e^0$ we can pick $\langle a_{ij} : i \neq j < n \rangle$ forming an e-number up to R_ι. In other words, e-numbers are plentiful.*

Proof. Let $a_R \mathop{\smile\!\!\!\!\!\!\;|}\, e$. By 4.2.3 there are $a_i \in_\iota a * e_i$ with $a_j \in_\iota a_i * e_{ij}$ and ${}_ia \in_\iota e_i^{-1} * a$ with ${}_ia \in_\iota e_{ij} * {}_ja$, for some R_ι.

Since $(e_i^{-1} * a) * e_j =_R e_i^{-1} * (a * e_j)$, ${}_ia * e_j \approx_\iota e_i^{-1} * a_j$ (for a higher R_ι), so take a_{ij} in the intersection.

By $(e_i^{-1} * a_k) * e_{kj} =_R e_i^{-1} * (a_k * e_{kj})$, $a_{ik} * e_{kj} \approx_\iota e_i^{-1} * a_j \ni_\iota a_{ij}$ (for some R_ι); take h in the intersection. If we can show that $a_{jR} \downarrow a_{ijR} a_{ikR} e_{kjR}$, we will have that h and a_{ij} are core-related and thus equal modulo some coarser R_ι.

Since $a_{ij} \in_R {}_ia * e_j$, $a_{ik} \in_R {}_ia * e_k$, $e_{kj} \in_R e_k^{-1} * e_j$, it is enough to check that $a_{jR} \downarrow {}_ia_R e_{jR} e_{kR}$. By assumption, $e_{iR} \downarrow a_R e_{jR} e_{kR}$, so ${}_ia_R \downarrow e_{jR} e_{kR} a_{jR}$, but $a_{jR} \downarrow e_{jR} e_{kR}$ so everything is independent and $a_{jR} \downarrow {}_ia_R e_{jR} e_{kR}$.

The fact that $a_{ij} \in_R e_{ik} * a_{kj}$ is proved symmetrically. □

Here is a criterion allowing us to recognise an e-number intrinsically.

Lemma 4.2.5. *Let $n \geq 4$. If a tuple $\tilde{a} = \langle a_{ij} : i \neq j < n \rangle$ satisfies*

(1) *for every $i \neq j < n$, $a_{ijR} \downarrow e$;*
(2) *for every $i < n$, $\{a_{ijR} : j \neq i\}$ (and dually, $\{a_{jiR} : j \neq i\}$) is independent;*
(3) *for every i, j, k, $a_{ij} \in_\iota a_{ik} * e_{kj}$ and $a_{ij} \in_\iota e_{ik} * a_{kj}$,*

(for some R_ι), there is an $a \in S_0$, $a_R \downarrow e$, witnessing that $\tilde{a}$ is an e-number. Furthermore, such a is unique up to some R_κ and we obtain a graded map $\pi : \tilde{a} \mapsto a$.

Proof. We just give a quick sketch of a proof. It is possible to find ${}_ia \in_\kappa \bigcap_j a_{ij} * e_j^{-1}$ and $a_j \in_\kappa \bigcap_i e_i * a_{ij}$ for some big enough κ. Since $(e_i * a_{ij}) * e_j^{-1} =_R e_i * (a_{ij} * e_j^{-1})$, $a_j * e_j^{-1} \approx_\kappa e_i * {}_ia$ for every i, j (and some κ). Finally, by given independencies, all things in such intersections are R_κ-equivalent, for high enough κ, and that gives the sought for a; for example, observe $a_1 * e_1^{-1} \approx_\kappa e_0 * {}_0a \approx_\kappa a_2 * e_2^{-1}$, and let h be in the first, k in the second intersection. Thus, $h, k \in_\kappa e_0 * {}_0a$, and $e_{0R} \downarrow h_R k_R$ (since $e_{0R} \downarrow a_{31R} e_{1R} e_{2R} e_{3R}$). □

An analogous characterization of e-tuples is even easier, so we omit the proof.

Lemma 4.2.6. *Let $n \geq 2$. If $\tilde{x}$ is a tuple in X_0 such that*

(1) *for every $i < n$, ${}_ix_Q \downarrow e$;*
(2) *$\{{}_ix_Q : i < n\}$ is independent;*
(3) *for every $i \neq j < n$, ${}_ix \in_\xi e_{ij} * {}_jx$,*

(for some Q_ξ), there is an $x \in X_0$, $x_Q \downarrow e$ witnessing that $\tilde{x}$ is an e-tuple. This x is unique up to Q and we obtain a graded map $\pi : \tilde{x} \mapsto x$.

The previous two lemmas show that the blowup of our original polyspace is actually a bounded covering of it.

Proposition 4.2.7. *Let $n \geq 4$. Let $\tilde{a}$ and $\tilde{b}$ be two e-numbers (up to some R_ι), $a_R \downarrow_e b_R$. There is a unique e-number $\tilde{c}$ (up to some coarser R_κ depending only on R_ι) such that $c_{ij} \in_\kappa \bigcap_{k \neq i,j} a_{ik} * b_{kj}$. Furthermore, $\pi(\tilde{c}) \in_\kappa \pi(\tilde{a}) * \pi(\tilde{b})$.*

Proof. We give the proof for $n = 4$, which is the only case we need, and at some point, we choose particular values for indices to enhance readability.

Firstly, the product is unique, since if e.g. $c_{10}, c'_{10} \in_\kappa a_{12} * b_{20} \cap a_{13} * b_{30}$, then they are core-related because $a_{12R} \downarrow a_{13R} b_{30R}$.

For the existence, by associativity, it's easily seen that $\bigcap_{k \neq i,j} a_{ik} * b_{kj} \neq_\iota \emptyset$, so we can pick c_{ij} inside. We claim that this c is an e-number up to some R_κ. Since $(a_{01} * b_{12}) * e_{23} =_R a_{01} * (b_{12} * e_{23})$, $c_{02} * e_{23} \approx_\kappa a_{01} * b_{13} \ni c_{03}$ (for some κ). Take h in the intersection; now $h, c_{03} \in_\kappa a_{01} * b_{13}$ and $a_{01R} \downarrow h_R c_{03R}$, so c_{03} is core-related to h, i.e. they are equal with respect to some coarser R_κ.

To see the last independence, it is enough to check that $a_{01R} \mathop{\smile\!\!\!\!\!\mid}\ c_{02R}e_{23R}c_{03R}$, or $a_{01R} \mathop{\smile\!\!\!\!\!\mid}\ a_{02R}a_{03R}b_{23R}b_{32R}$. If we let $a =_R \pi(\tilde{a})$ and $b =_R \pi(\tilde{b})$ the above follows from $e_{1R} \mathop{\smile\!\!\!\!\!\mid}\ a_R b_R e_{0R} e_{2R} e_{3R}$.

For the 'furthermore' part, let $c =_\kappa \pi(\tilde{c})$ (for high enough R_κ), so $c_{01} \in_\kappa e_0^{-1} * c * e_1$ (some κ); let ${}_0c$ be as in the proof of 4.2.5 such that $c_{01} \in_\kappa {}_0c * e_1$. Since also $c_{01} \in_\kappa a_{02} * b_{21}$, we can find ${}_kb$ such that ${}_0c * e_1 \approx_\kappa a_{02} * {}_2b * e_1$, and this implies, since $e_{1R} \mathop{\smile\!\!\!\!\!\mid}\ {}_0c_R a_{02R} {}_2b_R$ (follows from $e_{1R} \mathop{\smile\!\!\!\!\!\mid}\ a_R b_R e_{0R} e_{2R} e_{3R}$), that ${}_0c \in_\kappa a_{02} * {}_2b$ (for some higher κ). But then $e_0^{-1} * c \approx_\kappa e_0^{-1} * a_2 * {}_2b$, and since $e_{iR} \mathop{\smile\!\!\!\!\!\mid}\ c_R a_{2R} {}_2b_R$ (follows from $e_{0R} \mathop{\smile\!\!\!\!\!\mid}\ a_R b_R e_{1R} e_{2R} e_{3R}$), $c \in_\kappa a_2 * {}_2b$, or $c * b^{-1} * e_2 \approx_\kappa a * e_2$, and since $e_2 \mathop{\smile\!\!\!\!\!\mid}\ a_R b_R c_R$ (follows from $e_{2R} \mathop{\smile\!\!\!\!\!\mid}\ a_R b_R e_{0R} e_{1R} e_{3R}$), $c \in_\kappa a * b$ (for some κ). □

Proposition 4.2.8. *If a is an e-number (up to R_ι), x an e-tuple (up to Q_ξ) and $a_R \mathop{\smile\!\!\!\!\!\mid}_e x_Q$, there is a unique e-tuple y (up to some Q_ζ depending only on R_ι and Q_ξ) such that $y_i \in_\zeta a_{ij} * x_j$.*

Proof. Suppose $y_0, y_0' \in_\zeta a_{01} * x_1 \cap a_{02} * x_2$. Since $a_{01R} \mathop{\smile\!\!\!\!\!\mid}\ a_{02R} x_{2Q}$, we have that $y_0 Q y_0'$ so we have uniqueness.

For the existence, since $(a_{01} * e_{12}) * x_2 =_Q a_{01} * (e_{12} * x_2)$, we get that $a_{02} * x_2 \approx_\zeta a_{01} * x_1$, (for a high enough Q_ζ) and take h in the intersection. Similarly, we can find $k \in_\zeta a_{03} * x_3 \cap a_{01} * x_1$. By all the independencies, $\{a_{03R}, x_{3Q}, e_{32R}, e_{31R}\}$ is independent, so $e_{31R} \mathop{\smile\!\!\!\!\!\mid}\ a_{03R} x_{3Q} e_{32R}$ and thus $a_{01R} \mathop{\smile\!\!\!\!\!\mid}\ a_{03R} x_{3Q} e_{32R}$, yielding $a_{01R} \mathop{\smile\!\!\!\!\!\mid}\ a_{02R} x_{2Q} a_{03R} x_{3Q}$ and therefore $h Q_\zeta k$ for course enough Q_ζ and we can define y_0. Similarly for the remaining y_i's. □

The reader should recall the definition of type 3 morphisms between polygroups from 1.4.12 before proceeding to the next result.

Theorem 4.2.9. *Let $(S, X) = (S_0/R, X_0/Q)$ be a gradedly almost hyperdefinable polyspace chunk with a fixed n-frame e for $n \geq 4$. The set of e-numbers $\tilde{S}_e$ and the set of e-tuples $\tilde{X}_e$ with the multiplication of numbers and action of numbers on tuples is a gradedly almost hyperdefinable space chunk (over e) and the map $\pi : (\tilde{S}_e, \tilde{X}_e) \to (S_e, X_e)$ is a gradedly almost hyperdefinable generic epimorphism of type 3 with bounded fibres.*

Proof. We will show just the group chunk part, since checking the properties of the action is completely analogous. Let R_0 and R_1 be such that e-numbers exist up to R_0, and the product of two numbers up to R_0 is a number up to R_1. From the construction of numbers, it is clear that there is a fixed R_2 such that for every number a up to R_1, there is a number $a' R_2 a$ which is a number up to R_0. Let S_1 be the type-definable set of e-numbers up to R_1. Let $\cdot$ be the type-definable relation on S_1^3 given by: for $a, b \in S_1$, $c \in a \cdot b$ if there are $a' R_2 a$, $b' R_2 b$ with c in the product of a and b as in 4.2.7 (this is clearly nonempty for $a_R \mathop{\smile\!\!\!\!\!\mid}_e b_R$ by the choice of R_0, R_1 and R_2, and all the c's in the product of a and b are R_3-related for some R_3). The gradedness of the induced map is clear.

For generic independence, let $a \mathop{\smile\!\!\!\!\!\mid}_e b$ and $c \in a \cdot b$. In particular, $a_{ij} \mathop{\smile\!\!\!\!\!\mid}\ be$, and since $c_{ik} \in_R a_{ij} * b_{jk}$, by generic independence of the original polygroup chunk, $c_{ikR} \mathop{\smile\!\!\!\!\!\mid}\ be$, so $c_{ikR} \mathop{\smile\!\!\!\!\!\mid}_e b$, and as $c_R \in \mathrm{bdd}(c_{ikR} e)$, $c_R \mathop{\smile\!\!\!\!\!\mid}_e b$. Similarly we get $c_R \mathop{\smile\!\!\!\!\!\mid}_e a$.

For generic associativity, take $\{a_R, b_R, c_R\}$ e-independent e-numbers, let $x_R = a_R \cdot b_R$, $z_R = b_R \cdot c_R$, $u_R = x_R \cdot c_R$ and $v_R = a_R \cdot z_R$. We need to show $u_R = v_R$.

Let us start with e.g. $(a_{10} * b_{02}) * c_{23} =_R a_{10} * (b_{02} * c_{23})$; it implies that $x_{12} * c_{23} \approx_i a_{10} * z_{03}$ (for some R_i), so take h in both. Now, we have $h, v_{13} \in_i a_{10} * z_{03}$ and we would like to show that h and v_{13} are core-related. This will hold if $a_{10R} \mathop{\smile\!\!\!\!\!\mid}\ h_R v_{13R}$, but since $v_{13} \in_i a_{12} * z_{23}$ as well, it is enough to show $a_{10R} \mathop{\smile\!\!\!\!\!\mid}\ a_{13R} b_{32R} c_{23R} a_{12R} b_{21R} c_{13R}$, which follows from $a_{10R} \mathop{\smile\!\!\!\!\!\mid}\ a_{12R} b_{21R} c_{13R} e_{12R} e_{23R}$. As $\{a_{12R}, b_{21R}, c_{13R}, e_R\}$ is independent, and $\{e_{20R}, e_{12R}, e_{23R}\}$ is independent, we have $e_{20R} \downarrow a_{12R}b_{21R}c_{13R}e_{12R}e_{23R}$, which clearly implies what is required. In a similar way, we can get that u_{13} and h are core-equivalent and thus also u_{13} and v_{13}.

Generic invertibility follows directly from uniqueness and the corresponding property of S, and the fact that π is a generic homomorphism is shown above.

To see that π is generically of type 3, it is enough to show that if $c \in_R a * b$, for some $a_R \downarrow_e b_R \in S_e$ and $\pi(\tilde{c}) =_R c$, we can find $\tilde{a}$ and $\tilde{b}$ with $\pi(\tilde{a}) =_R a$, $\pi(\tilde{b}) =_R b$ and $\tilde{c} =_R \tilde{a} \cdot \tilde{b}$. We will work modulo R, and it is just a technicality to show that everything can be done gradedly. Thus, let $\tilde{c}$ with $\pi(\tilde{c}) = c$ and $c \in_R a * b$ be given. Reconstruct c_i and ${}_ic$ as in 4.2.5 such that ${}_ic \in_R e_i^{-1} * c$, $c_i \in_R c * e_i$ and $c_{ij} \in_R {}_ic * e_j \cap e_i^{-1} * c_j$ (this can be done uniquely). By the second form of associativity applied to $e_i^{-1} * c * b^{-1}$, find ${}_ia \in_R e_i^{-1} * a \cap {}_ic * b^{-1}$. We claim that ${}_ia \in_R e_{ij} * {}_ja$. Again by the second form of associativity applied to $e_i^{-1} * e_j * {}_ja$, we get ${}_ia \in_R e_i^{-1} * a \approx_R e_{ij} * {}_ja$, and let h be in the intersection. To show that ${}_ia$ and h are core-related, it is enough to check that $e_{iR} \downarrow {}_ia_{Rj}a_R e_{ijR}$, or, since ${}_ia \in_R {}_ic * b^{-1}$, that $e_{iR} \downarrow {}_ic_{Rj}c_R b_R e_{ijR}$. In fact, as ${}_jc \in_R e_{ji} * {}_ic$, it is enough to see that $e_{iR} \downarrow {}_ic_R b_R e_{ijR}$, but that follows from independence of $\{b_R, c_R, e_{iR}, e_{jR}\}$. Similarly (by observing $a^{-1} * c * e_j$) we can choose $b_j \in b * e_j \cap a^{-1} * c_j$. Also, if we choose any $a_0 \in a * e_0$, there will be unique (up to R) ${}_0b \in e_0^{-1} * b$ and $a_i \in a * e_i$, ${}_ib \in e_i^{-1} * b$ such that $c \in_R a_i * {}_ib$ and $a_i \in_R a_j * e_{ji}$ and ${}_ib \in_R e_{ij} * {}_jb$. From this data, as in 4.2.4, we can get a_{ij} and b_{ij} such that they form e-numbers $\tilde{a}$ and $\tilde{b}$. To check that $\tilde{c} =_R \tilde{a} * \tilde{b}$, let us first show that $c_{ij} \in_R {}_ia * b_j$. By associativity, from $e_i^{-1} * a * b_j$ and the construction above, where $c_j \in_R a * b_j$, we get that ${}_ia * b_j \approx e_i^{-1} * c_j \ni c_{ij}$, so pick h in the intersection. We claim that $c_{ij}Rh$. From independence of $\{a_R, b_R, e_{iR}, e_{jR}\}$, we have that $e_{iR} \downarrow a_R b_R c_R e_{jR}$, so $e_{iR} \downarrow a_R c_{jR} b_{jR}$, and it is easily seen that $a_R \downarrow c_{jR} b_{jR}$ so $a_R \downarrow e_{iR} c_{jR} b_{jR}$, $a_R \downarrow c_{ijR} b_{jR} e_{iR}$ and $a_{iR} \downarrow c_{ijR} b_{jR} e_{iR}$. But, as $e_{iR} \downarrow c_{ijR} b_{jR}$, we have that $e_{iR} \downarrow c_{ijR} {}_ia_R b_{jR}$, implying $e_{iR} \downarrow c_{ijR} h_R$ so $c_{ij}Rh$ by corelessness. The fact that $c_{ij} \in_R a_{ik} * b_{kj}$ is shown in a similar way using the independence of $\{a_R, b_R, e_{iR}, e_{jR}, e_{kR}\}$. □

Remark 4.2.10. When dealing with the coreless polygroups and polygroup chunks, we did not need to discuss the generalised associativity properties (unlike in the hyperdefinable case in the previous section) as they are trivially satisfied in this context.

4.3. Universal blowup

It is much easier to develop the mini-blowup described in Section 4.1 modulo R and get the group chunk of similar characteristics as in the previous section. Let us shortly explain how it should go at the level of ultraimaginaries (the full details for the gradedly almost hyperdefinable version are given in [BTW]), without worrying about the graded almost hyperdefinability issues, since our aim in this section is much higher, to construct a blowup with the universal property.

Let $(P, *)$ be the coreless polygroup chunk, and pick an $e \in P$. Let $P_e := \{a \in P : a \downarrow e\}$. The group chunk $(\tilde{P}_e, *)$ consists of tuples $\tilde{a} = \langle a, {}_ea, a_e \rangle$, where $a \in P_e$, ${}_ea \in e^{-1} * a$, $a_e \in a * e$, and the operation $*$ is defined as follows: if $\tilde{a} \downarrow_e \tilde{b}$ (which implies $\{a, b, e\}$ is independent), let $c \in a * b \cap a_e * {}_eb$, ${}_ec \in e^{-1} * c \cap {}_ea * b$, $c_e \in c * e \cap a * b_e$. It is easy to check that these values are unique (up to the core relation) and we say that $\tilde{c} = \tilde{a} * \tilde{b}$. The map $\pi : \tilde{P}_e \to P_e$ given by $\langle a, {}_ea, a_e \rangle \mapsto a$ is a bounded covering (a generic epimorphism of type 3 with bounded fibres).

Let e_i, $i < \lambda$ be a long Morley sequence and let e_{ij} form a frame (as in the previous section), i.e. $e_{ij} \in e_i^{-1} * e_j$ and the usual $e_{ik} \in e_{ij} * e_{jk}$ and $e_{ji} = e_{ij}^{-1}$.

Let P_i denote $\{a \in P : a \perp e_i\}$. Then obviously $P = \bigcup_i P_i$. Observe mini-blowups $\tilde{P}_i$ and let $P^\dagger = \bigcup_i \tilde{P}_i$. The projection π can clearly be extended to the whole of $P^\dagger$. We have transition maps f_{ij} between some parts of $\tilde{P}_i$ and $\tilde{P}_j$: if $\tilde{a} = \langle a, {}_i a, a_i \rangle \in \tilde{P}_i$, and $\tilde{a} \perp e_j$ (which amounts to $a \perp e_i e_j$), then we can uniquely find $a_j \in a * e_j \cap a_i * e_{ij}$ (the intersection is nonempty by the second form of associativity applied to $a * e_i * e_{ij}$) and similarly ${}_j a \in e_j^{-1} * a \cap e_{ji} * {}_i a$, so let $f_{ij} : (a, {}_i a, a_i) \mapsto (a, {}_j a, a_j)$.

Lemma 4.3.1. *Transition maps are partial generic homomorphisms:*

(1) *If $\tilde{a} \in \tilde{P}_i$ is independent of $e_j e_k$, then $f_{jk}(f_{ij}(\tilde{a})) = f_{ik}(\tilde{a})$.*
(2) *For $e_j \perp \tilde{a} \in \tilde{P}_i$, $f_{ji}(f_{ij}(\tilde{a})) = \tilde{a}$.*
(3) *For $\tilde{a} \perp_{e_i} \tilde{b} \in \tilde{P}_i$, if $e_k \perp \tilde{a}\tilde{b}$, then $f_{ik}(\tilde{a} * \tilde{b}) = f_{ik}(\tilde{a}) * f_{ik}(\tilde{b})$.*

Proof.
(1) For example, let $a_j = a * e_j \cap a_i * e_{ij}$ and $a_k = a * e_k \cap a_j * e_{jk}$, and, on the other hand, $a'_k = a * e_k \cap a_i * e_{ik}$. Since $a \perp e_i e_j e_k$, we have that $a_k \perp e_{ij} e_{ik} e_k$, but, as $e_k \perp e_{ij} e_{ik}$, $a_k \perp a_k e_{ij} e_{ik}$. Furthermore, as $e_{jk} \in e_{ij}^{-1} * e_{ik}$, $a_j \in a_k * e_{kj}$, $a_i \in a_j * e_{ij}$, we get that $e_k \perp a_i e_{ik} a_j e_{jk}$ and thus $e_k \perp a'_k a_k$, so we can conclude that a_k and a'_k are core-related.
(2) is easy.
(3) The assumption that $\tilde{a} \perp_{e_i} \tilde{b}$ and $e_k \perp \tilde{a}\tilde{b}$ amounts to the independence of $\{a, b, e_i, e_k\}$. Let $\tilde{c} = \tilde{a} * \tilde{b}$; so in particular $c_i \in c * e_i \cap a * b_i$. Suppose c'_k is the third coordinate of $f_{ik}(\tilde{c})$; $c'_k \in c * e_k \cap c_i * e_{ik}$. On the other hand, $b_k \in b * e_k \cap b_i * e_{ik}$, and the third coordinate of $f_{ik}(\tilde{a}) * f_{ik}(\tilde{b})$ is $c_k \in c * e_k \cap a * b_k$. As both $c_k, c'_k \in a * e_k$, if we can show that $e_k \perp c_i e_{ik} a b_k$, we would have $c_k = c'_k$ by corelessness. It clearly suffices to show, as $c_i \in a * b_i$ that $e_k \perp a b_i e_{ik} a b_k$, but since $b_k \in b_i * e_{ik}$, it is enough to check $e_k \perp a b_i e_{ik}$, which easily follows from the independence of $\{a, b, e_i, e_k\}$ and generic independence. □

It is much clearer now what we need to do in order to get a generic operation:

Definition 4.3.2. We shall say that $\tilde{a} \in \tilde{P}_i$ is $\sim$-related to $\tilde{a}' \in \tilde{P}_j$ if there is e_k, $\tilde{a} \perp e_k$, $\tilde{a}' \perp e_k$ such that $f_{ik}(\tilde{a}) = f_{jk}(\tilde{a}')$.

Lemma 4.3.3. *We have $\tilde{a} \sim \tilde{a}'$, for $\tilde{a} \in \tilde{P}_i$ and $\tilde{a}' \in \tilde{P}_j$ if and only if for every e_l with $\tilde{a} \perp e_l$, $\tilde{a}' \perp e_l$, $f_{il}(\tilde{a}) = f_{jl}(\tilde{a}')$.*

Proof. Let e_k witness that $\tilde{a} \sim \tilde{a}'$, and consider any $e_l \perp \tilde{a}$, $e_l \perp \tilde{a}'$.

If we take any e_m with $e_m \perp e_k \tilde{a}$ and $e_m \perp e_k \tilde{a}'$, in particular we have that $e_m \perp e_k e_i a$ and $e_m \perp e_k e_j a$, and thus, since $a \perp e_i e_k$ and $a \perp e_j e_k$, we have $a \perp e_i e_k e_m$ and $a \perp e_j e_k e_m$, so, by 4.3.1, $f_{km}(f_{ik}(\tilde{a})) = f_{km}(f_{jk}(\tilde{a}))$ implies that $f_{im}(\tilde{a}) = f_{jm}(\tilde{a})$.

Now, we can choose $e_m \perp e_k e_l \tilde{a}\tilde{a}'$. By the previous paragraph, we get $f_{im}(\tilde{a}) = f_{jm}(\tilde{a}')$. Furthermore, $e_l \perp e_m \tilde{a}$ and $e_l \perp e_m \tilde{a}'$, so again by the previous paragraph, $f_{il}(\tilde{a}) = f_{jl}(\tilde{a}')$. □

Corollary 4.3.4. *The relation $\sim$ is an equivalence relation.*

Let us denote the $\sim$-class of $\tilde{a}$ by $[\tilde{a}]$.

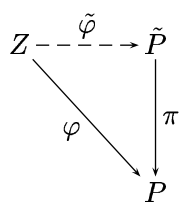
Remark 4.3.5. Even though we said at the beginning of this section that we shall not worry about the almost hyperdefinability of our objects, we feel obliged to provide the reader with a hint how to show graded almost hyperdefinability of $\sim$. We claim that $\tilde{P}_i \ni \tilde{a} \sim \tilde{a}' \in \tilde{P}_j$ if and only if there is $e^\dagger \perp \tilde{a}\tilde{a}'$ such that $f_{i\dagger}(\tilde{a}) = f_{j\dagger}(\tilde{a}')$ (where, of course, the fact that $\{e_i, e_j, e^\dagger\}$ is independent implies that it can be completed to a frame e', provided we choose $e'_{ij} := e_{ij}$ and any $e'_{i\dagger} \in e_i^{-1} * e^\dagger$; this gives a meaning to $f_{i\dagger}$ and $f_{j\dagger}$). The latter relation is clearly a good candidate for becoming almost hyperdefinable in view of our considerations around the core relation.

Suppose that $e^\dagger$ is as above, and find k_0 such that $\{e_i, e_j, e^\dagger, e_k : k \geq k_0\}$ is independent. After choosing $e''_{ij} := e_{ij}$, $e''_{ik} := e_{ik}$ for $k \geq k_0$ and $e''_{i\dagger} := e'_{i\dagger}$, we can complete this to a frame e''. However, since $e''_{kl} \in e''_{ki} * e''_{il} \cap e_k^{-1} * e_l = e_{ki} * e_{il} \cap e_k^{-1} * e_l \ni e_{kl}$, by all the independencies, $e''_{kl} = e_{kl}$ (core related) for $k, l \geq k_0$ or $k \geq k_0$, $l = j$. Then, since $e^\dagger$ is in e'', by the previous lemma, $f''_{ik}(\tilde{a}) = f''_{jk}(\tilde{a}')$ for some $k \geq k_0$, but as $f''_{ik} = f_{ik}$ and $f''_{jk} = f_{jk}$, $\tilde{a} \sim \tilde{a}'$ with respect to the original frame e. The converse is trivial by 4.3.3.

Definition 4.3.6. Let $\tilde{P} := P^\dagger/{\sim}$, and let $[\tilde{a}] \perp [\tilde{b}] \in \tilde{P}$. Without loss of generality, $\tilde{P}_i \ni \tilde{a} \perp \tilde{b} \in \tilde{P}_j$. We define $[\tilde{a}]*[\tilde{b}] := [f_{ij}(\tilde{a})*\tilde{b}](= [\tilde{a}*f_{ji}(\tilde{b})])$. Also, let $\pi([\tilde{a}]) := \pi(\tilde{a})$. We call $\tilde{P}$ the *universal* blowup.

Theorem 4.3.7 (Universal blowup). *Let $\tilde{P} \xrightarrow{\pi} P$ be the universal blowup of a coreless polygroup chunk P with respect to some frame $e = \langle e_i, e_{ij}\rangle$. Then, $(\tilde{P}, *)$ is a group chunk and π is a (generic) epimorphism of type 3 with bounded fibres.*

Furthermore, for every group chunk Z with a frame f and every epimorphism $\varphi : Z \to P$ such that $\varphi(f) = e$ (i.e. $\varphi(z * t) \in \varphi(z) * \varphi(t)$ for $z \perp t$ and $\varphi(z^{-1}) = \varphi(z)^{-1}$), there is a unique $\tilde{\varphi} : Z \to \tilde{P}$ such that $\varphi = \pi \circ \tilde{\varphi}$.



Proof. The multiplication is well-defined by 4.3.3 and 4.3.1: let us say we have $\tilde{P}_i \ni \tilde{a} \perp \tilde{b} \in \tilde{P}_j$, and let $\tilde{a} \sim \tilde{a}' \in \tilde{P}_k$, witnessed by some $e_l \perp \tilde{a}\tilde{a}'\tilde{b}$, i.e. $f_{il}(\tilde{a}) = f_{kl}(\tilde{a}')$, and suppose $\tilde{a}' \perp \tilde{b}$. Let $\tilde{c} := f_{ij}(\tilde{a}) * \tilde{b}$, and let $\tilde{c}' := f_{kj}(\tilde{a}') * \tilde{b}$. Now, applying f_{jl} to the last equation, we get that

$$f_{jl}(\tilde{c}') = f_{jl}(f_{kj}(\tilde{a}')) * f_{jl}(\tilde{b}) = f_{kl}(\tilde{a}') * f_{jl}(\tilde{b}) = f_{il}(\tilde{a}) * f_{jl}(\tilde{b}).$$

Applying f_{lj} to both sides yields $\tilde{c}' = f_{ij}(\tilde{a}) * \tilde{b} = \tilde{c}$.

For generic independence, let $\tilde{P}_i \ni \tilde{a} \perp \tilde{b} \in \tilde{P}_j$ and let $\tilde{c} := f_{ij}(\tilde{a}) * \tilde{b}$. We have that $ae_i \perp be_j$, so $b \perp ae_ie_j$ and by generic independence of P, since $c \in a * b$, $c \perp ae_ie_j$, so $ce_j \perp ae_i$ and $\tilde{c} \perp \tilde{a}$. To see that $[\tilde{a}] \cdot [\tilde{b}] \perp \tilde{b}$, we can check in a similar way as above that $\tilde{a} * f_{ji}(\tilde{b}) \perp \tilde{b}$. Associativity, invertibility and the fact that π is of type 3 follow from the corresponding properties on each $\tilde{P}_i$.

For the 'furthermore' part, let $z \in Z$ and let $z_i := z * f_i$, ${}_iz := f_i^{-1} * z$ and put $\tilde{\varphi}(z) := [\langle \varphi(z), \varphi({}_iz), \varphi(z_i)\rangle]$ for big enough i. It is clear that $\tilde{\varphi} : Z \to \tilde{P}$ since $\varphi(z_i) = \varphi(z * f_i) \in \varphi(z) * e_i$ and similarly $\varphi({}_iz) \in e_i^{-1} * \varphi(z)$. □

Remark 4.3.8. In the statement of the theorem, if we require that φ be a generic epimorphism of type 3, we can omit the assumption that there is a frame f in Z with $\varphi(f) = e$ because e can be pulled back via φ to give such a frame.

Proof. Let the frame $e = \langle e_i, e_{ij}\rangle$ be given and pick an arbitrary f_0 such that $\varphi(f_0) = e_0$. Since for every i, $e_0 \in e_i * e_{i0}$, using the fact that φ is generically of type 3, we can find f_i and f_{i0} such that $f_0 = f_i * f_{i0}$ and $\varphi(f_i) = e_i$, $\varphi(f_{i0}) = e_{i0}$. As Z is the group chunk, this extends uniquely to a frame by $f_{ji} = f_{ij}^{-1}$ and $f_{ij} = f_{i0} * f_{0j}$. □

CHAPTER 5

Applications and geometric simplicity

The aim of this chapter is to demonstrate how the group configuration theorem obtained through the previous chapters gives rise to 'geometric simplicity theory', i.e. the set of methods commonly referred to as geometric stability can be partially extended to simple theories.

The fact that the group configuration at the moment produces an almost hyperdefinable and not hyperdefinable groups creates some difficulties, and our hope is that they are not significant. Much more serious obstacles come from the lack of knowledge on groups in simple theories; for example, whether an SU-rank 1 group hyperdefinable in a simple theory has a 'large' abelian part.

In Section 5.1, we solve what might be considered a fundamental problem in the classical theory of hypergroups (see the extensive literature in [Co]) for the class of almost hyperdefinable polygroups in simple theories: given a polygroup of that class, we find a group in the same class which is 'closely' related to it. We were quite surprised that the classical literature lacks the ability to (nontrivially) answer such questions and our hope is that this result will initiate some development. This problem, posed by myself, was first approached by Wagner and myself using stabilisers and the idea of using the universal property of the group chunk theorem is by Ben-Yaacov.

In Section 5.2 we finally decide to recover the space corresponding to the original partial generic multiaction and not just the group, and use it in Section 5.5 to prove that pseudolinearity implies one-basedness in an ω-categorical simple theory. The action was originally defined by myself in a similar fashion for the case of a polygroup, but there I was lacking the generic faithfulness. The present state of these two sections comes from collaboration with Wagner.

In Section 5.3 we show that at least in an ω-categorical theory, our group configuration theorem gives an interpretable group. It is due to myself.

Section 5.4 shows that an (ω-categorical) one-based, SU-rank 1 structure interprets a vector space over a (finite field) division ring as a stable reduct, thus partially proving the stable forking conjecture mentioned in Section 1.2. Also, we argue that this is the first (two) case(s) of a Zil'ber-type trichotomy for simple theories. I presented a proof of this in [To], but the proof had an omission which was later patched up with help of Wagner. Independently, the same result was obtained by de Piro and Kim in [dPK] by an ad hoc method and partially by Vassiliev in [Va] using generic pairs.

5.1. Getting a group from a polygroup

Definition 5.1.1. Two polygroups P_1 and P_2 are said to be *polyisogenous*, if there is a subpolygroup $H < P_1 \times P_2$ such that the first projection of H has a bounded index in P_1 and the second projection of H has a bounded index in P_2, and kernels of both projections are bounded.

Intuitively, H induces a bounded-to-bounded correspondence (almost a homomorphism) between big parts of P_1 and P_2.

Remark 5.1.2. In a simple theory, with every gradedly almost hyperdefinable polygroup we can associate a gradely almost hyperdefinable group which is polyisogenous to it.

Let P be the polygroup in question, coreless without loss of generality, and observe its generic part X, which is a polygroup chunk by 3.3.15. We can blow it up, obtain a group chunk and use it to generate a group G. In particular, there will be a generic element $g' \in G$ interbounded with a generic element $g \in P$ (over an independent parameter which we suppress for the moment).

Look at $H := \mathrm{stab}(gg') < P \times G$. Since $\mathrm{stab}(g) \times \mathrm{stab}(g') < H$, it is clear (by 3.6.5 and 3.6.6) that the first projection of H is of bounded index in P, and the second is of bounded index in G and the fact that the kernels of both projections are bounded should follow from the fact that g and g' are interbounded, so H will realise a polyisogeny between P and G.

Unfortunately, the polyisogeny constructed above need not be gradedly almost hyperdefinable, but we do not attempt to refine the result since a much more explicit correspondence can be obtained using the universal property of the group chunk as follows.

Theorem 5.1.3. *Let P be a coreless gradedly almost hyperdefinable polygroup in a simple theory. There is a gradedly almost hyperdefinable group G which is a bounded covering of P, i.e. with an epimorphism $\pi : G \to P$ of type 3 with bounded fibres.*

Proof. We show how to get the group using e.g. the blowup from Section 4.2 (it would be more 'universal' to use the blowup from 4.3, but then the group would be 'manifold-like'); let X be the generic part of P and let $\pi : \tilde{X}_e \to X_e$ be a blowup with respect to some frame e. By the universal property from 3.5.1 and 3.5.2, if G_e is the group generated by $\tilde{X}_e$, π lifts to $\pi^2 : G_e \to P$ of type 3.

Let us show that $\pi^2 : G_e \to P$ is surjective (we work at the level of ultraimaginaries). Let $a \in P$ and let $g \mathop{\smile\!\!\!\!\!\mid} ae$, $g' \mathop{\smile\!\!\!\!\!\mid} age$ be generics of P (in particular, $g, g' \in X_e$). Let $h \in g^{-1} * a$ (so $h \mathop{\smile\!\!\!\!\!\mid} ae$) and $h' \in g'^{-1} * a$ (so $h' \mathop{\smile\!\!\!\!\!\mid} aghe$). Clearly $h, h' \in X_e$, too. Then, $a \in g * h \cap g' * h'$, so by the transposition property (associativity), there is $d \in g^{-1} * g' \cap h * h'^{-1}$, and $d \in X_e$. By surjectivity of π, there is $\tilde{d} \in \tilde{X}_e$ with $\pi(\tilde{d}) = d$, and, since π is generically of type 3, we can find $\tilde{g}, \tilde{g}', \tilde{h}, \tilde{h}' \in \tilde{X}_e$ with $\pi(\tilde{g}) = g$, $\pi(\tilde{g}') = g'$, $\pi(\tilde{h}) = h$, $\pi(\tilde{h}') = h'$ and $\tilde{d} = \tilde{g}^{-1} \cdot \tilde{g}' = \tilde{h} \cdot \tilde{h}'^{-1}$. Let $\tilde{a} := \tilde{g} \cdot \tilde{h} = \tilde{g}' \cdot \tilde{h}'$. Then, $\pi^2(\tilde{a}) \in \pi^2(\tilde{g}) * \pi^2(\tilde{h}) \cap \pi^2(\tilde{g}') * \pi^2(\tilde{h}') = g * h \cap g' * h'$, and therefore must be core-equivalent to a.

For boundedness of fibres, let $\tilde{a} \in G_e$, and pick a generic $\tilde{g} \mathop{\smile\!\!\!\!\!\mid}_e \tilde{a}$. Let $\pi^2(\tilde{a}) = a$; if $\langle \tilde{a}_i : i < \kappa \rangle$ is an unbounded sequence of ae-conjugates of $\tilde{a}$, we may refine it and get an $a\tilde{g}e$-indiscernible sequence $\langle \tilde{a}_i : i < \omega \rangle$ such that $\pi^2(\tilde{a}_i) = a$ (when we speak about indiscernible sequences of ultraimaginaries over ultraimaginaries, it is understood that it holds for some representatives). Then, writing $\tilde{g}'_i := \tilde{g} * \tilde{a}_i$, we have that $\langle \tilde{g}'_i : i < \omega \rangle$ is $a\tilde{g}e$-indiscernible, but since $a \in \pi(\tilde{g})^{-1} * \pi(\tilde{g}'_0)$, and $\tilde{g}$ is bounded over $\pi(\tilde{g})$, we get that $\tilde{g}'_0$ is bounded over $a\tilde{g}e$ and the sequence must be constant. □

Remark 5.1.4. (Added in proof). In the above, since π is onto of type 3, we get a (weak form) of a structure theorem for gradedly almost hyperdefinable coreless polygroups in simple theories: $P \cong G \,/\!/\, \pi$, see [Cr] for further analysis of quotients of polygroups by homomorphisms of type 3. By analysing the structure of the blowup further, Ben-Yaacov in [BY1] obtains much more: $P \cong G \,/\!/\, H$ for some $H < G$.

5.2. Reconstructing the action

Thus far, we have always constructed a group starting with a good enough partial generic multiaction $\pi(x, y, z)$, where $\pi(f, a, b)$ intuitively means that f acts on a's and b's. However, we have 'forgotten' about reconstructing this original action, even though we have proved the space chunk theorem and have formulated some blowup constructions for polyspace chunks. There is a good reason for postponing the discussion of the action part until now: it is hard to do at the level of the polygroup chunk (we cannot control ranks of objects involved, and cannot obtain the action generically faithful), so we choose to do it after the blowup procedure has been done.

Let us start with a hyperdefinable polygroup chunk obtained in 4.1, i.e. (after adding the parameters to the language), we will be working with $(P_0, *)$, $P_0 = \mathrm{Germ}(\pi)$, such that $y, y' \in f(x)$, for $f \in P_0$, $x \in \mathrm{Arg}(\pi)$, $f \downarrow x$, we have that $\mathrm{bdd}(y) = \mathrm{bdd}(y')$, and if $h, h' \in f * g$, $f \downarrow g \in P_0$, $\mathrm{bdd}(h) = \mathrm{bdd}(h')$. Thus, P_0 is a group chunk modulo the core relation denoted by R in this section.

Definition 5.2.1. Let $X_0 := \{(g,x) : g \in P_0, x \in \mathrm{Arg}(\pi), g \downarrow x\}$. We shall write $(g,x) \sim_i (g',x')$ if there are $h, h' \in P_0$, each independent of $gxg'x'$, with $h(x) \approx h'(x')$ and $g * h^{-1} R_i g' * h'^{-1}$. It is clear that every $\sim_i$ is symmetric, and for every $g \downarrow x$ we can find $h \downarrow gx$ such that $x \in \mathrm{dom}(h)$, so $\sim_i$ is reflexive as well. Let $\sim$ be the direct limit of $\sim_i$'s.

Lemma 5.2.2. *The relation $\sim$ is a gradedly almost hyperdefinable equivalence relation.*

Proof. To see almost hyperdefinability, let R_0 witness the almost-hyperdefinability of R, and let $g/R \subseteq \bigcup_{\alpha<\mu} g_\alpha/R_0$. Suppose $(g',x') \sim_i (g,x)$; there will be $h \downarrow gxg'x'$ and $h' \downarrow gxg'x'$ with $h(x) \approx h'(x')$ and $g * h^{-1} R_i g' * h'^{-1}$. We may assume that $hh' \downarrow_{gxg'x'} \bar{g}$. Let $g'' R_j g$ such that $g'' * h^{-1} R_1 g' * h'^{-1}$ for some fixed R_1, and let $g_\alpha R_0 g''$. Then $g_\alpha * h^{-1} R_2 g' * h'^{-1}$ and h and h' are independent of $g_\alpha x g' x'$, so $(g_\alpha, x) \sim_2 (g', x')$ and therefore $\sim_2$ witnesses the almost hyperdefinability of $\sim$.

For graded transitivity, if $(g,x) \sim_i (g',x')$ and $(g,x) \sim_j (g'',x'')$, let $h \downarrow gxg'x'$, $h' \downarrow gxg'x'$ with $y \in h(x) \cap h'(x')$ and $g * h^{-1} R_i g' * h'^{-1}$, and $h_1 \downarrow gxg''x''$, $h_2 \downarrow gxg''x''$ with $y' \in h_1(x) \cap h_2(x'')$ and $g * h_1^{-1} R_j g'' * h_2^{-1}$. We may assume $yhh' \downarrow_{gxg'x'} g''x''$ and $y'h_1h_2 \downarrow_{gxg''x''} g'x'hh'y$. Since $\{x, h, h_1\}$ is independent, $y'xy \models h^\dagger := \mathrm{Cb}(y'y/hh_1) \in h * h_1^{-1}$. Also, $x''y'y \models h'' := \mathrm{Cb}(x''y/h^\dagger h_2) \in h^\dagger * h_2$, since $y' \downarrow h^\dagger h_2$ (from above, we can get $h \downarrow y'h_1h_2$, so by generic independence and $h^\dagger \in h * h_1^{-1}$, $h^\dagger \downarrow y'h_2$). Then, $y \in h'(x') \cap h''(x'')$ and

$$\begin{aligned} g' * h'^{-1} &=_i g * h^{-1} =_1 g * (h_1^{-1} * h^{\dagger -1}) =_0 (g * h_1^{-1}) * h^{\dagger -1} \\ &=_{j'} (g'' * h_2^{-1}) * h^{\dagger -1} =_0 g'' * (h_2^{-1} * h^{\dagger -1}) =_1 g'' * h''^{-1}, \end{aligned}$$

for some constant 1 and j' depending only on j, so $(g',x') \sim_k (g'',x'')$ for some k depending only on i and j. □

We denote the $\sim$-class of $(g,x) \in X_0$ by $[g,x]$.

Lemma 5.2.3. *The hyperdefinable relation, given by $(k,x) \in f * (g,x)$ if and only if $f \downarrow gx$ and $k \in f * g$, induces a gradedly almost hyperdefinable generic action $* : P_0/R \otimes X_0/\sim \to X_0/\sim$.*

Proof. It is enough to check that if e.g. $(g,x) \sim_i (g',x')$ and $f \downarrow gx$, $f \downarrow g'x'$, then there is j such that for every $k \in f * g$ and $k' \in f * g'$, $(k,x) \sim_j (k',x')$. Let $h \downarrow gxg'x'$ and $h' \downarrow gxg'x'$ with $h(x) \approx h'(x')$ and $g * h^{-1} R_i g' * h'^{-1}$ and let $k \in f * g$, $k' \in f * g'$. We may assume that $hh' \downarrow_{gxg'x'} f$, which implies that $h \downarrow kxk'x'$ and $h' \downarrow kxk'x'$ and $f \downarrow gh$, $f \downarrow g'h'$. Thus, by regularity of R, $f * (g * h^{-1}) R_{i'} f * (g' * h'^{-1})$ for some i' and $k * h^{-1} R_j k' * h'^{-1}$ for some higher j, showing what was required. □

Lemma 5.2.4. *If $xy \models f$, $x'y \models f'$, then $[f,x] = [f',x']$.*

Proof. Find $g \mathop{\smile\!\!\!\!\!\!\;\;|}\, ff'xx'y$ such that $yt \models g$ for some t. Then, $xyt \models h := \mathrm{Cb}(xt/fg) \in g{*}f$ and $x'yt \models h' := \mathrm{Cb}(x't/f'g) \in g{*}f'$. Thus, $t \in h(x) \cap h'(x')$ and $f{*}h^{-1}Rg^{-1}Rf'{*}h'^{-1}$, witnessing $(f,x) \sim (f',x')$. □

Proposition 5.2.5. $(P, X) := (P_0/R, X_0/{\sim}, *)$ *is a gradedly almost hyperdefinable space chunk where the action is generically faithful and generically transitive.*

Proof. Generic associativity and invertibility of the action follow easily. For generic faithfulness, let $[g,x] \downarrow f_R f'_R$ such that $f_R * [g,x] = f'_R * [g,x]$. We may assume $gx \downarrow ff'$ and let $k \in f * g$, $k' \in f' * g$ such that e.g. $[k,x] = [k',x]$. Let $h \downarrow kxk'$ and $h' \downarrow kxk'$ such that $h(x) \approx h'(x)$ and $k * h^{-1}Rk' * h'^{-1}$. Since $x \downarrow g$, it follows that $x \downarrow gff'$ and $x \downarrow kk'$, which, together with $h \downarrow kxk'$ implies $x \downarrow hkk'$, and, as $h \in \mathrm{bdd}(hkk')$, $x \downarrow hh'$ and therefore by $h(x) \approx h'(x)$ and reducedness, we conclude $h = h'$. Then, $k * h^{-1}Rk' * h^{-1}$ implies that kRk' and finally fRf'.

In order to check generic transitivity, let $x \downarrow z \models \mathrm{Arg}(\pi)$. We can find $xy \models a$ and $zy' \models b$ for some $a, b \in P$. Since $\mathrm{Arg}(\pi)$ is a Lascar strong type, $y \equiv^L y'$, $y \downarrow x$, $y' \downarrow z$ and $x \downarrow z$ so by the Independence Theorem we may assume (renaming a and b) that $y' = y \downarrow xz$ and $xy \models a$, $zy \models b$. By 5.2.4, it follows that $[a,x] = [b,z]$, which clearly implies generic transitivity. □

Lemma 5.2.6. *For any $g \downarrow x$, $[g,x]$ and x are interbounded over the independent parameter g.*

Proof. Claim: If $xy \models f$, then $[f,x] \in \mathrm{bdd}(y)$.
Let $f'x'$ realise the nonforking extension of $\mathrm{tp}(fx/y)$ to fxy. Thus, $f'x' \downarrow_y fx$ and $x'y \models f'$ so by 5.2.4, $[f,x] \downarrow_y [f,x]$ implying $[f,x] \in \mathrm{bdd}(y)$.
Claim: For any g and x, $[g,x] \downarrow g$.
For any $[g,x]$ it is possible to find $f \downarrow gx$ such that $xy \models f$ for some $y \in \mathrm{Arg}(\pi)$. Then, for $k \in f * g^{-1}$, $k \downarrow gx$ and $[f,x] \in k * [g,x]$. Notice that if $xy \models f$, then $[f,x] \downarrow f$ since $[f,x] \in \mathrm{bdd}(y)$. But this will again be true for any $[g,x]$: from $x \downarrow_f g$ we get that $[f,x] \downarrow fgk$ as $[f,x] \downarrow f$, and then $[g,x] \downarrow g$ as $[g,x] \downarrow k$.

Now we will be able to obtain $x \in \mathrm{bdd}([g,x],g)$: if $x' \equiv_{[g,x]g} x$, then $[g,x] = [g,x']$, so there are h and h', each independent from gxx' with $h(x) \approx h'(x')$ and $g * h^{-1}Rg * h'^{-1}$. Therefore hRh', implying that h and h' are interbounded and eventually $\mathrm{bdd}(x) = \mathrm{bdd}(x')$. In fact, xQx', the $(I \times \omega)$-graded relation Q being the direct limit of relations $Q_{i,n}$, where $Q_{i,n}$ is the n-th transitive closure of $Q_{i,1}$ and $xQ_{i,1}x'$ if there are hR_ih', $hh' \downarrow xx'$ with $h(x) \approx h'(x')$. So $x_Q \in \mathrm{dcl}([g,x],g)$ and thus $[g,x]$ and x are interbounded over the independent parameter g. □

Applying the space chunk theorem 3.5.3 yields:

Corollary 5.2.7. *Starting with a partial generic multiaction $\pi(x,y,z)$ as in 4.1, we can get a gradedly almost hyperdefinable space (G, X) where the action is faithful and transitive. Furthermore, over some independent parameters a generic element of G is interbounded with an element of* $\mathrm{Germ}(\pi)$ *and a generic element of X is interbounded with an element of* $\mathrm{Arg}(\pi)$.

5.3. Group configuration in ω-categorical theories

Since our result in general is not completely satisfactory (the group living on almost-hyperimaginaries and not necessarily hyperimaginaries), we might attempt to improve it under the assumption of ω-categoricity. Extra care is required, however, to stay within a finite power of $\mathfrak{C}^{\mathrm{eq}}$, as expounded below.

My aim here is to show that it is indeed possible, starting from a good enough configuration on finite tuples, to get an interpretable group.

Remark 5.3.1. Recall that in an ω-categorical theory, if a, b are finite tuples in $\mathfrak{C}^{\mathrm{eq}}$, then $\mathrm{Cb}(a/b)$ will be finite, and also it is possible to replace b by an (interalgebraic) finite tuple b_0 such that $\mathrm{tp}(a/b_0)$ is equivalent to $\mathrm{lstp}(a/b)$. Furthermore, this finiteness is uniform: if a's belong to a sort S_1 and b's belong to a sort S_2, since there is only finitely many types $\mathrm{tp}(ab)$, all the possible canonical bases of the form $\mathrm{Cb}(a/b)$ are contained in finitely many sorts.

Proof. The relation $R(a, a', b)$ iff $a \equiv^L_b a'$ (for some finite tuples a, a', b), is an invariant relation so it is definable by ω-categoricity. Then the definable set $R(x, a, b)$ is clearly equivalent to $\mathrm{lstp}(a/b)$ and for b_0 we can take the name for it. □

We will profit from this remark in the ω-categorical case for the following reason: in order to use ω-categoricity, we need to keep working over (uniformly) finite tuples of imaginaries. In the general case, however, to achieve completeness in y of some generic action $\pi(x, y, z) = \mathrm{lstp}(fab)$, needed in 2.3.10, we typically replace a by $\mathrm{bdd}(a)$, which would take us into considering infinite tuples even if f, a, b were finite. But in ω-categorical case, it is possible to replace a by a finite tuple a_0 (by 5.3.1) depending on $\mathrm{tp}(fb/a)$ so the completion in y will still live on finite tuples. We formalise the above discussion as follows.

Lemma 5.3.2. *Completion with respect to any of the variables is a finitary operation, i.e. if $\pi(x, y, z)$ is a type-definable partial generic multiaction on finite tuples of imaginaries, the completion in any of the variables is on finite tuples of imaginaries. Furthermore, reduction with respect to any of the variables is a finitary operation as well.*

Proof. Let us consider e.g. the completion in x. The relation $\mathrm{LS}(yz, x, y'z', x')$, true if $x = x'$ and $yz \equiv^L_x y'z'$ is definable by ω-categoricity. Then, the first variable in the completion $\underline{\pi}$ (compare to 2.2.1) is of the sort $(yz, x)/\mathrm{LS}$ and thus imaginary.

For reduction, if e.g. $\pi(x, y, z)$ is complete in x, observe the transitive closure $\sim$ of $f \sim_1 f'$ if there is $x \mathop{\smile\!\!\!\!\!\!\mid} ff'$ with $f(x) \approx f'(x)$. This is a definable relation by ω-categoricity and therefore the first variable in the reduction $\bar{\pi}$ is of the imaginary sort $x/\!\sim$. □

Theorem 5.3.3. *An ω-categorical simple theory with an algebraic quadrangle on finite tuples of imaginaries interprets an infinite group.*

Proof. Let us start with an algebraic quadrangle (a, b, c, x, y, z) on finite tuples. By the above lemma, we may assume the generic partial multiaction $\pi := \mathrm{lstp}(byz)$ is complete in all the variables without destroying finiteness. Now, as in 2.4.2, it follows that $\pi^{-1} \circ \pi$ is generic and we can apply 2.3.10 to it and obtain a polygroup chunk $(P, *)$.

Since $P = \mathrm{Germ}(\pi^{-1} \circ \pi)$ is obtained by composing two multiactions, completing the result in the first variable and then reducing it, and all these operations are finitary, we obtain an interpretable polygroup chunk $(P, *)$. The core relation on P, being invariant, will become definable by ω-categoricity, hence the blowup (choose e.g. the classical blowup of 4.2 which only requires finitely many parameters), and later the construction of the group from the group chunk can be done definably. □

Using the results from Section 5.2 and arguing in a similar way, we get:

Corollary 5.3.4. *An ω-categorical simple theory with an algebraic quadrangle on finite tuples of imaginaries interprets an infinite space.*

Lemma 5.4.1. *In a one-based theory, for every partial generic multiaction* π, $\pi^{-1} \circ \pi$ *is generic.*

Proof. Clearly, if $f \in \mathrm{Germ}(\pi)$ and $ab \models f$, then $f = \mathrm{Cb}(ab/f) \in \mathrm{bdd}(ab)$ by one-basedness (1.2.22). Now, let $abc \models h \in \widehat{g \circ f}$, so $a \mathop{\smile\!\!\!\!\!\mid} fgh$. In particular, since $f \mathop{\smile\!\!\!\!\!\mid} g$, we get $f \mathop{\smile\!\!\!\!\!\mid}_a g$ and thus $b \mathop{\smile\!\!\!\!\!\mid}_a c$. Then, as above, $h \in \mathrm{bdd}(ac)$ and $f \in \mathrm{bdd}(ab)$, so $h \mathop{\smile\!\!\!\!\!\mid}_a f$ and $h \mathop{\smile\!\!\!\!\!\mid} f$. Similarly $h \mathop{\smile\!\!\!\!\!\mid} g$. □

We will make significant use of the following result ([W], 4.8.18), which relies on the fact that every group in a one-based theory is bounded-by-abelian ([W], 4.8.4):

Theorem 5.4.2. *Let* G *be a hyperdefinable connected group in a simple theory, and suppose that a generic type* p *of* G *is locally modular and regular. If* R *denotes the division ring of* p*-endogenies of* G*, then a tuple* $(g_0, \ldots, g_n)$ *in* G *is dependent if and only if there are* $r_i \in R$*, not all zero, with* $\sum_{i=0}^{n} r_i(g_i) \subset cl_p(\emptyset)$.

Theorem 5.4.3. *Let* T *be an* ω*-categorical,* SU*-rank 1, one-based nontrivial. Then* T *interprets a vector space over a finite field (as a stable 'reduct').*

Proof. Since T is one-based nontrivial, we can get a pairwise independent, non-independent triple $\{a, b', c\}$ such that each element is bounded over the other two, as in [BH]. By ω-categoricity, find a finite tuple b such that $\mathrm{tp}(ac/b)$ is equivalent to $\mathrm{lstp}(ac/b')$.

Then, $\pi := stp(abc)$ is a generic action, invertible, complete in the second variable, $Arg(\pi)$ is a (Lascar) strong type and, by 5.4.1, $\pi^{-1} \circ \pi$ is generic. Using the group configuration machinery (noting that all the germs will be (uniformly) finite tuples by choice of π and ω-categoricity as in 5.3) we get a group G (interpretable over finitely many parameters) with $\mathrm{SU}(G) = 1$. Thus, $G^0 := G^0_\emptyset$ is a connected group of SU-rank 1 and by 5.4.2, the ring of endogenies is a division ring and induces a vector space structure on G^0.

In what follows, we will show that endogenies have boundedly finite order, so by Wedderburn's theorem the ring of endogenies will actually be a finite field. Since every endogeny is induced by a definable subgroup of $G^0 \times G^0$, we can identify it with an imaginary element (the code of the subgroup). So let r be an endogeny, and let $v \in G^0$ be a nonzero vector. By ω-categoricity, there are only finitely many types among $\{\mathrm{tp}(r^i v/rv) : i \in \omega\}$. But, since all r^i are defined over r, we get that there are m, n such that $r^m = r^n$ and so r is of finite order. Pick now any two endogenies r and s. If $\sigma : rv \equiv_v sv$, then $\sigma(r)v = sv$, so since we're in a vector space, $\sigma(r) = s$ and it follows that r and s are of the same order. By ω-categoricity again, there can be only finitely many orders of endogenies, as required. □

Remark 5.4.4. Notice that ω-categoricity is not essential in the above. Reproving 5.4.2 on 1-based groups for 1-based gradedly almost hyperdefinable groups gives directly a gradedly almost hyperdefinable vector space over a division ring.

We can thus view this result as a first (nontrivial locally modular) case of a Zil'ber-type trichotomy for simple theories. The remaining non-modular case would involve studying a generalisation of Zariski structure framework from [HZ] to simple structures. I have already done some preliminary investigation in the direction of *pseudo-Zariski* structures, aimed at characterising Zariski closed sets over pseudofinite (or bounded PAC) fields.

Furthermore, the result can also be viewed as a first satisfactory step towards the *stable forking* conjecture, since inside a simple structure we have found a stable one where the independence is clearly governed by the stable structure (independence coincides with linear independence).

Remark 5.4.5. A careful reader might have noticed that we have not used the full potential of 5.4.2 in the above, since the group we obtained was of SU-rank 1, so its generic type was also of SU-rank 1 and thus regular, but the theorem will work for any group with a regular generic.

Let p be nontrivial regular locally modular type in a (ω-categorical) simple theory. Then, rewriting all of the group configuration techniques from this thesis in the language of regular types as in [P3], Chapter 7, will give a (definable) group G with a regular generic type, which will again yield a vector space by 5.4.2. The details of this will appear in [TW].

Remark 5.4.6. The purpose of this remark is to persuade the model-theoretic community that, at least in an ω-categorical case, references to the classical results about pregeometries, e.g.

- ([Ar]) a projective geometry of dimension not less than 4, in which each closed set of dimension 2 contains at least 3 elements, is isomorphic to a projective geometry over some division ring;
- ([DH]) a locally projective (i.e. non-trivial and locally modular), locally finite geometry of dimension greater than 4, in which all closed sets of dimension 2 have the same size, is an affine or projective geometry over a finite field;

can be replaced by a model-theoretic construction. For example, if D is a solution set of an SU-rank 1 Lascar strong type, and it is nontrivial, then [dPK] have shown that for each independent $a, b \in D$, $\mathrm{cl}(a) \cup \mathrm{cl}(b)$ is properly contained in $\mathrm{cl}(ab)$, so it makes sense to define $a * b := cl(ab) - (\mathrm{cl}(a) \cup \mathrm{cl}(b))$. If D is modular, it is easy to check that $*$ is generically associative, and Steinitz exchange implies generic invertibility (where $a^{-1} = a$), so $(D, *)$ will be a definable polygroup chunk by ω-categoricity. We can make it coreless by dividing out by the core relation and then apply the blowup procedure to obtain a group chunk, and eventually a group and even a division ring.

The construction from this section is an adaptation of two famous classical constructions: firstly, the group configuration corresponds to the (Hilbert), Veblen-Young and von Neumann construction of the division ring from a projective geometry, and then, once an abelian group has been obtained, one might study the nearring of endomorphisms of it; if one is lucky (typically in some linear-like framework), this will be a ring or even a field. In this case, however, we weren't lucky, we had one-basedness.

5.5. Pseudolinearity implies linearity

We will need this technical result:

Lemma 5.5.1. *If x and x' are interbounded, and $e := \mathrm{Cb}(x/B)$, $e' := \mathrm{Cb}(x'/B)$, then e and e' are interbounded.*

Proof. We have $x \mathop{\smile\!\!\!\!\!\!|}_e B$ and $x' \mathop{\smile\!\!\!\!\!\!|}_{e'} B$, but since $\mathrm{bdd}(x) = \mathrm{bdd}(x')$, from the first we get $x' \mathop{\smile\!\!\!\!\!\!|}_e B$ which implies $e' \in \mathrm{bdd}(e)$ and from the second $x \mathop{\smile\!\!\!\!\!\!|}_{e'} B$ which implies $e \in \mathrm{bdd}(e')$. □

Lemma 5.5.2. *If D is k-linear, then $G(D)$ (as defined in 1.3.8) is k-linear.*

We include this lemma just for completeness, as we will not need the full power of it because we will only deal with plane curves of the form $\mathrm{lstp}(ab'/C)$, where $b' \in \mathfrak{C}^{\mathrm{eq}}$ is actually interalgebraic with some real b, so the fact that $\mathrm{SU}(\mathrm{Cb}(ab'/C)) \leq k$ will follow from the previous lemma and k-linearity.

Proof. Let $x, y \in G(D)$, $B \subseteq G(D)$ such that $\mathrm{SU}(xy/B) = 1$, $x \mathop{\smile\!\!\!\!\!\!\mid}\, y$. Without any loss of generality, $B \subseteq D$. For xy, find an independent finite set $F \cup \{a, b\} \subseteq D$ such that xy is interalgebraic with ab over F, $xy \mathop{\smile\!\!\!\!\!\!\mid}\, F$. Take F' realising a nonforking extension of $\mathrm{tp}(F/xy)$ to $xyabBF$; then $F' \mathop{\smile\!\!\!\!\!\!\mid}\, xyabBF$. Since $F \equiv_{xy} F'$, let $a'b'$ such that $abF \equiv_{xy} a'b'F'$. Now $\mathrm{acl}^{eq}(xyF') = \mathrm{acl}^{eq}(a'b'F')$.
Claim: $\mathrm{SU}(a'b'/F'B) = 1$. To prove the claim, note first that $a'b' \mathop{\smile\!\!\!\!\!\!\mid}_{F'} B$; otherwise, we would get that $xy \mathop{\smile\!\!\!\!\!\!\mid}\, B$ which would be a contradiction. If $\mathrm{SU}(a'b'/F'B) = 0$, then $x, y \in \mathrm{acl}(F'B)$ and subsequently $x, y \in \mathrm{acl}(B)$, again a contradiction.

Denote by $e := \mathrm{Cb}(xy/B) = \mathrm{Cb}(xy/BF')$ (so $e \in \mathrm{bdd}(B)$), and $e' := \mathrm{Cb}(a'b'/F'B)$. By the above claim and linearity of D, $\mathrm{SU}(e') \leq k$. By an argument similar to 5.5.1, since $a'b' \mathop{\smile\!\!\!\!\!\!\mid}_{e'} F'B$, in particular $a'b' \mathop{\smile\!\!\!\!\!\!\mid}_{e'F'} B$ so by interboundedness over F', $xy \mathop{\smile\!\!\!\!\!\!\mid}_{e'F} B$ so $e \in \mathrm{bdd}(e'F')$. Therefore, $\mathrm{SU}(e/F') \leq k$, but since $e \in \mathrm{bdd}(B)$ and $B \mathop{\smile\!\!\!\!\!\!\mid}\, F'$, $\mathrm{SU}(e) = \mathrm{SU}(e/F') \leq k$ and we are done. □

Let us state some results of that shall be used significantly below, allowing us to find large abelian parts in our group.

Theorem 5.5.3.
(1) ([Mp], [W], 6.2.35). An ω-categorical simple group is nilpotent-by-finite.
(2) ([EW], [W], 6.2.32). An ω-categorical supersimple group is finite-by-abelian-by-finite.

Theorem 5.5.4. *A pseudolinear set D in an ω-categorical simple theory is linear.*

Proof. Let D be k-linear, and pick a plane curve $q = lstp(bc/a)$ (a, b, c finite) such that for $a_0 := \mathrm{Cb}(q)$, $\mathrm{SU}(a_0) = k$. By ω-categoricity, find b' such that $\mathrm{tp}(ac/b')$ is equivalent to $\mathrm{lstp}(ac/b)$. Then, by 5.5.1, $\mathrm{SU}(\mathrm{Cb}(b'c/a))$ is still k, and it easily follows that $\pi := \mathrm{lstp}(a, b', c)$ is a partial generic multiaction, invertible, complete in the second variable, $\pi^{-1} \circ \pi$ generic, $Arg(\pi)$ is a Lascar strong type, the only possibly non-trivial claim being that $\pi^{-1} \circ \pi$ is generic, shown as follows: pick $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\, a_2 \models Germ(\pi)$ and $a_3 \in \widehat{a_1^{-1} \circ a_2}$. Then $\mathrm{SU}(a_3) \leq k$ by 5.5.2, as a_3 is again a canonical base of a plane curve in $G(D)$. From rank considerations, we get $a_3 \mathop{\smile\!\!\!\!\!\!\mid}\, a_1$ and $a_3 \mathop{\smile\!\!\!\!\!\!\mid}\, a_2$.

Thus, it gives rise to an interpretable group (similar remarks as before about getting this interpretable on finite tuples apply); moreover, using Section 5.2 and the space chunk theorem 3.5.3, we can get a group G of SU-rank k acting on a set X of SU-rank at most 1; in fact, below we will only need the generic part of that action and the fact that it can be obtained generically faithful.

By 5.5.3(2), G is finite-by-abelian-by-finite, and in fact we may assume, after dividing by a finite normal subgroup that G is actually abelian-by-finite, so in particular $Z(G^0) = G^0$ ($G^0 := G^0_\emptyset$) is infinite. We continue as in [P3], after Lemma 2.4.21.
Claim: If $f \in G^0$ and $x \in X$, $x \mathop{\smile\!\!\!\!\!\!\mid}\, f$, then $f \in \mathrm{dcl}(x, f \cdot x)$.
Let $g \in G^0$ be such that $tp(g/x, f \cdot x) = tp(f/x, f \cdot x)$. In particular $g \cdot x = f \cdot x$. Let $f' \in Z(G^0)$ such that $\mathrm{SU}(f') \geq 1$ (as $Z(G^0)$ is infinite) and $f' \mathop{\smile\!\!\!\!\!\!\mid}\, fgx$.

Let $y := f' \cdot x$. We claim that $f' \not\mathop{\smile\!\!\!\!\!\!\mid}\, xy$. Suppose otherwise; then $f' \mathop{\smile\!\!\!\!\!\!\mid}\, xy$ implies $f' \mathop{\smile\!\!\!\!\!\!\mid}_x y$ so $y \mathop{\smile\!\!\!\!\!\!\mid}_x y$ and thus $y \in \mathrm{bdd}(x)$. Similarly we get $x \in \mathrm{bdd}(y)$. Now pick f'' realising the nonforking extension of $\mathrm{lstp}(f'/x)$ to xf'. Since $y \in \mathrm{bdd}(x)$, $f'' \cdot x = y$ too, and thus $(f'^{-1} \cdot f'') \cdot x = x = 1 \cdot x$, and $x \mathop{\smile\!\!\!\!\!\!\mid}\, f'f''$ so by generic faithfulness of the action, $f'^{-1} \cdot f'' = 1$, implying $f' \in \mathrm{bdd}(\emptyset)$ which contradicts $\mathrm{SU}(f') \geq 1$.

From this, it easily follows that $f' \cdot x \mathop{\smile\!\!\!\!\!\!\mid}\, fg$ and $x \mathop{\smile\!\!\!\!\!\!\mid}\, ff'$, $x \mathop{\smile\!\!\!\!\!\!\mid}\, gf'$. Then

$$\begin{aligned} f \cdot y = f \cdot (f' \cdot x) &= (f \cdot f') \cdot x = (f' \cdot f) \cdot x = f' \cdot (f \cdot x) \\ &= f' \cdot (g \cdot x) = (f' \cdot g) \cdot x = (g \cdot f') \cdot x = g \cdot (f' \cdot x) = g \cdot y. \end{aligned}$$

Thus, by generic faithfulness, $f = g$, proving the claim.

In the above, let $f \in G^0$ with $\mathrm{SU}(f) = k$. By the claim and $f \downarrow x$, we have $\mathrm{SU}(f) = \mathrm{SU}(f/x) \leq \mathrm{SU}(f \cdot x/x) \leq 1$, which implies $k = 1$. □

Using the fact that linearity is equivalent to one-basedness (1.3.10), we obtain:

Corollary 5.5.5. *A pseudolinear set D in an ω-categorical simple theory is one-based.*

Remark 5.5.6. Again, if we are willing to rewrite the group configuration techniques in the language of regular types of [P3], Chapter 7, we can rephrase the above result as follows.

Let p be a regular type (over $\emptyset$) in an ω-categorical simple theory, k-linear in the sense that whenever $q(x, y)$ is a p-minimal extension of $p(x) \cup p(y)$ with $\mathrm{SU}_p(q) = 1$ (plane curve), then $\mathrm{SU}_p(\mathrm{Cb}(q)) \leq k$, and weight k is achieved for some plane curve. Then, an analogous proof as above in the framework of regular types would yield $k = 1$ and p is locally modular (where the appeal to 5.5.3(2) is replaced by the fact that if p is a finitely based regular type, a p-connected group of finite SU_p-rank is (central bounded-by-abelian)-by-bounded; it is conceivable that, with some additional work, even 5.5.3(1) should suffice). This will be expounded in [TW].

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