5 PROSTORI SOBOLEVA

- POKAZUJE SE PRAVI OKVIR ZA ANALIZU PDJ
- POSEBNO KROZ PRIMJENU FUNKCIONALNE ANALIZE
- OPĆA IDEJA: 1) ZAPISATI RUBNU (INICIJALNU) ZADAĆU ~~PO~~ POMOĆU

$$A : X \to Y$$

A - OPERATOR , X, Y - PROSTORI FUNKCIJA

2) PRIMJENITI OPĆE REZULTATE FUNKCIONALNE ANALIZE NA A.

- DELIKATNO : IZBOR X, Y, A
- PROSTORI SOBOLEVA ⟶ IZBOR ZA X, Y
- MNOGO KLASIČNIJI PRISTUP: HÖLDEROVI PROSTORI

5.1. HÖLDEROVI PROSTORI

$\mathbb{R}^n \supset U$ OTVOREN, $\gamma \in \langle 0,1]$

DEF: $u : U \to \mathbb{R}$ JE LIPSCHITZ NEPREKIDNA (LIPSCHITZOVA) FUNKCIJA AKO:

$$\exists C > 0 \quad \forall x,y \in U \qquad |u(x) - u(y)| \le C|x-y|$$

NAP: L. $\Rightarrow$ NEPREKIDNA & UNIFORMNO NEPREKIDNA

DEF: $u : U \to \mathbb{R}$ JE HÖLDER NEPREKIDNA S EKSPONENTOM γ AKO

$$\exists C > 0 \quad \forall x,y \in U \qquad |u(x) - u(y)| \le C|x-y|^{\gamma}$$

DEF: (i) ZA $u : U \to \mathbb{R}$ OGRANIČENA I NEPREKIDNA DEF:

$$\|u\|_{C(\bar{U})} := \sup_{x \in U} |u(x)|$$

(ii) γ-HÖLDER POLUNORMA:

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|}$$

(iii) γ-HÖLDER NORMA:

$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$$

(iv) k,γ-HÖLDER NORMA:

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \le k} \|D^{\alpha} u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^{\alpha} u]_{C^{0,\gamma}(\bar{U})}$$

DEF: HÖLDEROV PROSTOR

$$C^{k,\gamma}(\bar{U}) = \left\{ u \in C^k(\bar{U}) : \|u\|_{C^{k,\gamma}(\bar{U})} < +\infty \right\}$$

- PROSTOR C^k FUNKCIJA ČIJE SU PARCIJALNE DERIVACIJE HÖLDEROVE S EKSPONENTOM γ.

TEOREM 1: $C^{k,\gamma}(\bar{U})$ JE BANACHOV PROSTOR

DOKAZ: ZA ZADAĆU //

TREBA POKAZATI: 1) $\| \ \|_{C^{k,\gamma}(\bar{U})}$ JE NORMA

2) POTPUNOST PROSTORA (SVAKI C. NIZ KONV.)

NAP: ZA ANALIZU PDJ U HÖLDEROVIM PROSTORIMA TREBAJU NAM OCJENE U ODGOVARAJUĆOJ NORMI I GLATKOĆA.

TO OBIČNO NEMAMO.

TREBAMO VEĆE PROSTORE (ALI NE PREVELIKE)

5.2 PROSTORI SOBOLEVA

5.2.1. SLABA DERIVACIJA

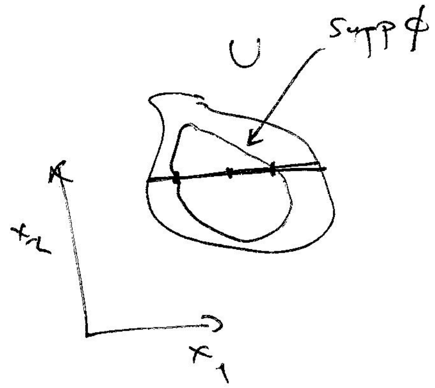
$C_c^\infty(U)$ – ∞ DFB. FJE S KOMPAKTNIM NOSAČEM U U

$\phi : U \to \mathbb{R}$ – TEST FUNKCIJA

MOTIVACIJA: NEKA $u \in C^1(U)$, $\phi \in C_c^\infty(U)$

P.I. PO x_i:

$$\int_U u \phi_{x_i} \, dx = - \int_U u_{x_i} \phi \, dx$$



I OPĆENITIJE: $u \in C^k(U)$, α - MULTINDEKS

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U D^\alpha u \, \phi \, dx, \quad \forall \phi \in C_c^\infty(U)$$

IMA SMISLA VEĆ ZA $u \in L^1(U)$ $\qquad \exists\, v \in L^1(U)$

DEF: NEKA SU $u, v \in L^1_{loc}(U)$, α - MULTINDEKS.
v JE α-SLABA DERIVACIJA OD u $\left(v = D^\alpha u\right)$ AKO

$$(*) \quad \int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx, \quad \forall \phi \in C_c^\infty(U)$$

NAP: ~~ZA~~ DANI u AKO $\exists\, v$ T.D $(*)$ VRIJEDI ~~VRIJEDI~~
TADA JE $v = D^\alpha u$ U SLABOM SMISLU

v JE α-SLABA DERIVACIJA OD u

NAP: ZAPRAVO JE STANDARDNO PRVO GLEDATI 1. DERIVACIJE

AKO ZA $u \in L^1_{loc}(U)$ $\exists$ $v \in L^1_{loc}(U)$ T.D

$$\int_U u\, \phi_{x_i} dx = -\int_U v\, \phi\, dx \quad , \forall \phi \in C_c^\infty(U)$$

$\Rightarrow$ 1) u IMA x_i SLABU DERIVACIJU

2) $u_{x_i} = v$ U SLABOM SMISLU

LEMA (JEDINSTVENOST SLABE DERIVACIJE): AKO POSTOJI SLABA α-PARCIJALNA DERIVACIJA JEDINSTVENA JE DO NA SKUP MJERE 0.

DOK: NEKA SU $v, \tilde{v} \in L^1_{loc}(U)$ α-SLABE DERIV. OD u:

$$\int_U u\, D^\alpha \phi\, dx = \underbrace{(-1)^{|\alpha|} \int_U v\, \phi\, dx = (-1)^{|\alpha|} \int_U \tilde{v}\, \phi\, dx}\ , \quad \forall \phi \in C_c^\infty(U)$$

$$\Rightarrow \int_U (v - \tilde{v})\, \phi\, dx = 0 \ , \quad \forall \phi \in C_c^\infty(U)$$

$$\Rightarrow v = \tilde{v} \quad \text{s.s.}$$

NAP: NEKA JE $u \in C^1(U)$, $v \in C(U)$, ~~$v = D_{x_i}$ SLABA DER.~~

v x_i SLABA DERIVACIJA OD u:

$$\forall \phi \in C_c^\infty(U) \int_U u\, \phi_{x_i}\, dx = -\int_U v\, \phi\, dx$$

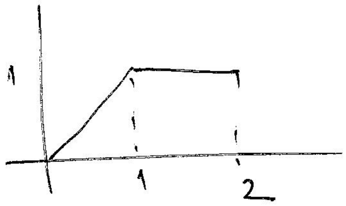
$\parallel$ P.I

$$-\int_U u_{x_i} \phi\, dx \quad \Rightarrow \quad \int_U (\overbrace{u_{x_i} - v}^{\text{NEPREKIDNA}})\, \phi\, dx = 0 \ , \quad \phi \in C_c^\infty(U)$$

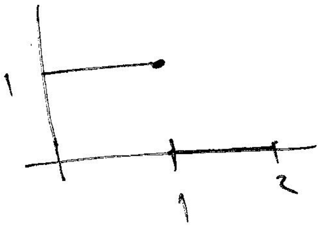
$\Rightarrow$ $v = u_{x_i}$ KLASIČNO!

PR 1: $n=1$, $U=\langle 0,2\rangle$

$$u(x)=\begin{cases} x, & x\in\langle 0,1] \\ 1, & x\in[1,2\rangle \end{cases}$$



$$v(x)=\begin{cases} 1, & x\in\langle 0,1] \\ 0, & x\in\langle 1,2] \end{cases}$$



TV: $v=u'$ U SLABOM SMISLU

DOK: TREBA POKAZATI:

$$\forall \phi\in C_c^\infty(U) \qquad \int_0^2 u\phi'\,dx = -\int_0^2 v\phi\,dx$$

RAČUNAMO:

$$\int_0^2 u\phi'dx = \int_0^1 x\phi'\,dx + \int_1^2 \phi' dx = -\int_0^1 \phi\,dx + x\phi(x)\Big|_0^1 + \phi(2)-\phi(1)$$

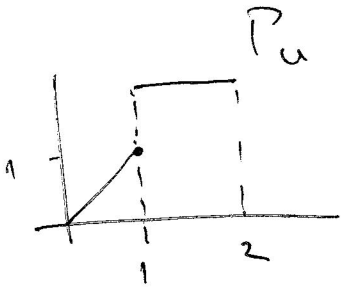
$$= -\int_0^1 \phi\,dx + \cancel{\phi(1)} - \underbrace{0\cdot\phi(0)}_{=0} + \underbrace{\phi(2)}_{=0} - \cancel{\phi(1)}$$

$$= -\int_0^1 \phi\,dx = -\int_0^2 v\phi\,dx$$

NAP: KAKO JE v DEF U POJEDINOJ TOČKI (SPEC. U 1) NIJE BITNO ZA INTEGRAL

PR 2: $n=1, \quad U=(0,2)$

$$u(x) = \begin{cases} x, & x \in \langle 0,1] \\ 2, & x \in \langle 1,2] \end{cases}$$



AKO POSTOJI SLABA DERIVACIJA OD u (ZOVEM JE v) TADA

$$\forall \phi \in C_c^\infty(U) \quad -\int_0^2 v\phi\,dx = \int_0^2 u\phi'\,dx$$

RAČUNAM D.S.

$$\int_0^2 u\phi'\,dx = \int_0^1 x\phi'\,dx + 2\int_1^2 \phi'\,dx = -\int_0^1 \phi\,dx + x\phi\Big|_0^1 + 2(\phi(2)-\phi(1))$$

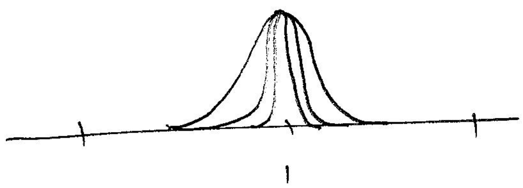
$$= -\int_0^1 \phi\,dx + \phi(1) - 2\phi(1) = -\int_0^1 \phi\,dx - \phi(1)$$

AKO v POSTOJI:

$$\forall \phi \in C_c^\infty(U) \quad -\int_0^2 v\phi\,dx + \int_0^1 \phi\,dx = -\phi(1)$$

ODABEREM $(\phi_n)_n \subseteq C_c^\infty(U)$:

$$0 \le \phi_n \le 1, \quad \phi_n(1)=1, \quad \phi_n(x) \to 0, \quad x \ne 1$$



$$\forall n \in \mathbb{N} \quad \underset{1}{\underset{\|}{\phi_n(1)}} = \underset{0}{\underset{\downarrow}{\int_0^2 v\phi_n\,dx}} - \underset{0}{\underset{\downarrow}{\int_0^1 \phi_n\,dx}}$$

$\Rightarrow\Leftarrow$

NE POSTOJI SLABA DERIVACIJA!

5.2.2. DEFINICIJA PROSTORA SOBOLEVA

$p \in [1, +\infty]$, $k \in \mathbb{N} \cup \{0\}$

DEF: (PROSTORI SOBOLEVA)

$$W^{k,p}(U) = \{ u \in L^p(U) : D^\alpha u \in L^p(U), \; |\alpha| \le k \}$$

SLABA DERIVACIJA

NAP: $p=2$: $H^k(U) = W^{k,2}(U)$, $k \in \mathbb{N} \cup \{0\}$

HILBERTOV !

$$H^0(U) = L^2(U)$$

NAP: ŠIRI OKVIR SU LEBESGUEOVI PROSTORI

FJE IDENTIFICIRANE DO NA SKUP MJERE 0

DEF: ZA $u \in W^{k,p}(U)$ DEF:

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p dx \right)^{1/p}, & p \in [1, \infty\rangle \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup}_U |D^\alpha u|, & p = \infty \end{cases}$$

NAP: TO JE NORMA. (DOK: KASNIJE)

NAP: $(u_m) \subseteq W^{k,p}(U)$, $u \in W^{k,p}(U)$

$$u_m \to u \text{ u } W^{k,p}(U) \iff \lim_{m \to +\infty} \|u_m - u\|_{W^{k,p}(U)} = 0$$

$$u_m \to u \text{ u } W^{k,p}_{loc}(U) \iff u_m \to u \text{ u } W^{k,p}(V) \quad \forall V \subset\subset U$$

komp.

NAP: $k=1$, $\|u\|_{W^{1,p}(U)} = \left(\|u\|^p_{L^p(U)} + \|\nabla u\|^p_{L^p(U)} \right)^{1/p}$

$$\|u\|_{W^{1,\infty}(U)} = \|u\|_{L^\infty(U)} + \sum_{i=1}^n \|u_{x_i}\|_{L^\infty(U)}$$

DEF: $W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{W^{k,p}(U)}$

$\Rightarrow$ $u \in W_0^{k,p}(U)$ ~~AKO~~ $\exists$ $u_m \in C_c^\infty(U)$ T.D $u_m \to u$ u $W^{k,p}(U)$

NAP: FJE IZ $W^{k,p}(U)$ T.D "$D^\alpha u = 0$ ~~NA~~ ∂U", $|\alpha| \le k-1$

NAP: $H_0^k(U) = W_0^{k,2}(U)$

PR: $U = B(0,1) \subseteq \mathbb{R}^n$

$$u(x) = |x|^{-\alpha}, \quad x \in U, \ x \neq 0$$

$u \in W^{1,p}(U)$?

u JE GLATKA I OGRANIČENA IZVAN $B(0,\varepsilon)$

$u_{x_i}(x) = -\alpha \dfrac{x_i}{|x|^{\alpha+2}}$ - GLATKO IZVAN $B(0,\varepsilon)$

$\Rightarrow$ $|Du(x)| = \dfrac{|\alpha|}{|x|^{\alpha+1}}$ $\leftarrow$ OGRANIČENO IZVAN $B(0,\varepsilon)$

$u \in L^p(U)$:

$$\infty > \int_{B(0,1)} u(x)^p \, dx = \int_{B(0,1)} \frac{1}{|x|^{\alpha p}} \, dx \approx \int_0^1 \frac{1}{r^{\alpha p}} r^{n-1} \, dr$$

$$= \int_0^1 r^{n-1-\alpha p} \, dr = \frac{r^{n-\alpha p}}{n-\alpha p} \Big|_0^1$$

OGR ZA $n - \alpha p > 0 \iff \boxed{\alpha < \frac{n}{p}}$

KAD POSTOJE SLABE DERIVACIJE?

$\phi \in C_c^\infty(U)$, $\varepsilon > 0$

I-TA KOMP. J. NORMALE

$$\int_{U \setminus B(0,\varepsilon)} u \phi_{x_i} \, dx = - \int_{U \setminus B(0,\varepsilon)} u_{x_i} \phi \, dx + \int_{\partial B(0,\varepsilon)} u \phi \nu^i \, dS$$

KADA $\alpha < n$ $u \in L^1(U)$

$$\int_U u \phi_{x_i}$$

KADA $\alpha + 1 < n$ $\Downarrow$ $|Du| \in L^1(U)$

$$\int_U u_{x_i} \phi \, dx$$

? 0

$$\left| \int_{\partial B(0,\varepsilon)} u \phi \nu^i \, dS \right| \leq \|\phi\|_{L^\infty} \int_{\partial B(0,\varepsilon)} u(x) \, dS = \|\phi\|_{L^\infty} \int_{\partial B(0,\varepsilon)} \varepsilon^{-\alpha} \, dS$$

$$\leq C \varepsilon^{n-1-\alpha} \longrightarrow 0$$

$n-1-\alpha > 0$

$\Rightarrow$ ZA $\boxed{\alpha < n-1}$ $\exists$ SLABA DERIVACIJA TJ.

$$\int_U u \phi_{x_i} \, dx = - \int_U u_{x_i} \phi \, dx \quad \forall \phi \in C_c^\infty(U)$$

$$|Du(x)| \in L^p(U) \iff \boxed{\alpha + 1 < \frac{n}{p}}$$

$$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}}$$

3 UVJETA

NAJSTROŽI!

PR. NEKA JE $\{r_k : k\in\mathbb{N}\} \subseteq \underset{U}{\underset{\parallel}{B(0,1)}}$ GUST

$$u(x) := \sum_{k\in\mathbb{N}} \frac{1}{2^k} |x - r_k|^{-\alpha}, \quad u\in U$$

$u\in W^{1,p}(U) \iff \alpha < \frac{n-p}{p}$

ZA $0<\alpha<\frac{n-p}{p}$ $\quad u\in W^{1,p}(U)$ & u NEOGRANIČENA NA SVAKOM OTVORENOM $\subseteq U$

$n=1$ $\quad 0<\alpha<\frac{1-p}{p}, \quad p\geq 1 \implies$ TAKVIH NEMA!

$n=2$ $\quad 0<\alpha<\frac{2-p}{p}, \quad p\geq 1$ IMA IH! (MOŽDA)

5.2.3 OSNOVNA SVOJSTVA

TM (SVOJSTVA SLABE DERIVACIJE): NEKA SU $u,v\in W^{k,p}(U)$, $|\alpha|\leq k$

TADA

(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$, $\quad D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$

$\forall\ \alpha,\beta, \quad |\alpha|+|\beta|\leq k$

(ii) $\forall\ \lambda,\mu\in\mathbb{R}, \quad \lambda u+\mu v\in W^{k,p}(U)$

$$D^\alpha(\lambda u+\mu v) = \lambda D^\alpha u + \mu D^\alpha v, \quad |\alpha|\leq k$$

(iii) $\forall\ V\subset U$ OTVOREN $\implies u\in W^{k,p}(V)$

(iv) AKO JE $\zeta\in C_c^\infty(U)$, TADA JE $\zeta u\in W^{k,p}(U)$ I

$$D^\alpha(\zeta u) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta\zeta\, D^{\alpha-\beta} u \qquad \text{(LEIBNIZOVA F.)}$$

DOK: (i) $\phi \in C_c^\infty(U) \Rightarrow D^\beta \phi \in C_c^\infty(U)$

$$|\alpha|+|\beta| \le k \qquad \int_U D^\alpha u \, D^\beta \phi \, dx \overset{\text{DEF. S.D.}}{\underset{\downarrow}{=}} (-1)^{|\alpha|} \int_U u \, D^\alpha (D^\beta \phi)\, dx = (-1)^{|\alpha|} \int_U u \, D^{\alpha+\beta} \phi \, dx$$

$$\overset{\text{DEF. S.D.}}{\underset{\downarrow}{=}} (-1)^{|\alpha|} (-1)^{|\alpha|+|\beta|} \int_U D^{\alpha+\beta} u \, \phi \, dx$$

$$= (-1)^{|\beta|} \int_U D^{\alpha+\beta} u \, \phi \, dx$$

$$\Rightarrow \quad D^\beta (D^\alpha u) = D^{\alpha+\beta} u \qquad \text{(SLIČNO I OBRATNO)}$$

(ii) $|\alpha| \le k \quad \phi \in C_c^\infty(U)$

$$\int_U (\lambda u + \mu v) D^\alpha \phi \, dx = \int_U \lambda u D^\alpha \phi + \mu v D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U \underbrace{(\lambda D^\alpha u + \mu D^\alpha v)}_{} \phi \, dx$$

$$\Rightarrow \exists \; \& \; D^\alpha(\lambda u + \mu v) = \; "$$

$$\Rightarrow \quad \lambda u + \mu v \in W^{k,p}(U)$$

(iii) $u \in W^{k,p}(U)$, $V \subset U$ ot.

$\Downarrow$

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U D^\alpha u \, \phi \, dx \qquad \forall \phi \in C_c^\infty(U)$$

$\phi \in C_c^\infty(V)$ PROŠIRIM S 0 NA U $\Rightarrow$ $\phi \in C_c^\infty(U)$

ISTO NA V!

(iv) INDUKCIJOM PO $|\alpha|$. BAZA $|\alpha| = 1$. $\phi \in C_c^\infty(U)$

$$\int_U \zeta u D^\alpha \phi \, dx = \int_U u \left(D^\alpha(\zeta \phi) - (D^\alpha \zeta) \phi \right) dx$$

$$= - \int_U (D^\alpha u \, \zeta + u D^\alpha \zeta) \phi \, dx \quad \dots$$

TM 2 $\forall k \in \mathbb{N}$, $p \in [1, +\infty]$ $W^{k,p}(U)$ JE BANACHOV PROSTOR.

DOK: 1. SVOJSNA NORME

i) $\|u\| = 0 \iff u = 0$

ii) $\|\lambda u\| = |\lambda| \|u\|$

iii) $\|u+v\| \leq \|u\| + \|v\|$

$p \in [1, +\infty)$ $$\|u+v\|_{W^{k,p}(U)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u + D^\alpha v\|^p_{L^p(U)} \right)^{1/p}$$

H. TROKUTA ZA L^p $$\leq \left(\sum_{|\alpha| \leq k} \left(\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)} \right)^p \right)^{1/p}$$

MINKOWSKI NA $\mathbb{R}^N$ $$\leq \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(U)} \right)^{1/p} + \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|^p_{L^p(U)} \right)^{1/p}$$

$$= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}$$

$p = \infty$ $$\|u+v\|_{W^{k,\infty}(U)} = \sum_{|\alpha| \leq k} \|D^\alpha(u+v)\|_{L^\infty(U)}$$

$$\leq \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(U)} + \|D^\alpha v\|_{L^\infty(U)}$$

$$= \|u\|_{W^{k,\infty}(U)} + \|v\|_{W^{k,\infty}(U)}$$

2. POTPUNOST:

$(u_m)_m \subseteq W^{k,p}(U)$ C-NIZ

$\forall \varepsilon > 0$ $\exists m_0 \in \mathbb{N}$, $m, k \geq m_0$ $\|u_m - u_k\|_{W^{k,p}(U)} < \varepsilon$

$$\| \quad \sum_{|\alpha| \leq k} \|D^\alpha(u_m - u_k)\|_{L^p(U)}$$

$\Rightarrow$ $(D^\alpha u_m)_m \subseteq L^p(U)$ C-NIZ U $L^p(U)$

L^p POTPUN $\Rightarrow$ $\exists u_\alpha \in L^p(U)$ T.D.

$D^\alpha u_m \longrightarrow u_\alpha$ u $L^p(U)$, $|\alpha| \leq k$

POSEBNO: $u_m \longrightarrow u_{(0,0,\dots,0)} =: u$ u $L^p(U)$

TV: $u \in W^{k,p}(\Omega)$, $u_\alpha = D^\alpha u$ i $u_m \longrightarrow u$ u $W^{k,p}(U)$

DOK: $\phi \in C_c^\infty(U)$, $|\alpha| \le k$

$$\int_U u_m D^\alpha \phi \, dx \longrightarrow \int_U u D^\alpha \phi \, dx$$

$$\| \qquad\qquad\qquad \|$$

JEDINSTVENOST LIMESA

$$(-1)^{|\alpha|} \int_U D^\alpha u_m \phi \, dx \longrightarrow (-1)^{|\alpha|} \int_U u_\alpha \phi \, dx$$

$$\Rightarrow \quad \exists D^\alpha u \quad \& \quad \boxed{u_\alpha = D^\alpha u}$$

$$\Rightarrow \quad D^\alpha u \in L^p(U), \quad |\alpha| \le k$$

$$\Rightarrow \quad \boxed{u \in W^{k,p}(U)}$$

$$D^\alpha u_m \longrightarrow D^\alpha u \quad \text{u } L^p(\Omega), \quad |\alpha| \le k$$

$$\| u - u_m \|_{W^{k,p}(U)} = \left(\sum_{|\alpha| \le k} \| D^\alpha u_m - D^\alpha u \|_{L^p(U)}^p \right)^{1/p} \longrightarrow 0$$