7.3. HIPERBOLIČKI SUSTAVI 1. REDA

7.3.1. DEFINICIJE

$$u_t + \sum_{j=1}^{n} B_j\, u_{x_j} = f \qquad \text{u } \mathbb{R}^n \times \langle 0, +\infty \rangle$$

$$u = g \qquad \text{NA } \mathbb{R}^n \times \{0\}$$

ZADANO:

$f : \mathbb{R}^n \times [0, +\infty\rangle \to \mathbb{R}^m \qquad f(x,t)$

$g : \mathbb{R}^n \to \mathbb{R}^m \qquad g(x)$

$B_j : \mathbb{R}^n \times [0, +\infty\rangle \to M^{m \times m} \qquad B_j(x,t)$

TRAŽIMO:

$u : \mathbb{R}^n \times [0, +\infty\rangle \to \mathbb{R}^m \qquad u(x,t)$

DEF: SUSTAV JE HIPERBOLIČKI AKO JE

$$B(x,t;y) := \sum_{j=1}^{n} y_j B_j(x,t)$$

DIJAGONALIZABILNA ZA SVE $x, y \in \mathbb{R}^n$, $t \geq 0$. (NAD $\mathbb{R}$)

NAP: $\forall x, y, t$ $B(x,t;y)$ IMA BAZU SVOJSTVENIH VEKTORA

DEF: (i) HIPERBOLIČKI SUSTAV JE SIMETRIČAN AKO SU $B_j(x,t)$ SIMETRIČNI

(ii) SUSTAV JE STROGO HIPERBOLIČKI AKO $\forall x, y \in \mathbb{R}^n$, $y \neq 0$, $t \geq 0$ $B(x,t;y)$ IMA m RAZLIČITIH REALNIH SV. VRIJEDNOSTI.

NAP: $\quad u_{tt} - u_{xx} = 0$

$$u_1 = u_x, \quad u_2 = u_t \quad \Big\} \quad \begin{aligned} u_{1t} - u_{2x} &= 0 \\ u_{2t} - u_{1x} &= 0 \end{aligned}$$

$$u_t + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} u_x = 0$$

NAP: HIPERBOLIČKI ~ RJEŠENJA SU VALOVI

PRETP: $f \equiv 0$, $B_j = \text{const}$, $B(y) = \sum_{j=1}^{n} y_j B_j$

KADA ĆE FUNKCIJE OBLIKA

$$u(x,t) = v(y \cdot x - \sigma t)$$

za $v: \mathbb{R} \to \mathbb{R}^n$ (PROFIL)

BITI RJ. SUSTAVA

$$0 = u_t + \sum_{j=1}^{n} B_j u_{x_j} = -\sigma v'(y \cdot x - \sigma t) + \sum_{j=1}^{n} B_j \, y_j \, v'(y \cdot x - \sigma t)$$

$$= \left(-\sigma I + \sum_{j=1}^{n} y_j B_j\right) v'(y \cdot x - \sigma t)$$

$$\Rightarrow \sigma \in \sigma\left(\sum_{j=1}^{n} y_j B_j\right), \quad v'(y \cdot x - \sigma t) \text{ SV. VEKTOR ZA } \sigma.$$

y FIX ... $v(y \cdot x - \sigma t)$, za $x \perp y$ JE KONSTANTNA

RAVNINSKI VAL

VAL SE ŠIRI U SMJERU y ... BRZINOM $\sigma \frac{y}{|y|}$

HIPERBOLIČNOST: $\forall y$ ... n RAVNINSKIH VALOVA

7.3.2. SIMETRIČNI HIPERBOLIČKI SUSTAVI

$B_j(x,t)$ SIMETRIČNO

$B_j \in C^2(\mathbb{R}^n \times [0,T]; M^{m\times m})$

$$\sup_{\mathbb{R}^n \times [0,T]} \left(|B_j|, |D_{x,t} B_j|, |D^2_{x,t} B_j| \right) < \infty$$

$g \in H^1(\mathbb{R}^n, \mathbb{R}^m)$

$f \in H^1(\mathbb{R}^n \times \langle 0,T \rangle, \mathbb{R}^m)$

$$\begin{cases} u_t + \sum_{j=1}^{n} B_j u_{x_j} = f & \text{u } \mathbb{R}^n \times \langle 0,T] \\ u = g & \text{NA } \mathbb{R}^n \times \{0\} \end{cases}$$

NAP:

$$B_0 u_t + \sum_{j=1}^{n} B_j u_{x_j} = f$$

SU OD INTERESA, ZA B_j SIMETRIČNE, $j=0,\dots,n$.
ZA DALJNJU TEORIJU VAŽNO JE DA JE B_0 POZITIVNO DEFINITNA (UNIFORMNO)

NAP: $$v_{tt} - \cancel{A} \cdot \overset{D^2_x v}{H_{v}} = 0 \quad , \quad A = A^T$$

DEF: $u = (u^1, \dots, u^{n+1}) := (v_{x_1}, v_{x_2}, \dots v_{x_n}, v_t)$

$$B_0 = \begin{bmatrix} a^{11} & \dots & a^{1n} & 0 \\ & & & \vdots \\ a^{1n} & \dots & a^{nn} & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right]$$

$$B_0 u_t = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} \nabla v_t \\ v_{tt} \end{bmatrix} = \begin{bmatrix} A \nabla v_t \\ \hline v_{tt} \end{bmatrix} = \begin{bmatrix} A \nabla(v_t) \\ \hline A \cdot H_v \end{bmatrix}$$

$$= \begin{bmatrix} A \nabla u^{n+1} \\ \hline A \cdot \nabla_x(\nabla_x v) \end{bmatrix}$$

$$B_0 u_t = \begin{bmatrix} A \nabla u^{n+1} \\ \hline A \cdot \nabla \begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a^j u^{n+1}_{x_j} \\ \hline \sum_{j=1}^{n} a^j \cdot \begin{bmatrix} u^1_{x_j} \\ \vdots \\ u^n_{x_j} \end{bmatrix} \end{bmatrix} = \sum_{j=1}^{n} \begin{bmatrix} a^j u^{n+1}_{x_j} \\ \hline a^j \cdot \begin{bmatrix} u^1_{x_j} \\ \vdots \\ u^n_{x_j} \end{bmatrix} \end{bmatrix}$$

$$= \sum_{j=1}^{n} \underbrace{\begin{bmatrix} & a^j \\ a^{jT} & \end{bmatrix}}_{= -B_j} u_{x_j}$$

UNIFORMNA HIPERBOLIČNOG ZA JDBU

$$v_{tt} - A \cdot H_v = 0$$

$\Rightarrow$ A UNIFORMNO POZ. DEF $\Rightarrow$ B_0 UNIF. POZ. DEF.

DEF:

$$B[u,v;t] := \int_{\mathbb{R}^n} \sum_{j=1}^{n} \left(B_j(\cdot,t)\, u_{x_j} \right) \cdot v \, dx$$

$t \in [0,T]$, $u,v \in H^1(\mathbb{R}^n;\mathbb{R}^m)$

DEF: FUNKCIJU

$u \in L^2(0,T; H^1(\mathbb{R}^n,\mathbb{R}^m))$, T.D. $u' \in L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^m))$

NAZIVAMO SLABIM RJEŠENJEM ~~ZA~~ C. ZADAĆU ZA SIMETRIČNI HIPERBOLIČKI SUSTAV

$$u_t + \sum_{j=1}^{n} B_j \cdot u_{x_j} = f \quad \text{u } \mathbb{R}^n \times (0,T]$$

$$u = g \quad \text{NA } \mathbb{R}^n \times \{0\}$$

AKO:

(i) $(u',v) + B[u,v;t] = (f,v)$, $v \in H^1(\mathbb{R}^n;\mathbb{R}^m)$, s.s. $t \in [0,T]$

(ii) $u(0) = g$

NAP: $u, u' \Rightarrow u \in C([0,T]; L^2(\mathbb{R}^n,\mathbb{R}^m))$.

METODA IŠČEZAVAJUĆE VISKOZNOSTI

DODAJEMO ČLAN $\varepsilon \Delta u$ U JEDNADŽBU $\rightsquigarrow$ MIJENJAMO TIP

$$\varepsilon\text{-PROBLEM} \begin{cases} u^\varepsilon_t - \varepsilon \Delta u^\varepsilon + \sum_{j=1}^n B_j u^\varepsilon_{x_j} = f & \text{u } \mathbb{R}^n \times \langle 0, T] \\ u^\varepsilon = g^\varepsilon & \text{u } \mathbb{R}^n \times \{0\} \end{cases}$$

$g^\varepsilon := \eta_\varepsilon * g \quad | \, \varepsilon \in \langle 0, 1]$.

PLAN:
1) POKAZATI DA RJEŠENJE ε-PROBLEMA POSTOJI
2) DOBITI OCJENE, UNIFORMNE PO ε
3) SLIJEDI KVG $u^\varepsilon \to u$
4) u ZADOVOLJAVA HIPERBOLIČKI SUSTAV

TEOREM 1 (EGZISTENCIJA ε-PROBLEMA)

$\forall \varepsilon > 0 \; \exists!$ RJEŠENJE (SLABO) ε-PROBLEMA SA SVOJSTVOM

$$u^\varepsilon \in L^2(0,T; H^2(\mathbb{R}^n; \mathbb{R}^m)), \quad u^{\varepsilon\prime} \in L^2(0,T; H^1(\mathbb{R}^n; \mathbb{R}^m))$$

DOK: KORISTIMO BANACHOV TEOREM O FIKSNOJ TOČKI.
(KONTRAKCIJA $K: X \to X$ IMA JEDINSTVENU FIKSNU TOČKU)

$$X = L^\infty(0,T; H^1(\mathbb{R}^n, \mathbb{R}^m))$$

ZA $v \in X$ DEF: $Kv := u$ GDJE JE u RJ

$$\begin{cases} u_t - \varepsilon \Delta u = f - \sum_{j=1}^n B_j v_{x_j} & \text{u } \mathbb{R}^n \times \langle 0, T] \\ u = g^\varepsilon & \text{NA } \mathbb{R}^n \times \{0\} \end{cases}$$

ZA OVU ZADAĆU DVIJE TEHNIKE:
i) KAO ZA $U \subseteq \mathbb{R}^n$, SKALARNA J.
ii) FUNDAMENTALNO RJEŠENJE

D.S. $f - \sum B_j v_{x_j} \in L^2(0,T; L^2(\mathbb{R}^n, \mathbb{R}^m))$

EGZISTENCIJA & REGULARNOST
$\Rightarrow u \in L^2(0,T; H^2(\mathbb{R}^n, \mathbb{R}^m))$, $u' \in L^2(0,T; L^2(\mathbb{R}^n, \mathbb{R}^m))$
$\Rightarrow u \in C([0,T]; H^1(\mathbb{R}^n, \mathbb{R}^m)) \subseteq L^\infty(\dots) = X$

KONTRAKCIJA: $\| K v - K \tilde{v} \|_X \underset{\uparrow}{=} \| K (v - \tilde{v}) \|_X$

LINEARNOST

OZNAČIMO: $\hat{v} = v - \tilde{v}$

$\hat{u} := K \hat{v}$ JE RJEŠENJE ZADAĆE

$$\begin{cases} \hat{u}_t - \varepsilon \Delta \hat{u} = - \sum_{j=1}^{n} B_j \hat{v}_{x_j} & u\ \mathbb{R}^n \times \langle 0, T] \\ \hat{u} = 0 & \text{NA } \mathbb{R}^n \times \{0\} \end{cases}$$

KAO U TM 5, § 7.1.3. (REGULARNOST) ILI IZ PRIKAZA POMOĆU FUNDAMENTALNOG RJ.

$$\operatorname*{ess\,sup}_{t \in [0,T]} \| \hat{u}(t) \|_{H^1(\mathbb{R}^n; \mathbb{R}^n)} \leq C(\varepsilon) \| \sum_{j=1}^{n} B_j \hat{v}_{x_j} \|_{L^2(0,T; L^2(\mathbb{R}^n; \mathbb{R}^n))}$$

$$\leq C(\varepsilon) \| \hat{v} \|_{L^2(0,T; H^1(\mathbb{R}^n; \mathbb{R}^n))}$$

$$\leq C(\varepsilon) T^{1/2} \| \hat{v} \|_{L^\infty(0,T; H^1(\mathbb{R}^n; \mathbb{R}^n))}$$

$$= C(\varepsilon) T^{1/2} \| \hat{v} \|_X$$

$$\Rightarrow \quad \underset{\| K \hat{v} \|}{\underset{\shortparallel}{\| \hat{u} \|_X}} \leq C(\varepsilon) T^{1/2} \| \hat{v} \|_X$$

AKO JE $C(\varepsilon) T^{1/2} \leq 1/2 \Rightarrow K$ JE KONTRAKCIJA

$\Rightarrow \exists!\ u$ T.D. $Ku = u$

AKO JE $C(\varepsilon) T^{1/2} > 1/2$ ODABEREM T_1 T.D.

$$C(\varepsilon) T_1^{1/2} = 1/2$$

$\Rightarrow$ DOBIJEM RJEŠENJE NA $[0, T_1]$

POHOVIM NA $[T_1, 2T_1]$ ....

PARABOLIČKA REGULARNOST, JER JE $f \in H^1(\dots)$

DAJE H^3 : H^1 ZA u^ε ~~$g \in H^1$~~

ENERGETSKE OCJENE

ŽELIMO $\varepsilon \to 0$.

TEOREM 2 $\exists\, C(n, B_j) > 0$ t.d. $\forall\, \varepsilon \in \langle 0, 1]$

$$\max_{t\in[0,T]} \|u^\varepsilon(t)\|_{H^1(\mathbb{R}^n,\mathbb{R}^m)} + \|u^{\varepsilon\prime}\|_{L^2(0,T;\,L^2(\mathbb{R}^n;\mathbb{R}^m))}$$

$$\leq C\left(\|g\|_{H^1(\mathbb{R}^n,\mathbb{R}^m)} + \|f\|_{L^2(0,T;\,H^1(\mathbb{R}^n;\mathbb{R}^m))} + \|f'\|_{L^2(0,T;\,L^2(\mathbb{R}^n;\mathbb{R}^m))}\right)$$

DOK: 1. KORAK

VRIJEDI

$$\frac{d}{dt}\left(\frac{1}{2}\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\right) = (u^\varepsilon(t), u^{\varepsilon\prime}(t)) = \Big(u^\varepsilon(t), f^\varepsilon(t) - \sum_{j=1}^n B_j \cdot u^\varepsilon_{x_j}(t) + \varepsilon\Delta u^\varepsilon(t)\Big)$$

OCJENJUJEMO DESNU STRANU

* $$|(u^\varepsilon(t), f^\varepsilon(t))| \leq \|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

* $$(u^\varepsilon(t), \varepsilon\Delta u^\varepsilon(t)) \overset{\text{P.I.}}{=} -\varepsilon\,\|Du^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;M^{m\times n})} \leq 0 \quad \text{NEMA } \varepsilon!$$

* $$\Big|\Big(u^\varepsilon(t), \sum_{j=1}^n B_j \cdot u^\varepsilon_{x_j}(t)\Big)\Big| \leq C\,\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

DOK: ZA $v \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m)$

SIMETRIČNI

$$\Big(v, \sum_{j=1}^n B_j \cdot v_{x_j}\Big) = \sum_{j=1}^n \int_{\mathbb{R}^n} (B_j \cdot v_{x_j}) \cdot v\, dx = \sum_{j=1}^n \int v_{x_j} \cdot B_j v$$

$$= \frac{1}{2}\sum_j \underbrace{\int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}(B_j v \cdot v)}_{= 0} - \frac{1}{2}\sum_j \int_{\mathbb{R}^n} B_{j,x_j} v \cdot v$$

$$\Rightarrow \Big|\Big(v, \sum_{j=1}^n B_j \cdot v_{x_j}\Big)\Big| \leq C\,\|v\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

PO GUSTOĆI VRIJEDI TVRDNJA.

DOBIVAMO:

$$\frac{d}{dt}\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \leq C\left(\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f(t)\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^m)}\right)$$

GRONWALL:

$$\max_{t\in[0,T]} \|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \leq C\left(\|g\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))}\right)$$

JER JE $\|g^\varepsilon\|_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \leq \|g\|_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$

3. KORAK TREBA NAM BOLJE PO x

$k\in\{1,\dots,n\}$, $v^k := u^\varepsilon_{x_k}$. DERIVIRAM JDBU PO x_k:

$$v^k_t - \varepsilon A v^k + \sum_{j=1}^n B_j v^k_{x_j} = f_{x_k} - \sum_{j=1}^n B_{j,x_k} u^\varepsilon_{x_j}, \quad \mathbb{R}^n\times\langle 0,T]$$

$$v^k = g^\varepsilon_{x_k} \quad \mathbb{R}^n\times\{0\}$$

ISTA ZADAĆA KAO GORE

$$\Rightarrow \frac{d}{dt}\|v^k(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \leq C\left(\|v^k(t)\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^m)} + \|f_{x_k}(t) - \sum B_{j,x_k}u^\varepsilon_{x_j}(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\right)$$

SUMIRAM PO k:

$$\frac{d}{dt}\|Du^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n,M^{m\times n})} \leq C\left(\|Du^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n,M^{m\times n})} + \|Df(t)\|^2_{L^2(\mathbb{R}^n,M^{m\times n})}\right)$$

GRONWALL:

$$\max_{t\in[0,T]}\|Du^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n,M^{m\times n})} \leq C\left(\|Dg\|^2_{L^2(\mathbb{R}^n;M^{m\times n})} + \|f\|^2_{L^2(0,T;H^1(\mathbb{R}^n;\mathbb{R}^m))}\right)$$

OPET $\|Dg^\varepsilon\|_{L^2} \leq \|Dg\|_{L^2}$

DOBILI SMO 1. OCJENU.

4. KORAK : $v := u^{\varepsilon\prime}$, DERIVIRAMO PO t:

$$\begin{cases} v_t - \varepsilon\Delta v + \sum_{j=1}^{n} B_j v_{x_j} = f' - \sum_{j=1}^{n} B_{j,t} u^{\varepsilon}_{x_j} & \text{u } \mathbb{R}^n \times \langle 0,T] \\ v = f(0) - \sum_{j=1}^{n} B_j(\cdot,0)\, g^{\varepsilon}_{x_j} + \varepsilon\Delta g^{\varepsilon}, & \text{NA } \mathbb{R}^n \times \{0\} \end{cases}$$

KAO I PRIJE

$$\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \le C\Big(\|v(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f'(t) - \sum_{j=1}^{n} B_{j,t} u^{\varepsilon}_{x_j}(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\Big)$$

$$\le C\Big(\|v(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f'(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|Du^{\varepsilon}(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\Big)$$

$$\le C\Big(\|v(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f'(t)\|^2_{L^2} + \|Dg\|^2_{L^2} + \|f\|^2_{L^2(0,T;H^1)}\Big)$$

GRONWALL:

$$\max_{t\in[0,T]} \|u^{\varepsilon\prime}(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \le C\Big(\|f(0) - \sum_{j=1}^{n} B_j(\cdot,0) g^{\varepsilon}_{x_j} + \varepsilon\Delta g^{\varepsilon}\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$
$$+ \|f\|^2_{L^2(0,T;H^1(\mathbb{R}^n;\mathbb{R}^m))} + \|f'\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))} + \|Dg\|^2_{L^2}\Big)$$

$$\le C\Big(\|f(0)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|Dg\|^2_{L^2(\mathbb{R}^n;M^{m\times n})} + \varepsilon^2\|\Delta g^{\varepsilon}\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$
$$+ \|f\|^2_{L^2(0,T;H^1(\mathbb{R}^n;\mathbb{R}^m))} + \|f'\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))}\Big)$$

VRIJEDI

$$\|\Delta g^{\varepsilon}\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \le \frac{C}{\varepsilon^2}\|Dg\|^2_{L^2(\mathbb{R}^n;M^{m\times n})}$$

$$\|f(0)\|_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \le C\Big(\|f\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))} + \|f'\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))}\Big)$$

TEOREM 3 (EGZISTENCIJA SLABOG RJEŠENJA)

POSTOJI SLABO RJEŠENJE ZADAĆE

$$\left.\begin{aligned} u_t + \sum_{j=1}^n B_j u_{x_j} &= f \quad \text{u } \mathbb{R}^n \times \langle 0,T] \\ u &= g \quad \text{NA } \mathbb{R}^n \times \{0\} \end{aligned}\right\} \quad (*)$$

DOK: 1. KORAK ZNAMO: $\forall \varepsilon\ \exists\ u^\varepsilon$ RJEŠENJE

$$\left.\begin{aligned} u^\varepsilon_t - \varepsilon \Delta u^\varepsilon + \sum_{j=1}^n B_j u^\varepsilon_{x_j} &= f \quad \text{u } \mathbb{R}^n \times \langle 0,T] \\ u &= g^\varepsilon \quad \text{NA } \mathbb{R}^n \times \{0\} \end{aligned}\right\} \quad (*_\varepsilon)$$

U PROSTORU: $u^\varepsilon \in L^2(0,T; H^3(\mathbb{R}^n;\mathbb{R}^m))$
$u^{\varepsilon'} \in L^2(0,T; H^1(\mathbb{R}^n;\mathbb{R}^m))$

TE ZADOVOLJAVAJU OCJENE:

$$\| u^\varepsilon \|_{L^\infty(0,T; H^1(\mathbb{R}^n;\mathbb{R}^m))}, \| u^{\varepsilon'} \|_{L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^m))} \leq C$$

UNIFORMNO PO $\varepsilon \in \langle 0,1]$.

$\Longrightarrow\ \exists\ u \in L^2(0,T; H^1(\mathbb{R}^n,\mathbb{R}^m))$

T.D.

$u' \in L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^m))$

I PODNIZ $\varepsilon_k \to 0$ T.D.

$$u^{\varepsilon_k} \rightharpoonup u \quad L^2(0,T; H^1(\mathbb{R}^n;\mathbb{R}^m))$$
$$u^{\varepsilon_k'} \rightharpoonup u' \quad L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^m))$$

POKAZAT ĆEMO DA JE TADA u RJEŠENJE HIPERBOLIČKOG SUSTAVA (JEDNADŽBA).

2. KORAK NEKA JE $v \in C^1([0,T]; H^1(\mathbb{R}^n;\mathbb{R}^m))$. POMNOŽIM $(*_\varepsilon)$ ~~SKALARNO~~ SKALARNO S v I INTEGRIRAM $\int_0^T dt$

$$(**_\varepsilon) \quad \int_0^T (u^{\varepsilon'}(t), v(t))\, dt + \int_0^T \varepsilon\, Du^\varepsilon(t) \cdot Dv(t)\, dt + \int_0^T B[u^\varepsilon(t), v(t); t]\, dt$$
$$= \int_0^T (f(t), v(t))\, dt$$

UZMEMO PODNIZ $\varepsilon = \varepsilon_k$ I PUSTIMO $k \to +\infty$, SLIJEDI:

(NAIME $\varepsilon D u^\varepsilon \to 0$ $L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^n))$)

$$(**) \quad \int_0^T \Big((u'(t), v(t)) + B[u(t), v(t); t] \Big) dt = \int_0^T (f(t), v(t)) dt$$

OVO VRIJEDI I ZA $v \in L^2(0,T; H^1(\mathbb{R}^n;\mathbb{R}^n))$

ZA $v(x,t) = \varphi(t) w(x)$, $\varphi \in L^2(0,T)$, $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ DOBIVAMO (ZBOG PROIZVOLJNOSTI OD φ):

$$(u'(t), w) + B[u(t), w; t] = (f(t), w), \quad w \in H^1(\mathbb{R}^n;\mathbb{R}^n).$$

JEDNADŽBA

3. KORAK (POČETNI UVJETI)

NEKA JE v T.D $v(T)=0$. IZ $(**_\varepsilon)$ P.I. PO t:

$$-\int_0^T (u^\varepsilon(t), v'(t)) dt + \int_0^T \varepsilon D u^\varepsilon(t) \cdot D v(t) + B[u^\varepsilon(t); v(t); t] \, dt = \int_0^T (f(t), v(t)) dt + (g^\varepsilon, v(0))$$

PODNIZ, PUSTIMO LIMES $k \to +\infty$

$$-\int_0^T (u(t), v'(t)) dt + \int_0^T B[u(t), v(t); t] dt = \int_0^T (f(t), v(t)) dt + (g, v(0))$$

IZ $(**)$ P.I. PO t

$$-\int_0^T (u(t), v'(t)) dt + \int_0^T B[u(t), v(t); t] dt = \int_0^T (f(t), v(t)) dt + (u(0), v(0))$$

ODUZIMANJEM:

$$(g - u(0), v(0)) = 0, \quad \forall v \in C^1([0,T]; H^1(\mathbb{R}^n;\mathbb{R}^n))$$

$$\Rightarrow \underline{\underline{u(0) = g}}$$