7.14. PRINCIP MAKSIMUMA

$$Lu = -A \cdot H_u + b \cdot \nabla u + cu \qquad \text{(NEDIVERGENTNI OBLIK)}$$

A - SIMETRIČNA (B.S.O)

PARABOLIČKA GRANICA $\Gamma_T = \overline{U}_T \setminus U_T$

TEOREM 8 (SLABI PRINCIP MAKSIMUMA)

NEKA JE $u \in C^2_1(U_T) \cap C(\overline{U}_T)$ I $c \equiv 0$ U U_T.

(i) AKO JE

$$u_t + Lu \leq 0 \quad \text{U } U_T \qquad \text{(PODRJEŠENJE)}$$

TADA JE

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u$$

(ii) AKO JE

$$u_t + Lu \geq 0 \quad \text{U } U_T \qquad \text{(NADRJEŠENJE)}$$

TADA JE

$$\min_{\overline{U}_T} u = \min_{\Gamma_T} u$$

DOK.: 1. KORAK PRETP: $u_t + Lu < 0$ U U_T & $\exists (x_0,t_0) \in U_T$ T.D.

$$u(x_0,t_0) = \max_{\overline{U}_T} u$$

2. KORAK PRETP: $t_0 \in \langle 0, T \rangle$ TJ. $(x_0,t_0) \in \operatorname{Int} U_T$

$\Rightarrow$ SVE DERIVACIJE $= 0$. SPECIJALNO $u_t(x_0,t_0) = 0$

$\Rightarrow$ $Lu \geq 0$ (VIDI DOK TM1 §6.4 PRINCIP MAX ZA ELIPTIČKU J.)

$\Rightarrow$ $u_t + Lu \geq 0$ U (x_0,t_0) $\Rightarrow\Leftarrow$

3. KORAK $t_0 = T$

$\Rightarrow$ $u_t(x_0,t_0) \geq 0$ $\Rightarrow$ ISTO KAO GORE

4. KORAK $u_t + Lu \leq 0$ & $u(x_0,t_0) = \max_{\overline{U}_T} u$

DEF: $u^\varepsilon(x,t) := u(x,t) - \varepsilon t$ (OVDJE NE LAKŠE!)

$\Rightarrow$ $u^\varepsilon_t + Lu^\varepsilon = u_t + Lu - \varepsilon < 0$

A.2.3.K. $\Rightarrow$ $\max_{\overline{U}_T} u^\varepsilon = \max_{\Gamma_T} u^\varepsilon$

PUSTIM $\varepsilon \to 0$ $\Rightarrow$ TV.

5. KORAK u NADRJEŠENJE $\Rightarrow$ $-u$ PODRJEŠENJE (i) $\Rightarrow$ TV //

TEOREM 9 (SLABI PRINCIP MAKSIMUMA & $c \geq 0$)

NEKA JE $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ & $c \geq 0$ u U_T.

(i) AKO $u_t + Lu \leq 0$ u U_T $\Rightarrow$ $\max_{\overline{U}_T} u \leq \max_{\Gamma_T} u^+$

(ii) AKO $u_t + Lu \geq 0$ u U_T $\Rightarrow$ $\max_{\overline{U}_T} u \geq -\max_{\Gamma_T} u^-$

NAP. AKO JE $u_t + Lu = 0$ u U_T $\Rightarrow$ $\max_{\overline{U}_T} |u| = \max_{\Gamma_T} |u|$.

DOK. 1. KORAK; PRETP. u ZADOVOLJAVA

$$u_t + Lu < 0 \quad u \ U_T$$

& $\underset{\overset{\uparrow}{U_T}}{u(x_0,t_0)} = \max_{U_T} u > 0$ (MAX JE POZITIVAN) AKO NIJE TRIVIJALNO

KAO PRIJE $u_t(x_0,t_0) = 0$ & $(Lu)(x_0,t_0) \geq 0$ ZBOG $c \geq 0$ i $u(x_0,t_0) > 0$

$\Rightarrow$ $u_t + Lu \geq 0$ u (x_0,t_0) $\Rightarrow \Leftarrow$

2. KORAK PRETP. u ZADOVOLJAVA $u_t + Lu \leq 0$ u U_T

$u^\varepsilon(x,t) = u(x,t) - \varepsilon t$ & $\underset{\overset{\uparrow}{U_T}}{u(x_0,t_0)} = \max_{U_T} u > 0$

$\Rightarrow$ $u_t^\varepsilon + Lu^\varepsilon < 0$ u U_T

u^ε POSTIŽE POZITIVAN MAX. ZA ε DOVOLJNO MALI

$\Downarrow$

$\Uparrow$ KAO PRIJE

3. KORAK (ii) SLIČNO. //

NAP. ZA RAZLIKU OD ELIPTIČKIH JEDNADŽBI ČAK I ZA $c < 0$ NEĆE SE DOBITI PRINCIP MAKSIMUMA.

HARNACKOVA NEJEDNAKOST

TEOREM 10 (PARABOLIČKA HARNACKOVA NEJEDNAKOST)

NEKA JE $u \in C^2_1(U_T)$ RJ.

$$u_t + Lu = 0 \quad u \; U_T$$

I $u \geq 0$ u U_T.

NEKA JE $V \subset\subset U$ POVEZAN. TADA ZA SVAKI $0 < t_1 < t_2 \leq T$ POSTOJI $C(V, t_1, t_2, L) > 0$ T.D.

$$\sup_V u(\cdot, t_1) \leq C \inf_V u(\cdot, t_2).$$

JAKI PRINCIP MAKSIMUMA

TEOREM 11 (JAKI PRINCIP MAKSIMUMA)

NEKA JE $u \in C^2_1(U_T) \cap C(\overline{U_T})$, $C \equiv 0$ u U_T, U POVEZAN.

(i) AKO JE $u_t + Lu \leq 0$ u U_T & $u(x_0, t_0) = \max_{\overline{U_T}} u(x_0, t_0)$, $(x_0, t_0) \in U_T$

TADA JE u KONSTANTA NA U_{t_0}

(ii) AKO JE $u_t + Lu \geq 0$ u U_T & $u(x_0, t_0) = \min_{\overline{U_T}} u(x_0, t_0)$, $(x_0, t_0) \in U_T$

TADA JE u KONSTANTA NA U_{t_0}.

NAP. KAO I PRIJE TO POTVRĐUJE ∞ BRZINU PROPAGIRANJA POREMEĆAJA. g NEGDJE > 0 $\rightarrow$ RJ > 0 SVUDA ($f = 0$)

DOK: 1. KORAK $\quad u_t + Lu \le 0$ u U_T & $\exists (x_0,t_0)\in U_T$ $\quad u(x_0,t_0)=\max_{\overline{U_T}} u$

NEKA JE $W \subset\subset U$, $x_0 \in W$. DEF v:

$$v_t + Lv = 0 \quad \text{u } W_T$$
$$v = u \quad \text{NA } \Delta_T \quad (\text{PARABOLIČKA OD } W_T)$$

SLABI PRINCIP MAX ZA $u - v$:

$$(u-v)_t + L(u-v) = u_t + Lu - \overset{0}{\overset{\shortparallel}{(v_t+Lv)}} \le 0$$

$$\Rightarrow \max_{\overline{W_T}} (u-v) = \max_{\Delta_T}(u-v) = 0$$

$$\Rightarrow \quad u - v \le 0 \quad \Rightarrow \quad u \le v \quad \text{NA } \overline{W_T}$$

SLABI PRINCIP MAX ZA v:

$$\max_{\overline{W_T}} v \le \max_{\Delta_T} u \le M = \max_{\overline{U_T}} u$$

$$\Rightarrow \quad v \le M \quad \text{NA } \overline{W_T}$$

$\Rightarrow$ NA $\overline{W_T}$ $\quad u \le v \le M$

U TOČKI (x_0,t_0) $\quad u(x_0,t_0)=M \Rightarrow v(x_0,t_0)=M$

$\Rightarrow \tilde v = M - v \ge 0$

$\tilde v_t + L\tilde v = 0$ (JER JE $c \equiv 0$) $\quad \big|$ u W_T

UZMEMO $V \subset\subset W$, $x_0 \in V$, POVEZAN $\quad | \quad 0 < t < t_0$

HARNACKOVA NEJEDNAKOST

$$0 \le \max_V \tilde v(\cdot,t) \le C \inf_V \tilde v(\cdot,t_0) \le 0$$

$\Rightarrow \tilde v(\cdot,t) \equiv 0$ NA $V \times \{t\}$ $\quad 0 < t < t_0$

V PROIZVOLJAN $\Rightarrow \tilde v \equiv 0$ NA W_{t_0} $\Rightarrow v \equiv M$ NA W_{t_0} $\Rightarrow u \equiv M$ NA $\partial W \times \{t\}$

— //.D — $\quad W$ PROIZVOLJAN $\Rightarrow u \equiv M$ NA U_t

TEOREM 12 (JAKI PRINCIP MAKSIMUMA, $c \geq 0$)

NEKA JE $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ I $c \geq 0$ U U_T, U POVEZAN.

(i) AKO JE $u_t + Lu \leq 0$ U U_T I $\underset{\underset{U_T}{\Uparrow}}{u(x_0,t_0)} = \max_{\overline{U}_T} u \geq 0$

$\Longrightarrow$ $u \equiv \text{const}$ NA U_{t_0}

(ii) AKO JE $u_t + Lu \geq 0$ U U_T I $\underset{\underset{U_T}{\Uparrow}}{u(x_0,t_0)} = \min_{\overline{U}_T} u \leq 0$

$\Longrightarrow$ $u \equiv \text{const}$ NA U_{t_0}.

DOK: ZA ZADAĆU.