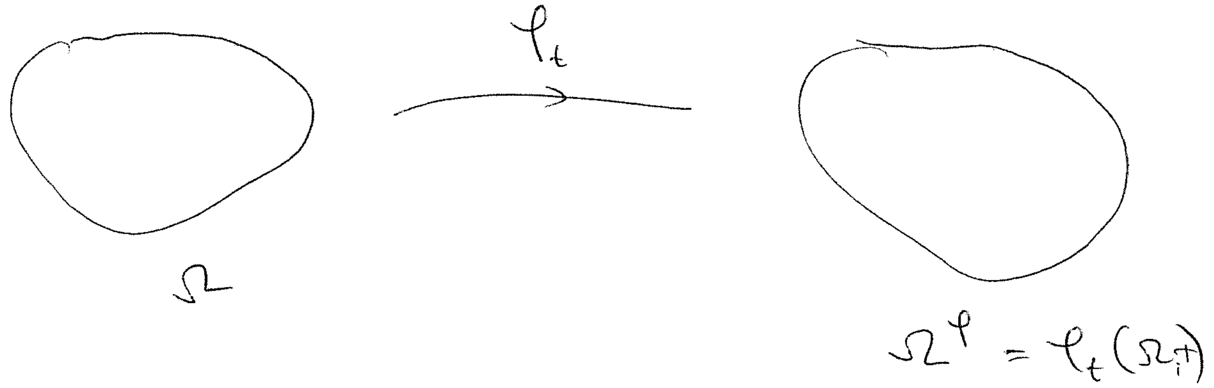
JEDNADŽBA GIBANJA U REFERENTNOJ KONFIGURACIJI



RADIMO ZAMJENU VARIJABLI U SLABOJ FORMULACIJI JEDNADŽBE GIBANJA U DEFORMIRANOJ KONFIGURACIJI

NOVA TEST FUNKCIJA:

$$\theta := \theta^{\varphi_t} \circ \varphi_t$$

$$\nabla\theta = \nabla\theta^\varphi \circ \varphi_t \, \nabla\varphi_t$$

$$\Rightarrow \nabla\theta^\varphi \circ \varphi_t = \nabla\theta \, (\nabla\varphi_t)^{-1}$$

$$\int_{\Omega^\varphi} T^\varphi \cdot \nabla\theta^\varphi \, dx^\varphi = \int_\Omega T^\varphi(\varphi_t(x), t) \cdot \nabla\theta^\varphi \circ \varphi_t(x) \det\nabla\varphi_t(x) \, dx$$

$$= \int_\Omega T^\varphi(\varphi_t(x), t) \cdot \nabla\theta(x) \left(\nabla\varphi_t(\cancel{x,t})\right)^{-1} \det\nabla\varphi_t(\cancel{x,t}) \, dx$$

DEFINIRAMO 1. PIOLA-KIRCHHOFFOV TENZOR NAPREZANJA (U REFERENTNOJ KONFIGURACIJI)

$$T(x,t) := \det\nabla\varphi_t(x,t) \, T^\varphi(\varphi_t(x), t) \, \nabla\varphi_t^{-T}(x)$$

$$T(x,t)^T = (\det \nabla\varphi_t(x))\, \nabla\varphi_t(x)^{-1} T^\varphi(\varphi_t(x),t)$$

$$= \underbrace{(\det \nabla\varphi_t(x))\, \nabla\varphi_t(x)^{-1} T^\varphi(\varphi_t(x),t)\, \nabla\varphi_t(x)^{-T}}_{T(x,t)} \nabla\varphi_t(x)^T$$

$$= \nabla\varphi_t(x)^{-1}\, T(x,t)\, \nabla\varphi_t(x)^T$$

$$\Rightarrow \boxed{\nabla\varphi_t(x)\, T(x,t)^T = T(x,t)\, \nabla\varphi_t(x)^T}$$

NEMA SIMETRIJE 1. P.-K. TENZORA

DEFINIRAMO 2. PIOLA-KIRCHHOFFOV TENZOR NAPREZANJA (DEF. U REFERENTNOJ KONFIGURACIJI)

$$\Sigma(x,t) := \nabla\varphi_t(x)^{-1}\, T(x,t)$$

VRIJEDI:

$$\Sigma(x,t)^T = \Sigma(x,t) \qquad \text{SIMETRIČAN!}$$

SILE (VOLUMNE)

$$\int_{\Omega^\varphi} f^\varphi \cdot \theta^\varphi \, dx^\varphi = \int_\Omega f^\varphi(\varphi_t(x),t) \cdot \underbrace{\theta^\varphi \circ \varphi_t(x)}_{=\,\theta(x)} \det\nabla\varphi_t(x)\, dx$$

DEF: $\quad f(x,t) := f^\varphi(\varphi_t(x),t) \det \nabla\varphi_t(x)$

$$= \int_\Omega f(x,t) \cdot \theta(x)\, dx$$

! f OVISI O φ I TO ZNAČAJNO (IAKO NOTACIJA NE SUGERIRA)

f - (VOLUMNA) GUSTOĆA VANJSKE SILE U REFERENTNOJ KONFIG.

SILA (POVRŠINSKA)

$$\int_{\Gamma_1^\varphi} g^\varphi \Theta^\varphi \, da^\varphi = \int_{\Gamma_1} \underbrace{g^\varphi(\varphi_t(x),t) \, \|\nabla \varphi_t(x)^{-T} n\| \det \nabla \varphi_t(x)}_{\overset{!!}{=}\, g(x,t)} \Theta(x) \, da$$

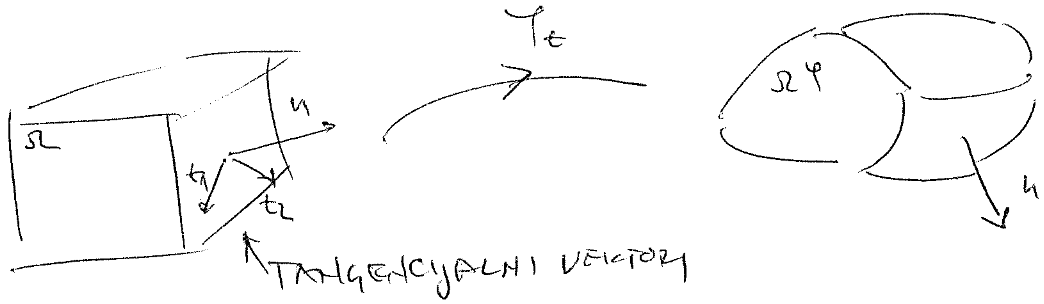
ZAMJENA VARIJABLI

n – NORMALA NA REFERENTNU KONFIGURACIJU

n^φ – NORMALA NA DEFORMIRANU KONFIGURACIJU

JEDINIČNE VANJSKE

ZA DOKAZ VIDI TM 1.7-1 (CIARLET)



$\nabla \varphi_t \, t_1$
$\nabla \varphi_t \, t_2$ RAZAPINJU TANGENCIJALNU RAVNINU U DEFORMIRANOJ KONFIGURACIJI

$$\Rightarrow \quad n^\varphi \cdot \nabla\varphi \, t_\alpha = 0, \quad \alpha = 1,2$$

$$\Rightarrow \nabla\varphi^T n^\varphi \cdot t_\alpha = 0, \quad \alpha = 1,2$$

$$\Rightarrow \nabla\varphi^T n^\varphi \parallel n$$

$$\Rightarrow \nabla\varphi^T n^\varphi = \alpha n$$

$$n^\varphi = \alpha \nabla\varphi^{-T} n$$

$$\Rightarrow n^\varphi = \frac{\nabla\varphi_t(x)^{-T} n}{\|\nabla\varphi_t(x)^{-T} n\|}$$

g – POVRŠINSKA GUSTOĆA KONTAKTNE SILE U REFERENTNOJ KONFIGURACIJI

INERCIJALNI ČLAN

$$\int_{\Omega^\varphi} \rho^\varphi \dot{v}^\varphi \cdot \theta^\varphi dx^\varphi = \int_\Omega \rho^\varphi(\varphi_t(x),t)\, \dot{v}^\varphi(\varphi_t(x),t) \cdot \theta^\varphi \circ \varphi_t(x) \det \nabla\varphi_t(x)\, dx$$

- KAO I PRIJE $\theta(x) = \theta^\varphi \circ \varphi_t(x)$
- ODNOS LAGRANGEOVE I EULEROVE BRZINE

$$\dot{v}^\varphi(\varphi_t(x),t) = \frac{\partial v}{\partial t}(x,t)$$

- REFERENTNA GUSTOĆA MASE

$$\rho(x) = \rho^\varphi(\varphi_t(x),t) \det \nabla\varphi_t(x)$$

$$= \int_\Omega \rho(x) \frac{\partial v}{\partial t}(x,t) \cdot \theta(x)\, dx$$

NAP: SAMO LINEARAN ČLAN U REFERENTNOJ KONFIGURACIJI

SLABA (VARIJACIJSKA) FORMULACIJA JEDNADŽBE GIBANJA U REFERENTNOJ KONFIGURACIJI

(PRINCIP VIRTUALNOG RADA U REFERENTNOJ KONFIGURACIJI)

$$\int_\Omega \rho(x) \frac{\partial v}{\partial t}(x,t) \cdot \theta(x) dx + \int_\Omega T(x,t) \cdot \nabla\theta(x) dx$$

$$= \int_\Omega f(x,t) \cdot \theta(x) dx + \int_{\Gamma_1} g(x,t) \cdot \theta(x) dx$$

$\forall\, \theta$ ~~DOVOLJNO~~ DOVOLJNO GLATKU

& $\theta|_{\Gamma_0} = 0$

DIFERENCIJALNA FORMULACIJA

$$\rho \frac{\partial v}{\partial t} = \operatorname{div} T + f \qquad \text{u } \Omega$$

$$\nabla\varphi_t T^T = T \nabla\varphi_t^T \qquad \text{u } \Omega$$

$$Tn = g \qquad \text{na } \Gamma_1$$

NAP: UZ PRETPOSTAVKU DOVOLJNE GLATKOĆE SLABA I DIFERENCIJALNA FORMULACIJA SU EKVIVALENTNE

NAP: OBJE FORMULACIJE MOGU SE NAPISATI POMOĆU 2. PIOLA - KIRCHHOFFOVOG TENZORA NAPREZANJA

NAP: NA Γ_0 $(=\partial\Omega - \Gamma_1)$ OBIČNO SE ZADAJE DIRICHLETOV RUBNI UVJET, TJ. PROPISUJEMO φ_t!

$$\varphi_t(x) = \varphi_0(x,t), \quad x \in \Gamma_0 \qquad (\varphi_0 \text{ ZADANA})$$

STOGA SU ZANIMLJIVE DEFORMACIJE IZ SKUPA

$$\Phi = \{ \psi : \bar{\Omega} \to \mathbb{R}^3 : \psi \text{ DOVOLJNO GLATKA}, \det \nabla\psi > 0 \text{ U } \bar{\Omega}, \; \psi = \varphi_0 \text{ NA } \Gamma_0 \}$$

– MNOGOSTRUKOST U PROSTORU FUNKCIJA

SJETIMO LI SE MINIMIZACIJSKE FORMULACIJE PROBLEMA DE ELASTIČNOSTI, ŽELIMO VARIRATI $\varphi \in \Phi$ DA OSTANEMO U Φ

VARIJACIJA JE STOGA U $T_\varphi \Phi$

$$T_\varphi \Phi = \{ \theta : \bar{\Omega} \to \mathbb{R}^3 : \theta \text{ DOVOLJNO GLATKA}, \; \theta|_{\Gamma_0} = 0 \}$$

– OTUD VARIJACIJSKA FORMULACIJA

PRIMJERI SILA

<u>DEF:</u> VANJSKU SILU NAZIVAMO "DEAD LOAD" AKO NJENA GUSTOĆA NA REFERENTNOJ KONFIGURACIJI NE OVISI O φ.

VAŽNO IZ MATEMATIČKE PERSPEKTIVE.
MALO JE TAKVIH!

<u>PR:</u> SILA TEŽA

$$f(x) = -\rho(x)\, g\, e_3, \quad x \in \Omega$$

<u>PR:</u> SLOBODNA GRANICA

$$g^\varphi = 0 \text{ na } \Gamma_1^\varphi \Rightarrow g = 0 \text{ NA } \Gamma_1$$

<u>PR:</u> TLAK

$$g^\varphi(x^\varphi, t) = -\underset{\substack{\uparrow \\ \text{KONSTANTA (TLAK)}}}{\pi}\, n^\varphi(x^\varphi, t), \qquad x^\varphi \in \Gamma_1^\varphi$$

$$g(x,t) = -\pi \det \nabla \varphi_t(x) \, \|\nabla \varphi_t(x)^{-T} n\| \, n^\varphi(\varphi_t(x), t)$$

$$= -\pi \det \nabla \varphi_t(x) \cancel{\|\nabla \varphi_t(x)^{-T} n\|} \frac{\nabla \varphi_t(x)^{-T} n}{\cancel{\|\nabla \varphi_t(x)^{-T} n\|}}$$

$$= -\pi \det \nabla \varphi_t(x) \quad \nabla \varphi_t(x)^{-T} n$$

<u>PR:</u> CENTRIFUGALNA SILA

- TIJELO ROTIRA OKO OSI e_1 KUTNOM BRZINOM ω

$$f^\varphi(x^\varphi, t) = \omega^2 \rho^\varphi(x^\varphi, t) \left(x_2^\varphi e_2 + x_3^\varphi e_3 \right)$$

$$\Rightarrow f(x,t) = f^\varphi(\varphi_t(x), t) \det \nabla \varphi_t(x)$$

$$= \omega^2 \rho^\varphi(\varphi_t(x), t) \left(\varphi_{t_2}(x) e_2 + \varphi_{t_3}(x) e_3 \right) \det \nabla \varphi_t(x)$$

$$= \omega^2 \rho(x) \left(\varphi_{t_2}(x) e_2 + \varphi_{t_3}(x) e_3 \right)$$

4. ZAKONI KONSTITUCIJE

- VEZA IZMEĐU TENZORA NAPREZANJA I MJERE DEFORMACIJE
  "SILA" "DEFORMACIJA"

UZROK $\leftrightarrow$ POSLJEDICA

- ZADAJU MODEL ZA KONKRETAN MATERIJAL
- TU JE RAZLIKA IZMEĐU FLUIDA I STRUKTURA
- OPĆENITA VEZA

$$T^{\varphi}(x^{\varphi}, t) = \mathcal{K}(\varphi_t(x), x), \quad x \in \Omega$$

<u>AKSIOMATIKA</u>: $\mathcal{K}$ ZADOVOLJAVA

a) PRINCIP KAUZALNOSTI: $\forall t$

$$\varphi^1 = \varphi^2 \text{ u } \Omega \times \langle -\infty, t \rangle \Rightarrow \mathcal{K}(\varphi^1) = \mathcal{K}(\varphi^2) \text{ u } \Omega \times \{t\}$$

b) PRINCIP LOKALNOSTI: $\forall x \in \Omega$, $\forall \mathcal{O}(x)$ OKOLINU OD x

$$\varphi^1 = \varphi^2 \text{ NA } \mathcal{O}(x) \times \mathbb{R} \Rightarrow \mathcal{K}(\varphi^1) = \mathcal{K}(\varphi^2) \text{ u } \{x\} \times \mathbb{R}$$

c) PRINCIP INVARIJANTNOSTI (O PROMATRAČU)

$\forall$ GIBANJE φ^1, $a : \mathbb{R} \to \mathbb{R}^3$, $Q : \mathbb{R} \to SO(3)$ JE

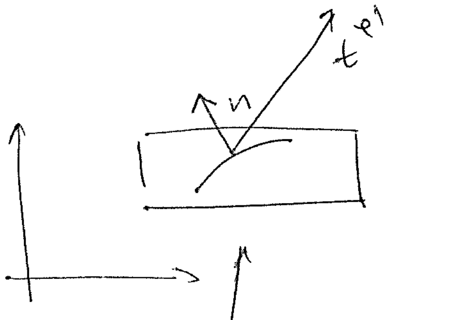
$$\mathcal{K}(\varphi^2) = Q\, \mathcal{K}(\varphi^1)\, Q^T$$

GDJE JE $\varphi^2(x,t) = a(t) + Q(t)\varphi^1(x,t)$

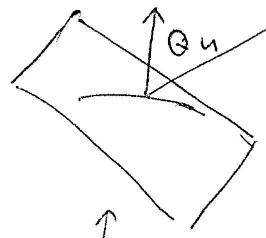
$\forall t$ KRUTA DEFORMACIJA OD φ^1

NAP OVISNOST O t JE NEBITNA U c)

$$\varphi^2 = a + Q\varphi^1 , \qquad a \in \mathbb{R}^3, \; Q \in SO(3)$$



PARAMETRIZIRAN s φ^1



$t^{\varphi^2} = Q t^{\varphi^1}$

TO JE SVOJSTVO!

PARAMETRIZIRAN s $\varphi^2 = a + Q\varphi^1$

$$K(\varphi^2) = Q K(\varphi^1) Q^T$$

$$\Rightarrow \quad T^{\varphi^2}(x^{\varphi^2}) = Q T^{\varphi^1}(x^{\varphi^1}) Q^T$$

$$\Rightarrow \quad T^{\varphi^2}(x^{\varphi^2}) Q = Q T^{\varphi^1}(x^{\varphi^1})$$

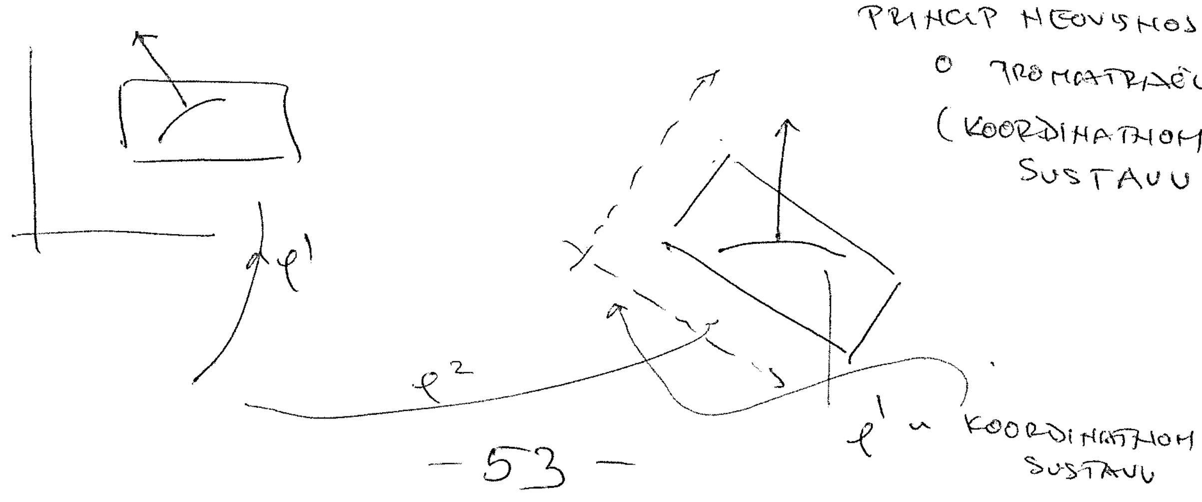
$$T^{\varphi^2}(x^{\varphi^2}) Qn = Q T^{\varphi^1}(x^{\varphi^1}) n$$

$$\begin{array}{ccc} \| & & \| \\ t^{\varphi^2}(x^{\varphi^2}, Qn) & & Q\, t^{\varphi^1}(x^{\varphi^1}, n) \end{array}$$

NAP.

(FRAME - INDIFFERENCE)

PRINCIP NEOVISNOSTI O PROMATRAČU (KOORDINATNOM SUSTAVU)



- IZ PRINCIPA LOKALNOST $T(x,t)$ ODREĐEN IZ $\varphi|_{\sigma(x)\times\mathbb{R}}$
- LOKALNO φ APROKSIMIRAN S $\varphi(x,t) + \nabla\varphi(x)(y-x)$ (TRANSLACIJA NEBITNA KOD NAPREZANJA)

DEF: MATERIJALNO TIJELO JE JEDNOSTAVNO AKO $\forall x \in \Omega$

$$\nabla\varphi^1(x,t) = \nabla\varphi^2(x,t), \quad t \in \mathbb{R} \Rightarrow \mathcal{K}(\varphi^1) = \mathcal{K}(\varphi^2) \text{ na } \{x\}\times\mathbb{R}$$

NAP:
- ZA JEDNOSTAVNO TIJELO $\mathcal{T}$ JE ODREĐEN S $\nabla\varphi$ NA $\Omega\times\langle-\infty,t\rangle$
- KADA JE OVISNOST O VREMENU NA $\langle-\infty,t\rangle$ GOVORIMO O DUGOJ MEMORIJI
- JEDNOSTAVAN MATERIJAL IMA KRATKU MEMORIJU (FADING MEMORY) AKO $\forall x \in \Omega, \forall \tau > 0$

$$\nabla\varphi^1(x,\cdot) = \nabla\varphi^2(x,\cdot) \text{ NA } \langle t-\tau, t\rangle$$

$$\Downarrow$$

$$\mathcal{K}(\varphi^1) = \mathcal{K}(\varphi^2) \text{ NA } \{x\}\times\mathbb{R}$$

- ZA MALENE τ PONAŠANJE U t APROKSIMIRANO DERIVACIJAMA PO t
- JEDNOSTAVAN MATERIJAL S KRATKOM MEMORIJOM DIFERENCIJALNOG TIPA $k \in \{0\}\cup\mathbb{N}$

$$\partial_t^r \nabla\varphi^1(x,t) = \partial_t^r \nabla\varphi^2(x,t), \ r=0,\dots,k \Rightarrow \mathcal{K}(\varphi^1) = \mathcal{K}(\varphi^2) \text{ u } (x,t)$$

- ZA TAKAV MATERIJAL $\exists \hat{T}$ - FUNKCIJA ODZIVA T.D.

$$T(x,t) = \hat{T}(x, \nabla\varphi(x,t), \partial_t\nabla\varphi(x,t), \dots, \partial_t^k\nabla\varphi(x,t)) \quad (*)$$

<u>DEF:</u>
JEDNOSTAVNI MATERIJAL BEZ MEMORIJE (DIFERENCIJALNOG TIPA 0) NAZIVAMO <u>ELASTIČNI MATERIJAL</u>. TADA $\exists\ \hat{T}$ FUNKCIJA ODZIVA T.D.

$$T(x,t) = \hat{T}(x, \nabla\varphi(x,t)), \qquad x \in \overline{\Omega} \times \mathbb{R}.$$

<u>NAP:</u>
IZ PIOLINE TRANSFORMACIJE SLIJEDI

$$T^{\varphi}(\varphi(x), t) = \frac{1}{\det \nabla\varphi(x)}\, T(x,t) \nabla\varphi(x,t)^{T}$$

$$= \frac{1}{\det \nabla\varphi(x)}\, \hat{T}(x, \nabla\varphi(x,t)) \nabla\varphi(x,t)^{T}$$

$$= \hat{T}^{D}(x, \nabla\varphi(x,t)),$$

GDJE JE

$$\hat{T}^{D}(x^{\varphi}, F) = \frac{1}{\det F}\, \hat{T}(x, F) F^{T}.$$

<u>NAP:</u>
KAKO JE $\quad \Sigma(x,t) = \nabla\varphi_t(x)^{-1} T(x,t)$

ZA FUNKCIJU $\quad \hat{\Sigma}(x,F) = F^{-1}\hat{T}(x,F)$

$$= \det F\ F^{-1}\hat{T}^{D}(x^{\varphi},F)\, F^{-T}$$

<u>NAP:</u> $\hat{T}^{D}$ I $\hat{\Sigma}$ SU FUNKCIJE ODZIVA ZA ODGOVARAJUĆE TENZORE

TEOREM 1) FUNKCIJA ODZIVA $\hat{T}^D : \bar{\Omega} \times M_+(3) \to Sym(3)$

ZADOVOLJAVA PRINCIP INVARIJANTNOSTI AKO I SAMO AKO

$$\hat{T}^D(x, QF) = Q\hat{T}^D(x,F)Q^T, \qquad x \in \bar{\Omega}, F \in M_+(3), Q \in SO(3)$$

2) FUNKCIJA ODZIVA $\hat{\Sigma} : \bar{\Omega} \times M_+(3) \to Sym(3)$ ZADOVOLJAVA PRINCIP INVARIJANTNOSTI AKO I SAMO AKO $\exists\, \tilde{\Sigma} : \bar{\Omega} \times Sym_+(3) \to Sym$

$$\hat{\Sigma}(x,F) = \tilde{\Sigma}(x, F^T F), \qquad x \in \bar{\Omega}, F \in M_+(3)$$

3) $\hat{T}^D$ ZADOVOLJAVA PRINCIP INVARIJANTNOSTI AKO I SAMO AKO

$$\hat{T}^D(x,F) = R\hat{T}^D(x,U)R^T, \qquad x \in \bar{\Omega}, \ F = RU \in M_+(3)$$

POLARNI RASTAV.

DOK: 1) P.I. $\iff T^{\varphi^2}(x^{\varphi^2}, t) = Q(t)\, T^{\varphi^1}(x^{\varphi^1}, t)\, Q(t)^T$

$$\iff \hat{T}^D(x, \nabla\varphi^2(x,t)) = Q(t)\,\hat{T}^D(x, \nabla\varphi^1(x,t))\,Q(t)^T$$

PRI TOME JE

$$\varphi^2(x,t) = a(t) + Q(t)\,\varphi^1(x,t)$$

$$\nabla\varphi^2(x,t) = Q(t)\,\nabla\varphi^1(x,t)$$

$$\iff \hat{T}^D(x, Q(t)\nabla\varphi^1(x,t)) = Q(t)\,\hat{T}^D(x, \nabla\varphi^1(x,t))\,Q(t)^T$$

ZA DANI $F \in M_+(3)$ I $Q \in SO(3)$ $\exists\, \varphi^1$ T.D. $\nabla\varphi^1(x) = F$, $Q(t) = Q$

$$\Rightarrow \hat{T}^D(x, QF) = Q\,\hat{T}^D(x,F)\,Q^T$$

OBRAT OČITO IZ OVOG ALGEBARSKOG SVOJSTVA

3) ~~ZA~~ $F \in M_+(3)$ $\quad F = RU$, $\quad R \in SO(3)$, $U \geq 0$

POLARNI RASTAV

IZ 1) P.I $\Longrightarrow$ $\hat{T}^D(x^e, F) = \hat{T}^D(x^e, RU) = R\,\hat{T}^D(x^e, U)\,R^T$

OBRATNO IZ 3)

$$\hat{T}^D(x, QF) = \hat{T}^D(x^e, (QR)U) \overset{3)}{=} QR\,\hat{T}^D(x^e, U)\,R^T Q^T$$

$$\overset{3)}{=} Q\,\hat{T}^D(x^e, RU)\,Q^T = Q\,\hat{T}^D(x^e, F)\,Q^T$$

$(QR)U$ JE POLARNI RASTAV OD QRU.

2) U POLARNOM RASTAVU F REGULARNA $\Rightarrow$ U REGULARNA

$\Rightarrow$ $R = FU^{-1}$ DOBRO DEFINIRANA

3) $\Rightarrow$ $\hat{T}^D(x^e, F) = R\,\hat{T}^D(x, U)\,R^T = FU^{-1}\,\hat{T}^D(x^e, U)\,U^{-T}F^T$

IZ NAPOMENE SA STR 50.

$$\hat{\Sigma}(x, F) = \det F \; F^{-1}\,\hat{T}^D(x^e, F)\,F^{-T}$$

$$= \det(RU)\; F^{-1}FU^{-1}\,\hat{T}^D(x^e, U)\,U^{-T}F^T F^{-T}$$

$$= (\det U)\,U^{-1}\,\hat{T}^D(x^e, U)\,U^{-T} =: \tilde{\Sigma}(F^T F)$$

JER JE $\quad U = (F^T F)^{1/2}$

OBRATNO

~~DEF~~ $\tilde{\Sigma}$ $\Rightarrow$ $\hat{T}^D(x^e, F) = \dfrac{1}{\det F}\, F\,\tilde{\Sigma}(x, F^T F)\,F^T$

$$= \frac{1}{\det U}\, RU\,\tilde{\Sigma}(x, U^2)\,U^T R^T$$

$$= R\left(\frac{1}{\det U}\, U\,\tilde{\Sigma}(x, U^T U)\,U^T\right)R^T$$

$$= R\,\hat{T}^D(x^e, U)\,R^T$$

NAP: SADA SMO DOBILI

$$\Sigma(x,t) = \tilde{\Sigma}(x, \underbrace{\nabla\varphi(x,t)^T \nabla\varphi(x,t)})$$

MJERA DEFORMACIJE (STRAIN)

2. PIOLA-KIRCHHOFFOV TENZOR NAPREZANJA (STRESS)

ZADANA FUNKCIJA

DAJE VEZU IZMEĐU NAPREZANJA I MJERE DEF (STRESS-STRAIN RELATIONSHIP)

ZAD U TERMINIMA $\hat{T}$ i $\hat{\Sigma}$ P.I. $\left(\hat{T}^D(x, QF) = Q\hat{T}^D(x,F)Q^T \quad F \in M_+(3),\ Q \in SO(3)\right)$ JE EKVIVALENTAN SA

$$\hat{T}(x, QF) = Q\hat{T}(x,F)$$

$$\hat{\Sigma}(x, QF) = \hat{\Sigma}(x,F)$$

SIMETRIJE U SVOJSTVIMA MATERIJALA

- OPĆENITO SVOJSTVA MATERIJALA SU RAZLIČITA U RAZLIČITIM SMJEROVIMA. TAKAV MATERIJAL NAZIVAMO <u>ANIZOTROPNIM</u>.
- MATERIJAL KOJI IMA ELASTIČNA SVOJSTVA JEDNAKA U SVIM SMJEROVIMA NAZIVA SE <u>IZOTROPNIM</u>
- IZMEĐU POSTOJE MATERIJALI S VIŠE ILI MANJE UGRAĐENIH SIMETRIJA, <u>ORTHOTROPIC</u> MATERIJAL JE ONAJ KOJEM ~~IMA 3 ORTOGONAL~~ SE SVOJSTVA RAZLIKUJU DUŽ TRI MEĐUSOBNO ORTOGONALNE OSI PRIMJER JE DRVO.

  TRANSVERZNO IZOTROPAN JE MATERIJAL KOJI SE U SMJEROVIMA JEDNE RAVNINE PONAŠA JEDNAKO, A DRUKČIJE U ODNOSU NA ORTOGONALAN SMJER. PRIMJER SU LAMINIRANI MATERIJALI (ŠPERPLOČA)

FORMALNA DEFINICIJA IZOTROPNIH MATERIJALA

$$\Omega^{\Theta} = \Theta(\Omega)$$

$$\Theta(y) = x + Q^T(y-x)$$

- DVIJE REFERENTNE KONFIGURACIJE
- PRIMIJENIMO DEFORMACIJE $\varphi : \Omega \to \mathbb{R}^3$

$$\tilde{\varphi} : \Omega^{\Theta} \to \mathbb{R}^3$$

$$\tilde{\varphi} = \varphi \circ \Theta^{-1}$$

- ISTI JE KONAČNI POLOŽAJ TIJELA

$$\nabla \tilde{\varphi}(y) = \nabla \varphi(\Theta^{-1}(y)) \nabla \Theta^{-1}(y) = \nabla \varphi(\Theta^{-1}(y)) Q$$

- ELASTIČNO TIJELO JE IZOTROPNO AKO JE

$$T^{\varphi}(x^{\varphi}) = T^{\tilde{\varphi}}(x^{\tilde{\varphi}}), \quad x^{\varphi} = x^{\tilde{\varphi}} = \varphi(x) = \tilde{\varphi}(x)$$

$$\begin{array}{ccc} T^{\varphi}(x^{\varphi}) & = & T^{\tilde{\varphi}}(x^{\tilde{\varphi}}) \\ \| & & \| \\ \hat{T}^D(x, \nabla\varphi(x)) & & \hat{T}^D(x, \nabla\tilde{\varphi}(x)) \\ & & \| \\ & & \hat{T}^D(x, \nabla\varphi(x) Q) \end{array}$$

$$\Rightarrow \boxed{\hat{T}^D(x, F) = \hat{T}^D(x, FQ), \quad Q \in SO(3),\ F \in M_+(3)}$$

ZAD ZA FUNKCIJE ODZIVA 1. I 2. PIOLA-KIRCHHOFFOVOG TENZORA NAPREZANJA TO JE EKVIVALENTNO SA

$$\hat{T}(x, FQ) = \hat{T}(x, F)\,Q, \qquad F \in M_+(3),\ Q \in SO(3)$$

$$\hat{\Sigma}(x, FQ) = Q^T \hat{\Sigma}(x, F)\,Q, \qquad F \in M_+(3),\ Q \in SO(3)$$

Rj:
$$\hat{T}(x, FQ) = \det(FQ)\,\hat{T}^D(x, FQ)(FQ)^{-T}$$
$$= \underbrace{\det F\ \hat{T}^D(x, F)\,F^{-T}}\,Q^{-T}$$
$$= \hat{T}(x, F)\,Q$$

$$\hat{\Sigma}(x, FQ) = \cancel{\det F}\,(FQ)^{-1}\,\hat{T}(x, FQ)$$
$$= Q^{-1}\,\underbrace{F^{-1}\,\hat{T}(x, F)}\,Q$$
$$= Q^T\,\hat{\Sigma}(x, F)\,Q$$

ZA IZOTROPAN MATERIJAL KOJI ZADOVOLJAVA PRINCIP INVARIJANTNOSTI VRIJEDI PUNO SPECIJALNIJI ZAKON PONAŠANJA

TEOREM ((KOROLAR) RIVLIN-ERICKSENOVOG TEOREMA REPREZENTACIJE (THEOREM 3.6-2 (CIARLET))

NEKA ELASTIČNI MATERIJAL ZADOVOLJAVA PRINCIP INVARIJANTNOSTI I NEKA JE IZOTROPAN U TOČKI $x \in \bar{\Omega}$.
ZA PROIZVOLJNU DEFORMACIJU $\varphi: \bar{\Omega} \to \mathbb{R}^3$
2. PIOLA-KIRCHHOFFOV TENZOR NAPREZANJA U TOČKI $x \in \bar{\Omega}$ DAN JE SA

$$\Sigma(x) = \hat{\Sigma}(x, \nabla\varphi(x)) = \tilde{\Sigma}(x, \nabla\varphi(x)^T \nabla\varphi(x))$$

GDJE JE FUNKCIJA ODZIVA $\tilde{\Sigma}(x, \cdot): Sym_+(3) \to Sym(3)$ OBLIKA

$$(*) \qquad \tilde{\Sigma}(x, C) = \gamma_0(x, \iota_C) I + \gamma_1(x, \iota_C) C + \gamma_2(x, \iota_C) C^2, \quad C \in Sym_+(3),$$

A $\gamma_0(x, \cdot), \gamma_1(x, \cdot), \gamma_2(x, \cdot)$ SU REALNE FUNKCIJE ~~PRIN~~ GLAVNIH INVARIJANTI MATRICE C.

$$\iota_C = (\iota_1(C), \iota_2(C), \iota_3(C)),$$

$$\iota_1(C) = \operatorname{tr} C = \lambda_1 + \lambda_2 + \lambda_3, \qquad (\lambda_1, \lambda_2, \lambda_3 \in \sigma(C))$$

$$\iota_2(C) = \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3,$$

$$\iota_3(C) = \det C = \lambda_1\lambda_2\lambda_3 = \frac{1}{6}\left((\operatorname{tr} C)^3 - 3 \operatorname{tr} C \operatorname{tr} C^2 + 2 \operatorname{tr} C^3\right).$$

ŠTOVIŠE VRIJEDI I OBRAT, TJ. AKO JE $\tilde{\Sigma}$ OBLIKA (*), MATERIJAL JE IZOTROPAN I ZADOVOLJAVA PRINCIP INVARIJANTNOSTI

ZAKON PONAŠANJA U OKOLINI REFERENTNE KONFIGURACIJE

- RADIMO ZA 2. PIOLA-KIRCHHOFFOV TENZOR I STANDARDNO PRETPOSTAVLJAMO DA JE ZADOVOLJEN PRINCIP INVARIJANTNOSTI, PA JE

$$\Sigma(x) = \hat{\Sigma}(x, \nabla\varphi) = \tilde{\Sigma}(x, \nabla\varphi^T \nabla\varphi)$$

$C = \nabla\varphi^T \nabla\varphi$ CAUCHY-GREENOV TENZOR

$E = \frac{1}{2}(C - I)$ KONAČNI TENZOR DEFORMACIJE
MJERI ODSTUPANJE OD KRUTE DEFORMACIJE

- $\tilde{\Sigma}(x, C) = \tilde{\Sigma}(x, I + 2E)$

- KAD JE $E = 0$ U KRUTOJ DEFORMACIJI SMO
- AKO SMO U OKOLINI REFERENTNE KONFIGURACIJE ZNAČI $E \approx 0$

- MOŽEMO KORISTITI TAYLOROV RAZVOJ

$$\tilde{\Sigma}(x, C) = \tilde{\Sigma}(x, I + 2E) = \tilde{\Sigma}(x, I) + 2 \sum_{i,j=1}^{3} \frac{\partial \tilde{\Sigma}}{\partial C_{ij}}(x, I) E_{ij} + o(E)$$

- KAKO JE $\tilde{\Sigma}(x, \cdot) : Sym(3) \to Sym(3)$
RADI SE KAO O PRESLIKAVANJU S $\mathbb{R}^6 \to \mathbb{R}^6$

- 36 $\frac{\partial \tilde{\Sigma}_{kl}}{\partial C_{ij}}$ NEOVISNO

- ZAPRAVO IH IMA 81, PRISUTNA SIMETRIJA!

DJELOVANJE $2\sum_{i,j=1}^{3} \frac{\partial \tilde{\Sigma}}{\partial C_{ij}}(X, I) E_{ij}$ ZAPISUJE SE

POMOĆU TENZORA $\mathcal{C} : \mathrm{Sym}(3) \longrightarrow \mathrm{Sym}(3)$ 4. REDA

$$(\mathcal{C} E)_{kl} = \sum_{i,j=1}^{3} \underset{\substack{\uparrow \\ 3^4 = 81 \text{ KOEFICIJENT}}}{\mathcal{C}^{klij}} E_{ij}$$

SIMETRIJE

$$\mathcal{C}^{klij} = \mathcal{C}^{lkij} = \mathcal{C}^{klji} = \mathcal{C}^{ijkl} \qquad i,j,k,l \in \{1,2,3\}$$

U SLUČAJU KADA JE MATERIJAL IZOTROPAN TENZOR ELASTIČNOSTI (LINEARNE) $\mathcal{C}$ POPRIMA VRLO JEDNOSTAVNU STRUKTURU. POSLJEDICA JE TO R-E TEOREMA.

TEOREM NEKA JE ELASTIČNI MATERIJAL IZOTROPAN I ZADOVOLJAVA PRINCIP INVARIJANTNOSTI U $x \in \Omega$. NEKA SU FUNKCIJE $\gamma_0(x,\cdot), \gamma_1(x,\cdot), \gamma_2(x,\cdot)$ DIFERENCIJABILNE U TOČKI $L_I = (3,3,1)$. TADA POSTOJE KONSTANTE $\pi(x), \lambda(x), \mu(x) \in \mathbb{R}$ T.D.

$$\tilde{\Sigma}(x, C) = -\pi(x) I + \lambda(x)(\operatorname{tr} E) I + 2\mu(x) E + o(E, x)$$

ZA SVE $C = I + 2E \in \mathrm{Sym}_+(3)$.

DOK:

$$\tilde{\Sigma}(x, C) = \gamma_0(x, L_C) I + \gamma_1(x, L_C) C + \gamma_2(x, L_C) C^2$$

ISPUŠTAMO x U NASTAVKU

$$C = I + 2E$$

$$C^2 = (I + 2E)^2 = I + 4E + 4E^2 = I + 4E + o(E)$$

$$\tilde{\Sigma}(x, C) = \big(\gamma_0(L_C) + \gamma_1(L_C) + \gamma_2(L_C)\big) I + \big(2\gamma_1(L_C) + 4\gamma_2(L_C)\big) E + o$$

IZ DFB OD $\gamma_i(3,3,1)$ SLIJEDI

$$\gamma_i(L_C) = \gamma_i(L_I) + D\gamma_i(L_I)(L_C - L_I) + o(L_C - L_I)$$

$$L_C = L_I + DL_I(2E) + o(E)$$

$$\iota_1(C) = \operatorname{tr} C = \operatorname{tr}(I+2E) = 3 + 2\operatorname{tr} E$$

$$\iota_2(C) = \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right) = \frac{1}{2}\left(\left(\operatorname{tr}(I+2E)\right)^2 - \operatorname{tr}(I+2E)^2\right)$$

$$= \frac{1}{2}\left((3+2\operatorname{tr} E)^2 - \operatorname{tr}(I+4E+4E^2)\right)$$

$$= \frac{1}{2}\left(9+12\operatorname{tr} E + o(E) - 3 - 4\operatorname{tr} E + o(E)\right)$$

$$= \frac{1}{2}\left(6 + 8\operatorname{tr} E + o(E)\right)$$

$$= 3 + 4\operatorname{tr} E + o(E)$$

$$\iota_3(C) = \frac{1}{6}\left((\operatorname{tr} C)^3 - 3\operatorname{tr} C \operatorname{tr} C^2 + 2\operatorname{tr} C^3\right)$$

$$= \frac{1}{6}\Big((3+2\operatorname{tr} E)^3 - 3(3+2\operatorname{tr} E)\operatorname{tr}(I+4E+4E^2)$$
$$+ 2\operatorname{tr}\,(I+4E+o(E))(I+2E)\Big)$$

$$= \frac{1}{6}\Big(27 + 54\operatorname{tr} E + o(E) - (9+6\operatorname{tr} E)(3+4\operatorname{tr} E + o(E))$$
$$+ 2\operatorname{tr}(I+6E+o(E))\Big)$$

$$= \frac{1}{6}\left(\cancel{27} + \cancel{54\operatorname{tr} E} - \cancel{27} - \cancel{54\operatorname{tr} E} + o(E) + 6 + 12\operatorname{tr} E + o(E)\right)$$

$$= 1 + 2\operatorname{tr} E + o(E)$$

$$\Rightarrow \iota_C - \iota_I = (2\operatorname{tr} E, 4\operatorname{tr} E, 2\operatorname{tr} E) + (o(E)^{*}, o(E), o(E))$$

$$\gamma_i(\iota_C) = \gamma_i(\iota_I) + \frac{\partial \gamma_i}{\partial \iota_1}(\iota_I)\, 2\operatorname{tr} E + \frac{\partial \gamma_i}{\partial \iota_2}(\iota_I)\, 4\operatorname{tr} E + \frac{\partial \gamma_i}{\partial \iota_3}(\iota_I)\, 2\operatorname{tr} E + o(E)$$

ОЗНАЧИМО ЛИ $\quad \mathring{\gamma}_i(\iota_I) := 2\frac{\partial \gamma_i}{\partial \iota_1}(\iota_I) + 4\frac{\partial \gamma_i}{\partial \iota_2}(\iota_I) + 2\frac{\partial \gamma_i}{\partial \iota_3}(\iota_I)$

$$\gamma_i(\iota_C) = \gamma_i(\iota_I) + \mathring{\gamma}_i(\iota_I)\operatorname{tr} E + o(E)$$

SAD UVRŠTAVAMO U $\hat{\Sigma}(x, C)$

$$\hat{\Sigma}(C) = \Big(\gamma_0(\iota_I) + \dot{\gamma}_0(\iota_I)\,\mathrm{tr}\,E + \gamma_1(\iota_I) + \dot{\gamma}_1(\iota_I)\,\mathrm{tr}\,E + \gamma_2(\iota_I) + \dot{\gamma}_2(\iota_I)\,\mathrm{tr}\,E + o(E)\Big) I$$

$$+ \Big(2\gamma_1(\iota_I) + 2\dot{\gamma}_1(\iota_I)\,\mathrm{tr}\,E + o(E) + 4\gamma_2(\iota_I) + 4\dot{\gamma}_2(\iota_I)\,\mathrm{tr}\,E + o(E)\Big) E + o(E)$$

$$= \big(\gamma_0(\iota_I) + \gamma_1(\iota_I) + \gamma_2(\iota_I)\big) I$$

$$+ \big(\dot{\gamma}_0(\iota_I) + \dot{\gamma}_1(\iota_I) + \dot{\gamma}_2(\iota_I)\big)(\mathrm{tr}\,E)\, I$$

$$+ \big(2\gamma_1(\iota_I) + 4\gamma_2(\iota_I)\big) E + o(E)$$

$$\Rightarrow \quad \pi(x) = \gamma_0(x, \iota_I) + \gamma_1(x, \iota_I) + \gamma_2(x, \iota_I)$$

$$\lambda(x) = \dot{\gamma}_0(x, \iota_I) + \dot{\gamma}_1(x, \iota_I) + \dot{\gamma}_2(x, \iota_I)$$

$$\mu(x) = \gamma_1(\iota_I) + 2\gamma_2(\iota_I)$$

<u>NAP:</u> $\tilde{\Sigma}(x, I) = -\pi(x)\, I$ — REZIDUALNO NAPREZANJE

$$\hat{\Sigma}(x, I) = \hat{T}(x, I) = \hat{T}^D(x^p, I)$$ NAPREZANJE U REFERENTNOJ KONFIGURACIJI

$$= T_R(x)$$ ZA IZOTROPAN MATERIJAL TO JE TLAK!

REFERENTNA KONFIGURACIJA KOD KOJE JE $T_R = 0$ NAZIVA SE PRIRODNA KONFIGURACIJA.

<u>TEOREM</u> NEKA VRIJEDE PRETPOSTAVKE PRETHODNOG TEOREMA, TE NEKA JE MATERIJAL HOMOGEN. TADA POSTOJE KONSTANTE $\lambda, \mu \in \mathbb{R}$ (LAMÉOVE KONSTANTE) T.D.

$$\tilde{\Sigma}(I+2E) = \lambda(\operatorname{tr} E)\, I + 2\mu E + o(E).$$

SVOJSTVA LAMÉOVIH KONSTANTI λ, μ

ε - MALI PARAMETAR, $\bar{\Omega}$ - ELASTIČNO TIJELO, HOMOGENO, IZOTROPNO
PRIRODNA KONFIGURACIJA!

$\varphi^\varepsilon : \bar{\Omega} \to \mathbb{R}^3$ MALA DEFORMACIJA OBLIKA

$$\varphi^\varepsilon(x) = x + u^\varepsilon(x) = x + \varepsilon \xi(x) + o(\varepsilon, x) \quad \& \quad G = \nabla \xi = \text{const}$$

$$\nabla \varphi^\varepsilon(x) = I + \varepsilon G + o(\varepsilon)$$

1. P-K. $$T(x) = \nabla \varphi^\varepsilon(x) \tilde{\Sigma}(\nabla \varphi^\varepsilon(x)^T \nabla \varphi^\varepsilon(x))$$

$$= (I + \varepsilon G + o(\varepsilon)) \tilde{\Sigma}((I + \varepsilon G + o(\varepsilon))^T (I + \varepsilon G + o(\varepsilon)))$$

$$= \underbrace{(I + \varepsilon G) \tilde{\Sigma}((I + \varepsilon G)^T (I + \varepsilon G))}_{\text{CONST}} + o(\varepsilon)$$

$\Rightarrow \operatorname{div} T = o(\varepsilon)$ TAKVA DEFORMACIJA POSLJEDICA KONTAKTNIH SILA

ODNOSNO $$(I + \varepsilon G)^T (I + \varepsilon G) = I + \varepsilon(G + G^T) + o(\varepsilon)$$

$$T(x) = I \tilde{\Sigma}(I) + \varepsilon \left[G \tilde{\Sigma}(I) + I \left(\lambda \operatorname{tr}\left(\tfrac{1}{2}(G + G^T)\right) I + 2\mu \tfrac{1}{2}(G + G^T) \right) \right] + o(\varepsilon)$$

$$\tilde{\Sigma}(I) = 0$$

$$\Rightarrow T(x) = \varepsilon \left[\lambda (\operatorname{tr} G) I + \mu (G + G^T) \right] + o(\varepsilon)$$

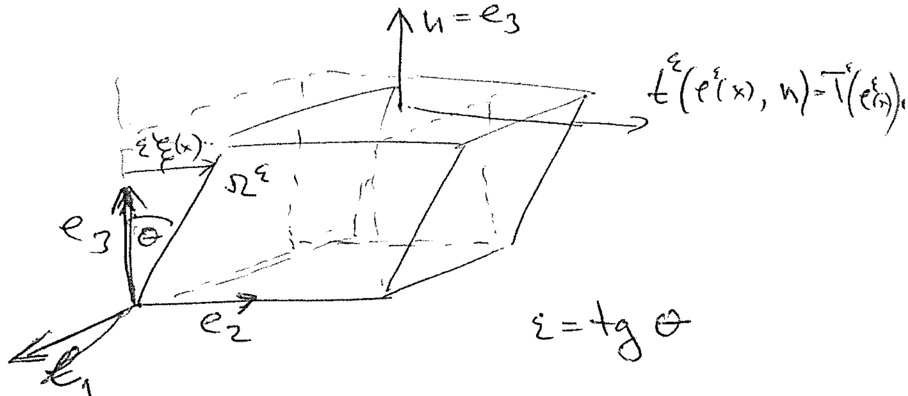
$$T^\varepsilon(\varphi^\varepsilon(x)) = \frac{1}{\det(I + \varepsilon G + o(\varepsilon))} T(x) (I + \varepsilon G + o(\varepsilon))^T$$

$$= \varepsilon \left[\lambda (\operatorname{tr} G) I + \mu (G + G^T) \right] + o(\varepsilon)$$

EKSPERIMENT 1:

$$\varphi^\varepsilon(x) = x + \varepsilon \underbrace{\begin{bmatrix} 0 \\ x_3 \\ 0 \end{bmatrix}}_{\xi} + o(\varepsilon)$$

JEDNOSTAVNO SMICANJE



$$G = \nabla \xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\varepsilon = \operatorname{tg} \theta$

$$T^\varepsilon(\varphi^\varepsilon(x)) = \varepsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{bmatrix} + o(\varepsilon)$$

$$\Rightarrow \quad t^\varepsilon(\varphi^\varepsilon(x), n) \cdot e_2 = T^\varepsilon(\varphi^\varepsilon(x))\, e_3 \cdot e_2 = \varepsilon\mu + o(\varepsilon) > 0$$

ZA OVU DEFORMACIJU

$$\mu = \frac{t^\varepsilon(\varphi^\varepsilon(x), e_3) \cdot e_2}{\varepsilon} = \frac{t^\varepsilon(\varphi^\varepsilon(x), e_3) \cdot e_2}{\operatorname{tg} \theta} > 0$$

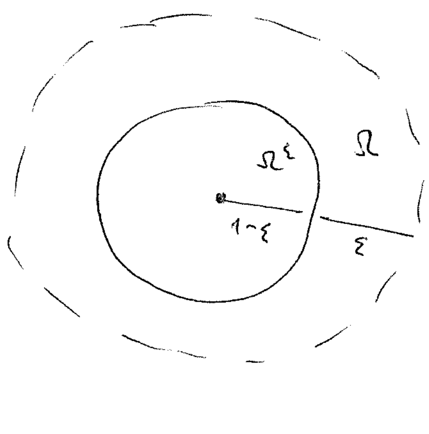
MODUL SMICANJA (PONEKAD OZNAČEN I S G)

EKSPERIMENT 2:

$$\varphi^\varepsilon(x) = x - \varepsilon \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{} + o(\varepsilon) = (1-\varepsilon)\, x + o(\varepsilon)$$

ČISTA KOMPRESIJA

$$\xi(x) = x \Rightarrow G = \nabla \xi = I$$



RELATIVNO

$$T^\varepsilon(e^\varepsilon(x)) = \varepsilon\left[-\lambda\, 3\, I - 2\mu I\right] + o(\varepsilon) = -(3\lambda+2\mu)\varepsilon I + o(\varepsilon)$$

$$T^\varepsilon(e^\varepsilon(x))\, n = -\underbrace{(3\lambda+2\mu)\,\varepsilon}_{=\pi}\, n + o(\varepsilon)$$

TLAK!

$\pi > 0$ STISKANJE!

$$\Rightarrow 3\lambda + 2\mu > 0$$

$$3\lambda + 2\mu = \frac{\pi}{\varepsilon}$$

OMJER TLAKA I RELATIVNE PROMJENE RADIJUSA

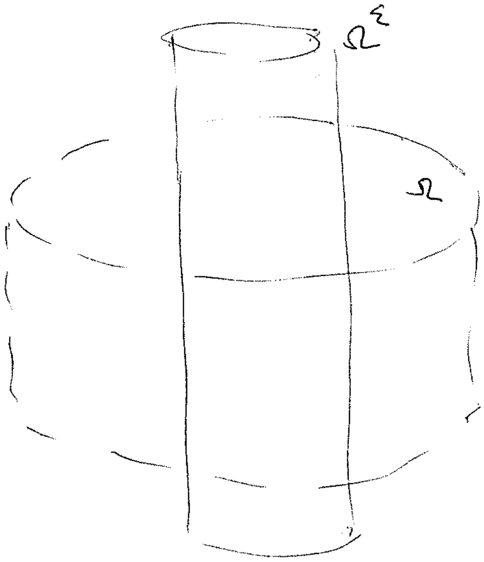
NAP: $\kappa = \frac{1}{3}(3\lambda + 2\mu)$ BULK MODULUS

EKSPERIMENT 3

$$e^\varepsilon(x) = x + \varepsilon \underbrace{\begin{bmatrix} -\nu x_1 \\ -\nu x_2 \\ x_3 \end{bmatrix}}_{=\xi} + o(\varepsilon)$$

ČISTA TENZIJA

$$G = \nabla \xi = \begin{bmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- RELATIVNO PRODULJENJE ε
- RELATIVNO SUŽENJE $\nu\varepsilon$
- PRIRODNO (OČEKIVANO) $\nu > 0$

$$T^{\varepsilon}(e^{\varepsilon}(x)) = \varepsilon\left(\lambda(1-2\nu)I + 2\mu\begin{bmatrix} -\nu & & \\ & -\nu & \\ & & 1\end{bmatrix}\right) + o(\varepsilon)$$

$$= \varepsilon\begin{bmatrix} \lambda(1-2\nu)-2\mu\nu & 0 & 0 \\ 0 & \lambda(1-2\nu)-2\mu\nu & 0 \\ 0 & 0 & \lambda(1-2\nu)+2\mu \end{bmatrix} + o(\varepsilon)$$

ČISTA TENZIJA $\Rightarrow$ NEMA NAPREZANJA OSIM UZDUŽNOG

$$\Rightarrow \lambda(1-2\nu) - 2\mu\nu = 0$$

$$\lambda - 2\nu(\lambda+\mu) = 0 \Rightarrow \nu = \frac{\lambda}{2(\lambda+\mu)}$$

POISSONOV BROJ (OMJER)

$\mu > 0 \quad 3\lambda + 2\mu > 0 \Rightarrow 3\lambda + 3\mu > 0 \Rightarrow \lambda + \mu > 0$

$\nu > 0 \Rightarrow \lambda = 2\nu(\lambda+\mu) > 0$ (POSTOJE MATERIJALI (UMJETNI) ZA KOJE TO NE VRIJEDI!)

$$T^{\varepsilon}_{33} = \varepsilon(\lambda(1-2\nu)+2\mu) + o(\varepsilon) = \varepsilon\left(\lambda\left(1-\frac{\lambda}{\lambda+\mu}\right)+2\mu\right) + o(\varepsilon)$$

$$= \varepsilon\left(\lambda\frac{\mu}{\lambda+\mu} + 2\mu\right) + o(\varepsilon) = \varepsilon\,\frac{\lambda\mu + 2\mu(\lambda+\mu)}{\lambda+\mu} + o(\varepsilon)$$

$$= \varepsilon\,\frac{3\lambda\mu + 2\mu^2}{\lambda+\mu} + o(\varepsilon) = \varepsilon\,\underbrace{\mu\,\frac{3\lambda+2\mu}{\lambda+\mu}}_{= E} + o(\varepsilon)$$

YOUNGOV MODUL ELASTIČNOSTI $E > 0$

POSLJEDICA

NAP: $0 < \nu = \dfrac{\lambda}{2(\lambda+\mu)} < \dfrac{1}{2}$

NAP: λ, μ LAMÉ $\longleftrightarrow E, \nu$