3. JEDNADŽBA GIBANJA

SLIJEDI IZ 2. NEWTONOVOG ZAKONA:

SUMA SVIH SILA = BRZINA PROMJENE KOLIČINE GIBANJA
(INERCIJALNA SILA, $m \cdot a$)

SUMA SVIH MOMENATA = BRZINA PROMJENE KUTNE KOLIČINE GIBANJA

SILE (VANJSKE, ZADANE)

- OPISUJU DJELOVANJE VANJSKOG SVIJETA NA TIJELO
- VANJSKA VOLUMNA SILA:

$$f^{\varphi} : \Omega^{\varphi} \times \mathbb{R} \to \mathbb{R}^3 \quad (\text{ZAPRAVO } f^{\varphi} : \mathcal{T} \to \mathbb{R}^3)$$

ZA $P \subseteq \Omega^{\varphi}$

$$\int_P f^{\varphi} dx^{\varphi}$$ – UKUPNA VOLUMNA SILA NA KOMAD P

f^{φ} – VOLUMNA GUSTOĆA

PRIMJER: SILA TEŽA

- VANJSKA KONTAKTNA SILA:

$$\Gamma_1^{\varphi} \subseteq \partial \Omega^{\varphi}$$

$$g^{\varphi} : \Gamma_1^{\varphi} \times \mathbb{R} \to \mathbb{R}$$ (PLOŠNA) GUSTOĆA VANJSKE KONTAKTNE SILE

ZA $A \subseteq \Gamma_1^{\varphi}$

$$\int_A g^{\varphi} da^{\varphi}$$ – UKUPNA VANJSKA KONTAKTNA SILA NA KOMAD A

$\psi : D \to A$ PARAMETRIZACIJA

$$\int_A g^{\varphi} da^{\varphi} = \int_D g^{\varphi} \circ \psi \sqrt{\det(\nabla\psi^T \nabla\psi)}\, dy$$

PRIMJER: —

— TLAK (NA BALON U VODI)
NPR.

GUSTOĆA MASE:

— $\rho^\varphi : \Omega^\varphi \times \mathbb{R} \to \mathbb{R}$ NA DEFORMIRANOJ KONFIGURACIJI

— $P \subseteq \Omega^\varphi$ $\quad \int_P \rho^\varphi da^\varphi$ — MASA KOMADA P

NAP: $b^\varphi : \Omega^\varphi \times \mathbb{R} \to \mathbb{R}$ DEF SA $b^\varphi = \dfrac{f^\varphi}{\rho^\varphi}$

GUSTOĆA VANJSKE SILE PO JEDINICI MASE

KOLIČINE GIBANJA:

- LINEARNA, DANA GUSTOĆOM $\rho^\varphi v^\varphi$
(LINEAR MOMENTUM)

— KUTNA, DANA GUSTOĆOM $x^\varphi \times \rho^\varphi v^\varphi$
(ANGULAR MOMENTUM)

AKSIOM (EULER, CAUCHY)

NEPREKIDNO

POSTOJI VEKTORSKO POLJE $t^\varphi : \overline{\Omega^\varphi} \times S^2 \times \mathbb{R} \to \mathbb{R}^3$,

$S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$, TAKVO DA JE

a) $t^\varphi(x^\varphi, n^\varphi, t) = g^\varphi(x^\varphi, t), \quad x^\varphi \in \Gamma_1^\varphi$

n^φ – JEDINIČNA NORMALA NA Γ_1^φ, VANJSKA NA Ω^φ

b) ZAKON OČUVANJA IMPULSA:

$\forall \mathcal{P}^\varphi \subseteq \overline{\Omega}^\varphi$ OTVOREN I POVEZAN

$$\frac{d}{dt}\int_{\mathcal{P}^\varphi} \varrho^\varphi(x^\varphi,t)v^\varphi(x^\varphi,t)\,dx^\varphi = \int_{\partial\mathcal{P}^\varphi} t^\varphi(x^\varphi, n^\varphi, t)\,da^\varphi + \int_{\mathcal{P}^\varphi} f^\varphi(x^\varphi,t)\,dx^\varphi$$

n^φ – JEDINIČNA NORMALA NA $\partial\mathcal{P}^\varphi$, VANJSKA NA $\mathcal{P}^\varphi$

c) ZAKON OČUVANJA MOMENTA IMPULSA

$\forall \mathcal{P}^\varphi \subseteq \overline{\Omega}^\varphi$ OTVOREN I POVEZAN

$$\frac{d}{dt}\int_{\mathcal{P}^\varphi} x^\varphi \times \varrho^\varphi(x^\varphi,t)\, v^\varphi(x^\varphi,t)\,dx^\varphi = \int_{\partial\mathcal{P}^\varphi} x^\varphi \times t^\varphi(x^\varphi, n^\varphi, t)\,da^\varphi + \int_{\mathcal{P}^\varphi} x^\varphi \times f^\varphi(x^\varphi,t)\,dx^\varphi$$

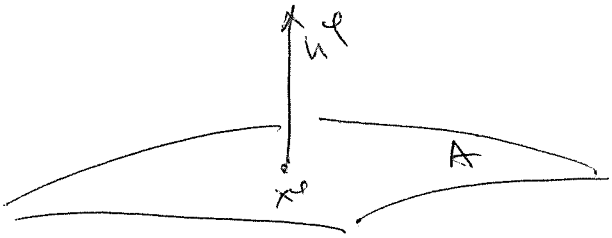
NAP: – t^φ SE ZOVE KONTAKTNA SILA

CAUCHYJEV VEKTOR NAPREZANJA

– ZAPRAVO JE GUSTOĆA

$\int_A t^\varphi\, da^\varphi$ UKUPNA KONTAKTNA SILA NA PLOHU A

$$\int_A t^\varphi\, da^\varphi \overset{!!}{=} P(A)$$



$P(A)$ – UKUPNA SILA KOJIM TIJELO "IZNAD" (U SMJERU NORMALE n^φ) DJELUJE NA TIJELO "ISPOD" PLOHE A

$$t^\varphi(x^\varphi, n^\varphi, t) \text{ "=" } \lim_{pov(A) \to 0} \frac{P(A)}{pov(A)}$$

– $\int_{\partial P^\varphi} t^\varphi + \int_{P^\varphi} f^\varphi$ – UKUPNA SILA NA KOMAD P^φ

– $\int_{\partial P^\varphi} x^\varphi \times t^\varphi + \int_{P^\varphi} x^\varphi \times f^\varphi$ – UKUPNI MOMENT NA KOMAD P^φ

– KONCEPT KONTAKTNE SILE POSLJEDICA JE FIZIKALNE ČINJENICE DA JE INTERAKCIJA RAZLIČITIH DIJELOVA TIJELA POSLJEDICA MEĐUMOLEKULARNIH SILA KOJE SU KRATKOG DOMETA

– POSTOJI RIGOROZNI NAČIN ~~OPRAVDANJA~~ DA t^φ OVISI U TOČKI x^φ O GEOMETRIJI SAMO KROZ n^φ!

– AKSIOM!

ZAKON OČUVANJA MASE

U REFERENTNOJ KONFIGURACIJI $\quad \varrho : \Omega \to \mathbb{R}_+$

U DEFORMIRANOJ KONFIGURACIJI Ω^φ, $\varrho^\varphi : \Omega^\varphi \to \mathbb{R}_+$

ZAKON OČUVANJA

$$\forall P \subset \Omega \text{ otv. pov.} \quad \int_{P^{\varphi_t}} \varrho^\varphi(x^\varphi, t)\, dx^\varphi = \int_P \varrho(x)\, dx$$

$$P^{\varphi_t} = \varphi_t(P) \longrightarrow \quad \Vert$$

$$\int_P \varrho^\varphi(\varphi(x,t),t) \det \nabla\varphi(x,t)\, dx$$

$$\Rightarrow \forall P \quad \int_P \left(\varrho(x) - \varrho^\varphi(\varphi(x,t),t) \det\nabla\varphi(x,t)\right) dx = 0$$

$$\Rightarrow \quad \varrho(x) = \varrho^\varphi(\varphi(x,t),t)\det\nabla\varphi(x,t) \quad \Big|\ \frac{\partial}{\partial t}$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial t}\left(\varrho^\varphi(\varphi(x,t),t)\det\nabla\varphi(x,t)\right)$$

KORISTIMO:

$$\frac{\partial}{\partial t}\left(\det\nabla\varphi(x,t)\right) = \left(\operatorname{div} v^\varphi\right)\circ\varphi(x,t)\det\nabla\varphi(x,t)$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial t}\left(\varrho^\varphi(\varphi(x,t),t)\right)\det\nabla\varphi(x,t) + \varrho^\varphi(\varphi(x,t),t)\left(\operatorname{div} v^\varphi\right)\circ\varphi(x,t) \det\nabla\varphi(x,t)$$

$$0 = \left(\dot{\varrho}^\varphi + \varrho^\varphi \operatorname{div} v^\varphi\right)\circ\varphi(x,t)\det\nabla\varphi(x,t)$$

$$\Rightarrow \quad 0 = \dot{\varrho}^\varphi + \varrho^\varphi \operatorname{div} v^\varphi \qquad \text{JEDNADŽBA KONTINUITETA}$$

TEOREM ZA DOVOLJNO GLATKO GIBANJE φ VRIJEDI

$$\frac{d}{dt}\left(\det \nabla \varphi(x,t)\right) = \left(\operatorname{div} v^{\varphi}\right)\circ \varphi(x,t)\, \det \nabla\varphi(x,t)$$

LEMA: $$D(\det)(A)(T) = \det A \ \operatorname{tr}(A^{-1}T), \quad A \in GL(\mathbb{R}^n)$$

DOK: det KAO POLINOM JE DFB.

$$\det X = \det(AA^{-1}X) = \det A \det(A^{-1}X)$$

$$\Rightarrow D(\det)(A)(T) = \det A \ D(\det)(I)(A^{-1}T)$$

TV: $D(\det)(I) = \operatorname{tr}$

DOK: KORIŠTENJEM DERIVACIJE U SMJERU

$$D(\det)(I)(T) = \lim_{\varepsilon\to 0} \frac{\det(I+\varepsilon T) - \det I}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{1+\varepsilon \operatorname{tr} T + O(\varepsilon^2) - 1}{\varepsilon} = \operatorname{tr} T \ //$$

$$\Rightarrow D(\det)(A)(T) = \det A \ \operatorname{tr}(A^{-1}T) \ //$$

DOKAZ TEOREMA: PRIMJENOM LEME DOBIVAMO

$$\frac{d}{dt}\left(\det \nabla\varphi(x,t)\right) = D(\det)\left(\nabla\varphi(x,t)\right)\left(\frac{\partial}{\partial t}\nabla\varphi(x,t)\right)$$

$$= \det \nabla\varphi(x,t) \ \operatorname{tr}\left(\nabla\varphi_t(x)^{-1} \frac{d}{dt}\nabla\varphi(x,t)\right)$$

S DRUGE STRANE $\quad v^{\varphi}(x^{\varphi},t) = \frac{\partial\varphi}{\partial t}\left(\varphi_t^{-1}(x^{\varphi}),t\right)$

$$\Rightarrow \nabla v^{\varphi}(x^{\varphi},t) = \frac{\partial}{\partial t}\nabla\varphi\left(\varphi_t^{-1}(x^{\varphi}),t\right)\nabla\varphi_t^{-1}(x^{\varphi})$$

$$\Rightarrow \operatorname{div} v^{\varphi}(x^{\varphi},t) = \operatorname{tr}\left(\frac{\partial}{\partial t}\nabla\varphi\left(\varphi_t^{-1}(x^{\varphi}),t\right)\nabla\varphi_t^{-1}(x^{\varphi})\right) = \operatorname{tr}\left(\nabla\varphi_t^{-1}(x^{\varphi})\frac{\partial}{\partial t}\nabla\varphi\left(\varphi_t^{-1}(x^{\varphi}),t\right)\right)$$

$$\& \quad \nabla\varphi_t\circ\varphi_t^{-1}\cdot\nabla\varphi_t^{-1} = I.$$

CILJ NAM JE IZVESTI DIFERENCIJALNU FORMULACIJU ZA JEDNADŽBU GIBANJA

LEMA : ZA SKALARNO ILI VEKTORSKO EULEROVO POLJE f : I $P \subseteq \mathcal{R}$ VRIJEDI TRANSPORTNA FORMULA $\quad (P^{\varphi_t} = \varphi_t(P))$

$$\frac{d}{dt} \int_{P^{\varphi_t}} f(x^\varphi, t)\, dx^\varphi = \int_{P^{\varphi_t}} \left(\dot{f}(x^\varphi, t) + f(x^\varphi, t) \operatorname{div} v^\varphi(x^\varphi, t) \right) dx^\varphi$$

DOK: ZAMJENOM VARIJABLI $x^\varphi = \varphi_t(x)$ INTEGRAL OVISAN O t PREBACIMO NA DOMENU P NEOVISNU O t.

$$\frac{d}{dt} \int_{\varphi_t(P)} f(x^\varphi, t)\, dx^\varphi = \frac{d}{dt} \int_P f(\varphi(x,t), t) \det \nabla\varphi(x,t)\, dx$$

$$= \int_P \frac{\partial}{\partial t}\Big(f(\varphi(x,t),t) \Big) \det\nabla\varphi(x,t) + f(\varphi(x,t),t) \frac{\partial}{\partial t}\left(\det\nabla\varphi(x,t) \right) dx$$

KORISTIMO

$$\frac{\partial}{\partial t}\left(\det\nabla\varphi(x,t) \right) = \left(\operatorname{div} v^\varphi \right) \circ \varphi(x,t) \det\nabla\varphi(x,t)$$

$$= \int_P \left(\frac{\partial}{\partial t}\Big(f(\varphi(x,t),t) \Big) + f(\varphi(x,t),t) \left(\operatorname{div} v^\varphi \right) \circ \varphi(x,t) \right) \det\nabla\varphi(x,t)\, dx$$

$$= \int_{P^{\varphi_t}} \left(\dot{f}(x^\varphi, t) + f(x^\varphi, t) \operatorname{div} v^\varphi(x^\varphi, t) \right) dx^\varphi$$

KOROLAR $\frac{d}{dt}\int_{P^{\varphi_t}} \rho^\varphi v^\varphi dx^\varphi = \int_{P^{\varphi_t}} \rho^\varphi \dot{v}^\varphi dx^\varphi$, $P \subseteq \Omega$, $P^{\varphi_t} = \varphi_t(P)$

DOK: PRIMJENOM TRANSPORTNE FORMULE DOBIVAMO

$$\frac{d}{dt}\int_{P^{\varphi_t}} \rho^\varphi v^\varphi dx^\varphi = \int_{P^{\varphi_t}} (\rho^\varphi v^\varphi)^\cdot + \rho^\varphi v^\varphi \operatorname{div} x^\varphi \, dx^\varphi$$

ZA EULEROVA POLJA f, g VRIJEDI

$$(fg)^\cdot(x^\varphi, t) = \frac{d}{dt}\Big(f(\varphi_t(x), t)\, g(\varphi_t(x), t)\Big) \circ \varphi_t^{-1}(x^\varphi)$$

$$= \frac{d}{dt}\big(f(\varphi_t(x), t)\big) \circ \varphi_t^{-1}(x^\varphi)\, g(x^\varphi, t)$$

$$+ f(x^\varphi, t) \frac{d}{dt}\big(g(\varphi_t(x), t)\big) \circ \varphi_t^{-1}(x^\varphi) = \dot{f} g + f \dot{g}$$

STOGA VRIJEDI

$$\frac{d}{dt}\int_{P^{\varphi_t}} \rho^\varphi v^\varphi dx^\varphi = \int_{P^{\varphi_t}} \rho \dot{v}^\varphi + \underbrace{\dot{\rho}^\varphi v^\varphi + \rho^\varphi v^\varphi \operatorname{div} x^\varphi}_{\substack{\| \\ 0}} \, dx$$

IZ JEDNADŽBE KONTINUITETA

TEOREM (CAUCHY):

NEKA SU $t^{\varphi} : \overline{\Omega}^{\varphi} \times S^2 \times \mathbb{R} \to \mathbb{R}^3$

$f^{\varphi} : \overline{\Omega}^{\varphi} \times \mathbb{R} \to \mathbb{R}^3$ NEPREKIDNE

$\varrho^{\varphi}, v^{\varphi} : \overline{\Omega}^{\varphi} \times \mathbb{R} \to \mathbb{R}^3$

I NEKA VRIJEDI ZAKON OČUVANJA IMPULSA

TADA $\exists\, T^{\varphi} : \overline{\Omega}^{\varphi} \times \mathbb{R} \to M_3(\mathbb{R})$ T.D.

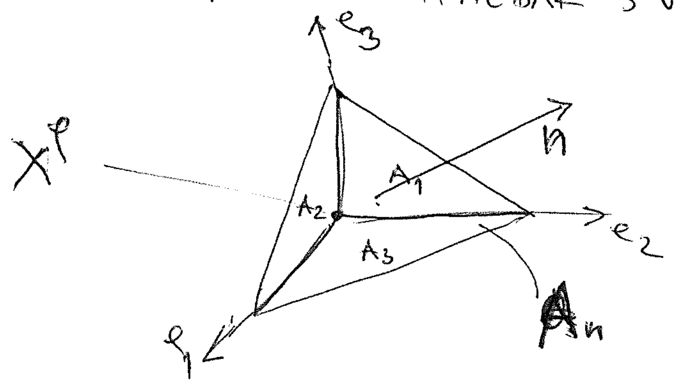
(CAUCHYJEV TENZOR NAPREZANJA)

$$t^{\varphi}(x^{\varphi}, n^{\varphi}, t) = T^{\varphi}(x^{\varphi}, t)\, n^{\varphi}, \quad x^{\varphi} \in \overline{\Omega}^{\varphi}, n^{\varphi} \in S^2.$$

DOK: NEKA JE $x^{\varphi} \in \Omega^{\varphi}$, $n \in S^2$, $n = n_1 e_1 + n_2 e_2 + n_3 e_3$

$n_1, n_2, n_3 > 0$

DEF: A^{φ} = TETRAEDAR S VRHOM U x^{φ} KAO NA SLICI



TRI STRANICE A_1, A_2, A_3 U KOORDINATNIM RAVNINAMA

A_n S NORMALOM n

NORMALA NA A_j .. $-e_j$, $j = 1,2,3$

$i \in \{1,2,3\}$

$$\int_{\partial A^{\varphi}} t_i^{\varphi}(x^{\varphi}, n^{\varphi}, t)\, da^{\varphi} = \int_{A_1} t_i^{\varphi}(x^{\varphi}, -e_1, t)\, da^{\varphi} + \int_{A_2} t_i^{\varphi}(y^{\varphi}, -e_2, t)\, da^{\varphi}$$

$$+ \int_{A_3} t_i^{\varphi}(y^{\varphi}, -e_3, t)\, da^{\varphi} + \int_{A_n} t_i^{\varphi}(y^{\varphi}, n, t)\, da^{\varphi}$$

JER ~~JE~~ t_i^φ NEPREKIDNA MOŽEMO PRIMJENITI TEOREM SREDNJE VRIJEDNOSTI ZA INTEGRALE, PA POSTOJE TOČKE

$$x^{ij} \in A_j,\ j=1,2,3,\qquad x^i \in A_n$$

TAKVE DA JE

$$\int_{\partial A^\varphi} t_i^\varphi(y^\varphi, n^\varphi, t)\,da^\varphi = t_i^\varphi(x^{i1}, -e_1, t)\,\mathrm{pov}(A_1) + t_i^\varphi(x^{i2}, -e_2, t)\,\mathrm{pov}(A_2)$$
$$+ t_i^\varphi(x^{i3}, -e_3, t)\,\mathrm{pov}(A_3) + t_i^\varphi(x^i, n, t)\,\mathrm{pov}(A_n)$$

GEOMETRIJSKI ARGUMENT $\Rightarrow$ $\mathrm{pov}(A_i) = n_i\,\mathrm{pov}\,A_n$

$$= \left(t_i^\varphi(x^{i1}, -e_1, t)n_1 + t_i^\varphi(x^{i2}, -e_2, t)n_2 + t_i^\varphi(x^{i3}, -e_3, t)n_3 + t_i(x^i, n, t)\right)\mathrm{pov}($$

PRIMJENA ZAKONA OČUVANJA IMPULSA SADA POVLAČI

$$\left| t_i^\varphi(x^{i1}, -e_1, t)n_1 + t_i^\varphi(x^{i2}, -e_2, t)n_2 + t_i^\varphi(x^{i3}, -e_3, t)n_3 + t_i^\varphi(x^i, n, t)\right|$$

$$\leq \frac{1}{\mathrm{pov}\,A_n}\left|\int_{A^\varphi}\left(\rho^\varphi \ddot{y}^\varphi - f^\varphi\right)dx^\varphi\right|$$

$\rho^\varphi \ddot{y}^\varphi - f^\varphi$ JE NEPREKIDNA NA $\overline{A^\varphi}$ $\Rightarrow$ OGRANIČENA S C

$$\leq C\,\frac{\mathrm{vol}(A^\varphi)}{\mathrm{pov}(A_n)} \longrightarrow 0 \quad \text{KAD} \quad A^\varphi \to \{x^\varphi\}$$

HOMOTETIČNO

S TIM I

$x^{ij} \to x^\varphi$

$x^i \to x^\varphi$

U LIMESU DOBIVAMO

$$t_i^\varphi(x^\varphi, n, t) = -\sum_{j=1}^{3} t_i^\varphi(x^\varphi, -e_j, t)\,n_j$$

JER TO VRIJEDI ZA SVAKU KOMPONENTU

$$t^{\varphi}(x^{\varphi},n,t) = -\sum_{j=1}^{3} t^{\varphi}(x^{\varphi},-e_j,t)\,n_j$$

JER JE t^{φ} NEPREKIDNA U LIMESU $n \to e_k$ $(n_j > 0)$

$$t^{\varphi}(x^{\varphi},e_k,t) = -t^{\varphi}(x^{\varphi},-e_k,t)$$

SADA JE

$$t^{\varphi}(x^{\varphi},n,t) = \sum_{j=1}^{3} t^{\varphi}(x^{\varphi},e_j,t)\,n_j \qquad (n_j > 0)$$

DEFINIRAMO:

$$T^{\varphi}(x^{\varphi},t)\,e_j := t^{\varphi}(x^{\varphi},e_j,t) \qquad j=1,2,3.$$

SADA VRIJEDI

$$\begin{aligned}
T^{\varphi}(x^{\varphi},t)\,n &= \sum_{j=1}^{3} n_j\, T^{\varphi}(x^{\varphi},t)\,e_j \\
&= \sum_{j=1}^{3} |n_j|\, T^{\varphi}(x^{\varphi},t)\,\mathrm{sign}(n_j)\,e_j \\
&= \sum_{j=1}^{3} |n_j|\,\mathrm{sign}(n_j)\, t^{\varphi}(x^{\varphi},e_j,t) \\
&= \sum_{j=1}^{3} |n_j|\, t^{\varphi}(x^{\varphi},\mathrm{sign}(n_j)\,e_j,t) \\
&= t^{\varphi}\Big(x^{\varphi},\sum_{j=1}^{3}|n_j|\,\mathrm{sign}(n_j)\,e_j,t\Big) \\
&= t^{\varphi}(x^{\varphi},n,t)
\end{aligned}$$

ZAKON OČUVANJA IMPULSA SADA GLASI

$$\int_{A^e} \rho^e \dot{v}^e \, dx^e = \int_{\partial A^e} T^e(x^e,t)\, n^e da^e + \int_{A^e} f^e \, dx^e$$

$$\forall A^e$$

TEOREM O DIVERGENCIJI OMOGUĆUJE PRELAZAK NA INTEGRAL PO DOMENI A^e

$$\int_{\partial A} T n \, da = \int_A \operatorname{div} T \, dx \qquad \left(T \text{ - MATRIČNA FUNKCIJA}\right)$$

(DIREKTNA POSLJEDICA KLASIČNOG TEOREMA UZ

$$\operatorname{div} T = \begin{bmatrix} \operatorname{div} t_1 \\ \operatorname{div} t_2 \\ \operatorname{div} t_3 \end{bmatrix}, \qquad T = \begin{bmatrix} t_1^T \\ t_2^T \\ t_3^T \end{bmatrix}$$

DOBIVAMO

$$\int_{A^e} \rho^e \dot{v}^e \, dx^e = \int_{A^e} \operatorname{div} T^e \, dx^e + \int_{A^e} f^e dx^e$$

$$\int_{A^e} \left(\rho^e \dot{v}^e - \operatorname{div} T^e - f^e\right) dx^e = 0, \qquad \forall A^e$$

PROIZVOLJNOST DOMENE

$$\boxed{\rho^e \dot{v}^e = \operatorname{div} T^e + f^e \qquad u \ \Omega^e}$$

DIFERENCIJALNI OBLIK ZAKONA OČUVANJA IMPULSA

JEDNADŽBA GIBANJA U EULEROVOJ FORMULACIJI

POSLJEDICE ZAKONA OČUVANJA MOMENTA IMPULSA

ZAKON:

$$\int_{A^\varphi} x^\varphi \times \rho^\varphi \ddot{x}^\varphi \, dx^\varphi = \int_{\partial A^\varphi} x^\varphi \times T^\varphi n^\varphi \, da^\varphi + \int_{A^\varphi} x^\varphi \times f^\varphi \, dx^\varphi$$

$$\Rightarrow \int_{A^\varphi} x^\varphi \times (\rho^\varphi \ddot{x}^\varphi - f^\varphi)\, dx^\varphi = \int_{\partial A^\varphi} x^\varphi \times T^\varphi n^\varphi \, da^\varphi$$

$\Vert$ ← JEDNADŽBA GIBANJA

$$\int_{A^\varphi} x^\varphi \times \operatorname{div} T^\varphi \, dx^\varphi$$

DESNU STRANU RAČUNAMO KORIŠTENJEM TEOREMA O DIVERGENCIJI

ZA $v \in \mathbb{R}^3 \longrightarrow A_v$ - ANTISIMETRIČNA S AKSIJALNIM VEKTOROM v

$$\int_{\partial A^\varphi} x^\varphi \times T^\varphi n^\varphi \, da^\varphi = \int_{\partial A^\varphi} (A_{x^\varphi} T^\varphi) n^\varphi \, da^\varphi = \int_{A^\varphi} \operatorname{div}(A_{x^\varphi} T^\varphi)\, dx^\varphi$$

$$= \int_{A^\varphi} \sum_i \partial_{x_i^\varphi} (A_{x^\varphi} T^\varphi e_i)\, dx^\varphi = \sum_i \int_{A^\varphi} \partial_{x_i^\varphi} (x^\varphi \times T^\varphi e_i)\, dx^\varphi$$

$$= \sum_i \int_{A^\varphi} e_i \times T^\varphi e_i + x^\varphi \times \partial_{x_i^\varphi} (T^\varphi e_i)\, dx^\varphi$$

$$= \int_{A^\varphi} \sum_i e_i \times T^\varphi e_i + x^\varphi \times \operatorname{div} T^\varphi \, dx^\varphi$$

GORNJA JEDNADŽBA POVLAČI

$$\int_{A^\varphi} \sum_i e_i \times T^\varphi e_i \, dx^\varphi = 0 \qquad \forall A^\varphi$$

PROIZVOLJNOST DOMENE POVLAČI

$$\sum_i e_i \times T^\varphi e_i = 0$$

$$\parallel$$

$$e_1 \times (T^\varphi_{11} e_1 + T^\varphi_{21} e_2 + T^\varphi_{31} e_3) + e_2 \times (T^\varphi_{12} e_1 + T^\varphi_{22} e_2 + T^\varphi_{32} e_3) + e_3 \times (T^\varphi_{13} e_1 + T^\varphi_{23} e_2 + T^\varphi_{33} e_3)$$

$$= T^\varphi_{21} e_3 - T^\varphi_{31} e_2 - T^\varphi_{12} e_3 + T^\varphi_{32} e_1 + T^\varphi_{13} e_2 - T^\varphi_{23} e_1 = 0$$

$$\Rightarrow \quad T^\varphi_{21} = T^\varphi_{12} \,, \quad T^\varphi_{31} = T^\varphi_{13} \,, \quad T^\varphi_{32} = T^\varphi_{23}$$

$$\Rightarrow \quad T^{\varphi T} = T^\varphi$$

DIO JEDNADŽBE GIBANJA

(CAUCHYJEV TENZOR NAPREZANJA JE SIMETRIČAN)

POSLJEDICE AKSIOMA:

DIFERENCIJALNA JEDNADŽBA GIBANJA U DEFORMIRANOJ KONFIGURACIJI

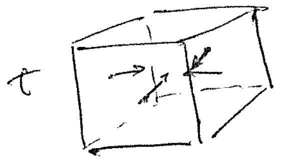
$$\boxed{\begin{aligned} &\rho^\varphi \ddot{x}^\varphi = \operatorname{div} T^\varphi + f^\varphi && \text{u } \Omega^\varphi \\ &T^{\varphi T} = T^\varphi && \text{u } \Omega^\varphi \\ &T^\varphi n^\varphi = g^\varphi && \text{na } \Gamma^\varphi_1 \end{aligned}}$$

PRIMJER (TLAK)

$$T^e(x^e) = -\bar{\pi} I, \quad \bar{\pi} \in \mathbb{R} \qquad T^e = \begin{bmatrix} -\bar{\pi} & & \\ & -\bar{\pi} & \\ & & -\bar{\pi} \end{bmatrix}$$

$t^e(x^e, n) = -\bar{\pi} n$ OKOMIT NA PLOHU NA KOJU DJELUJE

ZA $\bar{\pi} > 0$ U SUPROTNOM SMJERU OD n

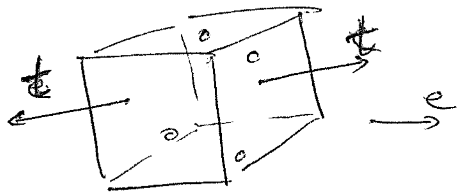


PRIMJER (ČISTA TENZIJA / KOMPRESIJA)

$$T^e(x^e) = \tau\, e \otimes e, \quad e \in \mathbb{R}^3, \ \|e\| = 1, \ \tau \in \mathbb{R}$$

$$t^e(x^e, n) = T^e(x^e) n = \tau (e \cdot n) e \qquad T^e = \begin{bmatrix} \tau & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

- PARALELAN S e
- $\tau > 0$ TENZIJA, $\tau < 0$ KOMPRESIJA   ZA $e = e_1$
- ZA $n \perp e$ $t^e = 0$

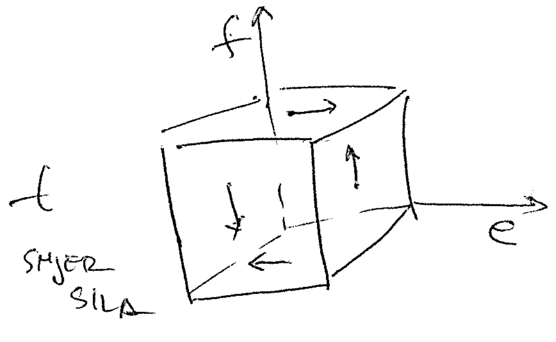


PRIMJER (ČISTO SMICANJE)

$$T^e(x^e, t) = \sigma (e \otimes f + f \otimes e), \quad \sigma \in \mathbb{R}, \ e, f \in \mathbb{R}^3, \ \|e\| = \|f\| = 1, \ e \cdot f = 0$$

$$t^e(x^e, n) = \sigma ((f \cdot n) e + (e \cdot n) f)$$

$$T^e = \begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} e = e_1 \\ f = e_2 \end{matrix}$$



SLABA FORMULACIJA

$$\rho^\varphi \ddot{x}^\varphi = \operatorname{div} T^\varphi + f^\varphi \qquad \text{u } \Omega^\varphi$$

$$T^{\varphi T} = T^\varphi \qquad \text{u } \Omega^\varphi$$

$$T^\varphi n^\varphi = g^\varphi \qquad \text{na } \Gamma_1^\varphi$$

POMNOŽIMO 1. JDBU S θ^φ DOVOLJNO GLATKOM I T.D.

$$\theta^\varphi\big|_{\Gamma_0^\varphi} = 0 \qquad \left(\Gamma_0^\varphi = \partial\Omega^\varphi \setminus \Gamma_1^\varphi \right)$$

I INTEGRIRAMO PO Ω^φ

$$(*) \qquad \int_{\Omega^\varphi} \rho^\varphi \ddot{x}^\varphi \cdot \theta^\varphi \, dx^\varphi = \int_{\Omega^\varphi} \operatorname{div} T^\varphi \cdot \theta^\varphi \, dx^\varphi + \int_{\Omega^\varphi} f^\varphi \cdot \theta^\varphi \, dx^\varphi$$

ŽELIMO PRIMJENITI TEOREM O DIVERGENCIJI

ZA $\eta = T\theta$ RAČUNAMO

$$\begin{aligned}
\operatorname{div} \eta &= \partial_1 \eta_1 + \partial_2 \eta_2 + \partial_3 \eta_3 \\
&= \partial_1 \left(T_{11}\theta_1 + T_{12}\theta_2 + T_{13}\theta_3 \right) + \partial_2 \left(T_{21}\theta_1 + T_{22}\theta_2 + T_{23}\theta_3 \right) \\
&\quad + \partial_3 \left(T_{31}\theta_1 + T_{32}\theta_2 + T_{33}\theta_3 \right) \\
&= \partial_1 T_{11}\, \theta_1 + \partial_1 T_{12}\, \theta_2 + \partial_1 T_{13}\, \theta_3 + \partial_2 T_{21}\, \theta_1 + \partial_2 T_{22}\, \theta_2 + \partial_2 T_{23}\, \theta_3 \\
&\quad + \partial_3 T_{31}\, \theta_1 + \partial_3 T_{32}\, \theta_2 + \partial_3 T_{33}\, \theta_3 \\
&\quad + T_{11}\, \partial_1\theta_1 + T_{12}\, \partial_1\theta_2 + T_{13}\, \partial_1\theta_3 + T_{21}\, \partial_2\theta_1 + T_{22}\, \partial_2\theta_2 + T_{23}\, \partial_2\theta_3 \\
&\quad + T_{31}\, \partial_3\theta_1 + T_{32}\, \partial_3\theta_2 + T_{33}\, \partial_3\theta_3 \\
&= \begin{bmatrix} \partial_1 T_{11} + \partial_2 T_{21} + \partial_3 T_{31} \\ \partial_1 T_{12} + \partial_2 T_{22} + \partial_3 T_{32} \\ \partial_1 T_{13} + \partial_2 T_{23} + \partial_3 T_{33} \end{bmatrix} \cdot \theta + T \cdot \begin{bmatrix} \partial_1\theta_1 & \partial_1\theta_2 & \partial_1\theta_3 \\ \partial_2\theta_1 & \partial_2\theta_2 & \partial_2\theta_3 \\ \partial_3\theta_1 & \partial_3\theta_2 & \partial_3\theta_3 \end{bmatrix}
\end{aligned}$$

$$\Rightarrow \operatorname{div}(T\theta) = (\operatorname{div} T^T)\cdot\theta + T^T\cdot\nabla\theta$$

ODNOSNO

$$(\times) \quad \operatorname{div}(T^T\theta) = (\operatorname{div} T)\cdot\theta + T\cdot\nabla\theta$$

SADA SE VRAĆAMO U (*):

$$\int_{\Omega^\varphi} \rho^\varphi \dot{v}^\varphi\cdot\theta^\varphi dx^\varphi = \int_{\Omega^\varphi} \operatorname{div}(T^{\varphi T}\theta^\varphi) - T^\varphi\cdot\nabla\theta^\varphi dx^\varphi + \int_{\Omega^\varphi} f^\varphi\cdot\theta^\varphi dx^\varphi$$

$$(\text{TEOREM O DIVERGENCIJI}) = \int_{\partial\Omega^\varphi} T^{\varphi T}\theta^\varphi\cdot n\, da^\varphi - \int_{\Omega^\varphi} T^\varphi\cdot\nabla\theta^\varphi dx^\varphi + \int_{\Omega^\varphi} f^\varphi\cdot\theta^\varphi dx^\varphi$$

$$\partial\Omega^\varphi = \Gamma_0^\varphi \cup \Gamma_1^\varphi$$

$$\int_{\partial\Omega^\varphi} \theta^\varphi\cdot T^\varphi n^\varphi da^\varphi = \int_{\Gamma_1^\varphi} \theta^\varphi\cdot T^\varphi n^\varphi da^\varphi = \int_{\Gamma_1^\varphi} \theta^\varphi\cdot g^\varphi da^\varphi$$

JER JE $\theta^\varphi\big|_{\Gamma_0^\varphi} = 0$ JER JE $T^\varphi n^\varphi\big|_{\Gamma_1^\varphi} = g^\varphi$

STOGA DOBIVAMO

$$(+) \quad \boxed{\int_{\Omega^\varphi} \rho^\varphi \dot{v}^\varphi\cdot\theta^\varphi dx^\varphi + \int_{\Omega^\varphi} T^\varphi\cdot\nabla\theta^\varphi dx^\varphi = \int_{\Omega^\varphi} f^\varphi\cdot\theta^\varphi dx^\varphi + \int_{\Gamma_1^\varphi} g^\varphi\cdot\theta^\varphi da^\varphi \quad \forall\theta^\varphi \text{ T.D. } \theta^\varphi\big|_{\Gamma_0^\varphi} = 0}$$

VARIJACIJSKA (SLABA) FORMULACIJA JEDNADŽBE GIBANJA U DEFORMIRANOJ KONFIGURACIJI

ILI

PRINCIP VIRTUALNOG RADA U DEFORMIRANOJ KONFIGURACIJI

NAP: ~~KAO~~ I OBIČNO IZ SLABE FORMULACIJE MOŽEMO SE VRATITI U DIFERENCIJALNU.

NEKA JE $\theta^{\varphi}\big|_{\partial\Omega^{\varphi}} = 0$. IZ (X) & (+) SLIJEDI

$$\int_{\Omega^{\varphi}} \rho^{\varphi} \dot{v}^{\varphi} \cdot \theta^{\varphi} dx^{\varphi} + \int_{\Omega^{\varphi}} \Big(\underbrace{\operatorname{div}\left(T^{\varphi T} \theta^{\varphi}\right)}_{\substack{\| \\ 0}} - \operatorname{div} T^{\varphi} \cdot \theta^{\varphi} \Big) dx^{\varphi} = \int_{\Omega^{\varphi}} f^{\varphi} \cdot \theta^{\varphi} dx^{\varphi}$$

IZ TEOREMA O DIVERGENCIJI

$$\Rightarrow \int_{\Omega^{\varphi}} \left(\rho^{\varphi} \dot{v}^{\varphi} - \operatorname{div} T^{\varphi} - f^{\varphi} \right) \cdot \theta^{\varphi} dx^{\varphi} = 0$$

PROIZVOLJNOST OD θ^{φ} $\Rightarrow$ DIF. J.

$$\rho^{\varphi} \dot{v}^{\varphi} = \operatorname{div} T^{\varphi} + f^{\varphi}$$

SADA UZMEMO $\theta^{\varphi}\big|_{\Gamma_0^{\varphi}} = 0$, PA IZ (X) I (+) DOBIVAMO

$$\int_{\Omega^{\varphi}} \rho^{\varphi} \dot{v}^{\varphi} \cdot \theta^{\varphi} dx^{\varphi} + \int_{\Omega^{\varphi}} \left(\operatorname{div}\left(T^{\varphi T} \theta^{\varphi}\right) - \operatorname{div} T^{\varphi} \cdot \theta^{\varphi} \right) dx^{\varphi}$$

$$= \int_{\Omega^{\varphi}} f^{\varphi} \cdot \theta dx^{\varphi} + \int_{\Gamma_1^{\varphi}} g^{\varphi} \cdot \theta^{\varphi} da^{\varphi}$$

IZ DIF. J.

$$\int_{\Omega^{\varphi}} \operatorname{div}\left(T^{\varphi T} \theta^{\varphi}\right) dx^{\varphi} = \int_{\Gamma_1^{\varphi}} g^{\varphi} \cdot \theta^{\varphi} da^{\varphi}$$

$$\|$$

$$\int_{\partial\Omega^{\varphi}} \theta^{\varphi} \cdot T^{\varphi} n^{\varphi} da^{\varphi} = \int_{\Gamma_1^{\varphi}} \theta^{\varphi} \cdot T^{\varphi} n^{\varphi} da^{\varphi} \qquad \Rightarrow T^{\varphi} n^{\varphi} = g^{\varphi}$$

TEOREM: RUBNA ZADAĆA

$$\varrho^\varphi \dot{v}^\varphi = \operatorname{div} T^\varphi + f^\varphi \quad u\ \Omega^\varphi$$

$$T^\varphi n^\varphi = g^\varphi \quad na\ \Gamma_1^\varphi$$

FORMALNO JE EKVIVALENTNA SLABOJ FORMULACIJI (+).

NAP: ZAKON OČUVANJA IMPULSA (VARIJABILNOST U DOMENI)

PRINCIP VIRTUALNOG RADA (VARIJABILNOST U TEST FUNKCIJI)
U NAŠEM IZVODU
IZMEĐU JE DIFERENCIJALNA FORMULACIJA,
KOJA ZAHTJEVA VEĆU REGULARNOST FUNKCIJA,
NO MOŽE SE ZAOBIĆI!