Zadaci za vježbu

Domena, limesi, derivacije

ZADATAK 1. Odredite prirodnu domenu funkcije f zadane s

a) $f(x) = \sqrt{\dfrac{x}{x-1}}$;

b) $f(x) = \dfrac{\sqrt{x+5}}{\ln(9-x)}$;

c) $f(x) = \log_{x-1}(x+1)$;

d) $f(x) = \log_2 \log_{0.5} \dfrac{x+1}{x-2}$;

e) $f(x) = \log_{\cos x} \sin x$;

f) $f(x) = \dfrac{\ln(x^2-14x+40)}{\sqrt{x^2-x-30}}$;

g) $f(x) = \sqrt{\mathrm{tg}(x+2)} + \sqrt[4]{\mathrm{ctg}(x-2)}$;

h) $f(x) = \sqrt[3]{x-3} + \sqrt{3^{x-3}} - \sqrt[6]{x^2-6x+9}$;

i) $f(x) = \mathrm{tg}\,(x+3) - \sqrt[8]{\ln\left(\sqrt{x^2-2}\right)}$;

j) $f(x) = \sqrt{\mathrm{ctg} x} + \mathrm{ctg}\left(x - \dfrac{\pi}{4}\right)$.

ZADATAK 2. Odredite nepoznate parametre $a, b \in \mathbb{R}$ tako da dana funkcija f bude neprekidna na cijelom $\mathbb{R}$.

a) $f(x) = \begin{cases} \dfrac{5^x-1}{x} + a, & x < 0, \\ b\sin x + \cos x, & 0 \leq x \leq \dfrac{\pi}{2}, \\ \ln(x+4), & x > 2. \end{cases}$

b) $f(x) = \begin{cases} ax + b, & x < 1, \\ -ax - b, & 1 \leq x \leq 2, \\ 2, & x > 2. \end{cases}$

ZADATAK 3. Odredite sljedeće limese:

a) $\lim\limits_{x\to\infty} \dfrac{2+3x-x^2}{3x^2+2x-1}$;

b) $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x^2+4}+\sqrt[6]{x^4+x^2}}{\sqrt[3]{x+3}+\sqrt[7]{8x^7-3}}$;

c) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+4}+\sqrt[6]{x^4+x^2}}{\sqrt[3]{x+3}+\sqrt[7]{8x^7-3}}$;

d) $\lim\limits_{x\to4} \dfrac{\sqrt[3]{x+4}-2}{x^2-5x+4}$;

e) $\lim\limits_{x\to0} \dfrac{\sin(\sin x)}{x}$;

f) $\lim\limits_{x\to7} \dfrac{\cos^2(x-7)-1}{(x-7)^2}$;

g) $\lim\limits_{x\to0^+} \dfrac{\cos x-5^x}{x^2}$;

h) $\lim\limits_{x\to0} \dfrac{8^x-7^x}{6^x-5^x}$;

i) $\lim\limits_{x\to0^+} \dfrac{\ln(\cos x)}{x^2}$;

j) $\lim\limits_{x\to0} (\cos x)^{\frac{1}{x^2}}$;

k) $\lim\limits_{x\to2} \left(1+\dfrac{(x-2)^2}{\text{tg}(x-2)}\right)^{\ln(x-1)}$.

Napomena: Limese možemo odrediti i L'Hospitalovim pravilom, no za zadatke b), c) i k) dobivamo znatno kompliciranija rješenja zbog složenih derivacija.

ZADATAK 4. Odredite derivaciju funkcije f danu s

a) $f(x) = \dfrac{\text{ctg}x}{x\ln x}+3xe^x$;

b) $f(x) = \text{tg}\left(\ln\dfrac{1-x}{1+x}\right)$;

c) $f(x) = \sqrt{4x-1}+\text{ctg}\sqrt{4x-1}$;

d) $f(x) = \ln \ln \left(x^4 + x\right)$;

e) $f(x) = e^{\sqrt{xe^x}} + \cos^6 x \text{tg}^7 x$;

f) $f(x) = x^x e^{x^2}$;

g) $f(x) = x^{\ln x} - (\cos x)^{-3x}$.

ZADATAK 5. Odredite derivaciju implicitno zadane funkcije y.

a) $\left(x^2 + 1\right) y + x \left(y^2 + 1\right) = \ln(x + y)$;

b) $y \sin x = x \sin y$.

ZADATAK 6. Odredite n-tu derivaciju funkcije f u točki x_0.

a) $f(x) = x^3 + x^2 + x + 1, x_0 = 1$;

b) $f(x) = e^{3x} - x^4, x_0 = 0$.

ZADATAK 7. Odredite prvu i drugu derivaciju parametarski zadane funkcije y:

a) $\begin{cases} x(t) = 2t - 1, \\ y(t) = t^3. \end{cases}$

b) $\begin{cases} x(t) = a\left(\cos t + t \sin t\right), \\ y(t) = a\left(\sin t - t \cos t\right). \end{cases}$

c) $\begin{cases} x(t) = \sqrt{t}, \\ y(t) = \sqrt[3]{t}. \end{cases}$

ZADATAK 8. Pomoću L'Hospitalovog pravila odredite sljedeće limese:

a) $\lim\limits_{x \to +\infty} \dfrac{e^x}{x^5}$;

b) $\lim\limits_{x \to 1} \dfrac{1 - x}{1 - \sin \frac{\pi x}{2}}$;

c) $\lim\limits_{x \to 0} (1 - \cos x) \, \text{ctg} x$;

d) $\lim\limits_{x \to 1} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x} \right)$;

e) $\lim\limits_{x \to 0^+} x^{\sin x}$;

f) $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} \right)^{\text{tg} x}$.

Rješenja

ZADATAK 1.

a) $\mathcal{D}_f = \langle -\infty, 0] \cup \langle 1, +\infty\rangle$;

b) $\mathcal{D}_f = [-5, 9\rangle \setminus \{8\}$;

c) $\mathcal{D}_f = \langle 1, +\infty\rangle \setminus \{2\}$;

d) $\mathcal{D}_f = \langle -\infty, -1\rangle$;

e) $\mathcal{D}_f = \bigcup_{k\in\mathbb{Z}} \left\langle 2k\pi, 2k\pi + \frac{\pi}{2}\right\rangle$;

f) $\mathcal{D}_f = \langle 4, 6\rangle$;

g) $\mathcal{D}_f = \bigcup_{k\in\mathbb{Z}} \left\langle 2 + k\pi, -2 + (k+1)\pi + \frac{\pi}{2}\right\rangle$;

h) $\mathcal{D}_f = \mathbb{R}$;

i) $\mathcal{D}_f = \left(\left\langle -\infty, -\sqrt{3}\right\rangle \cup \left\langle \sqrt{3}, \infty\right\rangle\right) \setminus \left\{-3 + \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \setminus \{0, 1\}\right\}$;

j) $\mathcal{D}_f = \bigcup_{k\in\mathbb{Z}} \left(\left\langle k\pi, k\pi + \frac{\pi}{2}\right\rangle \setminus \left\{\frac{\pi}{4} + k\pi\right\}\right)$.

ZADATAK 2.

a) $a = 1 - \ln 5, b = \ln\left(\frac{\pi}{2} + 4\right)$;

b) $a = -2, b = 2$.

ZADATAK 3.

a) $-\frac{1}{3}$;

b) $\frac{\sqrt{3}}{\sqrt[7]{8}}$;

c) $-\frac{\sqrt{3}}{\sqrt[7]{8}}$;

d) $\frac{1}{36}$;

e) 1;

f) -1;

g) $-\infty$;

h) $\dfrac{\ln\frac{8}{7}}{\ln\frac{6}{5}}$;

i) $-\dfrac{1}{2}$;

j) $\dfrac{3}{2}$;

k) e.

ZADATAK 4.

a) $f'(x) = \dfrac{\frac{x\ln x}{\cos^2 x} - \text{ctg}x\,(1+\ln x)}{x^2\ln^2 x} + 3e^x + 3xe^x$;

b) $f'(x) = \dfrac{-2x(1+x)}{\cos^2\left(\ln\frac{1-x}{1+x}\right)(1-x)^3}$;

c) $f'(x) = \dfrac{1}{2\sqrt{4x-1}}\left(1 - \dfrac{1}{\sin^2\sqrt{4x-1}}\right)$;

d) $f'(x) = \dfrac{4x^3+1}{(x^4+x)\ln(x^4+x)}$;

e) $f'(x) = \dfrac{e^{\sqrt{xe^x}}(1+x)e^x}{2\sqrt{xe^x}} - \dfrac{6\sin^8 x}{\cos^2 x} + \dfrac{7\sin^6 x}{\cos^2 x}$;

f) $f'(x) = x^x(\ln x + 1 + 2x)\,e^{x^2}$;

g) $f'(x) = 2x^{\ln x - 1}\ln x - 3(\cos x)^{-3x}(-\ln\cos x + x\text{tg}x)$.

ZADATAK 5.

a) $y' = \dfrac{-(x+y)(2xy+y^2+1)+1}{(x+y)(x^2+1+2xy)+1}$;

b) $y' = \dfrac{y\cos x - \sin y}{x\cos y - \sin x}$.

ZADATAK 6.

a) $f^{(n)}(1) = \begin{cases} 4, & n = 1, \\ 6, & n = 2, 4, \\ 8, & n = 3, \\ 0, & n \geq 5. \end{cases}$

b) $f^{(n)}(0) = \begin{cases} 3^n, & n \neq 4, \\ 3^4 - 4!, & n = 4. \end{cases}$

ZADATAK 7.

a) $y'(x) = \frac{3}{2}t^2, y''(x) = \frac{3}{2}t$;

b) $y'(x) = \operatorname{tg} t, y''(x) = \frac{1}{at\cos^3 t}$;

c) $y'(x) = \frac{2}{3\sqrt[6]{t}}, y''(x) = -\frac{8}{9\sqrt[3]{t}} + \frac{2}{3\sqrt[3]{t^2}}$.

ZADATAK 8.

a) $+\infty$;

b) $\pm\infty$;

c) 0;

d) $\frac{1}{2}$;

e) 1;

f) 1.

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