9. NEVARIJACIJSKE TEHNIKE ~~ZA~~ NELINEARNE JEDNADŽBE

9.1. BROWDER - MINTYJEVA METODA (MONOTONOST)

~~ZADAĆA~~:

$$(*)\begin{cases} -\operatorname{div} a(\nabla u) = f & \text{u } U \\ u = 0 & \text{na } \partial U \end{cases}$$

ZADANO:

$f \in L^2(U)$

$a : \mathbb{R}^n \to \mathbb{R}^n$

TRAŽIMO:

$u : \bar{U} \to \mathbb{R}$

NAP: AKO $\exists\ F : \mathbb{R}^n \to \mathbb{R}$ T.D. $a = \nabla F$

PROBLEM MOŽEMO NAPISATI MINIMIZACIJSKI

$$\min_u \frac{1}{2}\int_U F(\nabla u) - \int_{\partial U} f u$$

VARIJACIJSKE TEHNIKE (§ 8) (KONVEKSNOST ...)

OVDJE IDEMO OPĆENITO! ~~[illegible]~~

IPAK, ZA TRENUTAK NEKA JE $a = \nabla F$ & F KONVEKSNA

$$\big(a(p) - a(q)\big)\cdot(p-q) = \big(\nabla F(p) - \nabla F(q)\big)\cdot(p-q)$$

$$= \int_0^1 \frac{d}{dt}\Big(\nabla F(q + t(p-q))\Big)\,dt \cdot (p-q)$$

$$= \int_0^1 D^2F(q+t(p-q))\,(p-q)\cdot(p-q)\,dt \geq 0$$

TO NAM JE MOTIVACIJA ZA DEFINICIJU

DEF: VEKTORSKO POLJE $a : \mathbb{R}^n \to \mathbb{R}^n$ JE MONOTONO AKO JE

$$(a(p) - a(q)) \cdot (p - q) \geq 0, \quad p, q \in \mathbb{R}^n.$$

PRETPOSTAVLJAT ĆEMO:

1) a MONOTONO VEKTORSKO POLJE

2) $|a(p)| \leq C(1 + |p|)$ UVJET RASTA

3) $a(p) \cdot p \geq \alpha |p|^2 - \beta$ KOERCITIVNOST

ZA $C, \alpha > 0, \beta \geq 0$.

TEHNIKA: GALJERKINOVA METODA

$(w_k)_k \subseteq H^1_0(U)$ ONB

MOŽEMO UZETI DA SU SV. VEKTORI OD $-\Delta$ U $H^1_0(U)$

TRAŽIMO APROKSIMACIJU RJEŠENJA $(*)$ U OBLIKU

$$u_m = \sum_{k=1}^{m} d_m^k w_k$$

A KOJA ZADOVOLJAVA:

$$(*_m) \quad \int_U a(\nabla u_m) \cdot \nabla w_k \, dx = \int_U f w_k, \quad k = 1, \dots, m$$

PROBLEM $(*)$ PROJICIRAM NA $L\{w_1, \dots, w_m\}$.

PRVO MORAMO ZAKLJUČITI DA $(*_m)$ IMA RJEŠENJE.

LEMA: NEKA NEPREKIDNA FUNKCIJA $v:\mathbb{R}^n \to \mathbb{R}^n$ ZADOVOLJAVA:
$\exists\, r>0$ T.D.

$$v(x)\cdot x \geq 0, \quad |x|=r.$$

TADA $\exists\, x \in \overline{K(0,r)}$ T.D. $v(x)=0$.

POK: PRETPOSTAVIMO SUPROTNO:
$\exists\, r>0$ T.D. $\forall x,\ |x|=r \quad v(x)\cdot x \geq 0$
& $v(x) \neq 0,\ x \in \overline{\cancel{B}(0,r)}$

DEF: $w(x) := -\dfrac{r}{|v(x)|}v(x), \quad x \in \overline{K(0,r)}$

$$w : \overline{\cancel{B}(0,r)} \to \partial\cancel{B}(0,r)$$

KAKO JE $v(x)\neq 0$ NA $\overline{K(0,r)} \Rightarrow w$ NEPREKIDNA NA $\overline{K(0,r)}$

BROUWEROV TEOREM O FIKSNOJ TOČKI
(NEPR. FUNKCIJA SA ZATVORENE KUGLE U SEBE IMA FIKSNU T.)

$\Rightarrow \exists\, z \in \overline{K(0,r)}$ T.D. $w(z)=z$.

$\Rightarrow z \in \partial K(0,r)$

$$z = w(z) = -\frac{r}{|v(z)|}v(z)$$

$$\Rightarrow v(z)\cdot z = -\cancel{z}\,\frac{|v(z)|}{r}\cdot z < 0 \Rightarrow \Leftarrow$$

TEOREM 1: ZA SVAKI $m \in \mathbb{N}$ $\exists\, u_m = \sum_{k=1}^{m} d_m^k w_k$ T.D.

$$\int_U a(\nabla u_m)\cdot \nabla w_k \, dx = \int_U f w_k \, dx, \quad k=1,\dots,m.$$

DOK: 1. KORAK DEFINIRAMO $v : \mathbb{R}^m \to \mathbb{R}^m$, $v=(v^1,\dots,v^k)$ T.D.

$$v^k(d) = \int_U a\Big(\sum_{j=1}^{m} d_j \nabla w_j\Big)\cdot \nabla w_k - f w_k \, dx, \quad k=1,\dots,m,$$

ZA $d=(d_1,\dots,d_m)\in\mathbb{R}^m$.

$$v(d)\cdot d = \int_U \Big[a\Big(\sum_{j=1}^{m} d_j \nabla w_j\Big)\cdot\Big(\sum_{j=1}^{m} d_j \nabla w_j\Big) - f\Big(\sum_{j=1}^{m} d_j w_j\Big)\Big] dx$$

UVJET KOERCITIVNOSTI

$$\geq \int_U \Big[\alpha \Big|\sum_{j=1}^{m} d_j \nabla w_j\Big|^2 - \beta - f\Big(\sum_{j=1}^{m} d_j w_j\Big)\Big] dx$$

$$= \alpha \sum_{j,k=1}^{m} d_j d_k \underbrace{\int_U \nabla w_j \cdot \nabla w_k \, dx}_{\delta_{jk}} - \beta |U| - \overbrace{\sum_{j=1}^{m} d_j \int_U f w_j \, dx}^{\text{GEN. CAUCHY}}$$

$$\geq \alpha |d|^2 - \beta|U| - \frac{\alpha}{2}\sum_{j=1}^{m} d_j^2 - C \sum_{j=1}^{m} (f, w_j)^2_{L^2(U)}$$

$$= \frac{\alpha}{2}|d|^2 - \beta|U| - C\sum_{j=1}^{m} (f,w_j)^2_{L^2(U)}$$

NEKA JE $u \in H_0^1(U)$ RJEŠENJE $(-\Delta u = f)$

$$\int_U \nabla u \cdot \nabla w_j \, dx = \int_U f w_j, \quad j \in \mathbb{N}$$

$$\Rightarrow \sum_{j=1}^{m} (f,w_j)^2_{L^2(U)} = \sum_{j=1}^{m} (\nabla u, \nabla w_j)^2_{L^2(U)} \overset{\text{PARSEVAL}}{\leq} \|u\|^2_{H_0^1(U)} \overset{\text{KOREKTNOST}}{\leq} C\|f\|^2_{L^2(U)}$$

$$\Rightarrow v(d)\cdot d \geq \frac{\alpha}{2}|d|^2 - C.$$

ZA d, $|d|=r$ i $r>0$ DOVOLJNO VELIK $>0 \overset{\text{LEMA}}{\Longrightarrow} \exists d\in\mathbb{R}^m$ T.D. $\underline{v(d)=0}$

ZA LIMESIRANJE PROJICIRANE ZADAĆE TREBAMO UNIFORMNE OCJENE

TEOREM 2

$\exists\, C>0$, $C(U,a)$ T.D.

$$\|u_m\|_{H^1_0(U)} \leq C\left(1+\|f\|_{L^2(U)}\right), \quad m\in\mathbb{N}.$$

DOK:

$$\int_U a(\nabla u_m)\cdot\nabla w_k\, dx = \int_U f w_k, \quad k=1,\dots,m$$

MNOŽIMO S d_m^k , $\sum_{k=1}^m$

$$\int_U a(\nabla u_m)\cdot\nabla u_m = \int_U f u_m$$

KOERCITIVNOST $\Rightarrow$

GEN. CAUCHY

$$\alpha\int_U|\nabla u_m|^2-\beta \leq \int_U f u_m\, dx \leq \varepsilon\int_U u_m^2\, dx + \frac{1}{4\varepsilon}\int_U f^2\, dx$$

POINCARE

$$\leq \varepsilon C_p\|u_m\|^2_{H^1_0(U)} + \frac{1}{4\varepsilon}\int_U f^2$$

$$\Rightarrow (\alpha-\varepsilon C_p)\|u_m\|^2_{H^1_0(U)} \leq \beta + \frac{1}{4\varepsilon}\int_U f^2$$

ZA ε POVOLJNO MALI $\alpha-\varepsilon C_p>0 \Rightarrow$ OCJENA

ŽELIMO LIMESIRATI

$$\int_U \underline{a(\nabla u_m)} \cdot \nabla w_k = \int_U f w_k \quad , \quad k=1,\dots,m$$

ODOZGO $\exists$ PODNIZ u_{m_j} T.D. $u_{m_j} \rightharpoonup u \quad H^1_0(U)$

$$a(u_{m_j}) \longrightarrow ?$$

NE IDE. OPĆENITO ~~NE~~NELINEARNE FUNKCIJE NISU NEPREKIDNE U ODNOSU NA SLABE KONVERGENCIJE

UZ POMOĆ MONOTONOSTI IPAK STVAR PROĐE

TEOREM 3 (EGZISTENCIJA)

$\exists$ SLABO RJEŠENJE ZADAĆE $(*)$.

DOK: 1. KORAK $\exists$ u_{m_j} i $u \in H^1_0(U)$ T.D.

$$u_{m_j} \rightharpoonup u \quad H^1_0(U)$$

ZBOG OGRANIČENOSTI OD a:

$$\int_U a(\nabla u_{m_j})^2 \leq C \int_U (1+|\nabla u_{m_j}|)^2 \leq C\left(1+\|u_{m_j}\|^2_{H^1_0}\right) \leq C$$

$\Rightarrow$ $a(\nabla u_{m_j})$ OGRANIČEN U $L^2(U)$

$\Rightarrow$ $\exists$ PODNIZ (OPET OZNAČEN S u_{m_j}) i $\xi \in L^2(U)$ T.D.

$$a(\nabla u_{m_j}) \rightharpoonup \xi \quad L^2(U).$$

IZ $(*_m)$ SLIJEDI U LIMESU

$$\int_U \xi \cdot \nabla w_k = \int_U f w_k \quad , \quad k \in \mathbb{N}.$$

PO GUSTOĆI

$$\text{(POM)} \qquad \int_U \xi \cdot \nabla v = \int_U f v \quad , \quad v \in H^1_0(U)$$

3. KORAK MONOTONOST POVLAČI

$$\int_U \left(a(\nabla u_m) - a(\nabla w) \right) \cdot (\nabla u_m - \nabla w)\, dx \geq 0 \quad , \quad m \in \mathbb{N}, w \in H_0^1(U)$$

$$\parallel$$

$$\int_U a(\nabla u_m) \cdot \nabla u_m - \underline{a(\nabla w) \cdot \nabla u_m - a(\nabla u_m) \cdot \nabla w + a(\nabla w) \cdot \nabla w}$$

LIN. U u_m, PROĐE LIMES

$$\parallel \quad \Leftarrow \text{ IZ } (*_m)$$

$$\int_U f u_m$$

$$\Rightarrow \quad \int_U f u_m - a(\nabla w) \cdot \nabla u_m - a(\nabla u_m) \cdot \nabla w + a(\nabla w) \cdot \nabla w \geq 0$$

PUSTIMO $j \to +\infty$

$$\int_U f u - a(\nabla w) \cdot \nabla u - \xi \cdot \nabla w + a(\nabla w) \cdot \nabla w \geq 0$$

ZA ξ SMO DOBILI (POM): $\int_U \xi \cdot \nabla u = \int_U f u$ (ZA $v = u$)

$$\Rightarrow \quad \int_U \xi (\nabla u - \nabla w) - a(\nabla w) \cdot (\nabla u - \nabla w) \geq 0$$

$$\int_U \left(\xi - a(\nabla w) \right) \cdot (\nabla u - \nabla w) \geq 0 \quad , \quad w \in H_0^1(U)$$

4. KORAK $v \in H_0^1(U)$ FIKSAN, $w = u - \lambda v$, $\lambda > 0$

$$\int_U \left(\xi - a(\nabla u - \lambda \nabla w) \right) \cdot (+\lambda) \nabla v \geq 0 , \quad v \in H_0^1(U) \quad \Big| : \lambda$$

$$\int_U \left(\xi - a(\nabla u - \lambda \nabla w) \right) \cdot \nabla v \geq 0 , \quad v \in H_0^1(U)$$

PUSTIM $\lambda \to 0$

$$\int_U \left(\xi - a(\nabla u) \right) \cdot \nabla v \geq 0 , \quad v \in H_0^1(U)$$

$$\int_U (\xi - a(\nabla u)) \cdot \nabla v \geq 0 \quad , v \in H_0^1(U)$$

STAVIM $v = -v$

$$\Rightarrow \quad \int_U (\xi - a(\nabla u)) \cdot \nabla v \leq 0 \quad , v \in H_0^1(U)$$

$$\Rightarrow \quad \int_U (\xi - a(\nabla u)) \cdot \nabla v = 0 \quad , v \in H_0^1(U)$$

$$\Rightarrow \quad \int_U a(\nabla u) \cdot \nabla v = \int_U \xi \cdot \nabla v \overset{(\text{POM})}{=} \int_U f v \quad , v \in H_0^1(U).$$

<u>DEF:</u> a ZADOVOLJAVA UVJET STROGE MONOTONOSTI AKO $\exists \theta > 0$ T.D.

$$(a(p) - a(q)) \cdot (p - q) \geq \theta |p - q|^2 \quad , p, q \in \mathbb{R}^n.$$

<u>TEOREM 4</u> (JEDINSTVENOST) NEKA JE a STROGO MONOTONA. TADA POSTOJI TOČNO JEDNO RJEŠENJE ZADAĆE $(*)$.

<u>DOK:</u> NEKA SU u i $\tilde{u}$ DVA SLABA RJEŠENJA

$$\int_U a(\nabla u) \cdot \nabla v = \int_U a(\nabla \tilde{u}) \cdot \nabla v = \int_U f v \quad , v \in H_0^1(U).$$

$$\Rightarrow \quad \int_U (a(\nabla u) - a(\nabla \tilde{u})) \cdot \nabla v = 0 \quad , v \in H_0^1(U)$$

ZA $v = u - \tilde{u}$

$$\int_U (a(\nabla u) - a(\nabla \tilde{u})) \cdot (\nabla u - \nabla \tilde{u}) = 0$$

$$\geq$$

$$\theta \int_U |\nabla u - \nabla \tilde{u}|^2 \quad \Rightarrow u = \tilde{u} \text{ s.s. } U$$