3. RAVNOTEŽA NAPETE MEMBRANE

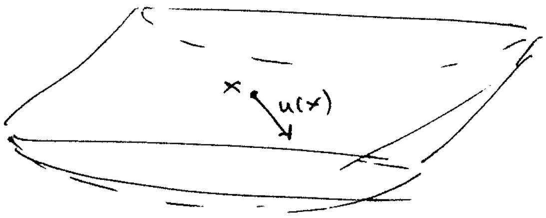
I. AGANOVIĆ, K. VESELIĆ, LINEARNE DIFERENCIJALNE JEDNADŽBE, PMF-MO, 1992.

- $\Omega \subseteq \mathbb{R}^2$ NEDEFORMIRANI POLOŽAJ MEMBRANE
- $x \in \Omega \xrightarrow{P} P(x) \in \mathbb{R}^3$ – POLOŽAJ DEFORMIRANE MEMBRANE
- $u(x) = P(x) - (x, 0)$ – POMAK, DOVOLJNO GLADAK

PRETPOSTAVKA

DEFORMACIJA JE MALA : $|\nabla u(x)| \ll 1$, $x \in \Omega$

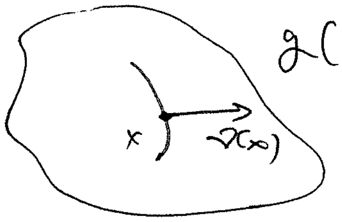
- SLIČNO KAO KOD ŽICE $\left| \frac{u(x)}{\operatorname{diam} \Omega} \right| \ll 1$



- KOMAD MEMBRANE $D \subset \Omega$ (ODNOSNO $P(D)$)
- f POVRŠINSKA GUSTOĆA VANJSKE NORMALNE SILE

$$f : \Omega \to \mathbb{R} \qquad \text{(U SMJERU } e_3\text{)}$$

- $g : \Omega \times S^2 \to \mathbb{R}$ KONTAKTNA SILA (U SMJERU e_3)



$g(x, \nu(x))$ – U TOČKI x NA ~~RUB~~ KRIVULJI S NORMALOM $\nu(x)$ JEDINIČNOM

PRINCIP RAVNOTEŽE

$$\forall D \qquad \int_{\partial D} g(x, \nu(x))\, ds + \int_D f(x)\, dx = 0$$

INTEGRALNI OBLIK

CAUCHYJEV TEOREM:

$$\exists\, \sigma : \Omega \to \mathbb{R}^2 \quad \text{T.D.} \quad g(x, \cancel{k}) = \sigma(x) k$$

$$\Rightarrow \quad \int_{\partial D} \sigma(x)\, \nu(x)\, ds + \int_D f(x)\, dx = 0$$

$\parallel$ TEOREM O DIVERGENCIJI

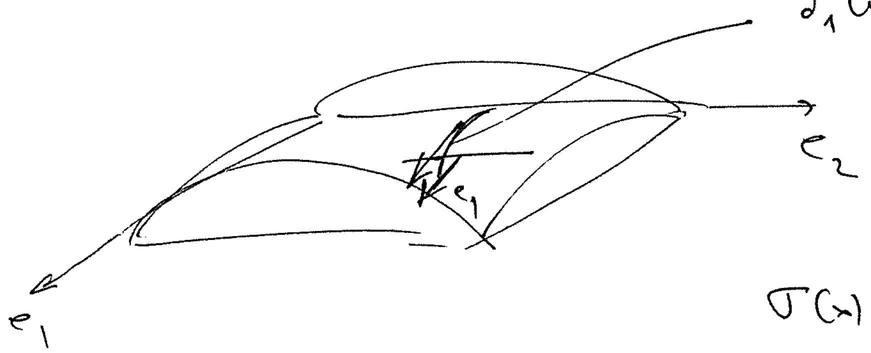
$$\int_D \operatorname{div} \sigma(x)\, dx$$

$$\Rightarrow \quad \int_D \left(\operatorname{div} \sigma(x) + f(x) \right) dx = 0 \quad \forall D$$

$$\Rightarrow \quad \operatorname{div} \sigma(x) + f(x) = 0$$

JEDNADŽBA RAVNOTEŽE

$$\sigma(x) e_1 = a\, \partial_1 u(x)$$



$$\sigma(x) e_2 = a\, \partial_2 u(x)$$

$\sigma(x) = a\nabla u(x)$ — ZAKON PONAŠANJA

— a NAPETOST

$\Rightarrow \operatorname{div}(a\nabla u) + f = 0$

$a\Delta u + f = 0$ J.R.

NAP: MOŽE I OPĆENITIJE

(*) $\operatorname{div}(A(x)\nabla u(x)) + f(x) = 0, \quad x \in \Omega$

$A(x)$ — NEHOMOGENO, ANIZOTROPNO NAPETO

RUBNA ZADAĆA

$u\big|_{\partial\Omega} = u_0$ DIRICHLET

$\sigma\cdot\nu = g\big|_{\partial\Omega} = g_0$ NEUMANN

$\left(A\nabla u\cdot\nu\big|_{\partial\Omega} = g_0\right)$

NEKA SU Γ_D, Γ_N T.D. $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \cup \Gamma_N = \partial\Omega$

$u\big|_{\Gamma_D} = u_0$

$\sigma\cdot\nu\; g\big|_{\Gamma_N} = g_0$

$g(x,\nu(x)) = -\mathcal{K}(x)\,u(x)$ ROBINOV

SLABA FORMULACJA

$$\begin{cases} \operatorname{div}\left(A(x)\nabla u(x)\right) + f(x) = 0 \\ u\big|_{\Gamma_D} = 0 \\ A\nabla u \cdot \nu\big|_{\Gamma_N} = g \end{cases}$$

$$V = \{ v \in H^1(\Omega) : \ v\big|_{\Gamma_D} = 0 \}$$

$$0 = \int_\Omega \left(\operatorname{div}(A\nabla u) + f \right) v = \int_\Omega \operatorname{div}(A\nabla u) v + \int_\Omega f v$$

$$= \int_\Omega \operatorname{div}(A\nabla u\, v) - A\nabla u \cdot \nabla v + \int_\Omega f v$$

$$= \int_{\partial\Omega} A\nabla u\, v \cdot \nu - \int_\Omega A\nabla u \cdot \nabla v + \int_\Omega f v$$

$$\cancel{\int_{\Gamma_D}} + \int_{\Gamma_N} \underbrace{A\nabla u \cdot \nu}_{g} \, v$$

$$\Rightarrow \boxed{\begin{array}{l} \text{Find } u \in V \text{ t.ż.} \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v = \int_\Omega f v + \int_{\Gamma_N} g v, \quad v \in V \end{array}} \quad (SF$$

NAP:

$$u|_{\Gamma_D} = u_0 \neq 0$$

ZA u_0 DOVOLJNO DOBAR $\Rightarrow \exists\ \bar{u}_0 \in H^1(\Omega)$ T.D. $Tr_{\Gamma_D} \bar{u}_0 = u_0$

PROMATRAMO ZADAĆU ZA $u - \bar{u}_0 \in V$

TEOREM

NEKA JE A SIMETRIČNA UNIFORMNO ELIPTIČAN TJ. $\exists \theta > 0$ T.D.

$$\forall_{x\in\Omega} \quad A(x)\xi\cdot\xi \geq \theta\,\xi^2 \quad , \xi \in \mathbb{R}^2$$

$A \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$.

TADA $\exists!$ RJEŠENJE (SF)

DOK:

V JE BANACHOV

$$L(v) = \int_\Omega f v + \int_{\Gamma_N} g v \quad \text{JE L.F}$$

$$|L(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} \quad \leftarrow \text{TM O TRAGU}$$

$$\leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} + \|g\|_{L^2(\Gamma_N)} \|v\|_{H^1(\Omega)}$$

NEPREKIDNOST

$B(u,v)$ JE SIMETRIČNA ZA A SIMETRIČNA

BILINEARNA FORMA

$$|B(u,v)| = \left|\int_\Omega A\nabla u\cdot\nabla v\right| \leq \|A\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

$$\leq \|A\|_{L^\infty} \|\nabla u\|_{H^1} \|\nabla v\|_{H^1}$$

NEPR.

$$B(u,u) = \int_\Omega A\nabla u\cdot\nabla u \geq \theta \|\nabla u\|^2_{L^2} \geq \theta\left(\frac{1}{2}C_P^2 \|u\|^2_{L^2} + \frac{1}{2}\|\nabla u\|^2\right)$$

$$\geq C \|u\|^2_{H^1(\Omega)} \quad \text{— KOERCITIVNOST}$$

ZAD: PROUČITE MODEL MEMBRANE U ELASTIČNOM SREDSTVU UZ NEUMANNOV RUBNI UVJET

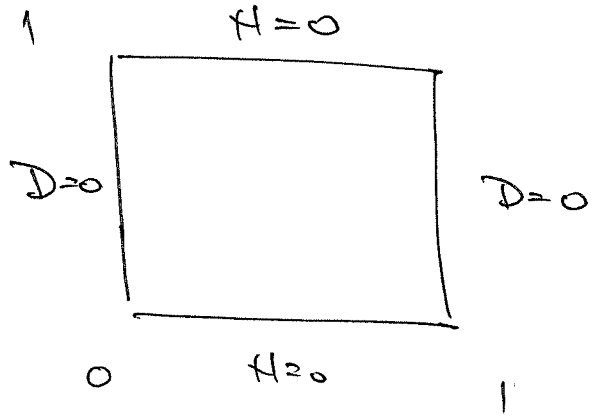
$$-\mathrm{div}\,(A\nabla u) + b u = f$$

ZAD: IZVEDITE OCJENU ZA KOREKTNOST ZADAĆE

$$\|u\|_{H^1(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)$$

ZAD: SVOJSTVENA ZADAĆA ZA LAPLACEA

$$-\Delta u = \lambda u$$



$$u(x,y) = X(x)\,Y(y)$$

$$-(X''Y + XY'') = \lambda XY \qquad \Big| : XY$$

$$-\frac{X''}{X} - \frac{Y''}{Y} = \lambda$$

$$-\frac{X''}{X} = \lambda + \frac{Y''}{Y} = \mu$$

$$\Rightarrow \boxed{\begin{aligned} &X'' + \mu X = 0 \\ &X(0) = X(1) = 0 \end{aligned}}$$

$$\Rightarrow \quad \mu_k = (k\pi)^2 \qquad k \in \mathbb{Z}$$

$$X_k(x) = \sin k\pi x$$

$$\Rightarrow \boxed{\begin{array}{l} Y'' + (\lambda - \mu_k) Y = 0 \\ Y'(0) = Y'(1) = 0 \\ \Rightarrow \lambda - \mu_k \geq 0 \end{array}}$$

$$\eta$$

$$\eta > 0$$

$$Y(y) = A \cos \sqrt{\eta}\, y + B \sin \sqrt{\eta}\, y$$

$$Y'(y) = -A\sqrt{\eta} \sin \sqrt{\eta}\, y + B\sqrt{\eta} \cos \sqrt{\eta}\, y$$

$$0 = Y'(0) = B\sqrt{\eta} \quad \Rightarrow \quad B = 0$$

$$0 = Y'(1) = -A\sqrt{\eta} \sin \sqrt{\eta}$$

$$\Rightarrow \quad \sin\sqrt{\eta} = 0 \quad \Rightarrow \quad \sqrt{\eta} = n\pi, \quad n \in \mathbb{N}$$

$$\eta_n = (n\pi)^2 \qquad n \in \mathbb{N}$$

$$Y_n(y) = \cos n\pi y$$

$$\eta = 0 \quad Y'' = 0 \quad \Rightarrow \quad Y(y) = Ay + B$$

$$Y'(y) = A \quad \Rightarrow \quad 0 = Y'(0) = A$$

$$\Rightarrow \quad Y_0(y) = 1$$

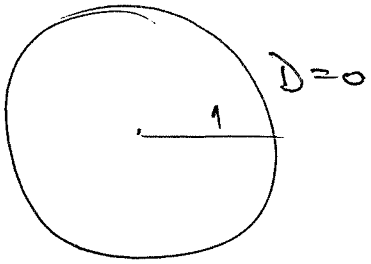
$$\Rightarrow \boxed{\begin{array}{l} \eta_n = (n\pi)^2 \\ Y_n(y) = \cos n\pi y \end{array} \quad n \in \mathbb{N} \cup \{0\}} \qquad \eta_0 = 0$$

$$\Rightarrow \quad \eta_n = \lambda - \mu_k \Rightarrow \lambda = \eta_n + \mu_k$$

$$\Rightarrow \boxed{\begin{array}{ll} \lambda_{nk} = (k^2+n^2)\pi^2 & k \in \mathbb{N} \\ & n \in \mathbb{N} \cup \{0\} \\ u_{nk}(x,y) = \sin k\pi x \cos n\pi y & \end{array}}$$

ZAD: SVOJSTVENA ZADAĆA ZA LAPLACEA ZA KRUG

$$-\Delta u = \lambda u$$



– POLARNE KOORDINATE

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right) = \lambda u$$

$$u(r,\varphi) = R(r)\,\Phi(\varphi)$$

$$-\left(\frac{1}{r}\Phi\,(rR')' + \frac{1}{r^2}R\,\Phi''\right) = \lambda R\Phi \quad \Big| : \frac{R\Phi}{r^2}$$

$$-\left(\frac{r\,(rR')'}{R} + \frac{\Phi''}{\Phi}\right) = \lambda r^2$$

$$-\frac{r\,(rR')'}{R} - \lambda r^2 \qquad \frac{\Phi''}{\Phi} = -\mu$$

$$\Phi'' + \mu \Phi = 0$$

Φ 2π – PERIODIČNA

$\mu_n = n^2$

$\Phi_n^1 = \cos n\varphi$, $n \in \mathbb{N} \cup \{0\}$

$\Phi_n^2 = \sin n\varphi$, $n \in \mathbb{N}$

$$+ r\,(\cancel{r}R')' + (\lambda r^2 - \mu_n)\,R = 0$$

$$\boxed{\begin{aligned} &(rR')' + \left(\lambda r - \frac{n^2}{r}\right)R = 0 \\ &R(1) = 0 \end{aligned}}$$

Rj 1: $\lambda = 0$ $\quad r^n, \; r^{-n}$, $n > 0$

$\quad 1, \; \ln r$, $n = 0$

SINGULARNA

NE PONIŠTAVAJU SE NA RUBU !

Rj 2: $\lambda > 0$ SUPSTITUCIJA $x = \cancel{r}\, r\sqrt{\lambda}$, $y(x) = R\left(\frac{x}{\sqrt{\lambda}}\right)$

$$(xy')' + \left(x - \frac{n^2}{x}\right)y = 0$$

$$x^2 y'' + x y' + (x^2 - n^2)\,y = 0$$

BESSELOVA JEDNADŽBA

Rj: CILINDRIČNE FUNKCIJE

(BESSELOVE)

$J_n(x)$ – REGULARNO U 0

DRUGO NIJE

$\Rightarrow R(1) = J_n(\sqrt{\lambda}) = 0$

$\Rightarrow \quad \lambda_{nj} = (x_{nj})^2 \qquad x_{nj}$ POZITIVNE NULTOČKE OD J_n

$j \in \mathbb{N}$

$R_{nj}(r) = J_n(r x_{nj})$

SV. VRIJEDNOSTI $\quad \lambda_{nj} = x_{nj}^2$

FUNKCIJE : $\quad \cos n\varphi \; J_n(r x_{nj}) \qquad n \in \mathbb{N} \cup \{0\},\ j \in \mathbb{N}$

$\sin n\varphi \; J_n(r x_{nj}) \qquad n \in \mathbb{N},\ j \in \mathbb{N}$

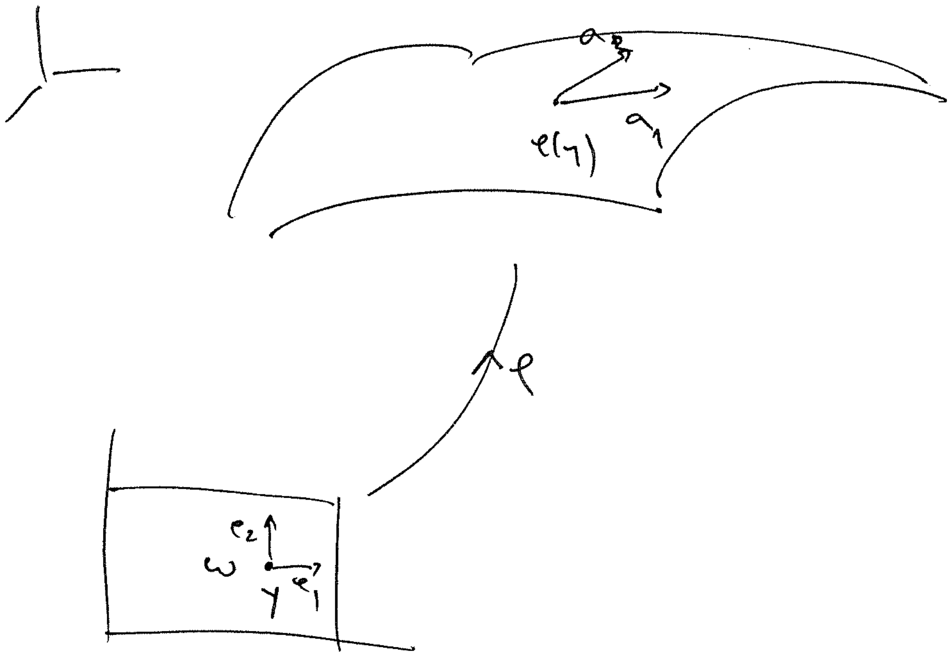
4. MODELI LJUSKI I PLOČA

4.1. MODELI LJUSKE

– $\varphi : \omega \to \mathbb{R}^3$ PARAMETRIZACIJA PLOHE U $\mathbb{R}^3$

DOVOLJNO GLATKA $\left(W^{1,\infty}(\omega; \mathbb{R}^3) \right)$

– $a_1 = \partial_1 \varphi$, $a_2 = \partial_2 \varphi$ – BAZA (KOVARIJANTNA) TANGENCIJALNOG PROSTORA



– $a_3 = \dfrac{a_1 \times a_2}{\| a_1 \times a_2 \|}$

– a_1, a_2, a_3 ~~DENA~~ BAZA

— a^1, a^2, a^3 BIORTOGONALNA (KONTRAVARIJANTNA) BAZA

$$a^i \cdot a_j = \delta^i_j, \qquad Q = (a^1\ a^2\ a^3)$$

— $A_c = (a_i \cdot a_j)_{i,j=1,2}$, $A^c = (a^i \cdot a^j)_{i,j=1,2}$

— $a = \det A_c$ — ELEMENT POVRŠINE

— h DEBLJINA LJUSKE

MODEL (NAGHDI):

NAĆI $(u, \omega) \in V_N = \{(v, w) \in H^1(\omega; \mathbb{R}^3) \times H^1(\omega; \mathbb{R}^3) : v|_{\Gamma_D} = w|_{\Gamma_D} = 0\}$

$$h\, B_{ms}((u,\omega),(v,w)) + h^3 B_f((u,\omega),(v,w)) = f((v,w)) \quad \forall (v,w) \in V_N$$

GDJE JE:

$$B_{ms}((u,\omega),(v,w)) = \int_\omega Q\, C_m\left(Q^T\left[\partial_1 u + a_1 \times \omega \quad \partial_2 u + a_2 \times \omega\right]\right) \cdot \left[\partial_1 v + a_1 \times w \quad \partial_2 v + a_2 \times w\right] \sqrt{a}\, dy$$

$$B_f((u,\omega),(v,w)) = \frac{1}{12}\int_\omega Q\, C_f\left(Q^T \nabla \omega\right) \cdot \nabla w \sqrt{a}\, dy$$

$\mathcal{C}_m, \mathcal{C}_f$ TENZORI ELASTIČNOSTI

$$\mathcal{C}_m \begin{bmatrix} C \\ c^T \end{bmatrix} \cdot \begin{bmatrix} D \\ d^T \end{bmatrix} = \frac{2\lambda\mu}{\lambda+2\mu} \operatorname{tr} C \operatorname{tr} D + 2\mu A_c C A^c \cdot D + \mu A^c c \cdot d$$

$$\mathcal{C}_f \begin{bmatrix} C \\ c^T \end{bmatrix} \cdot \begin{bmatrix} D \\ d^T \end{bmatrix} = a\left(\frac{2\lambda\mu}{\lambda+2\mu} A^c \cdot JC \; A^c \cdot JD + 2\mu A^c JCA^c \cdot JD \right) + a \mathcal{B}_f c \cdot d$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

NAGHDIJEV MODEL UKLJUČUJE:

- FLEKSIJSKU ENERGIJU $\quad$ U $\mathcal{B}_f$
- MEMBRANSKU ENERGIJU $\quad$ DIO U $\mathcal{B}_{mf}$
- SHEAR ENERGIJU (SMICANJA) $\quad$ DIO U $\mathcal{B}_{mf}$

(— UVRTANJE)

KOITEROV MODEL:

UMJESTO $V_H \longrightarrow V_K = \{ (v,w) \in V_N :$

$$w = \frac{1}{\sqrt{a}}\left((\partial_2 v \cdot a_3) a_1 - (\partial_1 v \cdot a_3) a_2 + \frac{1}{2}(\partial_1 v \cdot a_2 - \partial_2 v \cdot a_1) \right) \}$$

SMICANJA NEMA!

OKOMITOST POPREČNIH PRESJEKA

DIO $\quad [\partial_1 u + a_1 \times w \quad \partial_2 u + a_2 \times w] = 0$

OSTANE SIMETRIČAN $\quad$ 2x2 DIO

FLEKSIJSKI DIO

UMJESTO $V_M, V_K \longrightarrow V_F = \left\{(v,w)\in V_N : \partial_1 v + a_1\times w = \partial_2 v + a_2\times w = 0\right\}$

NEPRODULJIVOST & NESMIČLJIVOST !

OBLIK OD POMAKA

$$u_{KL}(y,y_3) = u(y) + y_3\, a_3(y)\times w(y) \quad !$$

DIFERENCIJALNA FORMULACIJA:

$$\operatorname{div}(\sqrt{a}\,p) + \sqrt{a}\,f = 0 \qquad p = [p_1 \ \ p_2]_{3\times 2}$$

$$\operatorname{div}(\sqrt{a}\,q) + \sqrt{a}\,a_\alpha \times p_\alpha = 0$$

$$q = M\nabla w,$$

$$p = N\,[\partial_1 u + a_1\times w \quad \partial_2 u + a_2\times w], \qquad M = \frac{h^3}{12} Q C_f Q^T, \qquad N = hQC_mQ^T$$

NAP: MEMBRANSKI MODEL LJUSKE $B_f = 0$ & NA V_K !

MODEL 1. REDA

SAMO RASTEZANJE U ENERGIJI

$$B_{ms}((u,w),(v,w)) = \int f\cdot v, \quad (v,w)\in V_K$$

$[\partial_1 u + a_1\times w \quad \partial_2 u + a_2\times w]$ SIMETRIČNA 2×2

4.2. MODEL PLOČE

SPECIJALIZIRAMO: $\varphi(y) = (y, 0)$, $\varphi : \omega \to \mathbb{R}^3$

$\Rightarrow$ $a_1 = e_1$, $a_2 = e_2$, $a_3 = e_3$ $\quad Q = I$, $A_c = I = A^c$, $a = 1$

PRETPOSTAVKA: $u(y) = (0, 0, u_3(y))$ $\quad \omega(y) = (\omega_1(y)\ \ \omega_2(y)\ \ 0)$

$$\begin{bmatrix} \partial_1 u + e_1 \times \omega & \partial_2 u + e_2 \times \omega \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 + \omega_2 & \partial_2 u_3 - \omega_1 \end{bmatrix}$$

~~ISTO ZA TEST FUNKCIJE~~

$$p = h\, C_m \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 + \omega_2 & \partial_2 u_3 - \omega_1 \end{bmatrix} = h\mu \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 + \omega_2 & \partial_2 u_3 - \omega_1 \end{bmatrix}$$

$$\cancel{= h\mu \begin{bmatrix} \partial_1 u_3 + \omega_2 \\ \partial_2 u_3 - \omega_1 \end{bmatrix}} =$$

$$q = \frac{h^3}{12} C_f \begin{bmatrix} \partial_1 \omega_1 & \partial_2 \omega_1 \\ \partial_1 \omega_2 & \partial_2 \omega_2 \\ 0 & 0 \end{bmatrix} = \frac{h^3}{12} \frac{2\lambda\mu}{\lambda + 2\mu} (\partial_1 \omega_2 - \partial_2 \omega_1) \begin{bmatrix} -J \\ 0 \quad 0 \end{bmatrix} + 2\mu \begin{bmatrix} \partial_1 \omega_1 & \partial_2 \\ \partial_1 \omega_2 & \partial_2 \\ 0 & 0 \end{bmatrix}$$

PRETPOSTAVKA: $\quad \partial_1 u_3 + \omega_2 = \partial_2 u_3 - \omega_1 = 0$

POPREČNI PRESJEK PRIBLIŽNO OKOMIT

$$\Rightarrow \quad \begin{aligned} \omega_1 &= \partial_2 u_3 \\ \omega_2 &= -\partial_1 u_3 \end{aligned}$$

$\Rightarrow \quad p^1_3 , p^2_3$ LAGRANGEOVI MULTIPLIKATORI

$$\mathrm{div} \begin{bmatrix} p^1_3 \\ p^2_3 \end{bmatrix} + f_3 = 0 \qquad (*)$$

$$0 = \partial_1 g^1 + \partial_2 g^2 + e_1 \times p^1 + e_2 \times p^2 = \begin{bmatrix} 0 \\ -p^1_3 \\ p^1_2 \end{bmatrix} + \begin{bmatrix} p^2_3 \\ 0 \\ -p^2_1 \end{bmatrix} + \partial_1 g^1 + \partial_2 g^2$$

$\mathrm{div}\, g$ IMA 0 u 3. RETKU

$$\Rightarrow \quad \begin{aligned} p^1_3 &= \partial_1 g^1_2 + \partial_2 g^2_2 \\ p^2_3 &= -\partial_1 g^1_1 - \partial_2 g^2_1 \end{aligned}$$

$$\frac{12}{h^3} g^1_1 = 2\mu \, \partial_{12} u_3$$

$$\frac{12}{h^3} g^2_1 = \frac{2\lambda\mu}{\lambda + 2\mu} \Delta u_3 + 2\mu \partial_2^2 u_3$$

$$\frac{12}{h^3} g^1_2 = -\frac{2\lambda\mu}{\lambda + 2\mu} \Delta u_3 - 2\mu \partial_1^2 u_3$$

$$\frac{12}{h^3} g^2_2 = -2\mu \, \partial_{12} u_3$$

$$p_3^1 = \frac{h^3}{12}\left(-\partial_1\left(\frac{2\lambda\mu}{\lambda+2\mu}\Delta u_3 + 2\mu\partial_1^2 u_3\right) - 2\mu\partial_{122} u_3\right) = -\frac{h^3}{12}\partial_1\left(\frac{2\lambda\mu}{\lambda+2\mu}+2\mu\right)\Delta u_3$$

$$p_3^2 = -\frac{h^3}{12}\left(2\mu\,\partial_{112}u_3 + \partial_2\left(\frac{2\lambda\mu}{\lambda+2\mu}\Delta u_3 + 2\mu\,\partial_2^2 u_3\right)\right) = -\frac{h^3}{12}\partial_2\left(\frac{2\lambda\mu}{\lambda+2\mu}+2\mu\right)\Delta u_3$$

Iz (*):

$$-\frac{h^3}{12}\,2\mu\,\frac{2\lambda+2\mu}{\lambda+2\mu}\,\Delta\Delta u_3 + f_3 = 0$$

$$\boxed{\frac{h^3}{12}\,\frac{E}{1-\nu^2}\,\Delta^2 u_3 = f_3}$$

JEDNADŽBA PLOČE

$\nu = \frac{1}{2}\frac{\lambda}{\lambda+\mu}$ POISSONOV OMJER

RUBNI UVJETI

$$\left.\begin{aligned} u_3\big|_{\Gamma_D} &= 0 \\ (\omega_1,\omega_2)\big|_{\Gamma_D} &= \nabla u_3\big|_{\Gamma_D} = 0 \end{aligned}\right\}\ \text{DIRICHLET RUBNI UVJETI}$$

$p\nu\big|_{\Gamma_N}$ – KONTAKTNA SILA

$q\nu\big|_{\Gamma_N}$ – KONTAKTNI MOMENT

} NEUMANNOVI RUBNI UVJETI

ZADAJEMO 2 OD OVA 4!

$$\frac{12}{h^3}\left(g^1_1 \nu_1 + g^2_1 \nu_2\right) = 2\mu\, \partial_{12} u_3\, \nu_1 + \left(\frac{2\lambda\mu}{\lambda+2\mu} \Delta u_3 + 2\mu\, \partial_2^2 u_3\right) \nu_2$$

$$\cancel{= 2\mu\, \partial_2}$$

$$\frac{12}{h^3}\left(g^1_2 \nu_1 + g^2_2 \nu_2\right) = \left(-\frac{2\lambda\mu}{\lambda+2\mu} \Delta u_3 - 2\mu\, \partial_1^2 u_3\right) \nu_1 - 2\mu\, \partial_{12} u_3\, \nu_2$$

KOMPONENTE KONTAKTNOG MOMENTA

$$\frac{12}{h^3}\left(p^1_3 \nu_1 + p^2_3 \nu_2\right) = \frac{12}{h^3}\left[\left(\partial_1 g^1_2 + \partial_2 g^2_2\right)\nu_1 - \left(\partial_1 g^1_1 + \partial_2 g^2_1\right)\nu_2\right]$$

$$= \left(-\frac{2\lambda\mu}{\lambda+2\mu}\, \partial_1 \Delta u_3 - 2\mu\, \partial_1^3 u_3 - 2\mu\, \partial_{122} u_3\right)\nu_1$$

$$- \left(2\mu\, \partial_{112} u_3 + \frac{2\lambda\mu}{\lambda+2\mu}\, \partial_2 \Delta u_3 + 2\mu\, \partial_{222} u_3\right)\nu_2$$

$$= -\partial_1 \Delta u_3 \left(\frac{2\lambda\mu}{\lambda+2\mu} + 2\mu\right)\nu_1 - \partial_2 \Delta u_3 \left(\frac{2\lambda\mu}{\lambda+2\mu} + 2\mu\right)\nu_2$$

$$= -\frac{E}{1-\nu^2}\, \nabla \Delta u_3 \cdot \nu$$

$$\Rightarrow \left(p \cdot \nu\right)_3 = -\frac{h^3}{12} \frac{E}{1-\nu^2}\, \nabla \Delta u_3 \cdot \nu$$

— KONTAKTNA SILA

ZADACIA:

$$\frac{h^3}{12}\frac{E}{1-\nu^2}\Delta^2 u = f \qquad u\ \Omega$$

$$u=0 \qquad \text{NA } \partial\Omega$$

$$\nabla u=0 \qquad \text{NA } \partial\Omega$$

SLABA FORMULACIJA : $V=\{v\in H^2(\Omega):\ v=0,\ \nabla v=0 \text{ NA } \partial\Omega\}=H^2_0(\Omega)$

$$\int_\Omega \frac{h^3}{12}\frac{E}{1-\nu^2}\underbrace{\Delta^2 u}_{\operatorname{div}\nabla\Delta u}\cdot v = \int_\Omega f v \quad ,\ v\in V$$

$$\parallel$$

$$\frac{h^3}{12}\frac{E}{1-\nu^2}\int_\Omega \operatorname{div}(\nabla\Delta u\, v) - \nabla\Delta u\cdot\nabla v$$

$$\parallel$$

$$\frac{h^3}{12}\frac{E}{1-\nu^2}\int_{\partial\Omega}\nabla\Delta u\cdot\nu\, \underset{0}{\underset{\parallel}{v}} - \frac{h^3}{12}\frac{E}{1-\nu^2}\int_\Omega \nabla\Delta u\cdot\nabla v$$

$$\parallel$$

$$-\frac{h^3}{12}\frac{E}{1-\nu^2}\int_\Omega \nabla\Delta u\cdot\nabla v$$

$$\parallel$$

$$-\frac{h^3}{12}\frac{E}{1-\nu^2}\int_\Omega \operatorname{div}(\Delta u\nabla v) - \Delta u\,\Delta v$$

$$\parallel$$

$$-\frac{h^3}{12}\frac{E}{1-\nu^2}\int_{\partial\Omega}\Delta u\, \underset{0}{\underset{\parallel}{\nabla v\cdot\nu}} + \frac{h^3}{12}\frac{E}{1-\nu^2}\int_\Omega \Delta u\,\Delta v$$

NAĆI $u \in V$

$$\int_\Omega \frac{h^3}{12} \frac{E}{1-\nu^2} \Delta u \, \Delta v = \int_\Omega f v \quad , \quad v \in V$$

- ANALIZA SLIČNA, NEŠTO SLOŽENIJA
- DRUGI RUBNI UVJETI, ...

EVOLUCIJSKI MODEL

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + f = 0$$

$$u\big|_{\Gamma_D} = 0, \quad \nabla u\big|_{\Gamma_D} = 0$$

$$u(t,x) = T(t) X(x)$$

$$T'' X + T \Delta^2 X = 0$$

$$T'' X = -T \Delta^2 X \qquad \Big| : TX$$

$$\frac{T''}{T} = -\frac{\Delta^2 X}{X} = -\lambda$$

$$T'' + \lambda T = 0$$

$$\begin{cases} \Delta^2 X = \lambda X \\ X\big|_{\Gamma_D} = 0 \\ \nabla X\big|_{\Gamma_D} = 0 \end{cases} \qquad \text{SVOJSTVENA ZADAĆA}$$

SLABA FORMULACIJA

NAĆI $\lambda \in \mathbb{R}$ & $X \in V$ T.D.

$$\int_\Omega \Delta X \cdot \Delta Y = \lambda \int_\Omega X Y \quad , \quad Y \in V$$

TEORIJA SLIČNA