INDEPENDENCE, MEASURE AND PSEUDOFINITE FIELDS

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Abstract. We give a measure-theoretic refinement of the Independence Theorem in pseudofinite fields.

1. Introduction

It was an important event in model theory when Hrushovski isolated the property of perfect bounded pseudo-algebraically closed fields (a class which includes pseudofinite fields) now known as the *Independence Theorem* in [13]. The generalisation of this theorem, by Kim and Pillay ([17], [16]), has given rise to the study of *simple theories*, a whole new area of model theory.

Suppose we work in a context with good dimension theory, like pseudofinite fields. The Independence Theorem says that, if we take a complete type p over a model E, and two extensions p_i of p to $K_i \supseteq E$ of the same dimension, $i = 1, 2$, with independent parameters, $K_1 \mathop{\smile\!\!\!\!\!\!\mid}_E K_2$, then the partial type $p_1 \cup p_2$ is again of the same dimension as p.

In the special case of pseudofinite fields, besides dimension, we have a way of *measuring* the definable sets, described in [2] and [10], and therefore a chance of giving a finer analysis of the situation. Our goal is to show that the types p_1 and p_2 above are in fact *independent as events* over p, when considered in a suitable probability space. Intuitively, the measure comes from uniformities in counting the number of rational points over finite fields, and probabilistic independence is a consequence of randomness and equidistribution phenomena related to finite fields.

Moreover, we wish to give a particular proof of the above, which emphasises the interplay between independence and fibre products. In some sense, our formulation of the Independence Theorem is an instance of the Künneth formula in cohomology. Such a development gives rise to speculations regarding the existence of a unified theory of independence, strongly related to fibre products. Of course, motivating examples such as linear independence in vector spaces, linear disjointness in fields and probabilistic independence all have interpretations in terms of fibre products.

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This paper may be of use to algebraists wishing to understand what the abstract Independence Theorem actually means in a familiar context. In our language, it becomes so natural that it is actually expected to hold. Certainly, we were very tempted by the possibility to generalise and conceptualise, until it all becomes trivial. A posteriori, our measure-theoretic formalism is essentially just a variant of the more precise cohomological methods developed for finite fields by the Grothendieck's school.

We develop a significant amount of theory for pseudofinite fields, notably various versions of Čebotarev's Theorem, L-functions and Dirichlet density, which we expect to be useful in other applications. For example, 6.8, will be indispensable in the study of 'random' reducts of pseudofinite fields, started in [21].

The underlying idea of the paper is relatively straightforward, and consists of these major steps:

(1) Definition of fibre products and independence of measure spaces.
(2) Finding the correspondence between the measure spaces of definable sets and certain Galois groups.
(3) In the context of the Independence Theorem, the appropriate Galois groups are independent, and thus, by step (2), the measure spaces of definable sets are independent as well.

On the other hand, concepts and techniques involved in precise statements of the results come from widely separated areas of mathematics (like model theory, representation theory, algebraic and Diophantine geometry, measure theory, functional analysis etc.), and this presented us with difficulties in the exposition of the material. In an attempt to make the paper as self-contained as possible, we give quick surveys of the necessary definitions and facts. The organisation of the paper is as follows.

Sections 2 and 3 contain introductory material regarding measure theory and representation theory, respectively. They are used throughout the paper. In particular, 2 completes step (1).

In Section 4, we give a description of types and definable sets in pseudofinite fields, our main objects of study, in terms of conjugacy classes in certain Galois groups.

Section 5 contains a short review of cohomological methods for studying finite fields, and shows how one can very elegantly obtain definability results over finite fields using Deligne's theory of weights. If the reader is willing to accept 6.1 as a fact, everything in this section except 5.1 and 5.5 can be completely avoided.

In Sections 6, 7 and 8 we develop the theory of integration related to the measure from [2] and [10]. In Section 6 we are able to integrate definable $\mathbb{C}$-valued functions on varieties (and on definable sets). Section 7 introduces definable L-functions in pseudofinite fields and the notion of Dirichlet density. The finitely-additive measure from these two sections are completed to the full measures in Section 8. The main results, establishing step (2), are 6.8, 7.7 and 8.3.

We prove our measure-theoretic Independence Theorem in Section 9, and show it is indeed a refinement of the original. This is step (3) mentioned above.

Lastly, in Section 10, we give a motivic interpretation of the Independence Theorem as a kind of a Künneth formula.

Throughout the text, we have attempted to use notation which is as standard as possible. One small exception is the complex conjugation, which we denote by $^\vee$, on one hand to stress the connection between the conjugated character and the contragredient representation, on the other to distinguish it from the notation $\bar{k}$ for the algebraic closure of a field k. The residue field at a point s of a scheme is denoted $\mathbf{k}(s)$. For a scheme S, a *variety over* S is a separated and reduced scheme of finite type over S. In most of our applications S will just be the spectrum of a field k and in that case we will just speak of a *variety over* k. We have given our best to write 'variety' instead of 'scheme' in the text, and the reader can always replace it by 'affine variety', or just imagine a set defined by a finite system of polynomial equations. Given a variety X over k, by $\bar{X}$ we shall denote the variety $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ (X considered over $\bar{k}$). *Geometric* properties of a variety X refer to the corresponding properties of $\bar{X}$. In particular, X is *geometrically irreducible (resp. normal, connected)* if the corresponding $\bar{X}$ is. For a scheme X and a field k, the set of k-rational points, $X(k)$, is the set of all morphisms $\mathrm{Spec}(k) \to X$. Whenever needed, we shall assume that the pseudofinite field we are working over is 'large': sometimes this can just mean 'uncountable', sometimes 'of large transcendence degree', and sometimes 'saturated' in the model-theoretic sense.

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2. Measure spaces

For the basic definitions and results of this section we refer the reader to [9]. Let X be a compact topological space (metrisable). A *measure* (or *complex measure*) on X is an element of the dual of the Banach space $\mathcal{C}_{\mathbb{C}}(X)$ of complex valued continuous functions on X. In other words, it is a linear form $f \mapsto \mu(f)$ on $\mathcal{C}_{\mathbb{C}}(X)$, bounded in the sense that, for some a,

$$|\mu(f)| \leq a\|f\|,$$

for all $f \in \mathcal{C}_{\mathbb{C}}(X)$, where $\|f\| = \sup_{x \in X} |f(x)|$.

Let now X be a locally compact space (metrisable and separable). For every compact subset K of X, let us denote by $\mathcal{K}_{\mathbb{C}}(X; K)$ the subspace of the vector space $\mathcal{C}_{\mathbb{C}}(X)$ consisting of functions with support contained in K. By $\mathcal{K}_{\mathbb{C}}(X)$ we denote the union of $\mathcal{K}_{\mathbb{C}}(X; K)$, where K ranges over all compact subsets of X, i.e. the vector space of complex valued continuous function on X with compact support.

A *measure* (or *complex measure*) on X is a linear form μ on $\mathcal{K}_{\mathbb{C}}(X)$ with the following property: for every compact subset K of X, there exists a number

$a_K \geq 0$ such that for all $f \in \mathcal{K}(X;K)$,

$$|\mu(f)| \leq a_K \|f\|.$$

When we wish to specify the variable of integration, we write $\int_X f(x)\ d\mu(x)$ instead of $\mu(f)$. We assume the reader is familiar with the usual way of extending the class of functions that can be measured (cf. [9]) and we freely use the notation $L^1(X)$ (resp. $L^p(X)$) for the Banach space of μ-integrable functions (resp. μ-measurable functions f with $|f|^p$ μ-integrable). We also use $L^1_{\mathrm{loc}}(X)$ for the space of locally integrable functions, with topology induced by the family of seminorms $f \to \int |f 1_K|$ ranging over the compacts K of X.

Remark 2.1. Let $\mathcal{B}$ be a base of clopen sets for the topology of a compact space X. Let us call a function $f \in \mathcal{C}_{\mathbb{C}}(X)$ $\mathcal{B}$-continuous, if for every open $V \subseteq \mathbb{C}$, $f^{-1}(V) \in \mathcal{B}$. Suppose we have a bounded linear form μ on $\mathcal{B}$-continuous functions. Then it is possible to extend it (uniquely) to a measure on X, by the following argument. Given $f \in \mathcal{C}_{\mathbb{C}}(X)$, there exists a sequence (f_n) of $\mathcal{B}$-continuous functions such that $f = \lim f_n$ in the sense of the norm $\|\cdot\|$. Then clearly $\mu(f) := \lim \mu(f_n)$ is linear and bounded, i. e., a measure.

Definition 2.2. Let (X, μ) be a measure space. For $f \in \mathcal{K}(X)$ and $g \in L^1_{\mathrm{loc}}(X)$, we write

$$(f, g)_\mu := \mu(f \cdot g^\vee).$$

If it is clear from the context which measure is being used, we may write $(f, g)_X$ or just (f, g). We extend this definition for $f \in L^p(X)$ and $g \in L^q(X)$, when $1/p + 1/q = 1$. In case $p = q = 2$, we get the scalar product making $L^2(X)$ into a Hilbert space.

We wish to define the fibre product of measures and therefore we will adopt a somewhat unusual functorial approach to spaces with measure. The advantage of this notation will become clear later when we encounter analogous operations on representations and sheaves. The following considerations are usually formulated in terms of *conditional expectation* and *conditional probability*. For example, our $\phi_* f$ below would usually be denoted as $\mathbb{E}[f|Y]$.

Theorem 2.3. *Let $\phi : X \to Y$ be a continuous map between measure spaces (X, μ) and (Y, ν). Let $\phi^* : \mathcal{K}(Y) \to \mathcal{K}(X)$ be the continuous map of algebras defined by $\phi^*(g) = g \circ \phi$, for $g \in \mathcal{K}(Y)$.*

Then there is a continuous linear map (unique satisfying the property (1) below) $\phi_* : L^1_{\mathrm{loc}}(X) \to L^1_{\mathrm{loc}}(Y)$, adjoint to ϕ^* in the following sense:

(1) $\mu(f \cdot \phi^* g) = \nu(\phi_* f \cdot g)$, *for all* $f \in \mathcal{K}(X)$, $g \in \mathcal{K}(Y)$;

We have the projection formula:

(2) $\phi_*(f \cdot \phi^* g) \approx \phi_* f \cdot g$, *for all* $f \in \mathcal{K}(X)$, $g \in \mathcal{K}(Y)$.

The operations * and $_*$ are functorial:

(3) $(\phi \circ \psi)^* = \psi^* \phi^*$, *and* $\mathrm{id}^* = \mathrm{id}$;
(4) $(\phi \circ \psi)_* = \phi_* \psi_*$, *and* $\mathrm{id}_* = \mathrm{id}$.

Proof. Let us show the existence of ϕ_* satisfying (1). Given an $f \in L^1_{\mathrm{loc}}(X)$, the linear form $g \mapsto \mu(f \cdot \phi^* g)$, for $g \in \mathcal{K}(Y)$, is a measure on Y equivalent to ν (they have the same negligible sets), so by the theorem of Radon-Nikodym-Lebesgue, there is an $\phi_* f \in L^1_{\mathrm{loc}}(Y)$ satisfying (1) for all $g \in \mathcal{K}(Y)$. The linearity, continuity and uniqueness are straightforward.

Property (2) is a direct consequence of (1); for an arbitrary $h \in \mathcal{K}(Y)$,

$$\nu(\phi_*(f \cdot \phi^* g) \cdot h) = \mu(f \cdot \phi^* g \cdot \phi^* h) = \mu(f \cdot \phi^*(g \cdot h)) = \nu(\phi_* f \cdot g \cdot h),$$

yielding that $\phi_*(f \cdot \phi^* g)$ and $\phi_* f \cdot g$ are equal almost everywhere.

Functoriality properties are obvious. □

Remark 2.4. Since $\phi^*(g^\vee) = (\phi^* g)^\vee$, the defining property of the operation $_*$ can be stated in terms of the 'scalar product':

$$(f, \phi^* g)_\mu = (\phi_* f, g)_\nu,$$

for all $f \in \mathcal{K}(X)$, $g \in \mathcal{K}(Y)$;

Definition 2.5. Suppose $\phi : (X, \mu) \to (Y, \nu)$ is a continuous map between measure spaces. We will call it a *map of measure spaces*, if $\phi(\mu) = \nu$, i. e., if $\mu(\phi^* g) = \nu(g)$ for all $g \in \mathcal{K}(Y)$. Equivalently, ϕ is a map of measure spaces if $\phi_* 1 = 1$.

By the usual arguments we get the following.

Corollary 2.6. *Let $p > 1$ and q such that $1/p + 1/q = 1$. There exist*

(1) *a covariant functor $_*$ from the category of measure spaces to the category of Banach spaces,*

$$\phi : (X, \mu) \to (Y, \nu) \;\mapsto\; \phi_* : L^p(X, \mu) \to L^p(Y, \nu),$$

(2) *a contravariant functor * from the category of measure spaces to the category of Banach algebras,*

$$\phi : (X, \mu) \to (Y, \nu) \;\mapsto\; \phi^* : L^q(Y, \nu) \to L^q(X, \mu),$$

extending the functors from 2.3, which are weakly adjoint in the ('decategorised') sense that, given a map of measure spaces ϕ as above, for all $f \in L^p(X, \mu)$ and $g \in L^q(Y, \nu)$,

$$\mu(f \cdot \phi^* g) = \nu(\phi_* f \cdot g) \text{ (alternatively, } (f, \phi^* g)_\mu = (\phi_* f, g)_\nu \text{)}.$$

Theorem 2.7. *Let $\phi_i : (X_i, \mu_i) \to (Y, \nu)$, $i \in \{1, 2\}$ be continuous maps, and let π_i be the projection $X_1 \times_Y X_2 \to X_i$, $i \in \{1, 2\}$. Given $f_i \in \mathcal{K}(X_i)$, let us write $f_1 \boxdot f_2$ for $\pi_1^* f_1 \cdot \pi_2^* f_2$.*

(1) *There is a unique measure μ on the topological fibre product $X_1 \times_Y X_2$ extending the rule*

$$\mu(f_1 \boxdot f_2) := \nu(\phi_{1*} f_1 \cdot \phi_{2*} f_2)$$

from $\mathcal{K}(X_1) \otimes_{\mathcal{K}(Y)} \mathcal{K}(X_2)$ to $\mathcal{K}(X_1 \times_Y X_2)$.

(2) *Given $f_i \in \mathcal{K}(X_i)$,*

$$(\phi_1 \times \phi_2)_*(f_1 \boxdot f_2) \approx \phi_{1*} f_1 \cdot \phi_{2*} f_2, \text{ for } i \in \{1, 2\}.$$

(3) (Base change). *For every* $f_1 \in \mathcal{K}(X_1)$,

$$\pi_{2*}\pi_1^* f_1 \approx \phi_2^* \phi_{1*} f_1.$$

Proof. (1) We repeat the well-known classical proof of the existence and uniqueness of the product measure in the relative setting. Uniqueness of μ satisfying the rule from above follows from the fact that the functions from $\mathcal{K}(X_1 \times_Y X_2)$ can be arbitrarily well approximated by functions from $\mathcal{K}(X_1) \otimes_{\mathcal{K}(Y)} \mathcal{K}(X_2)$.

For existence, given $h \in \mathcal{K}(X_1 \times_Y X_2)$, we consider the function

$$h_1(y_1) := \nu(\phi_{2*}[h(y_1, \cdot)]).$$

By continuity of ϕ_{2*} from 2.3, we conclude that h_1 is continuous and therefore it makes sense to define $\mu(h) := \nu(\phi_{1*} h_1)$. This turns out to be the correct definition.

(2) Let us take $f_i \in \mathcal{K}(X_i)$, $i \in \{1, 2\}$. Using 2.3 and the defining property of μ, for every $g \in \mathcal{K}(X)$, we get

$$\begin{aligned}\nu((\phi_1 \times \phi_2)_*(f_1 \boxdot f_2) \cdot g) &= \mu(\pi_1^* f_1 \cdot \pi_2^* f_2 \cdot \pi_1^* \phi_1^* g) = \mu((f_1 \cdot \phi_1^* g) \boxdot f_2) \\ &= \nu(\phi_{1*}(f_1 \cdot \phi_1^* g) \cdot \phi_{2*} f_2) = \nu(\phi_{1*} f_1 \cdot \phi_{2*} f_2 \cdot g),\end{aligned}$$

which gives the required identity $(\phi_1 \times \phi_2)_*(f_1 \boxdot f_2) \approx \phi_{1*} f_1 \cdot \phi_{2*} f_2$.

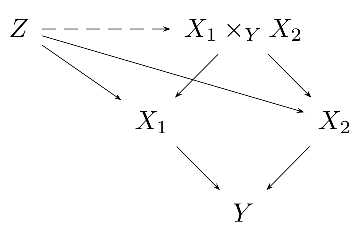
(3) For any $f_2 \in \mathcal{K}(X_2)$,

$$\mu_2(\pi_{2*}\pi_1^* f_1 \cdot f_2) = \mu(\pi_1^* f_1 \cdot \pi_2^* f_2) \overset{(2)}{=} \nu(\phi_{1*} f_1 \cdot \phi_{2*} f_2) = \mu_2(\phi_2^* \phi_{1*} f_1 \cdot f_2),$$

as required. □

The following definition from [1], fits our context remarkably well.

Definition 2.8. Let the notation be as in 2.7. Whenever we are given maps of measure spaces $(Z, \mu') \to (X_i, \mu_i)$, $i \in \{1, 2\}$ forming with the ϕ_i a commutative diagram, by the universal property of the fibre product of topological spaces, there is a unique continuous map $Z \to X_1 \times_Y X_2$, making the following diagram commutative:



We shall say that the measure spaces (X_1, μ_1) and (X_2, μ_2) are *independent over* (Y, ν) (*with respect to* (Z, μ')) and write

$$(X_1, \mu_1) \underset{(Y,\nu)}{\perp} (X_2, \mu_2),$$

if that map turns out to be a map of measure spaces $(Z, \mu') \to (X_1 \times_Y X_2, \mu)$.

3. Compact groups and representations

We wish to study in more detail the special case of compact groups with Haar measure, the framework where various Galois groups studied elsewhere in this paper naturally fit. We first recall the basic definitions and facts, assuming some familiarity with [20].

Definition 3.1. Let G be a locally compact group and let $f : G \to E$ be any function. For $g \in G$ we define the left and right shifts of f by g:

$$[\gamma(g)f](x) := f(g^{-1}x);$$
$$[\delta(g)f](x) := f(xg).$$

If μ is a measure on G, we say that μ is *left (right) invariant* if for all $f \in \mathcal{K}(G)$ and $g \in G$,

$$\mu(\gamma(g)f) = \mu(f) \text{ (resp. } \mu(\delta(g)f) = \mu(f)).$$

Fact 3.2. *Let G be a locally compact group. There exists a left invariant positive measure μ on G. All the other left invariant measures on G are proportional to it. It is called (the)* left Haar measure*. When G is compact, μ is also right invariant, we can normalise it so that $\mu(G) = 1$ and we speak of* the Haar measure.

From now on, we only work with compact groups and we always assume the Haar measure to be normalised.

Definition 3.3. Let G be a compact group and V a vector space of finite dimension over $\mathbb{C}$. A *linear representation* of G in V is a continuous homomorphism $\rho : G \to \mathbf{GL}(V)$. A *character* of G is the trace of a continuous representation.

A representation is called *irreducible*, if V does not contain proper nonempty G-stable subspaces. An *irreducible character* is a character of an irreducible representation. A sum of characters corresponds to a direct sum of representations, a product of characters to a tensor product of representations, and the conjugate $\alpha^\vee$ of a character α to the *contragredient* (or *dual*) $\rho^\vee$ of its representation ρ. This notation is consistent with our notation for complex conjugation.

A *central* (or *class*) function on G is a function from $L^2(G)$ which is invariant under conjugation in G. It is a well-known fact that irreducible characters form an orthonormal basis of the Hilbert space of central functions on G.

We are frequently interested in representations defined over fields other than $\mathbb{C}$, so we need to give a quick overview of the rationality issues. Let K be a field of characteristic zero contained in $\mathbb{C}$, with induced topology. Given a vector space V over K, let $V_{\mathbb{C}} := \mathbb{C} \otimes_K V$ be the vector space obtained from V by extension of scalars. Then, each continuous linear representation $\rho : G \to \mathbf{GL}(V)$ defines a continuous representation

$$\rho_{\mathbb{C}} : G \to \mathbf{GL}(V) \to \mathbf{GL}(V_{\mathbb{C}}).$$

The character is a continuous central function with values in K. Let $R_K(G)$ be the group generated by the characters of continuous representations over K. It is a subring of the ring $R(G)$ generated by the continuous characters over $\mathbb{C}$.

It is still true that $R_K(G)$ is generated by the characters of the irreducible continuous representations of G over K, which are mutually orthogonal (but not necessarily normalised).

We give appropriate generalisations of results of [20], Chapter 12, for profinite groups. Let $G = \varprojlim_i G_i$ be a profinite group with all the connecting maps $\phi_{ij} : G_i \to G_j$ surjective. Let $m_i = |G_i|$ and let L_i be the extension of K by the m_i-th roots of unity. The extension L_i/K is Galois and $\mathrm{Gal}(L_i/K)$ is a subgroup of the multiplicative group $(\mathbb{Z}/m_i\mathbb{Z})^*$ of invertible elements of $\mathbb{Z}/m_i\mathbb{Z}$. More precisely, if $\tau \in \mathrm{Gal}(L_i/K)$, there exists a unique element $t \in (\mathbb{Z}/m_i\mathbb{Z})^*$ such that

$$\tau(\omega) = \omega^t, \text{ if } \omega^{m_i} = 1.$$

We denote by Γ_K^i the image of $\mathrm{Gal}(L_i/K)$ in $(\mathbb{Z}/m_i\mathbb{Z})^*$, and if $t \in \Gamma_K^i$, we let τ_t^i denote the corresponding element of $\mathrm{Gal}(L_i/K)$. Hence we also get a similar correspondence $t \leftrightarrow \tau_t$ between $\Gamma_K := \varprojlim_i \Gamma_K^i$ and $\mathrm{Gal}(L/K)$, where $L = \bigcup_i L_i$. The group Γ_K acts on G in a natural way as a permutation group. We will say that $s, s' \in G$ are Γ_K-*conjugate,* if there is a $t \in \Gamma_K$ such that s' and s^t are conjugate in G. We define $\Psi^t(f)(s) := f(s^t)$, for f central on G.

Lemma 3.4. *Every continuous representation of a profinite group G over $\mathbb{C}$ factors through a finite quotient of G.*

Proof. Let $\rho : G \to \mathbf{GL}_r(\mathbb{C})$ be a continuous representation. Choose an open neighbourhood U of the identity in the Lie group $\mathbf{GL}_r(\mathbb{C})$ which does not contain any of its nontrivial subgroups. Let H be an open subgroup contained in $\rho^{-1}(U)$. Then clearly $\rho(H) = 1$ and ρ factors through G/H. □

Theorem 3.5. *Let f be a central function on a profinite group G with values in L. Then $f \in K \otimes_{\mathbb{Z}} R(G)$ if and only if $\tau_t(f) = \Psi^t(f)$, for all $t \in \Gamma_K$.*

Corollary 3.6. *Let f be a central function on a profinite group G with values in K. Then $f \in K \otimes R_K(G)$ if and only if f is constant on Γ_K-classes of G.*

Corollary 3.7. *The characters of the irreducible continuous representations of a profinite group G over K form a basis for the space of central functions on G which are constant on Γ_K-classes.*

In the special case of $K = \mathbb{Q}$, the Galois theory of cyclotomic extensions is well-known and we have:

Lemma 3.8. *In a profinite group G, two elements x and x' are $\Gamma_{\mathbb{Q}}$-conjugate, if and only if they (topologically) generate conjugate subgroups of G.*

Definition 3.9. Let G be a compact group with Haar measure μ.

(1) Let $C_{\mathbb{Q}}(G, \mathbb{C})$ be the space of continuous functions $f : G \to \mathbb{C}$ with the property that $f(x) = f(x')$ whenever $\langle x \rangle$ and $\langle x' \rangle$ are conjugate in G. We call such functions $\mathbb{Q}$-*central.*
(2) Let $C_{\mathbb{Q}}(G, \mathbb{Q})$ be the $\mathbb{Q}$-valued functions from $C_{\mathbb{Q}}(G, \mathbb{C})$.

(3) A subset of G is called $\mathbb{Q}$-central if its characteristic function is. In other words, it is a union of $\Gamma_{\mathbb{Q}}$-conjugacy classes.

Remark 3.10. We have shown in 3.7 that $\mathbb{C}_{\mathbb{Q}}(G, \mathbb{C})$ is generated by the characters of continuous irreducible representations of G over $\mathbb{Q}$ (which are, of course, $\mathbb{Q}$-central themselves), and thus, by 3.4, by characters of irreducible representations of finite quotients of G over $\mathbb{Q}$. Note, however, that an arbitrary $\mathbb{Q}$-central function need not factor through a finite quotient of G.

The functorial constructions regarding measure spaces happen to be especially natural when formulated in the context of compact groups with Haar measure, as the following results show.

Remark 3.11. Let $\phi : G \to G'$ be a continuous homomorphism of compact groups. Consider finite-dimensional representations ρ of G and ρ' of G'. Then clearly $\phi^*\rho'$ is a finite-dimensional representation of G (recall 2.3). On the other hand, it is possible, but not straightforward, to define a representation $\phi_*\rho$ on G', which may not be finite-dimensional any more, so that

$$\mathrm{Hom}(\rho, \phi^*\rho') \simeq \mathrm{Hom}(\phi_*\rho, \rho'),$$

and in that case * and $_*$ become adjoint functors (unlike just adjoint in the 'decategorised' sense as in 2.3 and 2.6). In the special case when ϕ is an inclusion, ϕ^* and ϕ_* are *restriction* and *induction* operations standard in representation theory.

These issues are well beyond the extent of this paper so we content ourselves by noting that, at the level of characters of finite-dimensional representations, ϕ^* and ϕ_* behave as in 2.3. Even better, we have an explicit description of these operations in 3.14.

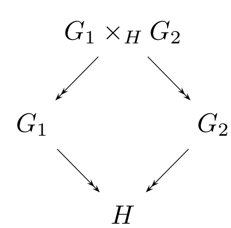
Lemma 3.12. *Let $\phi : (G, \mu) \to (H, \nu)$ be a continuous homomorphism of compact groups equipped with Haar measures.*

(1) $\phi^*[\gamma(\phi(x))h] = \gamma(x)\phi^*h$, *for all $x \in G$ and $h \in \mathcal{C}(H)$;*
(2) $\phi_*[\gamma(x)g] = \gamma(\phi(x))\phi_*g$, *for all $x \in G$ and $g \in \mathcal{C}(G)$.*

Proof. The lemma is a direct consequence of the definitions and the left invariance of the Haar measure. □

Proposition 3.13. (1) *Every continuous epimorphism of compact groups is a map of measure spaces.*

(2) *Consider the fibre product of compact groups as in the following diagram:*



Then $(G_1 \times_H G_2, \mu_{\text{Haar}}) \simeq (G_1, \mu_{G_1}) \times_{(H,\mu_H)} (G_2, \mu_{G_2})$*, where each group is equipped with its Haar measure.*

Proof. (1) Let $\phi : (G, \mu) \to (H, \nu)$ be a continuous epimorphism of compact groups, where μ and ν are their corresponding Haar measures. By the uniqueness of Haar measure, it is enough to show that the measure defined by $\nu'(h) := \mu(\phi^* h)$ for $h \in \mathcal{K}(H)$ is left invariant. This follows from the fact that $\phi^*[\gamma(y)h] = \gamma(x)\phi^* h$, for some x with $\phi(x) = y$, as mentioned in 3.12(1).

For (2), let us denote by $\phi_i : G_i \to H$, $i \in \{1, 2\}$ the continuous epimorphisms in question. The fibre product measure is determined by

$$\mu(f_1 \boxdot f_2) = \mu_H(\phi_{1*} f_1 \cdot \phi_{2*} f_2).$$

Again by uniqueness of the Haar measure, it is enough to show that μ is left invariant. Let $x_1 \in G_1$ and $x_2 \in G_2$ be such that $\phi_1(x_1) = \phi_2(x_2) =: y \in H$. By definition of μ, 3.12(2) and left invariance of μ_H, we get that

$$\begin{aligned}\mu(\gamma(x_1, x_2) f_1 \boxdot f_2) &= \mu_H(\phi_{1*}[\gamma(x_1) f_1] \cdot \phi_{2*}[\gamma(x_2) f_2]) = \\ &\mu_H(\gamma(\phi_1(x_1))\phi_{1*} f_1 \cdot \gamma(\phi_2(x_2))\phi_{2*} f_2) = \mu_H(\gamma(y)[\phi_{1*} f_1 \cdot \phi_{2*} f_2]) = \\ &\mu_H(\phi_{1*} f_1 \cdot \phi_{2*} f_2) = \mu(f_1 \boxdot f_2),\end{aligned}$$

as required. □

Remark 3.14. The following consideration of Haar measures of quotient groups is hidden in 3.13(1). Suppose we are given a short exact sequence of compact groups

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\phi} H \longrightarrow 1,$$

with Haar measures μ_K, μ_G, μ_H, respectively.

(1) For $f \in \mathcal{C}(G)$, the function
$$g(x) := \int_K f(x\, \iota(\xi))\, d\mu_K(\xi)$$
is continuous on G, with the property that $g(x\, \iota(\zeta)) = g(x)$ for every $\zeta \in K$. Therefore we can find $h \in \mathcal{C}(H)$ such that $g(x) = h(\pi(x))$. Clearly $h = \pi_* f$.

(2) Moreover, if χ is a character of a continuous representation Λ of G, then $\pi_*\chi$ is the character of the invariants Λ^K of K in Λ with the natural action of H.

(3) Suppose the above groups are profinite. Since the pushforward of a representation of a finite group over $\mathbb{Q}$ is again over $\mathbb{Q}$, it follows from 3.10 that
$$\pi_*(C_{\mathbb{Q}}(G, \mathbb{C})) \subseteq C_{\mathbb{Q}}(H, \mathbb{C}).$$

Example 3.15. In notation of 3.14, suppose C is a subset of G such that $\phi \restriction C$ has *fibres of constant size,* i. e., there exists a number m such that for every $A \subseteq H$, $\mu_G(C \cap \phi^{-1}(A)) = m\mu_H(\phi(C) \cap A)$. A direct verification (alternatively, a calculation using 3.14) shows that the pushforward $\phi_* 1_C$ of the characteristic function of C is given by

$$\phi_* 1_C = m\, 1_{\phi(C)}.$$

These considerations apply, for example, when C is a subgroup or a $\mathbb{Q}$-conjugacy class.

Example 3.16. With notation from the proof of 3.13(2), let $C_i \subseteq G_i$ be subsets with fibres of constant sizes (3.15) m_i, and suppose $C_0 := \phi_1(C_1) \cap \phi_2(C_2) \subseteq H$ is of nonzero measure.

We are interested in the measure of the subset $C_1 \times_H C_2$ of $G_1 \times_H G_2$. Using 3.13(2) and 3.15, it is equal to

$$\begin{aligned} \mu_H(\phi_{1*} 1_{C_1} \cdot \phi_{2*} 1_{C_2}) &= m_1 m_2\, \mu_H(C_0) \\ &= \frac{\mu_{G_1}(C_1 \cap \phi_1^{-1}(C_0))\mu_{G_2}(C_2 \cap \phi_2^{-1}(C_0))}{\mu_H(C_0)}. \end{aligned}$$

Dividing by $\mu_H(C_0)$ yields a more symmetric expression

$$\frac{\mu_H(\phi_{1*} 1_{C_1} \cdot \phi_{2*} 1_{C_2})}{\mu_H(C_0)} = \frac{\mu_{G_1}(C_1 \cap \phi_1^{-1}(C_0))}{\mu_H(C_0)} \cdot \frac{\mu_{G_2}(C_2 \cap \phi_2^{-1}(C_0))}{\mu_H(C_0)}.$$

The reader should find the above expression very suggestive, as it illustrates the connection between the $_*$-operation and classical conditional probability, as well as between independence of measure spaces and independence of events in the classical sense.

4. Types and formulae in pseudofinite fields

A field F is called *pseudofinite* if it is perfect, with absolute Galois group $\hat{\mathbb{Z}}$ and it is pseudo-algebraically closed, i.e. every geometrically irreducible variety over F has an F-rational point. The main examples of pseudofinite fields are nonprincipal ultraproducts of finite fields. In the rest of the paper we shall always consider a pseudofinite field F with a distinguished topological generator σ_F of its Galois group. Also, we shall implicitly identify the unique extension F_n of F of degree n with the fixed field of σ_F^n. As we shall see later, the important results will not depend on the choice of a particular σ_F.

The following is a folklore description of complete types in pseudofinite fields.

Theorem 4.1. *Let F be a pseudofinite field, and let k be a (small) subfield. Then two tuples a and b have the same type over k if and only if there is an isomorphism of the relative algebraic closures $F \cap \overline{k(a)}$ and $F \cap \overline{k(b)}$ of $k(a)$ and $k(b)$ fixing k pointwise and taking a to b.*

Remark 4.2. If we let X/k be the variety whose generic point is a (or b), the type of a over k can then be identified with a conjugacy class in the absolute Galois group $\mathrm{G}(k(X))$ (later written just as $\mathrm{G}(X)$) of a procyclic subgroup corresponding to the relative separable closure of the function field of X inside F. Intuitively, X is the quantifier-free positive part of the type.

Conversely, each conjugacy class of a procyclic subgroup of $\mathrm{G}(X)$ corresponds to a type.

A description of formulae in pseudofinite fields requires a bit more precision, based on the work in [11], [10]. We start with a general consideration of Galois stratification, adopting the more geometric language from [7] and [8].

Let A be an integral and normal variety. A morphism of varieties $C \to A$ is a *Galois cover*, if C is integral, h is étale, and there is a finite group $G = G(C/A)$ acting on C such that h induces the isomorphism $C/G \simeq A$.

A Galois cover $C \to A$ is *coloured*, if $G(C/A)$ is equipped with a family Con of subgroups stable by conjugation. In case when Con is a family of cyclic subgroups (the case that will occur in our applications), we may equivalently equip it with the $\mathbb{Q}$-central (see 3.9) conjugacy class of elements $\{\tau : \langle\tau\rangle \in \mathrm{Con}\}$.

Let S be an integral normal scheme and let $X \to S$ be a variety over S. A *normal stratification* of X,

$$\langle X, C_i/A_i : i \in I\rangle,$$

is a partition of X into a finite set of integral normal locally closed S-subschemes A_i, each equipped with a Galois cover $C_i \to A_i$. A *Galois stratification*

$$\mathcal{A} = \langle X, C_i/A_i, \mathrm{Con}(A_i) | i \in I\rangle$$

consists of a normal stratification in which each Galois cover C_i/A_i is coloured (by $\mathrm{Con}(A_i)$).

Let S be an integral and normal scheme and let $X \to S$ be a variety over S. For a point $s \in S$, associated with a canonical morphism $\mathrm{Spec}\,\mathbf{k}(s) \to S$, we denote by X_s the fibre of X over s ($X_s = X \times_S \mathbf{k}(s)$). Let $\mathcal{A} = \langle X, C_i/A_i, \mathrm{Con}(A_i) | i \in I\rangle$ be a Galois stratification of X, let $s \in S$, let K be a field containing $\mathbf{k}(s)$ and let $x \in A_{i,s}(K)$. The *Artin symbol*, $\mathrm{Ar}(C_i/A_i, s, x)$ is the conjugacy class of subgroups of $G(C_i/A_i)$ consisting of the decomposition subgroups at x. More precisely, considering the map corresponding to x, $\mathrm{Spec}(K) \to A_{i,s} \to A_i$, we have the induced map

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \pi_1(A_{i,s}) \to \pi_1(A_i) \to G(C_i/A_i),$$

and $\mathrm{Ar}(C_i/A_i, s, x)$ is its image, defined up to conjugacy (see 5.1 for the definition of π_1).

In case $\mathrm{Gal}(K^{\mathrm{sep}}/K) \simeq \hat{\mathbb{Z}}$ and we are given its distinguished topological generator σ_K, we shall write $\sigma_{K,x}$ for the image of σ_K by the above maps in either of $\pi_1(A_{i,s})$, $\pi_1(A_i)$, as well as in $G(C_i/A_i)$, and we have that $\mathrm{Ar}(C_i/A_i, s, x) = \langle\sigma_{K,x}\rangle$ (up to conjugacy). Sometimes it is also denoted $\mathrm{ar}(C_i/A_i, s, x)$. When $K = k$ is a finite field, we have a canonical generator of its Galois group, namely F_k, the geometric Frobenius automorphism. In that case, we speak of the *local Frobenius* $F_{k,x}$ (cf. 5.1).

When $S = \mathrm{Spec}(k)$ for a field k, we only have one meaningful parameter, the generic point of S, and in that case we suppress the parameter in the above notation. If $x : \mathrm{Spec}\, K \to S$ corresponds to the inclusion $k \subseteq K$, we write $\sigma_{K,k}$ in place of $\sigma_{K,x}$ in $\mathrm{G}(k)$.

Let $\mathcal{A}$ be a Galois stratification over S, as above. We will call an expression of the form

$$\mathcal{A}_s := \{x \in X_s : \mathrm{Ar}(x) \subseteq \mathrm{Con}(\mathcal{A})\}$$

a *Galois formula*, with parameters $s \in S$. It is a formula in the sense that it gives rise to a 'realisation' functor from the category of fields containing $\mathbf{k}(s)$ with elementary embeddings as morphisms into the category of sets. Indeed, for a field K containing $\mathbf{k}(s)$ and $x \in A_{i,s}(K)$, we write $\mathrm{Ar}(x) \subseteq \mathrm{Con}(\mathcal{A})$ for $\mathrm{Ar}(C_i/A_i, s, x) \subseteq \mathrm{Con}(C_i/A_i)$, and we can consider the *set of K-valued points of $\mathcal{A}_s$,*

$$\mathcal{A}_s(K) := \{x \in X_s(K) : \mathrm{Ar}(x) \subseteq \mathrm{Con}(\mathcal{A})\}.$$

The following result shows that in pseudofinite fields these are the only formulae that need be considered.

Theorem 4.3. *Let $\theta(Y_1, \ldots, Y_n)$ be a formula in the language of rings with coefficients in a field k. There exists a Galois stratification $\mathcal{A}$ of $\mathbb{A}^n_k$ such that for every pseudofinite field F containing k,*

$$\theta(F) = \mathcal{A}(F).$$

Conversely, each Galois stratification $\mathcal{A} = \langle X, C_i/A_i, \mathrm{Con}(C_i, A_i)\rangle$ over k, with every $\mathrm{Con}(C_i/A_i)$ a conjugacy domain of cyclic subgroups of $G(C_i/A_i)$, corresponds to a formula in the language of rings with coefficients in k.

Having in mind that set of realisations of a complete type over k is the intersection of definable sets over k that contain it, and this description of definable sets, it becomes transparent that 4.2 is the 'limit' stage of 4.3.

When we wish to transfer the uniform aspects of the behaviour of finite fields to pseudofinite fields, the following variant is of great interest.

Theorem 4.4. *Let $S = \mathrm{Spec}(R)$ be an affine variety over $\mathbb{Z}$, integral and normal. Let $\theta(Y_1, \ldots, Y_n)$ be a formula in the language of rings with coefficients in the ring R. There exists a nonzero $f \in R$ and a Galois stratification $\mathcal{A}$ of $\mathbb{A}^n_{S_f}$ such that for every closed point s in the localisation S_f,*

$$\theta_s(\mathbf{k}(s)) = \mathcal{A}_s(\mathbf{k}(s)).$$

Conversely, each Galois formula over S, where all the colourings are conjugacy domains of cyclic subgroups, corresponds to a formula in the language of rings with coefficients in R.

Remark 4.5. All the above results remain valid in the language of formulae of the form

$$\mathcal{A}_s := \{x \in X_s : \mathrm{ar}(x) \subseteq \mathrm{con}(\mathcal{A})\},$$

provided each conjugacy domain of subgroups $\mathrm{Con}(C_i/A_i)$ is replaced by the $\mathbb{Q}$-central (cf. 3.9) class $\mathrm{con}(C_i/A_i) := \{\tau : \langle\tau\rangle \subseteq \mathrm{Con}(C_i/A_i)\}$. When we wish to pass from the formulae of this form toward the formulae in the language of rings, it is important to require all the $\mathrm{con}(C_i/A_i)$ to be $\mathbb{Q}$-central.

5. Finite fields and cohomology

This section is based on masterful introductions to the subject given in [14] and [15], Chapter 9. We must emphasise here that the machinery of cohomology and Deligne's theory of weights is not strictly necessary for our purposes. One can obtain all the results using just Lang-Weil estimates and methods of [2], [10]. On the other hand, the author finds the conceptual clarity, gained from the more advanced methods, indispensable.

5.1. **Fundamental group.** Given any connected scheme X and any geometric point ξ of X (i.e., a point with values in some algebraically closed field), we have the profinite étale fundamental group $\pi_1(X,\xi)$, which classifies finite étale coverings of X. This gives a covariant functor on the category of pointed schemes (X,ξ). As in topology, varying ξ just changes $\pi_1(X,\xi)$ up to an inner automorphism. Thus we shall usually omit the base point, writing $\pi_1(X)$, when we only require calculations up to conjugacy. In the special case $X = \mathrm{Spec}(k)$, where k is a field, a geometric point ξ is just a choice of an algebraically closed overfield L of k, and $\pi_1(X,\xi)$ is just the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, where k^{sep} is the separable closure of k in L.

Another interesting special case is that of normal X. If a connected variety X is normal, it is irreducible, say with generic point η. Its function field K is the residue field $\mathbf{k}(\eta)$. If we view an algebraic closure $\bar{K}$ of K as a geometric generic point $\bar{\eta}$ of X, the group $\pi_1(X,\bar{\eta})$ is the quotient of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ which classifies those finite separable extensions L/K with the property that the normalisation of X in L is finite étale over X (i.e., unramified).

Given a connected scheme X, a field k and a k-valued point $x \in X(k)$, the associated morphism $\mathrm{Spec}\, k \to X$ induces a group homomorphism

$$\pi_1(\mathrm{Spec}(k)) = \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \pi_1(X),$$

well-defined up to conjugacy. If k is a finite field, the conjugacy class in $\pi_1(X)$ which is the image of the geometric Frobenius $F_k \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$ is denoted by $F_{k,x}$ and called the *local Frobenius at x*.

If X is geometrically irreducible, we have the short exact sequence for the fundamental group:

$$1 \longrightarrow \pi_1^{\mathrm{geom}}(X,\bar{\eta}) \longrightarrow \pi_1(X,\bar{\eta}) \longrightarrow \mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow 1,$$

where $\pi_1^{\mathrm{geom}}(X,\bar{\eta}) = \pi_1(X \times_k \bar{k},\bar{\eta})$ is the *geometric* fundamental group. As a contrast, π_1 is sometimes called the *arithmetic* fundamental group.

5.2. **Constructible sheaves.** Let X be a connected and normal scheme and l a prime invertible in X. A *lisse $\bar{\mathbb{Q}}_l$-sheaf* $\mathcal{F}$ of rank r on X is an r-dimensional continuous $\bar{\mathbb{Q}}_l$-representation of $\pi_1(X,\bar{\eta})$. A *constructible $\bar{\mathbb{Q}}_l$-sheaf* $\mathcal{F}$ on X is an étale sheaf such that X can be written as a union of finitely many locally closed subschemes U_i such that each $\mathcal{F} \restriction U_i$ is lisse.

Given a morphism $\phi : X \to Y$, we have the *inverse image* functor from sheaves on Y to sheaves on X, $\mathcal{G} \mapsto \phi^*\mathcal{G}$ on X, its right and left adjoints, the *direct image* functor $\mathcal{F} \mapsto \phi_*\mathcal{F}$ and the *direct image with proper support* $\mathcal{F} \mapsto \phi_!\mathcal{F}$, the *higher direct images* $R^i\phi_*$ as the derived functors of ϕ_* and the *higher direct images with proper supports* $R^i\phi_!$ (which are not quite the derived functors of $\phi_!$).

When $\mathcal{F}$ and $\mathcal{G}$ are lisse and X and Y are spectra of fields (one-point spaces), these notions coincide with the corresponding operations on representations, with respect to the map $\pi_1(\phi) : \pi_1(X) \to \pi_1(Y)$.

We fix a non-canonical embedding $\iota : \bar{\mathbb{Q}}_l \to \mathbb{C}$. Let X be a variety over $\mathbb{Z}[1/l]$ which is normal and connected, and let $\mathcal{F}$ be a lisse $\bar{\mathbb{Q}}_l$-sheaf on X, with associated representation ρ. For any real number w, we say that $\mathcal{F}$ is *ι-pure of weight w* if for every finite field k and every $x \in X(k)$, every eigenvalue of $\rho(F_{k,x})$ has, via ι, complex absolute value $|k|^{w/2}$.

If $\mathcal{F}$ is a constructible sheaf, it is said to be *punctually ι-pure of weight w* if there exists a partition of X into a finite number of locally closed subschemes U_i (each normal and connected), such that $\mathcal{F} \restriction U_i$ is ι-pure of weight w. A constructible sheaf $\mathcal{F}$ is *ι-mixed of weight $\leq w$* if it is a successive extension of finitely many constructible $\bar{\mathbb{Q}}_l$-sheaves $\mathcal{F}_i$ on X, with each $\mathcal{F}_i$ punctually ι-pure of some weight $w_i \leq w$.

Let k be a finite field. For any integer n, we denote by $\bar{\mathbb{Q}}_l(n)$ the lisse, rank one sheaf on $\mathrm{Spec}(k)$ which, as a character of $\mathrm{Gal}(\bar{k})$, takes the value $|k|^{-n}$ on F_k. Thus $\bar{\mathbb{Q}}_l(n)$ is ι-pure of weight $-2n$, for any ι. If X is a normal connected variety over k, we also denote by $\bar{\mathbb{Q}}_l(n)$ the sheaf on X obtained by pullback from k. Given another sheaf $\mathcal{F}$ on X, we write $\mathcal{F}(n)$ for $\mathcal{F} \otimes \bar{\mathbb{Q}}_l(n)$.

In our applications we only encounter sheaves with an additional finiteness property. We will say that a lisse $\bar{\mathbb{Q}}_l$-sheaf on X has property (FQ), if it factors through a finite quotient of $\pi_1(X)$, i. e., it comes from a finite Galois cover of X. A constructible sheaf on X is said to have the property (FQ) if it does on each piece of X where it is lisse. For such sheaves, passing between their $\bar{\mathbb{Q}}_l$- and $\mathbb{C}$-incarnations is not that 'non-canonical' (the values of characters are just sums of roots of unity). Moreover, there are no issues related to continuity with respect to different topologies. We must keep the formulations with l-adic sheaves since we wish to use the l-adic cohomology below, but the translation to $\mathbb{C}$ becomes 'second nature'. It is obvious that the property (FQ) is preserved by inverse images of sheaves, and, even though higher direct images do not preserve (FQ), we have the following.

Fact 5.1. *Let $\phi : X \to Y$ be a map of S-schemes of finite type and $\mathcal{F}$ a constructible (FQ)-sheaf on X. Then:*

(1) *$\phi_! \mathcal{F}$ is constructible with (FQ);*
(2) *$\phi_* \mathcal{F}$ is constructible with (FQ) over an open dense subset of S.*

Proof. The item (1) follows from the existence of Stein factorisation of proper maps and the fact that (FQ) is clearly preserved by direct images via finite maps and maps with geometrically connected fibres.

The item (2) can easily be extrapolated, using (1), from Deligne's proof of Théorème 1.9 from [4]. □

5.3. **Cohomology.** Now we consider an important class of examples of lisse sheaves. Let X be connected and normal over a finite field k with l invertible in k and let $\mathcal{F}$ be a constructible sheaf. We have ordinary *cohomology groups* and *cohomology groups with compact support*,

$$H^i(\bar{X}, \mathcal{F}) \quad \text{and} \quad H^i_c(\bar{X}, \mathcal{F}),$$

which are finite-dimensional $\bar{\mathbb{Q}}_l$-vector spaces on which $\mathrm{Gal}(\bar{k}/k)$ acts continuously, and which vanish unless $i \in \{0, \dots, 2d\}$. They can be considered as lisse sheaves on k. These groups are related by Poincaré duality. Namely, if $\mathcal{F}$ is lisse, X is smooth of dimension d, and $\mathcal{F}^\vee$ is the contragredient sheaf, the cup-product pairing

$$H^i_c(\bar{X}, \mathcal{F}) \times H^{2d-i}(\bar{X}, \mathcal{F}^\vee) \to H^{2d}_c(\bar{X}, \bar{\mathbb{Q}}_l) \simeq \bar{\mathbb{Q}}_l(-d)$$

is a $\mathrm{Gal}(\bar{k}/k)$-equivariant pairing.

The Diophantine interest of these cohomology groups is illustrated by the following. For a constructible sheaf $\mathcal{F}$ on X of dimension d over a finite field k, and every finite extension E/k, the Grothendieck-Lefschetz trace formula gives that

$$\sum_{x \in X(E)} \mathrm{Trace}(F_{E,x} | \mathcal{F}) = \sum_{i=0}^{2d} (-1)^i \mathrm{Trace}(F_E | H^i_c(\bar{X}, \mathcal{F})).$$

The L-function associated to this data is a fundamental Diophantine invariant. If we let $S_n := \sum_{x \in X(k_n)} \mathrm{Trace}(F_{k_n,x} | \mathcal{F})$, where k_n is the extension of k of degree n, it is defined as the formal power series

$$L(X/k, \mathcal{F}; T) := \exp\big(\sum_n \frac{S_n}{n} T^n\big).$$

Having in mind that $F_{k_n} = F_k^n$ and the above trace formula, we see that

$$L(X/k, \mathcal{F}; T) = \prod_{i=0}^{2d} \det(1 - TF_k | H^i_c(\bar{X}, \mathcal{F})),$$

i.e., it is a rational function. Poincaré duality yields a certain functional equation in case of a smooth and proper variety, and thus quickly establishes the first part of the Weil conjectures ([14]).

Moreover, Deligne has shown ([6]) that if we start with $\mathcal{F}$ ι-pure of weight $\leq w$, then H^i_c, considered as constructible sheaves on $\mathrm{Spec}(k)$, are ι-mixed of weight $\leq w + i$, which can be used to estimate the size of the above character sums. In particular, this trivially implies the Lang-Weil estimates, when applied to the constant sheaf $\bar{\mathbb{Q}}_l$. We do not expand this in more detail since we perform exactly this kind of calculation in 5.3 below.

The above machinery suffices to treat the case of a single variety X over a finite field k. What can we say in the case when we are given a family $X \to S$? The latter case is certainly of crucial importance for us, because in order to be able to lift our considerations to pseudofinite fields, we need to study uniformities in such families.

Suppose that S is a connected normal variety over $\mathbb{Z}[1/l]$, and let $\pi : X \to S$ be a normal and connected variety over S. Let $\mathcal{F}$ be a constructible sheaf on X. The answer to our question is given by the higher direct images of $\mathcal{F}$ with compact support, namely $R^i\pi_!\mathcal{F}$, which are constructible sheaves on S. Given a finite field k and any point $s \in S(k)$, by the proper base change theorem we have the *specialisation* property:

$$s^*(R^i\pi_!\mathcal{F}) = H^i_c(X_s, \mathcal{F}_s),$$

where X_s is the fibre over s, and $\mathcal{F}_s$ is the restriction (pullback) of $\mathcal{F}$ to X_s. Moreover, Deligne's main result from [6] shows that if $\mathcal{F}$ is ι-mixed of weight $\leq w$, then $R^i\pi_!\mathcal{F}$ are ι-mixed of weight $\leq w + i$. Thus we have all the necessary tools to treat our problem.

5.4. **Uniform estimates.**

Lemma 5.2. *Let $X \xrightarrow{\psi} \mathrm{Spec}(k)$ be a smooth connected variety of dimension d over a field k and let $\mathcal{F}$ be a lisse $\bar{\mathbb{Q}}_l$-sheaf on X, with corresponding character χ.*

(1) *$H^0(\bar{X}, \mathcal{F}) = \psi_*\mathcal{F}$, as a lisse sheaf on $\mathrm{Gal}(\bar{k}/k)$;*
(2) *$H^{2d}_c(\bar{X}, \mathcal{F}) = (\psi_*\mathcal{F}^\vee)^\vee(-d)$, the Tate-twisted dual of $H^0(\bar{X}, \mathcal{F}^\vee)$.*

When X is geometrically connected, the above can be written as

(1') *$H^0(\bar{X}, \mathcal{F}) = \mathcal{F}^{\pi_1^{geom}(X)}$, the $\pi_1^{geom}(X)$-invariants of $\mathcal{F}$, with the natural action of $\mathrm{G}(\bar{k}/k)$;*
(2') *$H^{2d}_c(\bar{X}, \mathcal{F}) = \mathcal{F}_{\pi_1^{geom}(X)}(-d)$, the Tate-twisted $\pi_1^{geom}(X)$-coinvariants of $\mathcal{F}$, with the natural action of $\mathrm{G}(\bar{k}/k)$.*

When $\mathcal{F}$ has (FQ), the character corresponding to $H^0(\bar{X}, \mathcal{F})$ is $\psi_*\chi$, and the character corresponding to $H^{2d}_c(\bar{X}, \mathcal{F})$ is $\psi_*\chi(-d)$.

Proof. Item (1) is by definition, and (2) follows from Poincaré duality. For (1') and (2'), we use the short exact sequence for the fundamental group of X, together with 3.14, while the statement regarding the (FQ)-property follows from 5.1. □

In the theorem below, we use k_n to denote the unique extension of degree n of a finite field k.

Theorem 5.3. *Let S be a connected normal variety over $\mathbb{Z}$ and let X be an S-scheme of finite type with normal fibres. For a prime l invertible in S, we fix a (noncanonical) embedding $\iota : \bar{\mathbb{Q}}_l \to \mathbb{C}$. Let $\mathcal{F}$ be a constructible $\bar{\mathbb{Q}}_l$-sheaf on X, which is (ι-)pure of weight 0 with $\mathbb{Q}$-central character χ.*

(1) *There is a constructible sheaf $\mathcal{A}$ with character α on S such that for every finite field k, every $s \in S(k)$ and every n,*

$$\left| \sum_{x \in X_s(k_n)} \iota\chi(F_{k_n,x}) - \iota\alpha(F^n_{k,s})|k_n|^{\dim(X_s)} \right| \leq C|k_n|^{\dim(X_s)-1/2}.$$

(2) *On each (of the finitely many) locally closed subscheme S' of S where $\mathcal{A}$ is locally constant, there are conjugacy classes C_i of procyclic subgroups D_i of $\pi_1(S')$ and characters $\alpha_i : \hat{\mathbb{Z}} \to \bar{\mathbb{Q}}_l$ such that for every finite field k, every $s \in S'(k)$ with $\langle F_s \rangle \in C_i$, in view of the canonical isomorphism $\mathrm{Gal}(\bar{k}/k) \simeq \hat{\mathbb{Z}}$, $F_k \mapsto 1$, we have $\alpha_s := s^*(\alpha) \simeq \alpha_i$. Moreover, for every n,*

$$\left| \sum_{x \in X_s(k_n)} \iota\chi_s(F_{k_n,x}) - \iota\alpha_i(n)|k_n|^{\dim(X_s)} \right| \le C|k_n|^{\dim(X_s)-1/2}.$$

(3) *When $\mathcal{F}$ has (FQ), the sheaf $\mathcal{A}$ from (1) has (FQ) over some localisation $\mathbb{Z}[1/m]$ of $\mathbb{Z}$. In particular, the statement of (2) becomes the following. There is a finite number of $\mathbb{Q}$-central characters α_j such that for all but finitely many finite fields k, and $s \in S(k)$, $\alpha_s \simeq \alpha_j$ for some j. Moreover, for each j there is a formula θ_{α_j} in the language of rings such that*

$$\{s \in S(k) : \alpha_s \simeq \alpha_j\} = \theta_{\alpha_j}(k).$$

Proof. (1) If we denote by π the structure map $X \to S$, the main result of Deligne Weil II tells us that $R^i\pi_!\mathcal{F}$ is constructible ι-mixed of weight i.

Furthermore, by specialisation,

$$s^*(R^i\pi_!\mathcal{F}) = H^i_c(\bar{X}_s, \mathcal{F}_s),$$

and $R^{2d}\pi_!\mathcal{F}$ is clearly ι-pure of weight $2d$. Let

$$\mathcal{A} := R^{2d}\pi_!\mathcal{F} \otimes \bar{\mathbb{Q}}_l(d),$$

and let α be its character. Clearly, for $s^*(\mathcal{A}) = \mathcal{A}_s$, $\mathcal{A}_s(F_{k_n}) = \mathcal{A}(F_{k_n,s})$.

Grothendieck's trace formula gives that for every $s \in S(k)$ and n,

$$\sum_{x \in X_s(k_n)} \chi(F_{k_n,x}) = \sum_{x \in X_s(k_n)} \chi_s(F_{k_n,x}) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(F_k^n | H^i_c(\bar{X}_s, \mathcal{F}_s)).$$

From this we get the estimate

$$\left| \sum_{x \in X_s(k_n)} \iota\chi(F_{k_n,x}) - \iota\alpha_s(F_k^n)|k_n|^d \right| \le \sum_{i=0}^{2d-1} |k_n|^{i/2} \dim(H^i_c(\bar{X}_s, \mathcal{F}_s)).$$

The standard constructibility and base change theorems for étale cohomology yield a uniform bound on the sum of the Betti numbers on the right, as desired.

Item (2) is essentially a self-explanatory restatement of (1). Since α is a $\mathbb{Q}$-central character on $\pi_1(S')$, various restrictions $\alpha \restriction_{\langle F_{k,s} \rangle}$ clearly depend only on the $\mathbb{Q}$-central class of $F_{k,s}$.

(3) Since we are only considering estimates, we may assume $X \to S$ is smooth. By Poincaré duality, $\mathcal{A}$ has (FQ), since, by 5.1, we know it for $R^0\pi_*\mathcal{F} = \pi_*\mathcal{F}$.

Thus, S can be written as a finite union of locally closed subschemes S' such that the number of possible α_i is finite and for every S', the set

$$\{s \in S'(k) : \alpha_s \simeq \alpha_i\} = \{s \in S'(k) : F_{k,s} \in C_i\}$$

is given by a coloured Galois covering. By renumbering the possible α_i for all S', we get finitely many α_j such that $\{s \in S(k) : \alpha_s \simeq \alpha_i\}$ is given by a Galois formula $\theta_{\alpha_i}(k)$. By 4.4, it is equivalent to a formula in the language of rings. □

5.5. **Generic constructible sheaves.** Since we shall only be interested in estimates as in 5.3, and measure-theoretic applications in pseudofinite fields as introduced in 6, it turns out that it suffices to consider only the information contained in the *generic stalks* of constructible (FQ)-sheaves, which leads us to the following definition.

Definition 5.4. Let X be a variety with connected components X_i. A *generic sheaf* $\mathcal{F}$ on X is a family of continuous complex finite-dimensional representations $\mathcal{F}_i$ of $\mathrm{G}(X_i) := \mathrm{Gal}(\mathbf{k}(X_i))$.

The usual operations with constructible sheaves apply here as well, except that they become easier to describe explicitly.

Definition 5.5. Let $\phi : X \to X'$ be a map of finite type, dominant on some component (let us denote the components of X by X_i, and those of X' by X'_j. Let $\mathcal{F}$ be a generic sheaf on X and $\mathcal{F}'$ on X'.

1. The *inverse image* $\phi^* \mathcal{F}'$ is a sheaf on X such that, if $\phi(X_i) \subseteq X'_j$, its restriction on X_i is obtained by the obvious pullback of representations via the induced map $\mathrm{G}(X_i) \to \mathrm{G}(X'_j)$.
2. We would like the *direct image* $\phi_* \mathcal{F}$ to satisfy the relation
$$\mathrm{Hom}(\mathcal{F}, \phi^* \mathcal{F}') = \mathrm{Hom}(\phi_* \mathcal{F}, \mathcal{F}'),$$
for every $\mathcal{F}'$ on X'. Clearly, on X'_j, $\phi_* \mathcal{F}$ is given as
$$\prod_{\phi(X_i) \subseteq X'_j} \phi_* \mathcal{F}_i,$$
where the $\phi_* \mathcal{F}_i$ above is the induced representation via $\mathrm{G}(X_i) \to \mathrm{G}(X'_j)$ in the sense of 3.11.

It is clear that for such an $\mathcal{F}$ and a generic geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$ of X (where $\Omega \supseteq \mathbf{k}(X)$ is separably closed) we can speak of the stalk $\mathcal{F}_{\bar{x}} = \bar{x}^* \mathcal{F}$. As already noted in 3.11, a completely arbitrary induction can become infinite-dimensional. The following result shows that this does not happen in our case.

Proposition 5.6. *The category of generic sheaves is stable by operations of inverse and direct image via (dominant) morphisms of finite type.*

Proof. The claim is trivial for the inverse image. For the direct image, we may reduce to the case where both X and X' are integral and the corresponding function field extension is separable. Then we can consider the tower

$$\mathbf{k}(X') \longrightarrow \overline{\mathbf{k}(X')} \cap \mathbf{k}(X) \longrightarrow \mathbf{k}(X)$$

where the first extension is finite, and the second regular, a 'baby case' of Stein factorisation. The direct images by the two types of maps preserve finite-dimensionality, as shown below. □

Lemma 5.7. *Let $\phi : X \to X'$ be a map of finite type between varieties and let V be the* $\mathrm{G}(X)$*-space associated with a generic sheaf $\mathcal{F}$ on X.*

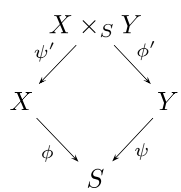
(1) *When ϕ is generically finite of degree d, the space associated with $\phi_* \mathcal{F}$ is V^d.*
(2) *When ϕ is dominant with geometrically connected fibres,*

$$\phi_* \mathcal{F} \simeq \mathcal{F}^K,$$

the K-invariants of $\mathcal{F}$, where K is the kernel of the surjection $\mathrm{G}(X) \to \mathrm{G}(X')$.

Proof. When ϕ is generically finite of degree d, the group $\mathrm{G}(X)$ is a subgroup of index d in $\mathrm{G}(X')$. When ϕ is dominant with geometrically connected fibres, the associated function field extension is regular so the map $\mathrm{G}(X) \to \mathrm{G}(X')$ is a surjection and we recall 3.14. □

Theorem 5.8 (Properties of generic sheaves)**.** *Suppose we have the diagram below, and let $\mathcal{F}$ be a generic sheaf on X, $\mathcal{G}$ on Y and $\mathcal{H}$ on S.*



Then we have:

(1) (Base change). $\psi^* \phi_* \mathcal{F} \simeq \phi'_* \psi'^* \mathcal{F}$;
(2) (Projection formula). $\phi_*(\phi^* \mathcal{H} \otimes \mathcal{F}) \simeq \mathcal{H} \otimes \phi_* \mathcal{F}$;
(3) (Künneth formula). *Writing $\mathcal{F} \boxtimes \mathcal{G}$ for $\psi'^* \mathcal{F} \otimes \phi'^* \mathcal{G}$,*

$$(\phi \times \psi)_*(\mathcal{F} \boxtimes \mathcal{G}) \simeq \phi_* \mathcal{F} \otimes \psi_* \mathcal{G}.$$

Proof. All these claims follow immediately from the known results for constructible sheaves. However, in view of the simple description of direct and inverse images in 5.5, it is possible to give straightforward proofs which we sketch here.

For (1), it suffices to prove the equality of generic geometric stalks, which reduces to the case when $Y = \mathrm{Spec}(\Omega)$ for a large algebraically closed field Ω. Using the factorisation of maps described in the proof of 5.6, we decompose ϕ into a composition of a generically finite map and a map with generically connected fibres. The claim follows since it is compatible with composition on the ϕ-side, and it is easy in each of the above cases.

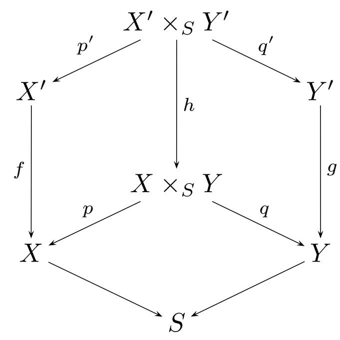


FIGURE 1. Commutative diagram for 5.9

Claim (2) is easy, since it already holds at the level of representations, as well as measures. The Künneth formula (3) follows formally from the previous two:

$$\phi_* \psi'_* (\psi'^* \mathcal{F} \otimes \phi'^* \mathcal{G}) \overset{(2)}{=} \phi_* (\mathcal{F} \otimes \psi'_* \phi'^* \mathcal{G}) \overset{(1)}{=} \phi_* (\mathcal{F} \otimes \phi^* \psi_* \mathcal{G}) \overset{(2)}{=} \phi_* \mathcal{F} \otimes \psi_* \mathcal{G}\,.$$

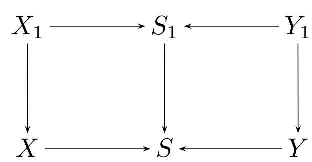
□

The Künneth formula is usually stated and used in the above form, but we can prove a more sophisticated variant. Let us fix the notation as in Figure 1. Given generic sheaves $\mathcal{K}$ on X, $\mathcal{K}'$ on X', $\mathcal{L}$ on Y and $\mathcal{L}'$ on Y', we shall write $\mathcal{K} \boxtimes \mathcal{L}$ for $p^*\mathcal{K} \otimes q^*\mathcal{L}$, and $\mathcal{K}' \boxtimes \mathcal{L}'$ for $p'^*\mathcal{K}' \otimes q'^*\mathcal{L}'$.

Theorem 5.9 (Relative Künneth)**.** *For each generic sheaf $\mathcal{F}$ on X' and $\mathcal{G}$ on Y',*

$$f_* \mathcal{F} \boxtimes g_* \mathcal{G} \simeq h_* (\mathcal{F} \boxtimes \mathcal{G}).$$

Proof. As in [5], let us consider a commutative diagram



and let $Z_1 = X_1 \times_{S_1} Y_1$, $X'_1 = X' \times_X X_1$, $Y'_1 = Y' \times_Y Y_1$, $Z'_1 = X'_1 \times_{S_1} Y'_1 = Z' \times_Z Z_1$, so that we obtain a morphism of diagrams

$$C : (S_1, X_1, Y_1, Z_1, X'_1, Y'_1, Z'_1) \to (S, X, Y, Z, X', Y', Z').$$

Using base change, we see that $C^*(f_* \mathcal{F} \boxtimes g_* \mathcal{G}) \simeq C^* h_* (\mathcal{F} \boxtimes \mathcal{G})$ if and only if $f_{1*} C^* \mathcal{F} \boxtimes g_{1*} C^* \mathcal{G} \simeq h_{1*}(C^* \mathcal{F} \boxtimes C^* \mathcal{G})$.

However, to prove the theorem, it is enough to verify it in every generic geometric point of Z. In other words, the above base change can be taken with S_1 a spectrum of some separably closed field, and X_1 and Y_1 isomorphic to S_1. After such a base change, the claim reduces to 5.8(3). □

6. Pseudofinite fields and motivic measures

In most of this paper we will use a certain *approximative motivic measure* on pseudofinite fields, and only in Section 10 we shall use the more sophisticated motivic measure defined by Denef and Loeser in [7]. There are two reasons for this two-step approach. On one hand, the approximative measure is much easier to understand and work with; on the other, the second approach is not developed in nonzero characteristics, mainly for the lack of resolution of singularities.

The following is a slight generalisation of the result from [2], allowing integration with respect to the measure defined there. For the proof, we refer the reader to 5.3(3).

Theorem 6.1. *Let S be a connected normal variety over $\mathbb{Z}$ and let $Y \to X$ be a Galois cover of S-varieties with group G. Let $\chi : G \to \mathbb{C}$ be an irreducible $\mathbb{Q}$-central character. Then there is a finite number of (continuous) $\mathbb{Q}$-central characters $\alpha_i : \hat{\mathbb{Z}} \to \mathbb{C}$ and a constant $C > 0$ such that for every finite field k, its extension k_n of degree n, and every parameter $s \in S(k)$, there exists an i with*

$$\left| \sum_{x \in X_s(k_n)} \chi_s(F_x) - \alpha_i(n) |k_n|^{\dim(X_s)} \right| \leq C |k_n|^{\dim(X_s) - 1/2}.$$

Moreover, for every i there is a formula θ_{α_i} in the language of rings which defines, in each finite field k, the set of $s \in S(k)$ for which the above estimate holds.

Our goal is to define a motivic measure capturing the uniform behaviour of summing over the finite fields from the theorem above. We start by defining the ring where it will take values.

Definition 6.2. Let K_0^+ be the *rig* ('ring without negatives') $\mathbb{Q}^+[\mathbb{L}]/\sim$, where $p \sim q$ if p and q have the same degree and leading coefficient. Let $\mathrm{K}_0^{\mathrm{approx}} := \mathrm{K}_0^+ \otimes \mathbb{C}$.

Intuitively, the formal variable $\mathbb{L}$ is a symbol for the 'approximative Lefschetz motive', representing the size of the affine line over F. It is a substitute for the cardinality q of the line in the finite field case.

The *coarse* (or *approximative*, to use the terminology from [18]) *Euler characteristic* on a pseudofinite field F will be defined as $\mathrm{K}_0^{\mathrm{approx}}$-valued map ν_F, which will evaluate integrals of *definable* $\mathbb{C}$*-valued functions* (functions with finitely many $\mathbb{C}$-values and definable level-sets). We proceed in stages, defining ν_F on more and more general functions.

Definition 6.3. (1) Let $Y_0 \to X_0$ be a Galois covering of a normal irreducible variety X_0 over a field $k \subseteq F$ (k of finite type over $\mathbb{Z}$), with group G. Let $\chi : G \to \mathbb{C}$ be a $\mathbb{Q}$-central irreducible character. We may consider it as the fibre $Y_\eta \to X_\eta$ of a Galois covering $Y \to X$ over S, where S can be assumed to be a normal variety over $\mathbb{Z}$, $\mathbf{k}(S) = k$, and η is the generic point of S. We let

$$\int_{X_0} \chi(\sigma_{F,x})\ d\nu_F(x) := \alpha(1)\,\mathbb{L}^{\dim(X)},$$

where α is the unique character such that $F \models \theta_\alpha(\eta)$ (the uniqueness follows by arguing over large finite fields).

(2) In the above notation, if $f : G \to \mathbb{C}$ is an arbitrary $\mathbb{Q}$-central function, then it can be written as $f = \sum_i \lambda_i \chi_i$, with $\lambda_i \in \mathbb{C}$, and χ_i irreducible $\mathbb{Q}$-central characters. We let

$$\int_{X_0} f(\sigma_{F,x})\ d\nu_F(x) := \sum_i \lambda_i \int_{X_0} \chi_i(\sigma_{F,x})\ d\nu_F(x).$$

(3) If we have a geometrically normal stratification $\langle X, C_i/A_i\rangle$ and a function $f : X \to \mathbb{C}$ is given so that on each A_i, $f\restriction_{A_i}(x) = f_i(\sigma_{F,x})$, where f_i is a $\mathbb{Q}$-central function on $G(C_i/A_i)$, we define

$$\int_X f\ d\nu_F(x) := \sum_i \int_{A_i} f_i(\sigma_{F,x})\ d\nu_F(x).$$

In view of the description of definable sets using Galois stratifications, we see that (3) above is the most general form of a definable $\mathbb{C}$-valued functions, and thus our definition is complete.

Remark 6.4. To tie in with the considerations from 5, the fundamental importance of (FQ)-constructible $\mathbb{Q}$-central sheaves lies in the fact that they correspond to definable $\mathbb{C}$-valued functions in the following way. Let χ be the character of such a sheaf on X, let ι be an embedding of $\bar{\mathbb{Q}}_l$ into $\mathbb{C}$ and let F be a pseudofinite (or finite) field. The function $X(F) \to \mathbb{C}$ given by $x \mapsto \iota\chi(\sigma_{F,x})$ has finite range and definable level sets (using 4.3, 4.4). Moreover, every definable function on X can be written as a linear combination of such functions. In this spirit, 5.1 is a *quantifier elimination* result.

Definition 6.5. (1) If f is a definable $\mathbb{C}$-valued function, we let $\mu_F(f) \in \mathbb{C}$ be the number such that $\nu_F(f) = \mu_F(f)\,\mathbb{L}^m$, i. e., the leading coefficient of $\nu_F(f)$.

(2) If f is a definable $\mathbb{C}$-valued function on X, let $\mu_{X,F}(f)$ be the coefficient of $\mathbb{L}^{\dim(X)}$ in $\nu_F(f)$. Note that if the support of f is lower-dimensional, then $\mu_{X,F}(f) = 0$.

Remark 6.6. The integral $\int_X f\ d\nu_F$, as defined above, clearly has an interpretation in terms of summing over finite fields like in 6.1. As a consequence:

(1) ν_F is well-defined, i. e., the definition does not depend on the choice of particular stratifications;

(2) when $f = 1_\theta$ is a characteristic function of a definable set θ, our definition of $\mu_F(\theta) := \mu_F(1_\theta)$ coincides with the measure from [2] and [10] on F. Thus we call it the *CDM-measure* from this point onwards.

The following is proved in [2], in a slightly different notation.

Fact 6.7. *The assignment $\nu := \nu_F$ factors through the Grothendieck ring of definable sets over F. In fact, we have the following.*

(1) *if θ and θ' are definably bijective,*
$$\nu(\theta) = \nu(\theta');$$
(2) *if θ and θ' are definable subsets of the same affine space,*
$$\nu(\theta \cup \theta') = \nu(\theta) + \nu(\theta') - \nu(\theta \cap \theta');$$
(3) *if $\tau : \theta' \to \theta$ is a definable map such that for every $t \in \theta$, $\nu(\tau^{-1}(t)) = \nu_0$, then*
$$\nu(\theta') = \nu_0 \nu(\theta).$$

Let $\pi : X \to k$ be a (normal) variety of dimension d over a field k contained in a pseudofinite field F (recall that we fix some generator σ_F of its Galois group). Let us consider a $\mathbb{Q}$-central character χ of a constructible sheaf $\mathcal{F}$ on X with (FQ). Let $\tilde{\chi} : \mathrm{G}(X) \to \mathbb{C}$ be the character of the generic sheaf $\tilde{\mathcal{F}}$ corresponding to $\mathcal{F}$. With this notation, we have the following.

Theorem 6.8 (Definable Grothendieck's trace formula)**.**
$$\int_X \chi(\sigma_{F,x})\, d\nu_F(x) = \pi_* \tilde{\chi}(\sigma_{F,k})\, \mathbb{L}^d.$$

Proof. As usual, we consider X as a generic geometric fibre X_η of some family $\pi : X \to S$, with $\mathbf{k}(S) = k$, $\eta : \mathrm{Spec}(k) \to S$. By 6.3 and the proof of 5.3, the required integral is in fact equal to $\alpha_\eta(\sigma_{F,k})\, \mathbb{L}^d$, where α_η is the character of
$$(R^{2d}\pi_* \mathcal{F})_\eta(d) = H^{2d}_c(\bar{X}_\eta, \mathcal{F}_\eta)(d).$$
If we denote by $\tilde{\mathcal{F}}_\eta$ the generic sheaf on $\tilde{X}_\eta := \mathrm{Spec}(\mathbf{k}(X_\eta))$, by (U_i) the system of smooth dense neighbourhoods in X_η, and $\mathcal{F}_i := \mathcal{F}_\eta \restriction U_i$, we get (using Poincaré duality, birational invariance of H^{2d}_c, and the (FQ) property):
$$\begin{aligned}\pi_{\eta*}\tilde{\mathcal{F}}_\eta = H^0(\tilde{X}_\eta, \tilde{F}_\eta) &= \varinjlim_i H^0(\bar{U}_i, \mathcal{F}_i) = \varinjlim_i H^{2d}_c(\bar{U}_i, \mathcal{F}_i^\vee)^\vee(d) \\ &= H^{2d}_c(\bar{X}_\eta, \mathcal{F}_\eta^\vee)^\vee(d) = H^{2d}_c(\bar{X}_\eta, \mathcal{F}_\eta)(d).\end{aligned}$$
□

Note that, because of the properties of $\mathbb{Q}$-central functions, the right-hand side does not depend on the choice of σ, as expected.

7. L-functions

For the purpose of this section, let us fix a pseudofinite field F with $\mathrm{Gal}(F) = \langle\sigma\rangle$ and let $F_n = \mathrm{Fix}(\sigma^n)$ be the unique extension of degree n of F. Let us write ν_n and μ_n in place of ν_{F_n} and μ_{F_n} and let the embedding of $\mathrm{K}_0^+(F_n) \to \mathrm{K}_0^+(F)$ be given by $\mathbb{L}_{F_n} \mapsto \mathbb{L}_F^n$.

Definition 7.1. The *(approximative) zeta function* of a formula θ is the following formal power series:

$$Z(\theta/F;T) = \exp\left(\sum_{n=1}^{\infty} \frac{\nu_n(\theta)}{n} T^n\right).$$

Definition 7.2. Let $Y \to X$ be a Galois cover over F with group G, and let $\chi : G \to \mathbb{C}$ be a $\mathbb{Q}$-central function. We let the corresponding *(approximative) L-function* be the following:

$$L(Y/X,\chi;T) = \exp\left(\sum_{n=1}^{\infty} \frac{\nu_n(\chi)}{n} T^n\right).$$

Similarly, if X is a variety over F and χ is a $\mathbb{Q}$-central continuous function on $\pi_1(X)$, the same formula can be used (after Section 8) to define the 'fancy' L-function $L(X,\chi;T)$. When we wish to speak about $L(Y/X,\chi)$ without explicitly mentioning X, we write $L(Y/G,\chi)$. Sometimes we substitute $T = \mathbb{L}^{-s}$ and write $L(X,\chi;s)$.

Let us list some formal properties of L-functions (cf. [12], [19]). Proofs are left as an exercise for the reader.

Proposition 7.3.

(1) $L(X,\chi+\chi') = L(X,\chi)L(X,\chi')$.

(2) *If Y is the disjoint union of the Y_i, with Y_i stable by G for each i,*

$$L(Y/G,\chi;s) = \prod_i L(Y_i/G,\chi;s),$$

with absolute convergence for $\Re(s) > \dim(X)$.

(3) *Let $\phi : G \to G'$ be a homomorphism, and let $\phi_* Y = Y \times^G G'$ be the scheme deduced from Y by 'extension of the structural group'. Let χ' be a character of G' and let $\phi^*\chi' = \chi' \circ \phi$ be the corresponding character of G. We have*

$$L(Y/G,\phi^*\chi') = L(\phi_* Y/G',\chi').$$

(4) *Let $\phi : G' \to G$ be a homomorphism, and let $\phi^* Y$ denote the scheme Y on which G' operates through ϕ. Let χ' be a character of G' and let $\phi_*\chi'$ be its direct image in the sense of 3.11, which is a character of G. We have*

$$L(Y/G,\phi_*\chi') = L(\phi^* Y/G',\chi').$$

(5) *Let $Y = \mathrm{Spec}(F_n)$, $X = \mathrm{Spec}(F)$, $G = \mathrm{Gal}(F_n/F) \simeq \mathbb{Z}/n\mathbb{Z}$, and χ an irreducible character of G. Then*

$$L(Y/X,\chi;T) = \frac{1}{1-\chi(\sigma)T},$$

where σ is the generator of G.

(6) *If $\chi = 1$ (unit character), $L(Y/X,1) = Z(X)$.*

(7) *If $\chi = r$ (character of the regular representation), $L(Y/X,r) = Z(Y)$.*

Theorem 7.4 (Near-rationality of Z and L-functions)**.**

(1) *If X is a geometrically irreducible variety over a pseudofinite field F,*

$$Z(X/F;T) = \frac{1}{1 - \mathbb{L}^{\dim(X)}\,T}.$$

(2) *If Y/X is a Galois cover over F with group G and $\chi \in C_{\mathbb{Q}}(G,\mathbb{Q})$ (3.9), there is an $l > 0$ such that $L(Y/X,\chi;T)^l$ is a rational function of T and $\mathbb{L}$.*

(3) *If $\theta(x)$ is a formula over F, there is an $l > 0$ such that $Z(\theta/F;T)^l$ is a rational function of T and $\mathbb{L}$.*

Proof. (1) is obvious, all the $\mu_n(X)$ being 1. (2) Since χ is $\mathbb{Q}$-central, we can write it as a rational linear combination of $\mathbb{Q}$-central characters, $\chi = \sum_i r_i\chi_i$. Thus, by 7.3(1), (4) and (6),

$$L(Y/X,\chi;T) = \prod_H L(X/Y,\chi_i;T)^{r_H},$$

so it clearly suffices to show rationality of $L(Y/X,\chi;T)$ when χ is a $\mathbb{Q}$-central character associated with a representation $\mathcal{F}$ of G. However, if $\mathcal{A}$ denotes the direct image of χ with respect to the structure map of X, the trace formula 6.8, together with $\sigma_{F_n} = \sigma_F^n$, easily implies that

$$L(Y/X,\chi;T) = \frac{1}{\det\Big(1 - \mathcal{A}(\sigma)\,\mathbb{L}^{\dim(X)}\,T\Big)}.$$

Item 3 follows from the description of formulae in terms of Galois formulae (4.3):

$$Z(\theta/F;T) = \prod_i Z(\langle C_i/A_i, \mathrm{Con}(C_i,A_i)\rangle;T) = \prod_i L(C_i/A_i, 1_{\mathrm{con}(C_i,A_i)};T).$$

□

Remark 7.5. Near-rationality of a series $\sum_n a_nT^n$ shows a strong regularity in the sequence of numbers (a_n). In particular, only a finite number of a_n's determine the whole sequence. As a consequence, L-functions are definable invariants in the same sense as the CDM-measure (6.1). We shall use this observation frequently.

Definition 7.6. Let $\theta(x)$ be a formula over a pseudofinite field F defining a subset of a variety X of dimension d. We define the *(definable) Dirichlet density* of θ with respect to X to be

$$\delta_{X,F}(\theta) := \lim_{s\to d} \frac{\ln Z(\theta/F,s)}{\ln Z(X/F,s)}.$$

Theorem 7.7 (Definable Čebotarev)**.** *Let X be a normal and connected variety over F. Let $f : \pi_1(X) \to \mathbb{C}$ be a continuous $\mathbb{Q}$-central function with (FQ). Then*

$$\int_X f(\sigma_x)\; d\delta_X(x) = \int_{\pi_1(X)} f\, d\mu_{\mathrm{Haar}}.$$

Proof. By 3.10, we reduce to proving the case when $f = \chi$ is the character of an irreducible $\mathbb{Q}$-central representation $\mathcal{F}$ of $\pi_1(X)$ with (FQ). By 7.4,

$$L(Y/X, \chi; T) = \frac{1}{\det\left(1 - \mathcal{A}(\sigma)\,\mathbb{L}^{\dim(X)}\,T\right)},$$

where $\mathcal{A}$ is the direct image of $\mathcal{F}$ via the structure map of X. Hence,

$$\int_X \chi(\sigma_x)\ d\delta_X(x) = \lim_{s\to\dim(X)} \frac{\ln L(Y/X,\chi;s)}{\ln Z(X/F;s)} = \begin{cases} 0, & \text{when } \chi \text{ is nontrivial,} \\ 1, & \text{when } \chi \text{ is trivial,} \end{cases}$$

as required. □

Remark 7.8. Let X be normal and geometrically connected over F. Let us call a $\mathbb{Q}$-central function f on $\pi_1(X)$ *geometric* if it has no $\pi_1^{\mathrm{geom}}(X)$-coinvariants, i.e., the direct image of f in $G(F)$ is just the part of f coming from the trivial character, i.e., $\int_{\pi_1(X)} f\, d\mu_{\mathrm{Haar}} \cdot 1$. It is clear then that for any n,

$$\int_X f(\sigma_x)\ d\delta_X(x) = \int_X f(\sigma_x)\ d\mu_n(x).$$

8. Completions

In this section we extend the CDM-'measures' $\mu_{X,F}$ and Dirichlet density $\delta_{X,F}$ to measures in the sense of Section 2. The simplest way of accomplishing this task is to exploit some notion of model-theoretic 'largeness'. More precisely, we shall say that a pseudofinite field $\mathfrak{F}$ is *large*, if it is saturated and homogeneous in cardinality much larger than all the other fields appearing in our considerations. In particular, for any given field k, it is at least $|k|^+$*-compact,* i. e., for every n, every covering of $\mathbb{A}^n(\mathfrak{F})$ by $|k|$ many definable subsets has a finite subcovering.

Let us write $\tau \sim \tau'$ in a profinite group G if $\langle\tau\rangle$ and $\langle\tau'\rangle$ are conjugate in G, i. e., if τ and τ' are $\Gamma_{\mathbb{Q}}$-conjugate (cf. 3.8). By a slight abuse of notation, let $x \sim x'$ in $X(\mathfrak{F})$ if the corresponding $\sigma_{\mathfrak{F},x} \sim \sigma_{\mathfrak{F},x'}$ in $\pi_1(X)$ or $\mathrm{G}(X)$.

Definition 8.1. Let X be a geometrically irreducible variety over a field k and let $\mathfrak{F}$ be a large pseudofinite field containing k.

Let $(X, \mu_{X,\mathfrak{F}})$ be the measure space described as follows. The underlying topological space is $X(\mathfrak{F})$, with topology induced by all the definable subsets of X with parameters from k. By largeness (in fact, $|k|^+$-compactness), we conclude that $X(\mathfrak{F})$ is compact. The measure $\mu_{X,\mathfrak{F}}$ is obtained by extending the already known function on the base via 2.1.

We define the measure space $(X, \delta_{X,\mathfrak{F}})$ analogously, the underlying topological spaces being formed on $\varinjlim_n X(\mathfrak{F}_n) = X(\tilde{\mathfrak{F}})$.

Definition 8.2. Let $X \xrightarrow{\pi} \operatorname{Spec} k$ be an irreducible variety over a field k contained in a pseudofinite field F. We wish to define a measure γ_F on $\mathrm{G}(X)$. If $f : \mathrm{G}(X) \to \mathbb{C}$ is a continuous function, we let

$$\gamma_F(f) := (\pi_* f)(\sigma_{F,k}).$$

Theorem 8.3 ('Fancy' Čebotarev)**.** *Let X be a geometrically irreducible variety and let $f : G(X) \to \mathbb{C}$ be a continuous $\mathbb{Q}$-central function. Then*

(1)
$$\int_X f(\sigma_{\mathfrak{F},x})\ d\delta_{X,\mathfrak{F}}(x) = \int_{G(X)} f\, d\mu_{\mathrm{Haar}};$$

(2)
$$\int_X f(\sigma_{\mathfrak{F},x})\ d\mu_{X,\mathfrak{F}}(x) = \int_{G(X)} f\, d\gamma_{\mathfrak{F}}.$$

Proof. First of all, let us remark that the function $x \mapsto f(\sigma_{F,x})$ is not defined everywhere, since $\sigma_{F,x} \in G(X)$ makes sense only for generic x. However, the integral is still meaningful, since the set where the function is not defined is lower-dimensional and therefore of zero measure. By 3.10 and linearity of both sides of the equality, we reduce (2) to the case of 6.8. Similarly, (1) reduces to 7.7. □

Remark 8.4. If we loosen the definition of maps of measure spaces to include maps defined up to a set of measure zero, the map $x \mapsto \sigma_x$ induces 'equivalences' of the following measure spaces:

(1) $(X, \delta_X)/\!\sim\ \simeq (G(X), \mu_{\mathrm{Haar}})/\!\sim$;
(2) $(X, \mu_{X,n})/\!\sim\ \simeq (G(X), \gamma_{\mathfrak{F}})/\!\sim$.

In view of 4.2, the left side of (1) and (2) is the Stone space of types in X. In particular, it follows that this very abstract model-theoretic object is *metrisable* and admits a measure.

9. Independence Theorem

In this section, all the 'completed' measures in the sense of the previous section are taken with respect to a fixed large pseudofinite field $\mathfrak{F}$. First of all, we need to study the behaviour of the CDM-measure with respect to the base change.

Lemma 9.1. *Let us consider the Cartesian square involving X, $S = \mathrm{Spec}(k)$, X_1, $S_1 = \mathrm{Spec}(k_1)$, $X_1 = X \times_S S_1$ like in Figure 2. Let us write $f_{1\natural}$ for the pushforward with respect to the CDM-measures μ_1 on X_1 and μ on X, to distinguish it from the direct image of generic sheaves. Then, for any generic sheaf $\mathcal{F}$ on X_1, and any representation $\mathcal{A}$ of the subgroup of $\mathrm{G}(k_1)$ generated by σ_{F,k_1},*

$$f_{1*}(\mathcal{F} \otimes \tau_1^* \mathcal{A}) = f_{1\natural}\, \mathcal{F} \otimes \tau^* \phi_{1*} \mathcal{A} = f_{1\natural}\, \mathcal{F} \otimes f_{1*} \tau_1^* \mathcal{A}.$$

Proof. Working with the character χ of $\mathcal{F}$, by definition of pushforwards and 6.8, for every character ϵ of a generic sheaf on X,

$$\tau_{1*}(\chi \cdot f_1^* \epsilon)(\sigma_{\mathfrak{F},k_1}) = \phi_1^* \tau_*(f_{1\natural} \chi \cdot \epsilon)(\sigma_{\mathfrak{F},k_1}),$$

or, generalising slightly, for any character α of $\langle \sigma_{\mathfrak{F},k_1} \rangle$, using the scalar products with respect to the Haar measures,

$$(\tau_{1*}(\chi \cdot f_1^* \epsilon), \alpha^\vee) = (\phi_1^* \tau_*(f_{1\natural} \chi \cdot \epsilon), \alpha^\vee).$$

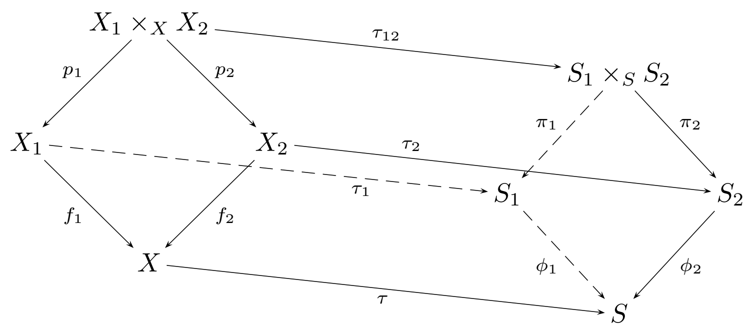


FIGURE 2. Commutative diagram for 9.2

After a short calculation using the adjointness of the direct and inverse image functors, the above reads:

$$(f_{1*}(\chi \cdot \tau_1^* \alpha), \epsilon^\vee) = (f_{1\natural}\chi \cdot \tau^* \phi_{1*}\alpha, \epsilon^\vee),$$

for arbitrary ϵ, and the desired conclusion follows. □

We are finally ready to state our refinement of the Independence Theorem in pseudofinite fields.

Theorem 9.2. *Let k, k_1, k_2 be (small) elementary subfields of $\mathfrak{F}$, with k relatively algebraically closed in $\mathfrak{F}$ and k_1 and k_2 free over k. Let X be a geometrically irreducible variety over k and let $X_i := X \times_k k_i$. Then we have independences of the following measure spaces:*

(1) $(X_1, \delta_{X_1}) \mathop{\smile\!\!\!\!\!\!|}_{(X,\delta_X)} (X_2, \delta_{X_2})$;
(2) $(X_1, \mu_{X_1}) \mathop{\smile\!\!\!\!\!\!|}_{(X,\mu_X)} (X_2, \mu_{X_2})$;

Proof. Since k is relatively algebraically closed, k_i/k are regular and so k_1 and k_2 are linearly disjoint over k. Thus $k_1[k_2] \cong k_1 \otimes_k k_2$. Let us write $S = \operatorname{Spec}(k)$, $S_i = \operatorname{Spec}(k_i)$. Using the properties of the fibre product,

$$X_{(k_1[k_2])} = X \times_S (S_1 \times_S S_2) = (X \times_S S_1) \times_X (X \times_S S_2) = X_1 \times_X X_2,$$

up to canonical isomorphisms. This shows that the function fields of X_1 and X_2 are linearly disjoint over the function field of X.

A model-theoretic proof of this fact: let x be a generic point of X over k. We may assume (by an automorphism) that x is free from k_1k_2 over k and thus, by assumptions, k_1 is free from $k_2(x)$ over k. However, both k_1 and $k_2(x)$ are regular extensions of k (by assumptions on k and geometric irreducibility), so in fact k_1 is linearly disjoint from $k_2(x)$ over k. This implies that $k_1(x)$ is linearly disjoint from $k_2(x)$ over $k(x)$, as required.

If by L_i we denote the function field of X_i and by L that of X, we get that

$$\mathrm{G}(L_1L_2)/\,\mathrm{G}(L_1^sL_2^s) \simeq \mathrm{Gal}(L_1^sL_2^s/L_1L_2) \simeq \mathrm{G}(L_1) \times_{\mathrm{G}(L)} \mathrm{G}(L_2).$$

In other words, $\mathrm{G}(X_1) \times_{\mathrm{G}(X)} \mathrm{G}(X_2)$ is a quotient of $\mathrm{G}\left(X_{(k_1 k_2)}\right)$. In view of 3.13 and 8.4, we have obtained (1).

(2) In order to show the required independence inside (X_{12}, μ_{12}), where X_{12} denotes the fibre product $X_1 \times_X X_2$, we need to show that μ_{12} is the fibre product of μ_1 and μ_2 over μ, i. e., that for any two generic sheaves $\mathcal{F}_i$ on X_i and their characters χ_i, with notation from Figure 2,

$$\mu_{12}(p_1^*\chi_1 \cdot p_2^*\chi_2) = \mu(f_{1\natural}\chi_1 \cdot f_{2\natural}\chi_2),$$

where by $f_{i\natural}\chi_i$ we denote the pushforward of χ_i with respect to the measures μ_i and μ. Clearly, it suffices to prove the following CDM-version of the Künneth formula:

$$(f_1 \times f_2)_\natural(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = f_{1\natural}\, \mathcal{F}_1 \otimes f_{2\natural}\, \mathcal{F}_2\,.$$

By the usual Künneth formula, for suitable $\mathcal{A}_1$ and $\mathcal{A}_2$ (as in 9.1), we get:

$$\begin{aligned} &f_{1*}\tau_1^*\mathcal{A}_1 \otimes f_{2*}\tau_2^*\mathcal{A}_2 \otimes (f_1 \times f_2)_*\left(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \otimes \tau_{12}^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2)\right) \\ &\quad = (f_1 \times f_2)_*(\tau_1^*\mathcal{A}_1 \boxtimes \tau_2^*\mathcal{A}_2) \otimes (f_1 \times f_2)_*\left((\mathcal{F}_1 \otimes \tau_1^*\mathcal{A}_1) \boxtimes (\mathcal{F}_2 \otimes \tau_2^*\mathcal{A}_2)\right) \\ &\qquad = (f_1 \times f_2)_*\tau_{12}^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \otimes f_{1*}(\mathcal{F}_1 \otimes \tau_1^*\mathcal{A}_1) \otimes f_{2*}(\mathcal{F}_2 \otimes \tau_2^*\mathcal{A}_2). \end{aligned}$$

The CDM-version of Künneth can be read off from the above using 9.1. □

At a first glance, this result does not resemble the classical form of the independence theorem. Let us derive the latter as a special case, after introducing some language.

We call a definable subset of a geometrically irreducible variety X *elementary*, if it corresponds to a single $\mathbb{Q}$-conjugacy class in the group G associated with a Galois cover of X. If μ is any measure and θ an event, by μ_θ we denote the conditional probability with respect to θ,

$$\mu_\theta(\alpha) := \frac{\mu(\alpha \cdot 1_\theta)}{\mu(\theta)}.$$

Corollary 9.3. *Let k, k_1, k_2 be (small) submodels of $\mathfrak{F}$, with k_1 and k_2 free over k. Let $\mathfrak{p}$ be a complete type over k and let $\mathfrak{p}_i$ be generic extensions of $\mathfrak{p}$ to k_i. Let δ be the Dirichlet density on the locus X of $\mathfrak{p}$ over E, and let $\mu := \mu_{\mathfrak{F}}$ be the CDM-measure.*

(1) *Given elementary formulae $\theta_i \in \mathfrak{p}_i$, there is a formula $\theta \in \mathfrak{p}$ such that*

$$\mu_\theta(\theta_1 \wedge \theta_2) = \mu_\theta(\theta_1)\mu_\theta(\theta_2).$$

(2) *Given elementary formulae $\theta_i \in \mathfrak{p}_i$, there is a formula $\theta \in \mathfrak{p}$ such that*

$$\delta_\theta(\theta_1 \wedge \theta_2) = \delta_\theta(\theta_1)\delta_\theta(\theta_2),$$

i. e., θ_1 and θ_2 are independent as events with respect to θ.

Proof. Let X/E be the variety associated with $\mathfrak{p}$ (i.e. the positive quantifier-free part of $\mathfrak{p}$) and let $\tilde{C}$ be the corresponding conjugacy class in $G(X)$. Since $\mathfrak{p}_i$ are generic extensions, their associated varieties are $X_i := X \times_{\mathrm{Spec}(E)} \mathrm{Spec}(K_i)$. Let us adopt the notation of the left side of the diagram in the proof of 9.2. The types $\mathfrak{p}_i$ then correspond to some conjugacy classes $\tilde{C}_i$ in $G(X_i)$, with $\tilde{C} = f_i(\tilde{C}_i)$.

Let now $\theta_i \in \mathfrak{p}_i$ be formulae, whose X_i-part corresponds to some $C_i \supseteq \tilde{C}_i$ which comes from a finite factor of $G(X_i)$. Then $f_1(C_1) \cap f_2(C_2) =: C \supseteq \tilde{C}$ also comes from a finite factor of $G(X)$ and thus corresponds to a formula $\theta \in \mathfrak{p}$ (essentially because intersection of subgroups of finite index is again of finite index). We have

$$\theta_1 \wedge \theta_2(F) = \{x \in X(F) : \sigma_x \in p_1^{-1}(C_1) \cap p_2^{-1}(C_2) = C_1 \times_{G(X)} C_2\},$$

where p_i are the maps $G(X_1 \times_X X_2) \to G(X_i)$. Part (2) follows directly from (3.16) (the sets C_i in question being $\mathbb{Q}$-conjugacy classes).

For (1), by the above, the measure of the intersection equals

$$\mu_{12}(1_{C_1} \boxdot 1_{C_2}) = \mu(f_{1\natural}1_{C_1} \cdot f_{2\natural}1_{C_2}).$$

Since for every $x \in X_i(\mathfrak{F})$, $\tau_i(\sigma_{\mathfrak{F},x}) = \sigma_{\mathfrak{F},k_i}$, we may assume $1_{C_i} \cdot \tau_i^* 1_{\sigma_{\mathfrak{F},k_i}} = 1_{C_i}$. Also, $\phi_{i*}1_{\sigma_{\mathfrak{F},k_1}} = \delta_i 1_{\sigma_{\mathfrak{F},k}}$, for some δ_i. Then, by 9.1 and 3.15,

$$f_{i\natural}1_{C_i} = \frac{1}{\delta_i \tau^* 1_{\sigma_{\mathfrak{F},k}}} m_i 1_{f_i(C_i)}$$

Thus

$$\mu(\theta_1 \wedge \theta_2) = \tau_* \left(\frac{m_1 m_2 1_{f_1(C_1) \cap f_2(C_2)}}{\delta_1 \delta_2 \tau^* 1_{\sigma_{\mathfrak{F},k}}} \right) (\sigma_{\mathfrak{F},k}) = \frac{m_1}{\delta_1} \frac{m_2}{\delta_2} \mu(1_C) = \mu_\theta(\theta_1)\mu_\theta(\theta_2)\mu(\theta),$$

and dividing both sides by $\mu(\theta)$ yields the claim.

□

Since any formula is a disjoint union of elementary formulae, the statement (1) of the above corollary implies that for arbitrary formulae $\theta_i \in \mathfrak{p}_i$, $\mu(\theta_1 \cap \theta_2) \neq 0$ and thus retrieves the usual form of the Independence Theorem, showing that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have a common refinement of the same dimension (a set of nonzero measure has the same dimension as the ambient space).

Example 9.4. This is in fact a non-example, showing how various Čebotarev theorems can be used when Independence Theorem cannot. In the Introduction we have mentioned the usefulness of 6.8, 7.7 and 8.3 when dealing with 'random' reducts of pseudofinite fields. In such applications, they are used as a much more precise substitute for the Independence Theorem, as we illustrate below.

Suppose then that we have a geometrically irreducible variety X over a pseudofinite field F equipped by two Galois covers X_1/X and X_2/X with groups G_1 and G_2, respectively. Assume X_i are also geometrically irreducible (the geometric case). Let $\theta_i := \{x \in X(F) : \sigma_x \in C_i\}$ be definable subsets of X, for $\mathbb{Q}$-rational conjugacy domains C_i. In applications we usually need to show that $\theta_1(F) \wedge \theta_2(F) \neq \emptyset$. Suppose we have some extra condition that guarantees that $X_1 \times_X X_2$ is irreducible (e.g., when $\mathbf{k}(X_1)$ and $\mathbf{k}(X_2)$ are linearly disjoint over

$\mathbf{k}(X)$). Then $X_1 \times_X X_2 \to X$ is a Galois cover with group $G_1 \times G_2$ and

$$\theta_1 \wedge \theta_2(F) = \{x \in X(F) : \sigma_x \in C_1 \times C_2\}.$$

Therefore, Čebotarev's Theorem implies that

$$\mu_X(\theta_1 \wedge \theta_2) = \mu_X(\theta_1)\mu_X(\theta_2) \neq 0,$$

allowing us to conclude that $\theta_1 \wedge \theta_2(F)$ is nonempty.

In many such cases the Independence Theorem cannot be applied. For example, $X_1 \times_X X_2$ can be irreducible even if fields of definitions of X_1 and X_2 are not (nontrivially) linearly disjoint over that of X.

10. Motives

In this section we give a quick overview of everything we have done so far in the language of motives and cohomology. The goal is to convince the reader that the measure-theoretic Independence Theorem for pseudofinite fields is essentially contained in the 'relative' Künneth formula for the cohomology of the fibre product. We use the notation from [7].

Let k be a field of characteristic zero for the rest of this section. Let Sch_k be the category of algebraic varieties over k. We denote by $\mathrm{Mot}_{k,\mathbb{C}}$ the category of Chow motives over a field k, with coefficients in $\mathbb{C}$, and by $K_0(\mathrm{Mot}_{k,\mathbb{C}})$ its Grothendieck group. Objects in $\mathrm{Mot}_{k,\mathbb{C}}$ are triples (X, p, n) with X a proper and smooth variety over k, p an idempotent correspondence with coefficients in $\mathbb{C}$ on X, and $n \in \mathbb{Z}$. There is a natural functor χ_c from Sch_k to $\mathrm{Mot}_{k,\mathbb{C}}$, given on smooth and proper X by $\chi_c(X) := (X, \mathrm{id}, 0)$.

Let G be a finite group. Let $X \in \mathrm{Sch}_k$ be endowed with a G-action. We say that X is a G-variety if the G-orbit of every closed point in X is contained in an affine open subscheme. This is a sufficient condition for the quotient X/G to exist and it is always satisfied when X is quasi-projective. Considered with the natural notion of G-morphism, this forms a category, and we may define the corresponding Grothendieck ring $K_0(\mathrm{Sch}_k, G)$.

Let α be the character of a representation $G \to \mathbf{GL}(V_\alpha)$ with $n_\alpha = \dim(V_\alpha)$, and let

$$p_\alpha := \frac{n_\alpha}{|G|} \sum_{g \in G} \alpha^{-1}(g)[g]$$

be the corresponding idempotent in $\mathbb{C}[G]$. There is a natural ring morphism Γ from $\mathbb{C}[G]$ to the ring of correspondences on X with coefficients in $\mathbb{C}$ sending a group element g onto the graph of multiplication by g. We recall the group $R(G)$ of virtual characters from 3.

The fundamental result relating these notions is the following result from [3].

Theorem 10.1. *Let G be a finite group. For every $\alpha \in R(G)$, there exists a unique morphism of rings*

$$\chi_c(\cdot, \alpha) : K_0(\mathrm{Sch}_k, G) \to K_0(\mathrm{Mot}_{k,\mathbb{C}})$$

such that

(1) *If X is projective and smooth with G-action and α is an irreducible character, $n_\alpha \chi_c([X], \alpha)$ is the class of the motive $(X, \Gamma(p_\alpha), 0)$ in $K_0(\mathrm{Mot}_{k,\mathbb{C}})$.*

(2) *For every G-variety X,*

$$\chi_c(X) = \sum_\alpha n_\alpha \chi_c(X, \alpha),$$

where α runs over the irreducible characters of G.

(3) *For every G-variety X, the function $\alpha \mapsto \chi_c(X, \alpha)$ is a group morphism $R(G) \to K_0(\mathrm{Mot}_{k,\mathbb{C}})$.*

This is a slight improvement of a result from [3] and [7]. Note the formal similarity to 7.3.

Proposition 10.2. (1) *Let $\phi : G \to G'$ be a homomorphism of finite groups, let Y be a G-variety, and let $\phi_* Y = Y \times^G G'$ be the G'-variety deduced from Y by 'extension of the structural group'. Let α' be a character of G' and let $\phi^* \alpha'$ be its pullback to G. Then*

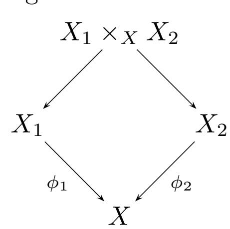
$$\chi_c(Y, \phi^* \alpha') = \chi_c(\phi_* Y, \chi').$$

(2) *Let $\phi : G' \to G$ a homomorphism of finite groups. Let Y be a G-variety and let $\phi^* Y$ denote the variety Y on which G' operates through ϕ. Let α' be a character of G' and let $\phi_* \alpha'$ be its direct image on G. Then*

$$\chi_c(Y, \phi_* \alpha') = \chi_c(\phi^* Y, \alpha').$$

Being able to associate the motive to a Galois cover with a given virtual character, Denef and Loeser [7] proceed and define the motives associated with arbitrary definable sets in pseudofinite fields. The procedure is similar to the stages in our definition of ν in 6.3. Thus, from this point on we assume that we have the motivic measure χ_c which can integrate definable $\mathbb{C}$-functions, and the integrals take values in $K_0(\mathrm{Mot}_{k,\mathbb{C}}) \otimes \mathbb{C}$.

Suppose now we have a diagram:



Let us also assume that we have Galois covers Y, Y_1 and Y_2 of X, X_1 and X_2 with groups G, G_1, G_2, respectively and that $Y_1 \times_Y Y_2$ is a Galois cover of $X_1 \times_X X_2$ with group $G_1 \times_G G_2$. Let us write φ_i for the maps $Y_i \to Y$ and $\varphi_\times$ for $\varphi_1 \times \varphi_2$. Let α_i be a character of G_i. We write $\alpha_1 \boxdot \alpha_2$ for the product of pullbacks of α_i to $G_1 \times_G G_2$. A repeated application of 10.2, as well as 3.13, yield

$$\begin{aligned}\chi_c(Y_1 \times_Y Y_2, \alpha_1 \boxdot \alpha_2) = \chi_c(\varphi_\times^* Y, \alpha_1 \boxdot \alpha_2) = \chi_c(Y, \varphi_{\times,*}(\alpha_1 \boxdot \alpha_2)) = \\ = \chi_c(Y, \varphi_{1*}\alpha_1 \cdot \varphi_{2*}\alpha_2).\end{aligned}$$

In a forthcoming paper we make an effort to write the right-hand side of the above as a product in a suitable category. This special case justifies the independence of spaces with finitely additive measure,

$$(X_1, \chi_c) \underset{(X,\chi_c)}{\perp} (X_2, \chi_c).$$

We argue below that the CDM-measure and Dirichlet density are just tiny invariants of χ_c, and therefore this independence implies all the previous results of the paper.

Indeed, it follows from 6.8 that the CDM-measure of a motive over a pseudofinite field F is calculated using its $H^{2d}(\cdot, \bar{\mathbb{Q}}_l)$. Let $\mathcal{F}_i$ be the lisse $\bar{\mathbb{Q}}_l$-sheaves on X_i corresponding to α_i via ι. Note that, given a lisse sheaf $\mathcal{F}$ on X corresponding to a representation of the Galois group G of the cover $Y \to X$ with character α,

$$H^r_c(X, \mathcal{F}) = H^r_c(\chi_c(Y, \alpha), \bar{\mathbb{Q}}_l).$$

Thus the calculations

$$H^{2d}(\chi_c(Y_1 \times_Y Y_2, \alpha_1 \boxdot \alpha_2), \bar{\mathbb{Q}}_l) = H^{2d}(X_1 \times_X X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2),$$

and

$$H^{2d}(\chi_c(Y, \varphi_{1*}\alpha_1 \cdot \varphi_{2*}\alpha_2), \bar{\mathbb{Q}}_l) = H^{2d}(X, \phi_{1*}\mathcal{F}_1 \otimes \phi_{2*}\mathcal{F}_2)$$

give the same result. However, the equality of the right-hand sides in the above is just a very special case of the Künneth formalism from [5]. If we let $\psi_\times := \psi \circ \phi_\times$, ψ being the structure map of X,

$$R\psi_{\times !}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = R\psi_! R\phi_{\times !}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = R\psi_!(R\phi_{1!}\mathcal{F}_1 \otimes R\phi_{2!}\mathcal{F}_2).$$

Thus, in a sense, the assumptions of the Independence Theorem just give a sufficient condition under which a diagram like above arises, and everything else is handled by the Künneth formula.

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