5.3. APROKSIMACIJE $W^{k,p}(U)$

5.3.1. LOKALNA APROKSIMACIJA

TM1 NEKA JE $k \in \mathbb{N}$, $p \in [1,+\infty\rangle$, $u \in W^{k,p}(U)$.

$$u^\varepsilon := \eta_\varepsilon * u \quad \text{NA } U_\varepsilon = \{x \in U : d(x, \partial U) > \varepsilon\}$$

TADA:

(i) $u^\varepsilon \in C^\infty(U_\varepsilon)$, $\varepsilon > 0$

(ii) $u^\varepsilon \longrightarrow u$ u $W^{k,p}_{loc}(U)$, KADA $\varepsilon \to 0$

DOK: (i) DOKAZALI SMO

(ii) IZ ZADAĆE:

$$D^\alpha u^\varepsilon(x) = D^\alpha \int_U \eta_\varepsilon(x-y)\, u(y)\, dy = \int_U D^\alpha_x \eta_\varepsilon(x-y)\, u(y)\, dy$$

$$= (-1)^{|\alpha|} \int_U D^\alpha_y \eta_\varepsilon(x-y)\, u(y)\, dy$$

$$\overset{P.I}{=} (-1)^{|\alpha|} (-1)^{|\alpha|} \int_U \eta_\varepsilon(x-y)\, D^\alpha u(y)\, dy \qquad \eta_\varepsilon \in C^\infty_c(U)$$

$$= \eta_\varepsilon * D^\alpha u(x)$$

NEKA JE $V \subset U$ ~~KOMPAKTNO~~ ULOŽEN (ZA ε DOVOLJNO MALI $V \subset U_\varepsilon$)

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \longrightarrow D^\alpha u \quad \text{u } L^p(V), \quad |\alpha| \le k$$

(SVOJSTVO IZGLAĐIVAČA)

$$\Longleftrightarrow$$

$$u^\varepsilon \longrightarrow u \quad \text{u } W^{k,p}_{loc}(U)$$

NAP: DAKLE SVAKU $W^{k,p}(U)$ FUNKCIJU MOŽEMO LOKALNO APROKSIMIRATI GLATKOM (C^∞) FUNKCIJOM U $W^{k,p}(U)$.

5.3.2 GLOBALNA APROKSIMACIJA

APROKS. U $W^{k,p}(U)$! TREBA PRIĆI RUBU.

NEMA PRETPOSTAVKE NA GLATKOĆU RUBA!

TM 2 NEKA JE U OGRANIČEN, $p \in [1, +\infty\rangle$, $u \in W^{k,p}(U)$.

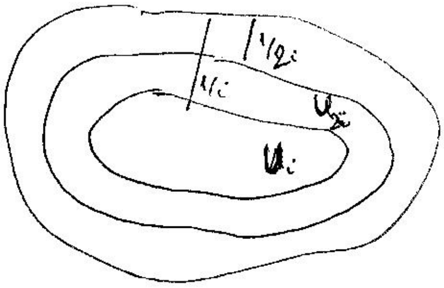
TADA $\exists\ u_m \in C^\infty(U) \cap W^{k,p}(U)$ T.D.

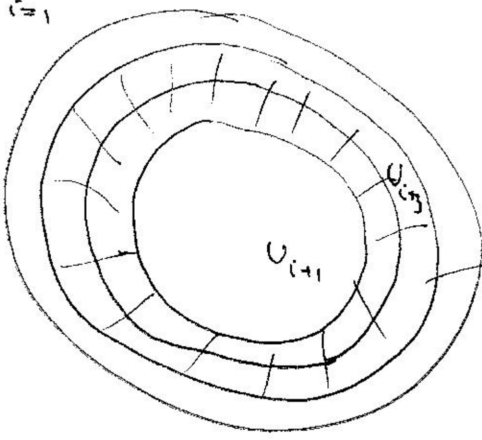
$$u_m \to u \quad \text{u } W^{k,p}(U)$$

DOK: DEF: $U_i = \{x \in U : d(x, \partial U) > 1/i\}$ $\quad i \in \mathbb{N}$

$$V_i = U_{i+3} - \overline{U}_{i+1}$$

TV: $U = \bigcup_{i=1}^{\infty} U_i$





V_i – OTVOREN

$V_i \subset U$ KOMPAKTNO ULOŽEN

SLOJEVI (PREKLAPAJU SE!)

DODAM $V_0 \subset U$ K.U. T.D.

$$U = \bigcup_{i=0}^{\infty} V_i$$

PARTICIJA JEDINICE: $\zeta_i \in C_c^\infty(V_i)$

$$0 \le \zeta_i \le 1$$

$$\sum_{i=0}^{\infty} \zeta_i = 1 \quad \text{NA } U$$

NEKA JE $u \in W^{k,p}(U)$

$$\Downarrow$$

$$\zeta_i u \in W^{k,p}(U) \quad , \quad supp(\zeta_i u) \subset V_i$$

FIX: $\delta > 0$

PROMATRAM: $u^i := \eta_{\varepsilon_i} * (\zeta_i u)$ C^∞ FUNKCIJA

— KONVOLUCIJA "PROŠIRI" NOSAČ ZA ε_i

$\Rightarrow$ ZA DOVOLJNO MALI ε_i SUPP $u^i \subset W_i := U_{i+4} - \overline{U}_i \supset V_i$

— u^i JE LOKALNA APROKSIMACIJA ZA $\zeta_i u$

KAD $\varepsilon_i \to 0$ $u^i \to \zeta_i u$ U $W^{k,p}(W_i)$

ALI I $W^{k,p}(U)$

$\Rightarrow$ ZA DOVOLJNO MALI ε_i:

$$\| u^i - \zeta_i u \|_{W^{k,p}(U)} \leq \frac{\delta}{2^{i+1}}$$

DEF: $v := \sum_{i=0}^{\infty} u^i$ — $C^\infty(U)$

— U SVAKOJ TOČKI SUMA KONAČNA

S DRUGE STRANE

$$u = \sum_{i=0}^{b} \zeta_i u \qquad (7.1)$$

NEKA JE $V \subset U$ K.U.

$$\|u - v\|_{W^{k,p}(V)} = \Big\| \sum_{i=0}^{\infty} (u^i - \zeta_i u) \Big\|_{W^{k,p}(V)} \qquad V \subset U$$

$$\leq \sum_{i=0}^{\infty} \| u^i - \zeta_i u \|_{W^{k,p}(V)} \qquad \text{N.T.}$$

$$\leq \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} = \delta$$

UZMI SUP PO $V \subset\subset U$ K.U. $\Rightarrow$ $\|u - v\|_{W^{k,p}(U)} \leq \delta$.

5.3.3. GLOBALNA APROKSIMACIJA 2

SAD APROKSIMIRAMO S $u_m \in C^\infty(\bar{U})$.

TREBA NAM GLATKOĆA RUBA

TM3: NEKA JE U OGRANIČEN, ∂U KLASE C^1, $p \in [1, +\infty\rangle$, $u \in W^{k,p}(U)$.

TADA $\exists\ u_m \in C^\infty(\bar{U})$ T.D

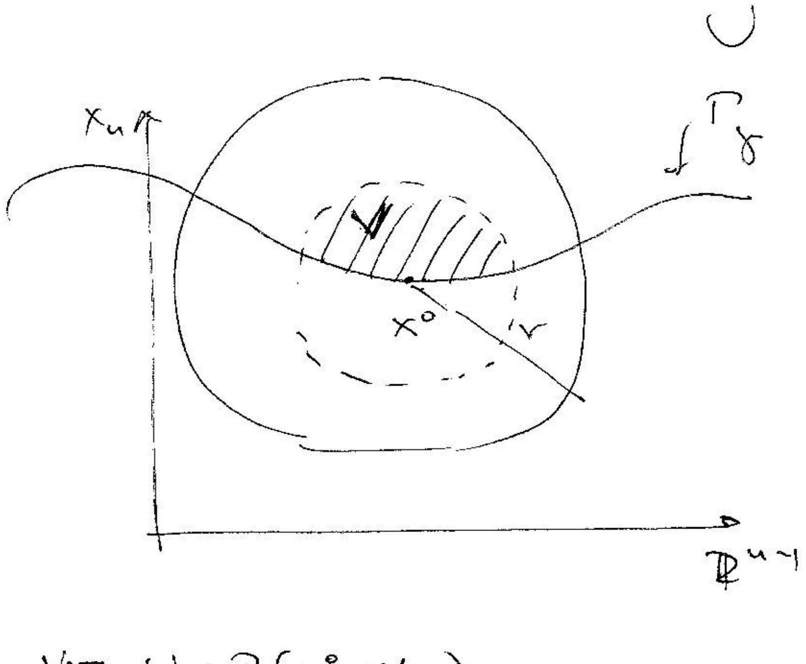
$$u_m \to u \quad u\ W^{k,p}(U).$$

DOK: ∂U JE $C^1 \Leftrightarrow \forall x^0 \in \partial U\ \exists r > 0 : \gamma \in C^1(\mathbb{R}^{n-1}; \mathbb{R})$ T.D.

$$U \cap B(x^0, r) = \{x \in B(x^0, r) : x_n > \gamma(x_1, \ldots, x_{n-1})\}$$

(AKO TREBA RENUMERIRAMO KOORDINATE)

- SKUP JE SAMO "S JEDNE STRANE" GRANICE



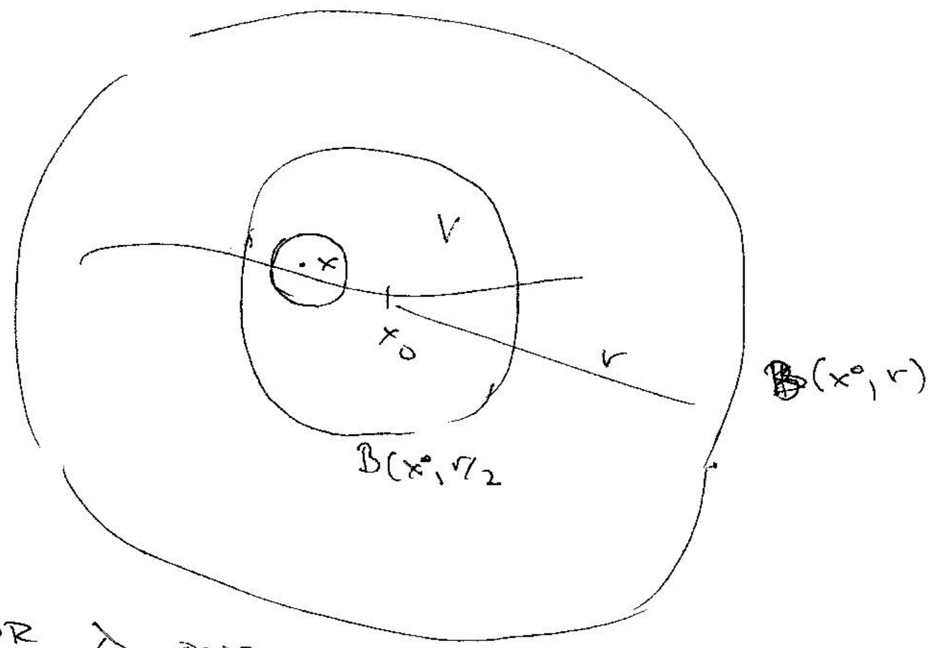
DEF $V := U \cap B(x^0, r/2)$

$$\varepsilon > 0, \quad y \in \mathbb{R}^n \xrightarrow{t^\varepsilon} y^\varepsilon := y + \lambda \varepsilon e_n = t^\varepsilon(y)$$

TRANSLACIJA "U UNUTRAŠNJOST"

NEKA JE $\lambda > 0$ DOVOLJNO VELIK.

TV: NEKA JE $x \in V$. TADA $\forall y \in K(x,\varepsilon) \Rightarrow y^\varepsilon \in U \cap B(x^0, r)$



IZBOR λ OVISI O γ!

ZA $\varepsilon < \frac{r}{2\lambda} \Rightarrow y^\varepsilon \in B(x^0, r)$

ZA $x \in V$ DEF: $v^\varepsilon := \eta_\varepsilon * (u \circ t^\varepsilon)$

$$v^\varepsilon(x) = \int_{K(x,\varepsilon)} \eta_\varepsilon(x-y)\, u(t^\varepsilon(y))\, dy$$

$$= \int_{K(x,\varepsilon)} \eta_\varepsilon(x-y)\, u(\underbrace{y + \lambda\varepsilon e_n}_{\in U})\, dy \quad \text{DOBRO DEF}$$

$v^\varepsilon \in C^\infty(\overline{V})$

TV: $v^\varepsilon \longrightarrow u$ U $W^{k,p}(V)$

Dok:

$$\|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} \leq \underbrace{\|D^\alpha v^\varepsilon - D^\alpha(u \circ t^\varepsilon)\|_{L^p(V)}}_{\downarrow \text{ KAO U TM1 } 0} + \|D^\alpha(u \circ t^\varepsilon) - D^\alpha u\|_{L^p(V)}$$

$$\|D^\alpha(u \circ t^\varepsilon) - D^\alpha u\|_{L^p(V)} = \|(D^\alpha u)\circ t^\varepsilon - D^\alpha u\|_{L^p(V)} \xrightarrow{\text{ZADAĆA}} 0$$

GLOBALNA SLIKA: FIX $\delta > 0$

∂U KOMPAKTAN $\Rightarrow$ $\exists\ x_i^0 \in \partial U \quad i=1,\dots,N$

$\exists\ V_i = U \cap B(x_i^0, \frac{r_i}{2})$

$\exists\ v_i \in C^\infty(\overline{V_i})$

T.D. $\partial U \subset \bigcup_{i=1}^{N} B(x_i^0, \frac{r_i}{2})$

$\|v_i - u\|_{W^{k,p}(V_i)} \leq \delta$

DODAM $V_0 \Subset U$ K.U. T.D.

$$U \subset \bigcup_{i=0}^{N} V_i$$

& $v_0 \in C^\infty(\overline{V_0})$ TM 1 ... $\|v_0 - u\|_{W^{k,p}(V_0)} \leq \delta$ (*)

NEKA JE ζ_i, $i=0,\dots,N$ P.1 KOJA ODGOVARA V_i, $i=0,\dots,N$

DEF:

$$v := \sum_{i=0}^{N} \zeta_i v_i \Rightarrow v \in C^\infty(\overline{U})$$

KAKO JE $u = \sum_{i=0}^{N} \zeta_i u$

$$\Rightarrow \|D^\alpha v - D^\alpha u\|_{L^p(U)} \leq \sum_{i=0}^{N} \|D^\alpha(\zeta_i v_i) - D^\alpha(\zeta_i u)\|_{L^p(V_i)}$$

LEIBNIZOVA FORMULA

$$\leq \sum_{i=0}^{N} \Big\| \sum_{\beta \leq \alpha} \underbrace{\binom{\alpha}{\beta} D^\beta \zeta_i}_{\text{OGRANIČENO U } L^\infty} D^\alpha(v_i - u) \Big\|_{L^p(V_i)}$$

$$\leq C \sum_{i=1}^{N} \|D^\alpha(v_i - u)\|_{L^p(V_i)}$$

$$\leq C \sum_{i=1}^{N} \|v_i - u\|_{W^{k,p}(V_i)} \leq CN\delta$$

(*)

5.4. PROŠIRENJE FUNKCIJA U $W^{k,p}$

ŽELIMO: $u \in W^{1,p}(U) \longrightarrow \tilde{u} \in W^{1,p}(\mathbb{R}^n)$

$$\tilde{u}\big|_U = u$$

NAIVNO: $\hat{u}(x) = \begin{cases} u(x), & x \in U \\ 0, & \text{INAČE} \end{cases}$

JE LI U $W^{1,p}(\mathbb{R}^n)$? MOGUĆ PREKID FJE

IMA LI UOPĆE SLABU DERIVACIJU

TM 1 (TEOREM PROŠIRENJA)

NEKA JE $p \in [1, +\infty]$, U OGRANIČEN, ∂U KLASE C^1.

V, $U \subset\subset V$ K.U. (OTVOREN) OGRANIČEN. TADA

$\exists$ OGRANIČEN LINEARAN OPERATOR

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

T.D. $\forall\, u \in W^{1,p}(U)$

(i) $Eu = u$ s.s. u U

(ii) $\operatorname{supp} Eu \subset V$

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)}$ (OGRANIČENOST)

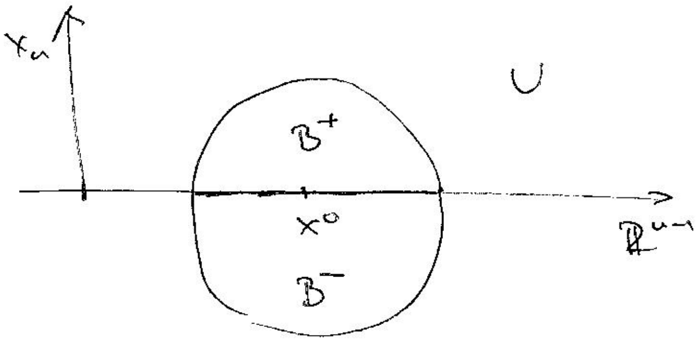
C OVISI SAMO O p, U i V.

DEF: AKO VRIJEDI (i) – (iii) Eu JE PROŠIRENJE OD u NA $\mathbb{R}^n$.

PRETP 0: $u \in C^\infty(\overline{U})$

DOK

PRETP 1: $x^0 \in \partial U$, ∂U RAVAN U OKOLINI x^0 OPISAN RAVNINOM $x_n = 0$



$B^+ := B \cap \{x_n \geq 0\} \subseteq \overline{U}$

$B^- := B \cap \{x_n \leq 0\} \subseteq \mathbb{R}^n \setminus U$

DEF:

$$\overline{u}(x) := \begin{cases} u(x) =: u^+(x) & , x \in B^+ \\ -3u(x_1, \dots, x_{n-1}, -x_n) + 4u(x_1, \dots, x_{n-1}, -\frac{x_n}{2}) =: u^-(x) & , x \in B^- \end{cases}$$

REFLEKSIJA OD u VIŠEG REDA

1) DEF DOBRA: $x \in B^+ \cap B^- \neq \emptyset$

$$u(x_1, \dots, x_{n-1}, 0)$$
$$-3u(x_1, \dots, x_{n-1}, 0) + 4u(x_1, \dots, x_{n-1}, 0) = u(x_1, \dots, x_{n-1}, 0)$$

$$\Rightarrow \overline{u} \in C(B)$$

2) $$\frac{\partial u^-}{\partial x_n}(x) = 3\frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, -x_n) - 2\frac{\partial u}{\partial x_n}\left(x_1, \dots, x_{n-1}, -\frac{x_n}{2}\right)$$

$$\Rightarrow \underline{\frac{\partial u^-}{\partial x_n}(x_1, \dots, x_{n-1}, 0)} = \frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, 0) = \underline{\frac{\partial u^+}{\partial x_n}(x_1, \dots, x_{n-1}, 0)}$$

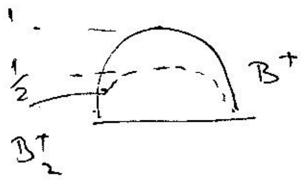
$$\frac{\partial u^-}{\partial x_i}(x) = -3\frac{\partial u}{\partial x_i}(x_1, \dots, x_{n-1}, -x_n) + 4\frac{\partial u}{\partial x_i}\left(x_1, \dots, x_{n-1}, -\frac{x_n}{2}\right)$$

$$\Rightarrow \underline{\frac{\partial u^-}{\partial x_i}(x_1, \dots, x_{n-1}, 0)} = \frac{\partial u}{\partial x_i}(x_1, \dots, x_{n-1}, 0) = \underline{\frac{\partial u^+}{\partial x_i}(x_1, \dots, x_{n-1}, 0)}$$

$\Rightarrow$ I SVE PARCIJALNE DERIVACIJE NEPREKIDNE NA B

$$\Rightarrow \overline{u} \in C^1(B)$$

DODATNO SLIJEDI:



$$\|\bar{u}\|_{W^{1,p}(B)} \leq C \|u\|_{W^{1,p}(B^+)}$$

ZA C KOJA NE OVISI O u.

SKICA: ZA L^p

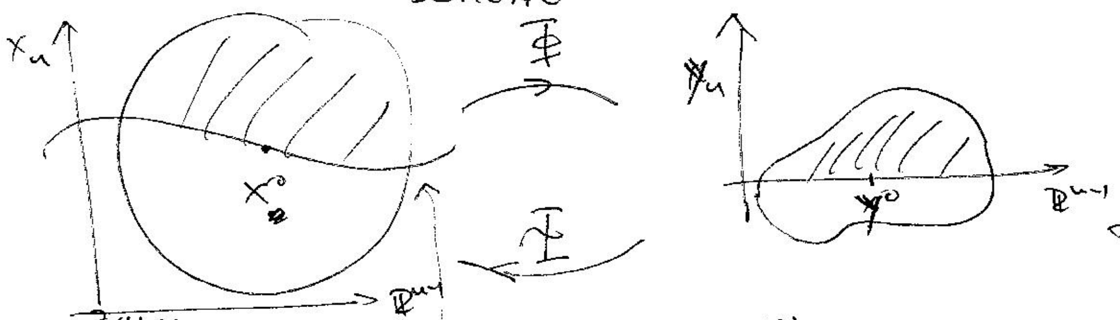
$$\|\bar{u}\|_{L^p(B^-)} = \|-3u\circ r_1 + 4u\circ r_2\|_{L^p(B^-)} \leq 3\|u\circ r_1\|_{L^p(B^-)} + 4\|u\circ r_2\|_{L^p(B^-)}$$

$$= 3\|u\|_{L^p(B^+)} + 4\|u\|_{L^p(B^+_2)}$$

$$\leq 3\|u\|_{L^p(B^+)} + 4\|u\|_{L^p(B^+)} = 7\|u\|_{L^p(B^+)}$$

PRETP2: ∂U NIJE RAVAN U OKOLINI x^0

IZRAVNAMO DOMENU



ZNAMO ZA RUB: $\exists r>0$, $\gamma: \mathbb{R}^{n-1} \to \mathbb{R}$ KLASE C^1 T.D.

$$U \cap B(x^0, r) = \{x \in B(x^0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}$$

Γ_γ

DEF: $\Phi^i(x) := x_i$

$$\Phi^n(x) := x_n - \gamma(x_1, \dots, x_{n-1})$$

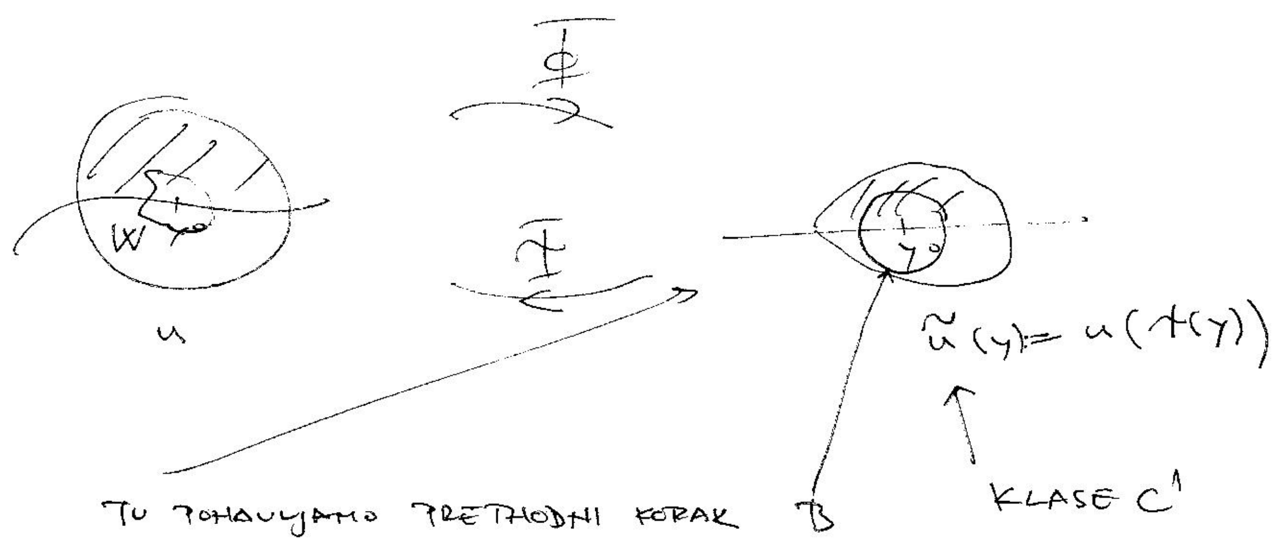
$$\Phi(x) = (\Phi^1(x), \dots, \Phi^n(x))$$ PRESLIKAVA ∂U

INVERZNA FUNKCIJA: RJEŠAVAMO $y = \Phi(x)$

$$\Rightarrow \left.\begin{array}{l} y_i = x_i \\ y_n = x_n - \gamma(x_1, \dots, x_{n-1}) \end{array}\right\} \Rightarrow \begin{array}{l} x_i = y_i \\ x_n = y_n + \gamma(y_1, \dots, y_{n-1}) \end{array}$$

$$\Phi(y) = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n + \gamma(y_1, \dots, y_{n-1}) \end{bmatrix}$$
$$\| \quad \Psi^{-1}(y)$$

$$\det \nabla \Phi = \det I = \det \nabla \Psi = 1 \qquad \text{(ČUVA VOLUMEN)}$$



$$\tilde{u} \xrightarrow{\text{PROŠIRIMO}} \bar{\tilde{u}} \in C^1(B)$$

$$\| \bar{\tilde{u}} \|_{W^{1,p}(B)} \leq C \| \tilde{u} \|_{W^{1,p}(B^+)}$$

<u>DEF!</u> $W := \Psi(B)$

DOBIVAMO PROŠIRENJE ~~OD~~ u NA W : $\bar{u} := \bar{\tilde{u}} \circ \Phi$ |

$$\| \bar{u} \|_{W^{1,p}(W)} \leq C \| u \|_{W^{1,p}(U)}$$

<u>SKICA:</u>

$$\| \bar{u} \|_{L^p(W)} = \| \bar{\tilde{u}} \circ \Phi \|_{L^p(W)} = \left(\int_W | \bar{\tilde{u}} \circ \Phi |^p dx \right)^{1/p} = \left| \begin{matrix} \text{SUPST} \\ y = \Phi(x) \\ \det \nabla \Phi = 1 \end{matrix} \right|$$

$$= \left(\int_B | \bar{\tilde{u}} |^p \right)^{1/p} = \| \bar{\tilde{u}} \|_{L^p(B)} \leq C \| \tilde{u} \|_{L^p(B^+)} = \left| x = \Psi(y) \right|$$

$$\leq C \| u \|_{L^p(W^+)} \leq C \| u \|_{L^p(U)}$$

DERIVACIJE SLIČNO

ČITAV RUB: ∂U KOMPAKTAN

$$\Rightarrow \exists\, x_i^0 \in \partial U,\ i=1,\dots,N, \quad W_i \text{ OTVORENE OKOLINE } x_i^0 \text{ TD. } \partial U \subset \bigcup_{i=1}^{N} W_i$$

$\bar{u}_i$: PROŠIRENJE u NA W_i (IZVAN $\bigcirc$ MOGU STAVITI)

DODAMO $W_0 \subset\subset U$ K.U. T.D. $U \subset \bigcup_{i=0}^{N} W_i$

$(\zeta_i)_{i=0,\dots,N}$ P. 1

DEF:
$$\bar{u} := \sum_{i=0}^{N} \zeta_i \bar{u}_i \qquad (\bar{u}_0 = u)$$

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} = \Big\| \sum_{i=0}^{N} \zeta_i \bar{u}_i \Big\|_{W^{1,p}(\mathbb{R}^n)}$$
$$\le \sum_{i=0}^{N} \|\zeta_i \bar{u}_i\|_{W^{1,p}(\mathbb{R}^n)}$$
$$= \sum_{i=0}^{N} \|\zeta_i \bar{u}_i\|_{W^{1,p}(W_i)}$$
$$\le C \sum_{i=0}^{N} \|\bar{u}_i\|_{W^{1,p}(W_i)}$$
$$\le C \sum_{i=0}^{N} \|u\|_{W^{1,p}(W_i^+)} \le C \|u\|_{W^{1,p}(U)}$$

NISU ISTE

C NE OVISI O u!

AKO BIRAM KUGLE KOJE POKRIVAJU RUB DOVOLJNO MALENOG RADIJUSA, UVIJEK SE MOGU UGURATI U $V \supset\supset U$.

OČITO JE $\bar{u}|_U = u$.

DEF: $Eu := \bar{u}$

OČITO LINEARNO PRESLIKAVANJE! OGRANIČEN (NEPREKIDAN)

PREPP 3: $u \in W^{1,p}(U)$

$$\exists\ u_m \in C^\infty(\overline{U}) \quad \text{T.D.} \quad u_m \longrightarrow u \quad \text{u } W^{1,p}(U).$$

RAČUNAM:

$$\|Eu_m - Eu_\ell\|_{W^{1,p}(\mathbb{R}^n)} \overset{\text{OGR.}}{\leq} C\,\|u_m - u_\ell\|_{W^{1,p}(U)}$$

$$\Rightarrow \left(Eu_m\right)_{m=1,\dots,\infty} \quad \text{JE C-NIZ u } W^{1,p}(U)$$

BANACHOV $\Rightarrow$ $Eu_m \longrightarrow \overline{u}$

DEF: $\underline{Eu := \overline{u}}$

IZ:

$$\|Eu_m\|_{W^{1,p}(\mathbb{R}^n)} \leq C\,\|u_m\|_{W^{1,p}(U)}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \qquad \|u\|_{W^{1,p}(U)}$$

$$\|$$

$$\|\overline{E}u\|_{W^{1,p}(\mathbb{R}^n)}$$

NAP: DEF NE OVISI O IZBORU NIZA u_m!

$u_m \to u$ $\qquad v_m \to u$

$Eu_m \to \overline{u}$ $\qquad Ev_m \to \overline{v}$ $\qquad$ TO JE ZAPRAVO PROŠIRENJE PO GUSTOĆI I NEPREKIDNOSTI

~~$\|\overline{u}-\overline{v}\| \leq \|Eu \dots$~~

$$\|\overline{u}-\overline{v}\| \leq \underbrace{\|\overline{v} - Ev_m\|}_{\downarrow\, 0} + \underbrace{\|E(v_m-u_m)\|}_{\leq C\|v_m-u_m\| \,\downarrow\, 0} + \underbrace{\|Eu_m - \overline{u}\|}_{\downarrow\, 0} \longrightarrow 0$$

$\Rightarrow \overline{u} = \overline{v}$

NAP: NEKA JE ∂U KLASE C^2.

- MOŽEMO PONOVITI KONSTRUKCIJU
- OPERATOR PROŠIRENJA NA KUGLU NIJE U $C^2(B)$, ALI JEST U $W^{2,p}(B)$

& $$\|\bar{u}\|_{W^{2,p}(B)} \leq C \|u\|_{W^{2,p}(B^+)}$$

- SLIJEDI $\exists$ E L.O.

$$\|Eu\|_{W^{2,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{2,p}(U)}$$

- KONSTRUKCIJA $\bar{u}$ NA B NE "RADI" ZA $W^{k,p}(U)$, $k \geq 2$

TREBA REFLEKSIJA VIŠEG REDA