2.2. LAPLACEOVA JEDNADŽBA

$$\Delta u = 0 \quad u \quad U \subseteq \mathbb{R}^n$$

$$\sum_{i=1}^{n} \partial_{x_i}^2 u = 0$$

NEHOMOGENA (POISSONOVA) J. : ZA $f: U \to \mathbb{R}$ ZADANU

$$-\Delta u = f \quad u \quad U$$

DEF: $u \in C^2(U)$ T.D. $\Delta u = 0$ ZOVE SE HARMONIJSKA FUNKCIJA

PORIJEKLO: u – GUSTOĆA SUPSTANCE (KONCENTRACIJA NEKOG KEMIJSKOG SPOJA)

F – FLUKS OD u: NPR.

$$F = -a\, Du \quad , \quad a > 0$$

SUSTAV U RAVNOTEŽI:

$$\int_{\partial V} F \cdot \nu \, dS = 0 \quad \text{ZA SVAKI } V \subset U$$

GAUSSOV TEOREM (TM O DIVERGENCIJI)

$$\int_V \operatorname{div} F \, dx = 0$$

PROIZVOLJNOST OD V

$$\operatorname{div} F = 0$$

$$-a \operatorname{div} D u = 0$$

$$\boxed{\Delta u = 0}$$

u

KEMIJSKA KONCENTRACIJA

TEMPERATURA

ELEKTRIČNI POTENCIJAL

FICKOV ZAKON DIFUZIJE

FOURIEROV ZAKON PROVOĐENJA TOPLINE

OHMOV ZAKON PROVOĐENJA

2.2.1 FUNDAMENTALNA RJEŠENJA

- TRAŽIMO NEKA SPECIJALNA RJEŠENJA
- SIMETRIJE

PR 1: $u \ldots \quad \Delta u = 0$

$$v(x) := u(x + c) \quad , \quad c \in \mathbb{R}^n$$

$$\Delta v(x) = \Delta u(x+c) = 0$$

PR 2: $u \ldots \quad \Delta u = 0$

$$Q \in O(n)$$

$$v(x) := u(Qx)$$

$$\partial_{x_i} v(x) = Du(Qx)\, Qe_i$$

$$\partial^2_{x_i} v(x) = D^2 u(Qx) Qe_i \cdot Qe_i = Q^T D^2 u(Qx) Q\, e_i \cdot e_i$$

$$\Delta v(x) = \operatorname{tr}\left(Q^T D^2 u(Qx) Q\right) = \operatorname{tr}\left(D^2 u(Qx)\right) \cdot$$

$$= \Delta u(Qx) = 0$$

LAPLACEOVA J. INVARIJANTNA NA ROTACIJE

TRAŽIMO RJ. ZA $U = \mathbb{R}^n$ OBLIKA

$$u(x) = v(|x|) \quad \left(v(r) \ , \ r(x) = |x| = \sqrt{\sum_{i=1}^n x_i^2} \right)$$

$$\frac{\partial r}{\partial x_i}(x) = \frac{1}{2}\left(\sum_{i=1}^n x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{r(x)} \quad , \quad x \neq 0$$

$$\partial_{x_i} u(x) = v'(r(x)) \frac{\partial r}{\partial x_i}(x) = v'(r(x)) \frac{x_i}{r(x)}$$

$$\partial^2_{x_i} u(x) = v''(r(x)) \frac{x_i^2}{r(x)^2} + v'(r(x)) \left(\frac{1}{r(x)} - \frac{x_i^2}{r(x)^3} \right)$$

$$\Delta u(x) = v''(r(x)) + v'(r(x)) \frac{n-1}{r(x)}$$

$$\Delta u(x) = 0 \iff v''(r(x)) + v'(r(x)) \frac{n-1}{r(x)} = 0$$

ZAMJENA VAR

$$\iff v''(r) + v'(r) \frac{n-1}{r} = 0$$

$$\frac{v''}{v'} = \frac{1-n}{r}$$

$$\parallel$$

$$(\log v')'$$

$$\Rightarrow \log v'(r) = (1-n)\log r + \log a$$

$$v'(r) = a r^{1-n} = \frac{a}{r^{n-1}}$$

DAKLE:

$$v(r) = \begin{cases} b \log r + c , & n = 2 \\ \frac{b}{r^{n-2}} + c , & n \geq 3 \end{cases}$$

<u>DEF:</u> FUNKCIJA

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log|x| , & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} , & n \geq 3 \end{cases}$$

NAZIVA SE FUNDAMENTALNO RJEŠENJE L.J.

- $\alpha(n)$ - VOLUMEN $K(0,1) \subseteq \mathbb{R}^n$.
- KONSTANTE SMO ODABRALI PRIGODNO

VRIJEDI:

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$$

$$|D^2\Phi(x)| \leq \frac{C}{|x|^n}$$

NEHOMOGENA J.

$x \neq 0 \qquad x \longmapsto \Phi(x)$ HARMONIJSKA

$x \neq y \qquad x \longmapsto \Phi(x-y)$ HARMONIJSKA

$f \in C_c^2(\mathbb{R}^n), \; x \neq y \qquad x \longmapsto \Phi(x-y)\, f(y)$ ~~HARMONIJSKA~~

DEF:

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy \quad , \quad x \in \mathbb{R}^n$$

NIJE HARMONIJSKA !

NAIME NE VRIJEDI

$$\Delta u(x) = \int_{\mathbb{R}^n} \Delta_x \Phi(x-y)\, f(y)\, dy$$

INTEGRAL NIJE DEFINIRAN

TM 1 u ZADOVOLJAVA

$$u \in C^2(\mathbb{R}^n)$$

$$-\Delta u = f \quad u \; \mathbb{R}^n$$

DOK: 1) $u \in C^2(\mathbb{R}^n)$

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-z)\, f(z)\, dz = \begin{vmatrix} y = x-z \\ dy = -dz \end{vmatrix} = \int_{\mathbb{R}^n} \Phi(y)\, f(x-y)\, dy$$

$$\Rightarrow D^2 u(x) = \int_{\mathbb{R}^n} \Phi(y)\, D_x^2 f(x-y)\, dy$$

2) $\varepsilon > 0$

$$\Delta u(x) = \int_{B(0,\varepsilon)} \Phi(y) \underbrace{\Delta_x f(x-y)}_{\leq C} dy + \int_{\partial B(0,\varepsilon)} \Phi(y) \underbrace{\frac{\partial f}{\partial \nu}(x-y)}_{\leq C} dS(y)$$

$$- \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y)\, f(x-y)\, dS(y) + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \underbrace{\Delta \Phi(y)}_{=0} f(x-y)\, dy$$

$2 \times$ P.I.

$$\frac{\partial \Phi}{\partial \nu}(y) = D\Phi(y) \cdot \nu(y)$$

VANJSKA JEDINIČNA NA $\partial B(0,\varepsilon)$

$$\nu(y) = -\frac{y}{|y|} = -\frac{y}{\varepsilon}$$

$$D\Phi(y) = \frac{1}{n(n-2)\alpha(n)}(-n+2)\frac{1}{|y|^{n-1}}\frac{y}{|y|} = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n} = -\frac{1}{n\alpha(n)}\frac{y}{\varepsilon^n}$$

$$\Rightarrow \frac{\partial \Phi}{\partial \nu}(y) = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$$

$$\Rightarrow -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,dS(y) = -\int \frac{1}{n\alpha(n)\varepsilon^{n-1}} f(x-y)\,dS(y)$$

$$= -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} f(y)\,dS(y)$$

$|S^{n-1}| \subset \mathbb{R}^n \leftarrow n\alpha(n)$

$$= -⨍_{\partial B(x,\varepsilon)} f(y)\,dS(y) \longrightarrow -f(x)$$

$$\Rightarrow \quad -\Delta u = f$$

FORMALNO: $\quad -\Delta \Phi = \delta_0 \quad$ u $\mathbb{R}^n$

$$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta_x \Phi(x-y) f(y)\,dy$$

$$= \int_{\mathbb{R}^n} \delta_x f(y)\,dy = f(x)$$

TM 2 (TEOREM SREDNJE VRIJEDNOSTI)

AKO JE $u \in C^2(U)$ HARMONIJSKA, TADA ZA SVAKU $K(x,r) \subset U$

$$u(x) = ⨍_{\partial B(x,r)} u \, dS = ⨍_{B(x,r)} u \, dy .$$

DOK: 1. JEDNAKOST

$$\phi(r) := ⨍_{\partial B(x,r)} u(y)\, dS(y) = ⨍_{\partial B(0,1)} u(x+rz)\, dS(z)$$

JEDINIČNA VANJSKA NORMALA

$$\phi'(r) = ⨍_{\partial B(0,1)} Du(x+rz)\cdot z \, dS(z) = ⨍_{\partial B(x,r)} Du(y) \frac{y-x}{r} dS(y)$$

$$= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\, dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\, dS(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy = 0$$

$\Rightarrow$ ϕ JE KONSTANTA

$$\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t \to 0} ⨍_{\partial B(x,t)} u(y)\, dS(y) = u(x)$$

$\Rightarrow$ 1. JEDNAKOST

2. JEDNAKOST

~~POLARNE KOORDINATE~~

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,s)} u \, dS \right) ds = \int_0^r u(x)\, n\alpha(n) s^{n-1}\, ds$$

$$= u(x)\, n\alpha(n) \frac{r^n}{n} = \underbrace{\alpha(n) r^n}_{V_n(r)} u(x)$$

$\square$

TM3 (OBRAT) AKO $u \in C^2(U)$ ZADOVOLJAVA

$$u(x) = ⨍_{\partial B(x,r)} u \, dS \quad , \quad B(x,r) \subset U$$

$\Rightarrow$ u JE HARMONIJSKA

DOK: $\Rightarrow$ $\phi(r)$ JE KONSTANTA.

PRETP. u NIJE HARMONIJSKA: $\Delta u \neq 0 \Rightarrow \exists B(x,r) \subset U$ T.D. $\Delta u > 0$

$$0 = \phi'(r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy > 0 \Rightarrow \Leftarrow$$

$\square$

TM 4 (PRINCIP MAKSIMUMA)

NEKA JE $U \subseteq \mathbb{R}^n$ OTVOREN I OGRANIČEN

NEKA JE $u \in C^2(U) \cap C(\bar{U})$ HARMONIJSKA U U.

TADA:

(i) $\max_{\bar{U}} u = \max_{\partial U} u$ PRINCIP MAKSIMUMA

(ii) AKO JE U POVEZAN I $\exists x_0 \in U$ T.D.

$$u(x_0) = \max_{\bar{U}} u$$

TADA JE u KONSTANTA NA U — JAKI PRINCIP MAKSIMUMA

NAP: TM VRIJEDI I ZA "MIN" $(-u)$

DOK: PRETP: $\exists x_0 \in U$ TD $u(x_0) = \max_{\bar{U}} u =: M$

PROMATRAMO: $S = \{x \in U : u(x) = M\} \ni x_0$

1) S RELATIVNO ZATVOREN U U

2) $x_0 \in S \Rightarrow S \neq \emptyset$

3) NEKA JE $z \in S$ T.J. $u(z) = M$

$$M = u(z) = \fint_{B(z,r)} u \, dy \leq M$$

(ZA r DOVOLJNO MALI DA JE $B(z,r) \subset U$)

KAKO VRIJEDI $=$ $\Rightarrow u \equiv M$ NA $B(z,r)$

$\Rightarrow B(z,r) \subset S \Rightarrow S$ JE OTVOREN

$\Rightarrow S = U$

NAP: U POVEZAN, $u \in C^2(U) \cap C(\bar{U})$: $\begin{cases} \Delta u = 0 & u \; U \\ u = g & na \; \partial U \end{cases}$

AKO JE $g \geq 0 \Rightarrow \min_{\partial U} g \geq 0$

$\overset{(i)}{\Rightarrow} \min_{\bar{U}} u = \min_{\partial U} u = \min_{\partial U} g \geq 0 \Rightarrow u \geq 0$ NA U

$\overset{(ii)}{\Rightarrow} \exists x_0 \in U$ $u(x_0) = 0 = \min_{\bar{U}} u$

$\Rightarrow u = 0$ NA $\bar{U}$

AKO JE g NEGDJE > 0 $\Rightarrow$

$\Rightarrow u > 0$ NA U

TM 5 (JEDNISTVENOST RUBNE ZADAĆE)

NEKA JE $g \in C(\partial U)$, $f \in C(U)$.

TADA $\exists$ NAJVIŠE JEDNO RJEŠENJE $u \in C^2(U) \cap C(\bar{U})$ OD

$$\begin{cases} -\Delta u = f & u \; U \\ u = g & na \; \partial U \end{cases}$$

DOK: $u, \tilde{u}$ ZADOVOLJAVAJU R. Z.

DEF: $w_\pm = \pm (u - \tilde{u})$

$$\Rightarrow \quad \begin{aligned} -\Delta w_\pm &= 0 & u \; U \\ w_\pm &= 0 & na \; \partial U \end{aligned}$$

PRINCIP MAKSIMUMA:

ZA w_+: $\quad \max\limits_{\bar{U}} w_+ = \max\limits_{\partial U} w_+ = 0$

ZA w_-: $\quad \max\limits_{\bar{U}} w_- = \max\limits_{\partial U} w_- = 0$

DAKLE:

$$\max_{\bar{U}} (u - \tilde{u}) = 0$$

$$\max_{\bar{U}} -(u - \tilde{u}) = 0$$
$$\shortparallel$$
$$-\min (u - \tilde{u})$$

$$0 = \min (u - \tilde{u}) \leq \max (u - \tilde{u}) = 0$$

$$\Rightarrow = \qquad \Rightarrow u = \tilde{u}$$

TM 6 (GLATKOĆA)

AKO $u \in C(U)$ ZADOVOLJAVA

$$u(x) = \fint_{\partial B(x,r)} u \, dS = \fint_{B(x,r)} u \, dy \qquad \forall B(x,r) \subset U$$

TADA JE $u \in C^{\infty}(U)$.

DEF: IZGLAĐIVAČ (MOLLIFIER) §C.4.

(i) NEKA $\eta \in C^{\infty}(\mathbb{R}^n)$

$$\eta(x) := \begin{cases} C\, e^{\frac{1}{|x|^2-1}} & , |x|<1 \\ 0 & , |x| \geq 1 \end{cases}$$

C ODABRAN DA JE $\int_{\mathbb{R}^n} \eta \, dx = 1$.

(ii) $\forall \varepsilon > 0$ DEF:

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$$

- η_ε IMA NOSAČ U $B(0,\varepsilon)$, $\operatorname{supp} \eta_\varepsilon \subseteq B(0,\varepsilon)$
- $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right) dx = \left| \begin{matrix} \frac{x}{\varepsilon} = y \\ dx = \varepsilon^n dy \end{matrix} \right| = \int_{\mathbb{R}^n} \eta(y) dy = 1$

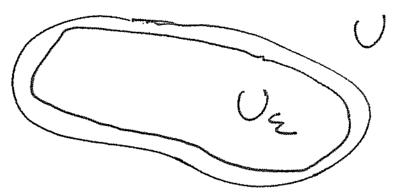
ZAD $f: U \to \mathbb{R}$ (NEPREKIDNA; LOKALNO INTEGRABILNA)

DEF IZGLAĐENJE

$$f^\varepsilon := \eta_\varepsilon * f \qquad \text{u } U_\varepsilon = \{x \in U : d(x, \partial U) > \varepsilon\}$$

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x-y) f(y) dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y) dy$$

$x \in U_\varepsilon$



SVOJSTVA:

(i) $f^\varepsilon \in C^\infty(U_\varepsilon)$

(ii) $f^\varepsilon \xrightarrow{\varepsilon \to 0} f$ s.s

(iii) AKO JE $f \in C(U) \Rightarrow f^\varepsilon \longrightarrow f$ UNIFORMNO NA KOMPAKTIMA U U

(iv) AKO JE $1 \le p < \infty$, $f \in L^p_{loc}(U) \Rightarrow f^\varepsilon \longrightarrow f$ u $L^p_{loc}(U)$

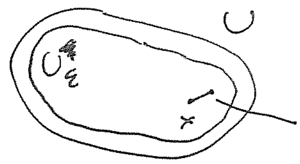
DOK (i) $x \in U_\varepsilon$ i h MALI T.D. $x + he_i \in U_\varepsilon$
$i \in \{1, \dots, n\}$

$$\frac{f^\varepsilon(x+he_i) - f^\varepsilon(x)}{h} = \int_U \frac{\eta_\varepsilon(x+he_i-y) - \eta_\varepsilon(x-y)}{h} f(y)\,dy$$

$$= \frac{1}{\varepsilon^n} \int_U \frac{1}{h}\left(\underbrace{\eta\left(\frac{x+he_i-y}{\varepsilon}\right)} - \eta\left(\frac{x-y}{\varepsilon}\right)\right) f(y)\,dy$$

NOSAČ U $B(x+he_i, \varepsilon)$

NOSAČ U $B(x, \varepsilon)$



$x + he_i \in U_\varepsilon$

DAKLE TREBA INTEGRIRATI PO NEKOM V OTV $\subset U$

$V \subset\subset U$

KOMPAKTNO SADRŽAN

JER JE η ~~ANALITIČKA~~ & V KOMPAKTAN

$$\frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right) \longrightarrow \frac{1}{\varepsilon}\frac{\partial \eta}{\partial x_i}\left(\frac{x-y}{\varepsilon}\right)$$

KAD $h \to 0$

UNIFORMNO ! (u sup normi)

$$\Rightarrow \frac{f^\varepsilon(x+he_i) - f^\varepsilon(x)}{h} \longrightarrow \frac{1}{\varepsilon^{n+1}} \int_U \frac{\partial \eta}{\partial x_i}\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy$$

$$\| $$

$$\int_U \frac{\partial \eta_\varepsilon}{\partial x_i}(x-y) f(y)\,dy =: \frac{\partial f^\varepsilon}{\partial x_i}(x)$$

$\exists$ &

I.T.D.

ДОК ТМ 6: DEF: $u^\varepsilon = \eta_\varepsilon * u$ в $U_\varepsilon = \{x \in U : d(x, \partial U) > \varepsilon\}$

ЗНАМО $u^\varepsilon \in C^\infty(U_\varepsilon)$

TV: $u = u^\varepsilon$ НА U_ε

Док:
$$u^\varepsilon(x) = \int_U \eta_\varepsilon(x-y)\, u(y)\, dy$$
$$= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\, dy$$
$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(x,r)} u\, dS \right) dr$$
$$\int_{\partial B(x,r)} u\, dS = u(x)\, |\partial B(x,r)| = u(x)\, n\alpha(n)\, r^{n-1}$$
$$= u(x) \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n\alpha(n) r^{n-1}\, dr$$
$$= u(x) \int_0^\varepsilon \eta_\varepsilon(r)\, n\alpha(n)\, r^{n-1}\, dr$$
$$= u(x) \int_{B(0,\varepsilon)} \eta_\varepsilon\, dy = u(x), \quad x \in U_\varepsilon$$

$\Rightarrow u \in C^\infty(U_\varepsilon),\ \varepsilon > 0$

TM 7 NEKA JE u HARMONIJSKA NA U. TADA

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}, \quad B(x_0,r)\subset U, \quad |\alpha|=k.$$

$$C_0 = \frac{1}{\alpha(n)}, \quad C_k = \frac{(2^{n+1} n k)^k}{\alpha(n)}, \quad k\in\mathbb{N}.$$

DOK: INDUKCIJOM PO k, ZAHVALJUJUĆI SVOJSTVU SREDNJE VRIJEDNOSTI

TM 8 (LIOUVILLEOV TEOREM)

NEKA JE $u:\mathbb{R}^n \to \mathbb{R}$ HARMONIJSKA, OGRANIČENA.

TADA JE u KONSTANTA.

DOK: $x_0\in\mathbb{R}^n$, $r>0$. TM 7 $\Rightarrow$

$$|Du(x_0)| \le \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \le \frac{C_1}{r^{n+1}} \|u\|_{L^\infty(B(x_0,r))} |B(x_0,r)|$$

$$\le \frac{C_1}{r^{n+1}} \|u\|_{L^\infty(\mathbb{R}^n)} \alpha(n) r^n = \frac{C_1\alpha(n)}{r} \underbrace{\|u\|_{L^\infty(\mathbb{R}^n)}}_{\le C} \to 0$$

$$\Rightarrow \quad Du(x_0)=0, \quad x_0\in\mathbb{R}^n \quad \Rightarrow \quad u = \text{const.}$$

TM 9 NEKA JE $f\in C_c^2(\mathbb{R}^n)$, $n\ge 3$.

TADA JE SVAKO OGRANIČENO RJEŠENJE

$$-\Delta u = f \quad \text{u } \mathbb{R}^n$$

OBLIKA (ZA NEKI $C\in\mathbb{R}$)

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy + C, \quad x\in\mathbb{R}^n.$$

DOK: ZA $n\ge 3$ Φ TRNE U ∞ $\Rightarrow$ INTEGRAL JE DOBRO ~~DEFINIRAN~~

$\Rightarrow$ u JE RJEŠENJE (OGRANIČENO)

NEKA JE $\tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy$

I u NEKO DRUGO OGRANIČENO RJ.

$\Rightarrow$ $w := u - \tilde{u}$ ZADOVOLJAVA: OGRANIČENA, $\Delta w = 0$

$\Rightarrow$ KONSTANTA

TM 10 (ANALITIČNOST)

NEKA JE u HARMONIJSKA NA U.

TADA JE u ANALITIČKA NA U.

DOK: $x_0 \in U$. TREBA POKAZATI: TAYLOROV RED KOG NA NEKOJ OKOLINI x_0.

$$T_{N-1}(x) = \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x-x_0)^\alpha$$

$$R_N(x) = u(x) - T_{N-1}(x) = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x-x_0))}{\alpha!} (x-x_0)^\alpha$$

TREBA NAM OCJENA NA

DEF:

$$r = \frac{1}{4} d(x, \partial U), \quad M := \frac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,2r))} < \infty$$

$B(x_0,2r) \subset U$

$$\forall x \in B(x_0,r) \Rightarrow B(x,r) \subset B(x_0,2r) \subset U$$

$$\text{TM 7} \Rightarrow \|D^\alpha u\|_{L^\infty(B(x_0,r))} = \sup_{x \in B(x_0,r)} |D^\alpha u(x)|$$

$$\le \sup_{x \in B(x_0,r)} \frac{(2^{n+1} n |\alpha|)^{|\alpha|}}{\alpha(n) r^{n+|\alpha|}} \underbrace{\|u\|_{L^1(B(x,r))}}_{\le \|u\|_{L^1(B(x_0,2r))}}$$

$$\le M \left(\frac{2^{n+1} n}{r}\right)^{|\alpha|} |\alpha|^{|\alpha|}$$

STIRLINGOVA FORMULA: $\lim_{k \to \infty} \frac{k^{k+\frac{1}{2}}}{k!\, e^k} = \frac{1}{\sqrt{2\pi}}$

$\Rightarrow k^k \le C e^k k!$ ZA DOVOLJNO VELIKI C i SVE $k \in \mathbb{N}$

$$\Rightarrow |\alpha|^{|\alpha|} \le C e^{|\alpha|} |\alpha|!$$

IZ $n^k = (1 + \cdots + 1)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \Rightarrow |\alpha|! \le n^{|\alpha|} \alpha!$

$$\Rightarrow \|D^\alpha u\|_{L^\infty(B(x_0,r))} \le M \left(\frac{2^{n+1} n}{r}\right)^{|\alpha|} C e^{|\alpha|} n^{|\alpha|} \alpha!$$

VRATIM SE ZA $x \in B(x_0, R)$, $R \leq r$

$$|R_N(x)| \leq \sum_{|\alpha|=N} M \left(\frac{2^{n+1} n^2 e}{r}\right)^N C \, |(x-x_0)^\alpha|$$

$$\leq \sum_{|\alpha|=N} MC \left(\frac{2^{n+1} n^2 e}{r}\right)^N R^N$$

$$= MC \left(\frac{2^{n+1} n^2 e}{r}\right)^N R^N n^N$$

$$= MC \left(\frac{2^{n+1} n^3 e}{r} R\right)^N \longrightarrow 0$$

$\Rightarrow$ R MORA ZADOVOLJAVATI

$$\frac{2^{n+1} n^3 e}{r} R < 1$$

$$R < \frac{r}{2^{n+1} n^3 e}$$

TM 11 (HARNACKOVA NEJEDNAKOST)

ZA SVAKI POVEZAN I OTVOREN SKUP $V \subset\subset U$

POSTOJI $C>0$ (OVISI SAMO O V) T.D.

$$\sup_V u \le C \inf_V u \quad \Big| \quad u \ge 0,\ \Delta u = 0 \text{ u } U$$

NAP:

$$\frac{1}{C} u(y) \le u(x) \le C u(y), \quad x,y \in V$$

NA CIJELOM V VRIJEDNOSTI SU "PRIBLIŽNE"

DOK: $r = \frac{1}{4} d(V, \partial U)$, $x, y \in V$, $|x-y| \le r$

$$u(x) = \fint_{B(x,2r)} u\,dz = \frac{1}{\alpha(n)(2r)^n} \int_{B(x,2r)} u(z)\,dz$$

$$\ge \frac{1}{\alpha(n)(2r)^n} \int_{B(y,r)} u(z)\,dz = \frac{1}{2^n} \fint_{B(y,r)} u\,dz$$

$$= \frac{1}{2^n} u(y)$$

$$2^n u(y) \ge u(x) \ge \frac{1}{2^n} u(y), \quad x,y \in V,\ |x-y| \le r$$

ISTA $x \leftrightarrow y$

$\overline{V}$ povezan i kompaktan: POKRIJEMO GA S KONAČNO OTVORENIH KUGLI RADIJUSA $\frac{r}{2}$, B_i, $i=1,\dots,N$ T.D. ~~koje~~ $B_i \cap B_{i+1} \neq \emptyset$.

TADA UZASTOPNOM PRIMJENOM NAJVIŠE N PUTA:

$$u(x) \ge \frac{1}{2^{nN}} u(y), \quad x,y \in V.$$

2.2.4. GREENOVA FUNKCIJA

RUBNA ZADAĆA: $\qquad U \subseteq \mathbb{R}^n$ OTVOREN, ∂U, C^1

NAĆI u T.D.

$$\begin{cases} -\Delta u = f & \text{u } U \\ u = g & \text{na } \partial U \end{cases}$$

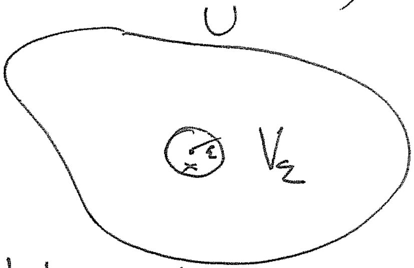
NEKA JE $u \in C^2(\overline{U})$ PROIZVOLJNA FUNKCIJA, $x \in U$, $\varepsilon > 0$ T.D. $B(x,\varepsilon) \subset U$. GREENOVA FORMULA ZA

$$V_\varepsilon := U \setminus B(x,\varepsilon) \quad \text{i} \quad u(y) \text{ i } \Phi(y-x)$$

klase $C^2(\overline{V_\varepsilon})$

$$\int_{V_\varepsilon} \Big(u(y)\, \overbrace{\Delta \Phi (y-x)}^{=0} - \Phi(y-x)\, \Delta u(y) \Big) dy$$

$$= \int_{\partial V_\varepsilon} \Big(u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y) \Big) dS(y)$$



ANALIZIRAMO DIO NA $\partial B(x,\varepsilon)$

$$\Big| \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \Big| \le C \max_{\partial B(0,\varepsilon)} |\Phi| \, \underbrace{|\partial B(0,\varepsilon)|}_{\sim \varepsilon^{n-1}} \longrightarrow 0$$

$\sim \log \varepsilon$

$\frac{1}{\varepsilon^{n-2}}$

DOK TM 1

$$\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) = \fint_{\partial B(x,\varepsilon)} u(y)\, dS(y) \longrightarrow u(x)$$

SLIJEDI

$$u(x) = \int_{\partial U} \Big(\Phi(y-x) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Phi}{\partial \nu}(y-x) \Big) dS(y) - \int_U \Phi(y-x)\, \Delta u(y)\, dy$$

U Z.7. IMAMO Δu, $u\big|_{\partial U}$, ALI FALI $\frac{\partial u}{\partial \nu}\big|_{\partial U}$!

NJEGA ŽELIMO ELIMINIRATI:

DEF: ϕ^x ZADOVOLJAVA

$$\begin{cases} \Delta \phi^x(y) = 0 & y \in U \\ \phi^x(y) = \Phi(y-x), & y \in \partial U \end{cases}$$

GREENOVA FORMULA: ϕ^x, u

$$\int_U u(y)\, \underbrace{\Delta \phi^x(y)}_{=0} - \phi^x(y) \Delta u(y)\, dy$$

$$= \int_{\partial U} u(y) \frac{\partial \phi^x}{\partial \nu}(y)\, dS(y) - \int_{\partial U} \underbrace{\phi^x(y)}_{=\Phi(y-x)} \frac{\partial u}{\partial \nu}(y)\, dS(y)$$

$$\int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) = \int_U \phi^x(y) \Delta u(y)\, dy + \int_{\partial U} u(y) \frac{\partial \phi^x}{\partial \nu}(y)\, dS(y)$$

$$\Rightarrow \quad u(x) = -\int_U \left(\Phi(y-x) - \phi^x(y)\right) \Delta u(y)\, dy - \int_{\partial U} u(y) \left(\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial \phi^x}{\partial \nu}(y)\right) dS(y)$$

DEF: GREENOVA FUNKCIJA ZA U:

$$G(x,y) := \Phi(y-x) - \phi^x(y), \quad x, y \in U,\ x \neq y$$

$$\Rightarrow \quad u(x) = -\int_U G(x,y) \Delta u(y)\, dy - \int_{\partial U} u(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y)$$

PRI ČEMU JE $\quad \frac{\partial G}{\partial \nu}(x,y) = D_y G(x,y) \cdot \nu(y)$

TM 12 (FORMULA RJEŠENJA)

NEKA $u \in C^2(\bar{U})$ ZADOVOLJAVA

$$-\Delta u = f \quad u\ U$$
$$u = g \quad na\ \partial U$$

TADA

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) + \int_U f(y)G(x,y)\,dy$$

$x \in U$

NAP: FORMALNO ZA $x \in U$ PROMATRAMO $y \mapsto G(x,y)$:

$$\begin{cases} -\Delta G = \delta_x & u\ U \\ G = 0 & na\ \partial U \end{cases}$$

TM 13 (SIMETRIČNOST)

$$G(y,x) = G(x,y) \quad , \quad x,y \in U \, , \, x \neq y$$

DOK SAMI

PRIMJERI

1) $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ POLUPROSTOR

$$G(x,y) = \Phi(y-x) - \phi^x(y)$$

$$\begin{cases} \Delta \phi^x = 0 & \text{u } \mathbb{R}^n_+ \\ \phi^x = \Phi(y-x) & \text{na } \partial\mathbb{R}^n_+ = \{x\in\mathbb{R}^n : x_n = 0\} \end{cases}$$

$$\phi^x(y) := \Phi(y - \tilde{x})$$

REFLEKSIJA

$\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$

IZVAN DOMENE $\Rightarrow$

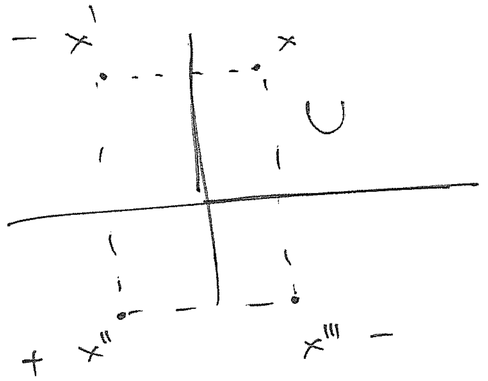
za $y \in \partial \mathbb{R}^n_+$ $\quad |y-x| = |y-\tilde{x}|$

2) KUGLA

$$\tilde{x} = \frac{x}{|x|^2} \quad \text{DUALNA TOČKA}$$

$$G(x,y) = \Phi(y-x) - \Phi\big(|x|(y-\tilde{x})\big)$$

3) KVADRANT



$$G(x,y) = \Phi(y-x) - \Phi(y-x') - \Phi(y-x''') + \Phi(y-x'')$$