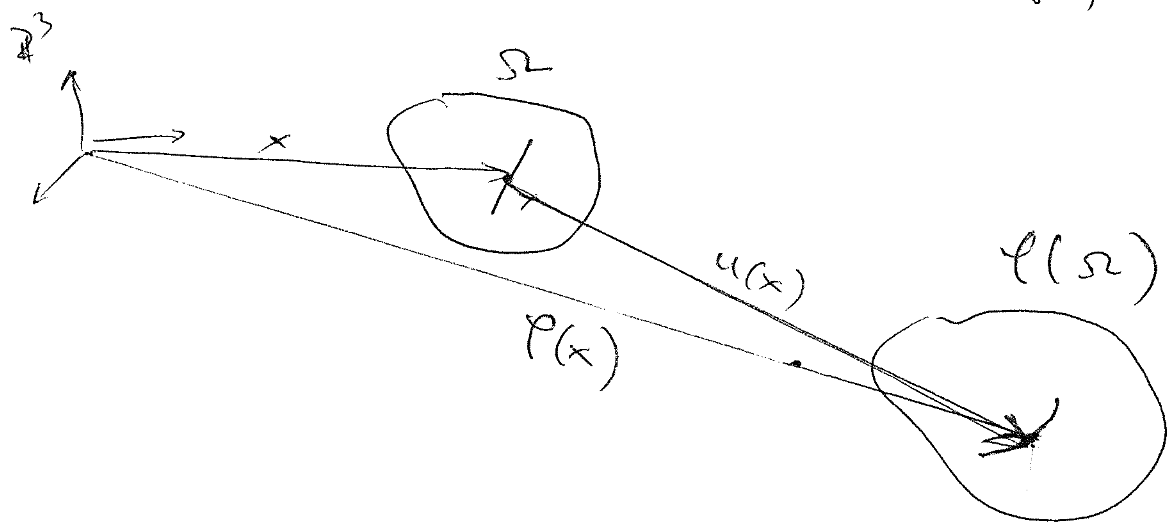
3D TEORIJA

- $\Omega \subseteq \mathbb{R}^3$ $(\mathbb{R}^2)$ OGRANIČEN, OTVOREN, POVEZAN SKUP

- $\bar{\Omega}$ - REPREZENTIRA VOLUMEN KOJI ZAUZIMA TIJELO PRIJE DEFORMACIJE - <u>REFERENTNA KONFIGURACIJA</u>

- $x \in \Omega$ MATERIJALNA TOČKA (M. KRIVULJA, M. PLOHA, ...)



- $\varphi : \bar{\Omega} \to \mathbb{R}^3$ <u>DEFORMACIJA</u> AKO JE INJEKTIVNA (OSIM MOŽDA NA $\partial\Omega$), DOVOLJNO GLATKA I $\det \nabla\varphi(x) > 0$, $x \in \bar{\Omega}$

- $\varphi(\bar{\Omega})$ - DEFORMIRANA KONFIGURACIJA

  $x^\varphi \in \varphi(\bar{\Omega})$

<u>NAP:</u> $\nabla\varphi(x) = \begin{pmatrix} \partial_1\varphi(x) & \partial_2\varphi(x) & \partial_3\varphi(x) \end{pmatrix} \in M_3(\mathbb{R})$

$\det \nabla\varphi(x) > 0 \Rightarrow \partial_1\varphi(x), \partial_2\varphi(x), \partial_3\varphi(x)$

ČINE DESNU BAZU

POSLJEDICE:

1) BAZA : LOKALNO INVERTIBILNO PRESLIKAVANJE NEMA KOMPRESIJE VOLUMENA U 0

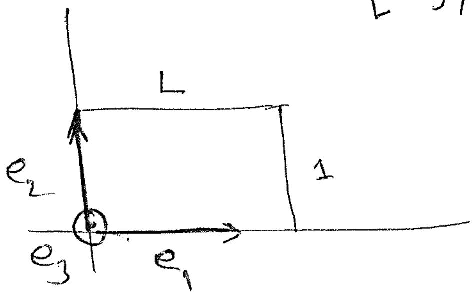
2) DESNA : ČUVA ORIJENTACIJU

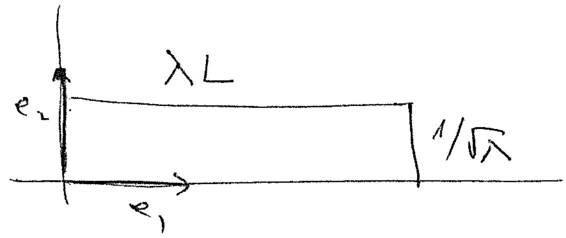
$-\ \varphi(x) = x + u(x)$, $u : \bar{\Omega} \to \mathbb{R}$ POMAK (VEKTOR POMAKA)

$-$ OZNAKA: $F(x) := \nabla\varphi(x) = I + \nabla u(x)$

GRADIJENT DEFORMACIJE

PR1: $\varphi(x) = \begin{bmatrix} \lambda x_1 \\ x_2/\sqrt{\lambda} \\ x_3/\sqrt{\lambda} \end{bmatrix}$, $\lambda > 0$ AKSIJALNA EKSTENZIJA





$$\nabla\varphi(x) = \begin{bmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$$

$\det \nabla\varphi(x) = 1 > 0 \Rightarrow$ DEFORMACIJA

$$\mathrm{vol}(\varphi(\Omega)) = \int_{\varphi(\Omega)} dx^{\varphi} = \int_{\Omega} \det\nabla\varphi(x)\,dx = \int_{\Omega} dx = \mathrm{vol}(\Omega)$$

ŠTOVIŠE $\forall P \subseteq \Omega$: $\mathrm{vol}(\varphi(P)) = \mathrm{vol}(P)$

$\Rightarrow$ DEFORMACIJA JE <u>IZOHORIČKA</u>

OPĆENITO: $\det \nabla\varphi(x) = 1 \Rightarrow \varphi$ JE IZOHORIČKA
LEMA

ZAD: VRIJEDI I OBRAT

RJ: PRETP $\forall P \subseteq \Omega$ $\qquad \displaystyle\int_{\varphi(P)} dx^{\varphi} = \int_P dx$

ZAMJENA VARIJABLI $\Rightarrow$ $\displaystyle\int_P \det \nabla\varphi(x)\, dx = \int_P dx$

$$\Rightarrow \quad \int_P \left(\det \nabla\varphi(x) - 1\right) dx = 0 \qquad \forall P \subseteq \Omega$$

OSNOVNA LEMA VARIJACIJSKOG RAČUNA
$\Rightarrow$ $\qquad \det \nabla\varphi(x) = 1$

TM: DEFORMACIJA φ JE IZOHORIČKA $\Leftrightarrow$ $\det \nabla\varphi(x) = 1$

NAP: $\displaystyle \operatorname{vol}(\varphi(P)) = \int_{\varphi(P)} dx^{\varphi} = \int_P \det \nabla\varphi(x)\, dx$

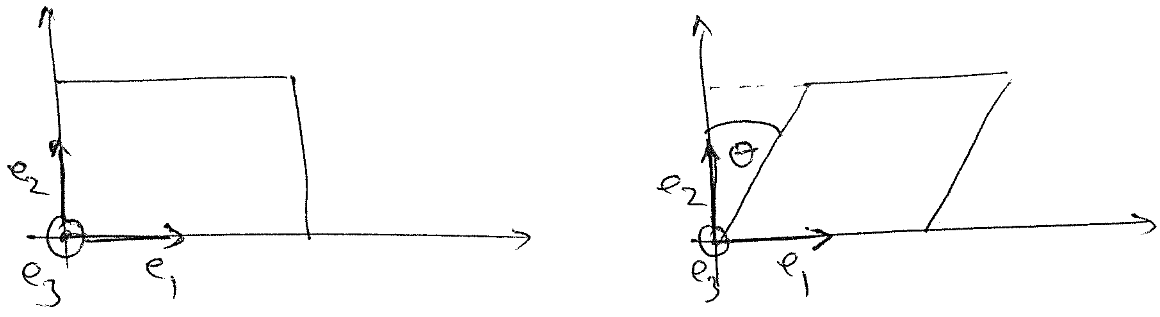
ZA $x^0 \in P$ TAYLOR $\Rightarrow$ $\nabla\varphi(x) = \nabla\varphi(x^0) + O(\|x - x^0\|)$

$$\Rightarrow \quad \operatorname{vol}(\varphi(P)) = \underbrace{\det \nabla\varphi(x^0)}_{\substack{\uparrow \\ \text{LOKALNA MJERA PROMJENE} \\ \text{VOLUMENA}}} \operatorname{vol}(P) + O(\|x - x^0\|)$$

> 0

PR 2: JEDNOSTAVNO SMICANJE

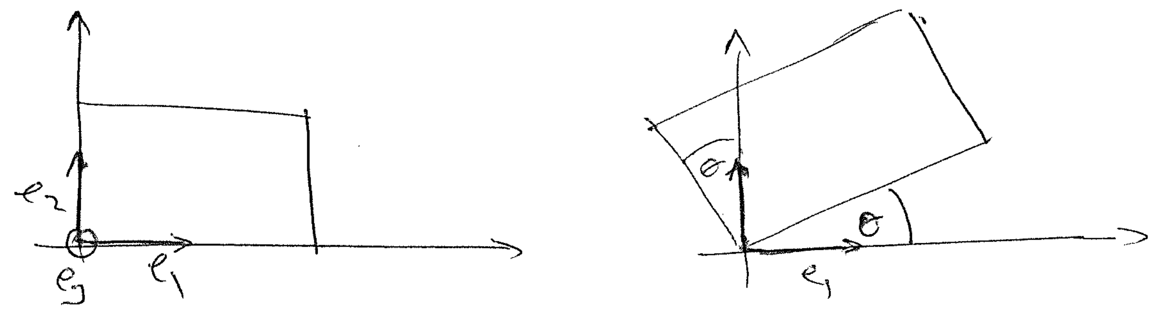
$$\varphi(x) = \begin{bmatrix} x_1 + \operatorname{tg}\theta \, x_2 \\ x_2 \\ x_3 \end{bmatrix}$$



JE LI DEFORMACIJA IZOHORIČKA ? $\det \nabla\varphi \ldots$

PR 3: ROTACIJA OKO e_3 ZA KUT θ (U POZITIVNOM SMJERU)

$$\varphi(x) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$



IZOHORIČKA ?

PR 4: OPĆA ROTACIJA $\quad f(x) = Rx, \quad R \in Orth^+(3) = SO(3)$

$$R^T R = I, \quad \det R = 1$$

PR 5: TRANSLACIJA $\quad f(x) = x + c, \quad c \in \mathbb{R}^3$

PR 6: HOMOGENA DEFORMACIJA $\quad f(x) = Ax + c$

$A \in M_3(\mathbb{R}), \quad c \in \mathbb{R}^3$

H.D. $\Rightarrow \nabla f(x) = A$ (KONSTANTNA MATRICA)

OBRATNO: $\nabla f(x) = A, \quad x \in \mathbb{R}$

DEF: $\varphi(x) = f(x) - Ax$

$$\Rightarrow \nabla \varphi(x) = \nabla f(x) - A = A - A = 0$$

$$\Rightarrow \varphi = \text{CONST.}$$

$$\Rightarrow \varphi(x) = f(x) - Ax = c \rightarrow f(x) = Ax + c$$

SVI PRIMJERI SU H.D.

$x(t) = at + b, \quad a, b \in \mathbb{R}^3$

PARAMETARSKI ZAPIS JEDNADŽBE PRAVCA

$$f(x(t)) = A(at + b) + c = (Aa)t + \underbrace{(Ab + c)}_{\text{PRAVAC}}$$

$\Rightarrow$ H.D. PRAVAC PRESLIKAVA U PRAVAC

— DEFORMACIJA JE <u>KRUTA</u> AKO JE

$$\forall x, y \in \Omega \qquad \|\varphi(x) - \varphi(y)\| = \|x - y\|$$

<u>PR:</u> $\varphi(x) = Rx + b, \quad R \in SO(3), \quad b \in \mathbb{R}^3$

$$\|Rx + \cancel{b} - (Ry + \cancel{b})\|^2 = \|R(x-y)\|^2 = R(x-y) \cdot R(x-y)$$

$$= R^T R(x-y) \cdot (x-y)$$

$$\|\varphi(x) - \varphi(y)\|^2 = \|x-y\|^2$$

$\Rightarrow$ KRUTA DEFORMACIJA

VRIJEDI I OBRAT.

<u>TM:</u> DEFORMACIJA JE KRUTA $\iff$ $\varphi(x) = Rx + b$
$R \in SO(3), \ b \in \mathbb{R}^3$

<u>DOK:</u> $\boxed{\Leftarrow}$ PRIMJER ✓

$\boxed{\Rightarrow}$ PRETP: $\|\varphi(x) - \varphi(y)\| = \|x - y\|, \quad x, y \in \Omega$

$$\Downarrow$$

$$(\varphi(x) - \varphi(y)) \cdot (\varphi(x) - \varphi(y)) = (x-y) \cdot (x-y), \quad x, y \in \Omega$$

$$\Downarrow$$

$$\varphi(x) \cdot \varphi(x) - 2\varphi(x) \cdot \varphi(y) + \varphi(y) \cdot \varphi(y) = x \cdot x - 2x \cdot y + y \cdot y, \quad x, y \in \Omega$$

$$\Downarrow \qquad \Big| \partial_{x_i} \partial_{y_j}$$

$$-2\, \partial_{x_i} \varphi(x) \cdot \partial_{y_j} \varphi(y) = -2\, \delta_{ij} \qquad i,j = 1,2,3$$

$$\Downarrow$$

$$\partial_{x_i} \varphi(x) \cdot \partial_{y_j} \varphi(y) = \delta_{ij} = (I)_{ij} \qquad i,j = 1,2,3$$

$$\Downarrow$$

$$(\nabla \varphi(x)^T \nabla \varphi(y))_{ij} = (\nabla \varphi(y)^T \nabla \varphi(x))_{ij} = (I)_{ij} \qquad i,j = 1,2,3$$

$\Downarrow$

$$\nabla\varphi(x)^T \nabla\varphi(y) = \nabla\varphi(y)^T \nabla\varphi(x) = I$$

1) ZA $y = x$ $\quad \nabla\varphi(x)^T \nabla\varphi(x) = I$

$\Rightarrow \quad \nabla\varphi(x) \in SO(3) \quad$ (JER JE $\det \nabla\varphi(x) > 0$)

2) $\quad \nabla\varphi(y)^{-1} = \nabla\varphi(x)^T \quad \forall x, y$

$\Rightarrow \quad \nabla\varphi(x) = \text{CONST}$

$\Rightarrow \quad \nabla\varphi(x) = R$

$\Rightarrow \quad \varphi(x) = Rx + b$

NAP: OD RANIJE ZNAMO $\varphi(x) = Rx + b \Leftrightarrow \nabla\varphi(x) = R$

STOGA VRIJEDI: DEFORMACIJA φ JE KRUTA $\Leftrightarrow \nabla\varphi(x) = R$
$R \in SO(3)$.

NAP: DEFORMACIJA JE KRUTA $\Rightarrow \nabla\varphi(x) = R, \ R \in SO(3)$

$\Rightarrow \nabla\varphi(x)^T \nabla\varphi(x) = I$

VRIJEDI I OBRAT, PA IMAMO

TM: φ JE KRUTA DEFORMACIJA $\Leftrightarrow \nabla\varphi(x)^T \nabla\varphi(x) = I$

DOK: $\boxed{\Rightarrow}$ ✓

$\boxed{\Leftarrow}$ CIARLET

1) φ JE IZOMETRIJA

2) $\nabla\varphi = \text{CONST}$ LOKALNO

3) Ω POVEZAN $\Rightarrow \nabla\varphi = \text{const}$ GLOBALNO

TM: X, Y - BANACHOVI, $\Omega \subseteq X$ OTVOREN, $f: \Omega \to Y$ T.D.

$$f \in C^1(\Omega; Y), \quad Df(x) \in GL(X; Y), \quad x \in \Omega$$

TADA JE $f(\Omega)$ OTVOREN U Y.

AKO JE f INJEKTIVNA, TADA JE

$f: \Omega \to f(\Omega)$ C^1-DIFEOMORFIZAM

DOK: TEOREM O INVERZNOM PRESLIKAVANJU.

BEZ GLATKOĆE TREBAMO INJEKTIVNOST.

TM (INVARIJANTNOST DOMENE U $\mathbb{R}^n$)

NEKA JE $\Omega \subseteq \mathbb{R}^n$ OTVOREN, $f \in C(\Omega; \mathbb{R}^n)$

INJEKCIJA. TADA JE $f(\Omega)$ OTVOREN.

DOK: CIARLET

POSLJEDICA

TM: NEKA JE $\Omega \subseteq \mathbb{R}^n$ OGRANIČEN, OTVOREN,

$f \in C(\bar{\Omega}; \mathbb{R}^n)$ I $f|_\Omega$ INJEKCIJA. TADA

$$f(\bar{\Omega}) = \overline{f(\Omega)} \qquad f(\Omega) \subseteq \operatorname{Int} f(\bar{\Omega}), \quad f(\partial\Omega) \supseteq \partial f(\bar{\Omega})$$

NAP: DAKLE: INTERIOR U INTERIOR

TIJELO OSTANE TIJELO

RUB SLIKE SADRŽAN U SLICI RUBA

DOK: 1) $\boxed{\subseteq}$ $y \in f(\overline{\Omega}) \Rightarrow \exists x \in \overline{\Omega}$ T.D. $f(x)=y$

$\Rightarrow \exists x_n \in \Omega \quad x_n \longrightarrow x$

$\Rightarrow f$ NEPR. $\underset{\underset{f(\Omega)}{\uparrow}}{f(x_n)} \longrightarrow f(x)=y$

$\Rightarrow y \in \overline{f(\Omega)}$

$\boxed{\supseteq}$ $\overline{\Omega}$ KOMPAKTAN $\Rightarrow f(\overline{\Omega})$ KOMPAKTAN

$\Rightarrow f(\overline{\Omega})$ ZATVOREN

$f(\Omega) \subseteq f(\overline{\Omega}) \Rightarrow \overline{f(\Omega)} \subseteq \overline{f(\overline{\Omega})} = f(\overline{\Omega})$

2) $f(\Omega) \subseteq f(\overline{\Omega}) \Rightarrow \underset{\underset{f(\Omega) \text{ – OTVOREN (IFTM)}}{\shortparallel}}{\mathrm{Int}\, f(\Omega)} \subseteq \mathrm{Int}\, f(\overline{\Omega})$

3) SLIČNO

TM: $\Omega \subseteq \mathbb{R}^n$ OGRANIČEN, OTVOREN I $\mathrm{Int}\,\overline{\Omega} = \Omega$.
$f \in C(\overline{\Omega}; \mathbb{R}^n)$ INJEKCIJA. TADA

$$f(\overline{\Omega}) = \overline{f(\Omega)}, \quad f(\Omega) = \mathrm{Int}\, f(\overline{\Omega}), \quad f(\partial\Omega) = \partial f(\Omega) = \partial f(\overline{\Omega})$$

DOK: SLIČNO

NAP: RAZLIKA GLOBALNA INJEKTIVNOST NA $\overline{\Omega}$ & Ω

MJERA DEFORMACIJE (STRAIN TENSOR)

- NISU NAM OD INTERESA KRUTE DEFORMACIJE (TRANS. & ROT.) VEĆ UPRAVO ODSTUPANJA OD NJE
- ZA KRUTU DEFORMACIJU φ VRIJEDI

$$\nabla\varphi(x)^T \nabla\varphi(x) = I$$

$$\Updownarrow$$

$$\nabla\varphi(x)^T \nabla\varphi(x) - I = 0$$

- MOGUĆA MJERA DEFORMACIJE:

$$E(x) = \frac{1}{2}\left(\nabla\varphi(x)^T \nabla\varphi(x) - I\right) = \frac{1}{2}\left(F^T F - I\right)$$

KONAČNI TENZOR DEFORMACIJE

- IZRAŽEN PREKO POMAKA

$$E(x) = \frac{1}{2}\left((I + \nabla u(x))^T (I + \nabla u(x)) - I\right)$$

$$= \frac{1}{2}\left(\nabla u(x) + \nabla u(x)^T + \nabla u(x)^T \nabla u(x)\right)$$

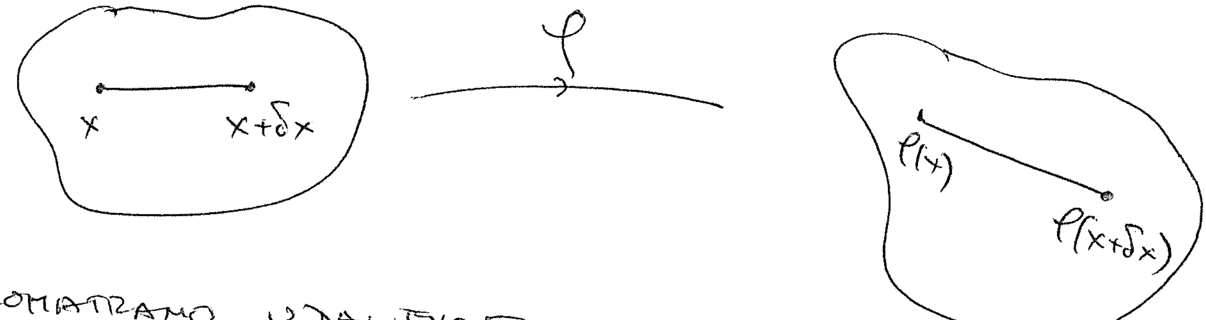
- $C(x) = \nabla\varphi(x)^T \nabla\varphi(x)$

DESNI CAUCHY - GREENOV TENZOR DEFORMACIJE

ODSTUPANJE C OD I MJERI ODSTUPANJE DEFORMACIJE OD KRUTE DEFORMACIJE

— E NIJE JEDINA MJERA ODSTUPANJA OD KRUTE D.

— ALI E IMA JOŠ NEKA ZGODNA SVOJSTVA



PROMATRAMO UDALJENOST SLIKA

$$\| \varphi(x+\delta x) - \varphi(x) \|^2$$

ZA δx MALEN, I DOVOLJNO GLADAK φ

$$\varphi(x+\delta x) = \varphi(x) + \nabla\varphi(x)\,\delta x + o(\|\delta x\|) \qquad \text{(TAYLOR)}$$

$$\Rightarrow \| \varphi(x+\delta x) - \varphi(x)\|^2 = \nabla\varphi(x)\,\delta x \cdot \nabla\varphi(x)\,\delta x + o(\|\delta x\|^2)$$

$$= \nabla\varphi(x)^T \nabla\varphi(x)\,\delta x \cdot \delta x + o(\|\delta x\|^2)$$

PROMJENA UDALJENOSTI (RELATIVNA)

$$\frac{\| \varphi(x+\delta x) - \varphi(x)\|^2 - \| x+\delta x - x\|^2}{\| x+\delta x - x\|^2}$$

$$= \frac{\nabla\varphi(x)^T \nabla\varphi(x)\,\delta x \cdot \delta x - \delta x \cdot \delta x + o(\|\delta x\|^2)}{\|\delta x\|^2}$$

$$= \underbrace{\left(\nabla\varphi(x)^T \nabla\varphi(x) - I\right)}_{E(x)} \frac{\delta x}{\|\delta x\|} \cdot \frac{\delta x}{\|\delta x\|} + o(1)$$

$E(x)$ — MJERI LOKALNU PROMJENU DULJINE!

VAŽNO SVOJSTVO:

<u>TM</u>: $\Omega \subseteq \mathbb{R}^n$ OTVOREN, POVEZAN, $\varphi, \psi \in C^1(\Omega; \mathbb{R}^n)$ T.D.

$$\nabla\varphi(x)^T \nabla\varphi(x) = \nabla\psi(x)^T \nabla\psi(x), \quad x \in \Omega$$

ψ INJEKTIVNA, $\det \nabla\psi(x) \neq 0, \; x \in \Omega.$

TADA $\exists\, a \in \mathbb{R}^n, R \in SO(3)$ T.D.

$$\varphi(x) = a + R\,\psi(x), \quad x \in \Omega.$$

<u>DOK</u>: ψ INJEKTIVNA $\Rightarrow$ ψ INVERTIBILNA

ψ KLASE C^1 & $\det \nabla\psi(x) \neq 0, x \in \Omega \to \psi$ C^1-DIFEOMORFIZAM

$$\psi^{-1} : \psi(\Omega) \to \Omega \quad C^1$$

$$\nabla\psi^{-1}(\psi(x))\, \nabla\psi(x) = I, \quad x \in \Omega$$

DEF: $\Theta = \varphi \circ \psi^{-1} : \psi(\Omega) \to \mathbb{R}^n$

$$\nabla\Theta(\psi(x)) = \nabla\varphi(\psi^{-1}(\psi(x)))\, \nabla\psi^{-1}(\psi(x))$$
$$= \nabla\varphi(x)\, \nabla\psi(x)^{-1}$$

$$\Rightarrow \nabla\Theta(\psi(x))^T \nabla\Theta(\psi(x)) = \nabla\psi(x)^{-T} \nabla\varphi(x)^T \nabla\varphi(x)\, \nabla\psi(x)^{-1}$$
$$= I$$

$\psi(\Omega)$ OTVOREN, POVEZAN (TM OD RANIJE)

$\Rightarrow$ Θ JE KRUTA DEFORMACIJA

$$\varphi(x) = \Theta \circ \psi(x) = a + R\,\psi(x), \quad x \in \Omega$$

NAP: $F = \nabla \varphi$ – ISKLJUČUJE TRANSLACIJU

POLARNA DEKOMPOZICIJA

$$F = RU = VR$$

ZA F REGULARAN, R, U, V JEDINSTVENI

R – ORTOGONALAN

U, V – SIMETRIČNE POZITIVNO DEFINITNE

$C = F^T F = U^T R^T R U = U^T U = U^2$ – DESNI C-G TENZOR

$B = F F^T = V R R^T V^T = V V^T = V^2$ – LIJEVI C-G TENZOR

$V = R U R^T$ – ORTOGONALNO SLIČNE

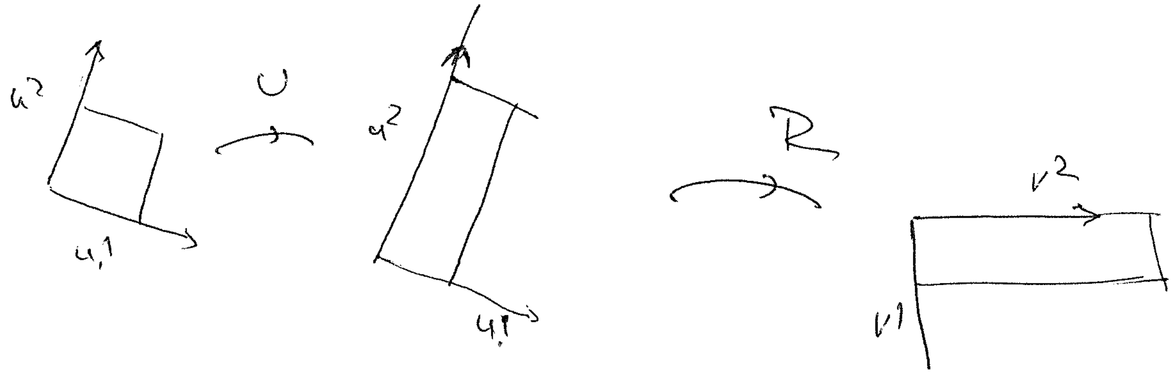
$B = V V^T = R U R^T R U R^T = R U^2 R^T = R C R^T$

– ORTOGONALNO SLIČNE

– IMAJU ISTE SVOJSTVENE VRIJEDNOSTI

– R^T PREVODI SV. VEKTORE OD B U SV. VEKTORE OD C

– SVAKA HOMOGENA DEFORMACIJA MOŽE SE DEKOMPONIRATI U ROTACIJU + RASTEZANJE + TRANSLACIJA



MOGLI SMO I PRVO ROTIRATI ZA R, PA RASTEZATI U SMJERU VEKTORA v^1, v^2

INFINITEZIMALNA DEFORMACIJA

PRETPOSTAVKA : $\|\nabla u\|$ MALEN

$\Rightarrow$ $\|\nabla u^T \nabla u\|$ MALEN U ODNOSU NA $\|\nabla u\|$

$$E = \frac{1}{2}(C - I) = \frac{1}{2}(F^T F - I) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u)$$

UVODIMO :

$$e = \frac{1}{2}(\nabla u + \nabla u^T)$$ - SIMETRIZIRANI GRADIJENT

- SIMETRIČNI DIO GRADIJENTA

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

NAP: IMALI SMO

φ JE KRUTA DEFORMACIJA $\iff \varphi(x) = Rx + b$, $R \in SO(3)$, $b \in \mathbb{R}^3$

$\iff \nabla\varphi(x) = R$, $R \in SO(3)$

$\iff \nabla\varphi(x) = \text{const}$ & $\nabla\varphi(x)^T \nabla\varphi(x) = I$

$\iff \nabla\varphi(x) = \text{const}$ & $E(x) = 0$

STOGA VRIJEDI :

u JE KRUTI POMAK $\iff \nabla u = \text{const}$ & $E(x) = 0$

STOGA DEFINIRAMO:

u JE <u>INFINITEZIMALNI KRUTI POMAK</u> AKO JE

$$\nabla u(x) = \text{const} \quad \& \quad e(x) = 0 \left(= \frac{1}{2}(\nabla u(x) + \nabla u(x)^T) \right)$$

NAP: PRIPADNA ROTACIJA NAZIVA SE INFINITEZIMALNA KRUTA DEFORMACIJA

$$e(x) = \begin{bmatrix} \partial_1 u_1(x) & \frac{1}{2}\left(\partial_1 u_2(x) + \partial_2 u_1(x)\right) & \frac{1}{2}\left(\partial_1 u_3(x) + \partial_3 u_1(x)\right) \\ & \partial_2 u_2(x) & \frac{1}{2}\left(\partial_2 u_3(x) + \partial_3 u_2(x)\right) \\ & & \partial_3 u_3(x) \end{bmatrix}$$

NIJE OČITO ŠTO ZNAČI $e(x) = 0$

POMOĆI ĆE NAM DA ZAKLJUČIMO DA SE UVJET $\nabla u(x) = \text{const}$ MOŽE IZBACITI IZ DEFINICIJE.

PRVO DOKAZUJEMO POMOĆNU LEMU.

__LEMA:__ $u : \Omega \to \mathbb{R}^3$ JE INFINITEZIMALNI KRUTI POMAK

$$\Updownarrow$$

$$(x-y)\cdot\left(u(x) - u(y)\right) = 0 \quad , \quad x, y \in \Omega.$$

__DOK:__ $\boxed{\Downarrow}$ TAYLOR $\Rightarrow$ $u(x) - u(y) = \nabla u(y)(x-y) + o(\|x-y\|)$

DEF $\Rightarrow \nabla u(y) = \text{CONST} \Rightarrow o(\|x-y\|) \equiv 0$

$$\overset{\text{CONST}}{\underset{W}{\|}}$$

DEF. $\Rightarrow 0 = e = \frac{1}{2}(W + W^T) \Rightarrow W^T = -W$ (ANTISIMETRIČNA)

$$\left.\begin{aligned}(x-y)\cdot\left(u(x) - u(y)\right) &= (x-y)\cdot W(x-y) \\ &= W^T(x-y)\cdot(x-y) \\ &= -W(x-y)\cdot(x-y)\end{aligned}\right) = 0$$

$\boxed{\Uparrow}$ $\quad (x-y)\cdot\left(u(x) - u(y)\right) = 0 \quad \Big| \partial_{x_i}\partial_{y_j}$

$$-e_j \cdot \partial_{x_i} u(x) - e_i \cdot \partial_{y_j} u(y) = 0$$

$$\partial_i u_j(x) + \partial_j u_i(y) = 0 \qquad i, j = 1, 2, 3$$

$$\nabla u(x) + \nabla u(y)^T = 0 \Rightarrow \nabla u(x) = \text{const} \quad i \quad y = x$$

$$\Downarrow$$

$$\nabla u(x) + \nabla u(x)^T = 0$$

TM: u JE INFINITEZIMALNI KRUTI POMAK $\Leftrightarrow$ $\nabla u(x) + \nabla u(x)^T = 0$

$$\nabla u(x) + \nabla u(x)^T = 2e(x)$$

DOK: $\boxed{\Rightarrow}$ OČITO

$\boxed{\Leftarrow}$ Ω OTVOREN, $K \subset \Omega$ KUGLA, $x, y \in K$

$\Rightarrow$ $\gamma : [0,1] \to K$ $\qquad \gamma(\lambda) = y + \lambda(x-y)$

KRIVULJNI INTEGRAL 2. VRSTE $\qquad$ GLADAK PUT OD y DO x

$$\int_\gamma \nabla u_i = \int_0^1 \nabla u_i(\gamma(\lambda)) \cdot \gamma'(\lambda)\, d\lambda = \int_0^1 \nabla u_i(y+\lambda(x-y)) \cdot (x-y)\, d\lambda$$

$$= \int_0^1 \frac{d}{d\lambda}(u_i \circ \gamma(\lambda))\, d\lambda = u_i(\gamma(1)) - u_i(\gamma(0))$$

$$= u_i(x) - u_i(y)$$

UZ DOGOVOR DA RAČUN VRIJEDI PO KOMPONENTAMA

$$u(x) - u(y) = \int_\gamma \nabla u$$

$$(x-y)\cdot(u(x)-u(y)) = (x-y)\cdot \int_0^1 \nabla u(\gamma(\lambda))(x-y)\, d\lambda$$

$$= \int_0^1 \nabla u(\gamma(\lambda))(x-y)\cdot(x-y)\, d\lambda$$

$$= \int_0^1 (x-y)\cdot \underbrace{\nabla u(\gamma(\lambda))^T}_{= -\nabla u(\gamma(\lambda))}(x-y)\, d\lambda$$

$$= -\int_0^1 \nabla u(\gamma(\lambda))(x-y)\cdot(x-y)\, d\lambda$$

$$= 0$$

LEMA $\Rightarrow$ TVRDNJU

KOROLAR: u JE INFINITEZIMALNI KRUTI POMAK

$$\Updownarrow$$

$\exists a, b \in \mathbb{R}^3$ T.D. $u(x) = a \times x + b$

DOK: $\boxed{\Uparrow}$ DEFINIRAMO $A := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$ - ANTISIMETRIČNA

TADA JE $Ax = a \times x$

$a \longleftrightarrow A$ BIJEKCIJA

a - AKSIJALNI VEKTOR MATRICE A

$\Rightarrow u(x) = Ax + b$

$\Rightarrow \nabla u(x) = A$

1) CONST

2) $\nabla u + \nabla u^T = A + A^T = 0$ (A ANTISIMETRIČ.)

$\Rightarrow u$ JE INFINITEZIMALNI KRUTI POMAK

$\boxed{\Downarrow}$ $\Rightarrow \nabla u = \text{const} \ \& \ \nabla u + \nabla u^T = 0$

$\nabla u =: A$, $\quad \Downarrow \quad A + A^T = 0$

~~TAYLOR~~ $\Rightarrow$

$v(x) := u(x) - Ax \Rightarrow \nabla v = \nabla u - A = A - A = 0$

$\Rightarrow v = \text{const} =: b$

$\Rightarrow b = u(x) - Ax \Rightarrow u(x) = Ax + b$

$A \longrightarrow a$ AKSIJALNI VEKTOR

$u(x) = a \times x + b$

НАП: TAYLOR: $u(x) - u(y) = \nabla u(y)(x-y) + o(\|x-y\|)$

$$\nabla u(y) = \underbrace{\frac{1}{2}\left(\nabla u(y) + \nabla u(y)^T\right)}_{e(y)} + \underbrace{\frac{1}{2}\left(\nabla u(y) - \nabla u(y)^T\right)}_{W(y)}$$

АНТИСИМЕТРИЧНА

$a(y)$ – АКСИЈАЛНИ ВЕКТОР

$$\Rightarrow u(x) = u(y) + e(y)(x-y) + \underbrace{W(y)(x-y)}_{a(y)\times(x-y)} + o(\|x-y\|)$$

ЛОКАЛНО: ТРАНСЛАЦИЈА

ИНФИНИТЕЗИМАЛНА ДЕФОРМАЦИЈА

ИНФИНИТЕЗИМАЛНА РОТАЦИЈА

НАП: $W(y) \longleftrightarrow a(y)$

$$W(y) = \begin{bmatrix} 0 & \frac{1}{2}(\partial_2 u_1 - \partial_1 u_2) & \frac{1}{2}(\partial_3 u_1 - \partial_1 u_3) \\ & 0 & \frac{1}{2}(\partial_3 u_2 - \partial_2 u_3) \\ & & 0 \end{bmatrix}$$

АНТИСИМЕТРИЧНА

$$a(y) = \frac{1}{2}\begin{bmatrix} \partial_2 u_3(y) - \partial_3 u_2(y) \\ \partial_3 u_1(y) - \partial_1 u_3(y) \\ \partial_1 u_2(y) - \partial_2 u_1(y) \end{bmatrix} = \frac{1}{2}\,\mathrm{rot}\, u(y)$$

NAP: A - ANTISIMETRIČNA, a - PRIPADNI AKSIJALNI VEKTOR

e^A - ORTOGONALNA

$$(e^A)^T e^A = e^{A^T} e^A = e^{-A} e^A = e^{-A+A} = I$$

$\uparrow$ $-A, A$ KOMUTIRAJU

$$e^A = \underbrace{I + A}_{} + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

LINEARNA APROKSIMACIJA ORTOGONALNE

$\cancel{A}$ JE SINGULARNA $\quad (Aa = a \times a = 0)$

$$e^A a = a + Aa + \frac{1}{2}A^2 a + \dots = a$$

$\Rightarrow$ a JE FIKSNA TOČKA ZA e^A - OS ROTACIJE

$\|A\| = \|a\|$ - KUT ROTACIJE

NAP: $J = \det \nabla \varphi = \det(I + \nabla u)$

AKO JE u INFINITEZIMALNA DEFORMACIJA $\|\nabla u\|^2$ ZANEMARUJEMO

$$J = \begin{vmatrix} 1+\partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & 1+\partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & 1+\partial_3 u_3 \end{vmatrix} \approx (1+\partial_1 u_1)(1+\partial_2 u_2)(1+\partial_3 u_3)$$

$$\approx 1 + \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 1 + \operatorname{tr} \nabla u$$

$$= 1 + \operatorname{tr} e$$

$$= 1 + \operatorname{div} u$$

$J \approx 1 + \operatorname{tr} e$ —— MJERI LOKALNU INFINITEZIMALNU PROMJENU VOLUMENA (ODSTUPANJE OD 0)

$\uparrow$

MJERI LOKALNU PROMJENU VOLUMENA (ODSTUPANJE OD 1)

$d = e - \frac{1}{3}(\operatorname{tr} e) I$ - DEVIJATORNI DIO TENZORA DEFORMACIJE (MJERI SMICANJE)

$\operatorname{tr} d = \operatorname{tr} e - \frac{1}{3}\operatorname{tr} e \cdot 3 = 0$

2. GIBANJE

DEF: <u>GIBANJE</u> JE DOVOLJNO GLATKO PRESLIKAVANJE

$\varphi : \Omega \times \mathbb{R} \to \mathbb{R}^3$ T.D. $\varphi(\cdot, t)$ DEFORMACIJA

NAP: - NA GIBANJE MOŽEMO GLEDATI KAO NA 1-PARAMETARSKU FAMILIJU DEFORMACIJA

$$t \longmapsto \varphi(\cdot, t) =: \varphi_t, \quad t \in \mathbb{R}$$

- t JE VRIJEME

- ZA FIKSNI $x \in \Omega$

$$t \longmapsto \varphi(x, t), \quad t \in \mathbb{R}$$

TRAJEKTORIJA MATERIJALNE TOČKE (KRIVULJA U $\mathbb{R}^3$)

- $\Omega_t = \Omega_t^{\varphi} := \varphi_t(\Omega)$ DEFORMIRANA KONFIGURACIJA U TRENUTKU t

DEF:

$$v = \frac{\partial \varphi}{\partial t} : \Omega \times \mathbb{R} \to \mathbb{R}^3$$ <u>LAGRANGEOVA BRZINA</u>

$$a = \frac{\partial v}{\partial t} = \frac{\partial^2 \varphi}{\partial t^2} : \Omega \times \mathbb{R} \to \mathbb{R}^3$$

<u>LAGRANGEOVO UBRZANJE</u>

NAP: SVA PRESLIKAVANJA $\Omega \times \mathbb{R}$ NAZIVAMO LAGRANGEOVIM POLJIMA.

DEF: TRAJEKTORIJA GIBANJA

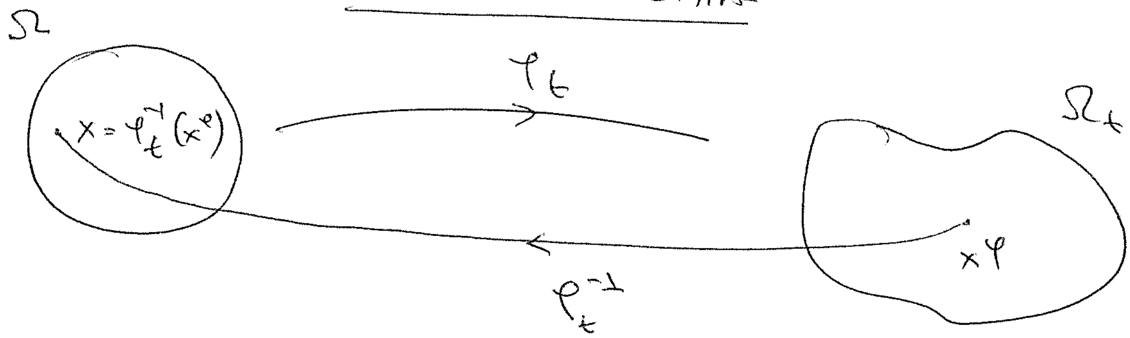
$$T_\varphi = T = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : x \in \Omega_t, t \in \mathbb{R}\}$$
$$= \bigcup_{t \in \mathbb{R}} \Omega_t \times \{t\}$$

SVA PRESLIKAVANJA NA T NAZIVAMO <u>EULEROVIM POLJIMA</u>

DEF:

$$v^\varphi(x^\varphi, t) = v(\varphi_t^{-1}(x^\varphi), t) = \frac{\partial \varphi}{\partial t}(\varphi_t^{-1}(x^\varphi), t)$$

<u>EULEROVA BRZINA</u>



ZA ZADANU LAGRANGEOVU BRZINU v I POČETNI POLOŽAJ $\varphi(\cdot, 0) = \varphi_0$ REKONSTRUIRAMO GIBANJE:

$$\frac{\partial \varphi}{\partial t}(x,t) = v(x,t)$$

$$\varphi(x,0) = \varphi_0(x)$$

ZA ZADANU EULEROVU BRZINU v^φ I POČETNI POLOŽAJ $\varphi(\cdot,0) = \varphi_0$

$$\frac{\partial \varphi}{\partial t}(x,t) = v^\varphi(\varphi(x,t), t)$$
$$\varphi(x,0) = \varphi_0(x)$$

DEF: $a^\varphi(x^\varphi, t) = a(\varphi_t^{-1}(x^\varphi), t)$ EULEROVO UBRZANJE

OPĆENITO:

1) ZA ZADANO f LAGRANGEOVO POLJE

$$f^\varphi(x^\varphi, t) := f(\varphi_t^{-1}(x^\varphi), t) \qquad \text{E. POLJE}$$

2) ZA ZADANO f^φ EULEROVO POLJE

$$f(x, t) := f^\varphi(\varphi_t(x), t) \qquad \text{L. POLJE}$$

DEF: MATERIJALNA DERIVACIJA EULEROVOG POLJA f^φ JE EULEROVO POLJE $\dot{f}^\varphi$ $\left(\frac{df^\varphi}{dt}\right)$ DEFINIRANO SA

$$\dot{f}^\varphi(x^\varphi, t) = \frac{df^\varphi}{dt}(x^\varphi, t) := \frac{\partial}{\partial t}\left(f^\varphi(\varphi_t(x), t)\right) \circ \varphi_t^{-1}(x^\varphi)$$

NEPRECIZNO

SAMO NA 1. VARIJABLU OD f^φ

$$\Rightarrow \dot{f}^\varphi(x^\varphi, t) = \left(\nabla_{x^\varphi} f^\varphi(\varphi_t(x), t)\,\frac{\partial \varphi}{\partial t}(x, t) + \frac{\partial f^\varphi}{\partial t}(\varphi_t(x), t)\right) \circ \varphi_t^{-1}(x^\varphi)$$

$$= \nabla_{x^\varphi} f^\varphi(x^\varphi, t)\,\underbrace{\frac{\partial \varphi}{\partial t}(\varphi_t^{-1}(x^\varphi), t)}_{v^\varphi(x^\varphi, t)} + \frac{\partial f^\varphi}{\partial t}(x^\varphi, t)$$

$$= \nabla_{x^\varphi} f^\varphi(x^\varphi, t)\, v^\varphi(x^\varphi, t) + \frac{\partial f^\varphi}{\partial t}(x^\varphi, t)$$

U DALJNJEM $\nabla_{x^\varphi} = \nabla$

$$\dot{f}^\varphi(x^\varphi,t) = \nabla_x f^\varphi(x^\varphi,t)\, v^\varphi(x^\varphi,t) + \frac{\partial f^\varphi}{\partial t}(x^\varphi,t)$$

$$\boxed{\dot{f}^\varphi = \nabla_x f^\varphi v^\varphi + \frac{\partial f^\varphi}{\partial t}}$$

PRIMJER

$$p^\varphi(x^\varphi,t) = x^\varphi \Rightarrow \nabla_{x^\varphi} p^\varphi(x^\varphi,t) = I$$

$$\frac{\partial p^\varphi}{\partial t}(x^\varphi,t) = 0$$

$$\Rightarrow \quad \dot{p}^\varphi(x^\varphi,t) = v^\varphi(x^\varphi,t)$$

PRIMJER

$$a^\varphi(x^\varphi,t) = a(\varphi_t^{-1}(x^\varphi),t) = \frac{\partial^2 \varphi}{\partial t^2}(\varphi_t^{-1}(x^\varphi),t)$$

$$= \frac{\partial v}{\partial t}(\varphi_t^{-1}(x^\varphi),t)$$

S DRUGE STRANE, IZ DEFINICIJE

$$\dot{v}^\varphi(x^\varphi,t) = \frac{\partial}{\partial t}\left(v^\varphi(\varphi_t(x),t)\right) \circ \varphi_t^{-1}(x^\varphi)$$

$$= \frac{\partial}{\partial t}\left(v(x,t)\right) \circ \varphi_t^{-1}(x^\varphi)$$

$$= \frac{\partial v}{\partial t}(\varphi_t^{-1}(x^\varphi),t) = a^\varphi(x^\varphi,t)$$

$$\Rightarrow \quad a^\varphi(x^\varphi,t) = \dot{v}^\varphi(x^\varphi,t) = \nabla_{x^\varphi} v^\varphi_{(x^\varphi,t)} v^\varphi_{(x^\varphi,t)} + \frac{\partial v^\varphi}{\partial t}(x^\varphi,t)$$

ZAD: EULEROVO POLJE f^φ ZADOVOLJAVA $\dot{f}^\varphi = 0$ U T

$$\Longleftrightarrow$$

f^φ = CONST NA TRAJEKTORIJI SVAKE MATERIJALNE TOČKE

Rj:

$$0 = \dot{f}^\varphi(x^\varphi, t) = \frac{\partial}{\partial t}\left(f^\varphi(\varphi_t(x), t)\right) \circ \varphi_t^{-1}(x^\varphi)$$

$$\Longleftrightarrow$$

$$\frac{\partial}{\partial t}\left(f^\varphi(\varphi_t(x), t)\right) = 0$$

$$\Longleftrightarrow$$

$t \longmapsto f^\varphi(\varphi_t(x), t)$ KONSTANTA

DEF: φ JE <u>KRUTO GIBANJE</u> AKO JE φ_t KRUTA DEFORMACIJA ZA SVAKI t.

LEMA: φ JE KRUTO GIBANJE

$$\Longleftrightarrow$$

$\nabla v^\varphi + \nabla v^{\varphi T} = 0$, $x^\varphi \in \Omega_t$ I $\varphi(\cdot, 0)$ JE KRUTO

DOK: $\boxed{\Rightarrow}$ φ_t JE KRUTA DEFORMACIJA $\forall t$.

$$\|\varphi_t(x) - \varphi_t(y)\| = \|x - y\|, \quad t, \; x, y \in \Omega$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\|\varphi_t(x) - \varphi_t(y)\|^2\right) = 0, \quad t, \; x, y \in \Omega$$

$$\Longleftrightarrow \left(\varphi_t(x) - \varphi_t(y)\right) \cdot \left(\frac{\partial \varphi}{\partial t}(x,t) - \frac{\partial \varphi}{\partial t}(y,t)\right) = 0, \quad t, \; x, y \in \Omega$$

$$\Longleftrightarrow (x^\varphi - y^\varphi) \cdot \left(v(x,t) - v(y,t)\right) = 0, \quad t, \; x^\varphi = \varphi_t(x), \; y^\varphi = \varphi_t(y), \; x, y \in \Omega$$

$$\Longleftrightarrow (x^\varphi - y^\varphi) \cdot \left(v(\varphi_t^{-1}(x^\varphi), t) - v(\varphi_t^{-1}(y^\varphi), t)\right), \quad t, \; x^\varphi, y^\varphi \in \Omega_t$$

$\overset{}{\underset{\text{LEMA}}{\Longleftrightarrow}}$ $(x^\varphi - y^\varphi) \cdot \left(v^\varphi(x^\varphi, t) - v^\varphi(y^\varphi, t) \right) = 0$, $t, x^\varphi, y^\varphi \in \Omega$

$\underset{\text{LEMA}}{\Longleftrightarrow}$ $\nabla v^\varphi(x^\varphi, t) = \mathrm{const}(t)$, $x^\varphi \in \Omega_t$ & $\nabla v^\varphi(x^\varphi, t) + \nabla v^\varphi(x^\varphi, t)^T = 0$, $x^\varphi \in \Omega_t$

$\Longleftrightarrow$ $\nabla v^\varphi(x^\varphi, t) + \nabla v^\varphi(x^\varphi, t)^T = 0$, $x^\varphi \in \Omega_t$, t

ZA OBRAT TREBA POKAZAT ~~PRVO~~ OBRAT PRVE IMPLIKACIJE

IZ $$\frac{\partial}{\partial t}\left(\| \varphi_t(x) - \varphi_t(y) \|^2 \right) = 0$$

$$\Rightarrow \quad \| \varphi_t(x) - \varphi_t(y) \| = \mathrm{CONST}(x, y) \quad , t \, , \, x, y \in \Omega$$

ZA $t = 0$ $\Rightarrow$ $\mathrm{CONST}(x,y) = \| \varphi_0(x) - \varphi_0(y) \|$

$= \| x - y \|$

JER JE φ_0 KRUTO //