7.1.3. REGULARNOST

MOTIVACIJA: NEKA JE $u(x,t)$ - DOVOLJNO GLATKA RJ.

$$\begin{cases} u_t - \Delta u = f & u \ \mathbb{R}^n \times (0,T] \\ u = g & \text{na } \mathbb{R}^n \times \{0\} \end{cases}$$
INICIJALNA ZADAĆA

1. KORAK: NEKA u DOVOLJNO BRZO TRNE ZA $|x| \longrightarrow \infty$

RAČUNAMO

$$\int_{\mathbb{R}^n} f^2 dx = \int_{\mathbb{R}^n} (u_t - \Delta u)^2 dx = \int_{\mathbb{R}^n} u_t^2 - 2\Delta u \, u_t + (\Delta u)^2 dx$$

$$\overset{P.I.}{=} \int_{\mathbb{R}^n} u_t^2 + \underbrace{2 Du \cdot Du_t}_{= \frac{d}{dt}(Du \cdot Du)} + (\Delta u)^2 dx \qquad \int_{\mathbb{R}^n} (\Delta u)^2 \overset{P.I.}{=} \int_{\mathbb{R}^n} |D^2 u|^2$$

$$= \int_{\mathbb{R}^n} u_t^2 + \frac{d}{dt} |Du|^2 + |D^2 u|^2 dx$$

INTEGRIRAM $\int_0^T dt$

$$\int_0^T \int_{\mathbb{R}^n} f^2 dx dt = \int_0^T \int_{\mathbb{R}^n} u_t^2 dx dt + \int_{\mathbb{R}^n} |Du|^2 dx \Big|_0^T + \int_0^T \int_{\mathbb{R}^n} |D^2 u|^2 dx dt$$

$$\Rightarrow \underbrace{\int_{\mathbb{R}^n} |Du|^2 dx}_{t=T} + \int_0^T \int_{\mathbb{R}^n} u_t^2 dx dt + \int_0^T \int_{\mathbb{R}^n} |D^2 u|^2 dx dt = \int_0^T \int_{\mathbb{R}^n} f^2 dx dt + \underbrace{\int_{\mathbb{R}^n} |Dg|^2 dx}_{t=0}$$

OCJENA

$$f \in L^2, \ Dg \in L^2 \Rightarrow \boxed{u_t \in L^2, \ D^2 u \in L^2, \ Du \in L^\infty(0,T; L^2(\mathbb{R}^n))}$$

BOLJE!

2. KORAK

DERIVIRAM INICIJALNU ZADAĆU PO t

$$u_{tt} - \Delta u_t = f_t \quad \text{na } \mathbb{R}^n \times \langle 0,T]$$
$$u_t = \Delta g + f \quad \text{na } \mathbb{R}^n \times \{0\}$$

IZ JEDNADŽBE ZA U

MNOŽIM S u_t; INTEGRIRAM PO $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} u_{tt} u_t - \Delta u_t u_t = \int_{\mathbb{R}^n} f_t u_t$$

P.I. $\frac{1}{2}\frac{d}{dt}|u_t|^2, \quad + Du_t Du_t$

$$\frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}|u_t|^2 + \int_{\mathbb{R}^n} |Du_t|^2 = \int_{\mathbb{R}^n} f_t u_t$$

GRONWALL / BEZ GRONWALLA

$$\Rightarrow \sup_{t\in[0,T]} \int_{\mathbb{R}^n} |u_t|^2 + \int_0^T \int_{\mathbb{R}^n} |Du_t|^2 \,dx\,dt$$
$$\leq C\left(\int_0^T \int_{\mathbb{R}^n} f_t^2 \,dx\,dt + \int_{\mathbb{R}^n} \left(|D^2 g|^2 + f(\cdot,0)^2\right)dx\right)$$

TM O TRAGU ZA $L^2(0,T;V)$ DAJE

$$\max_{t\in[0,T]} \|f(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq C\left(\|f\|_{L^2(\mathbb{R}^n\times(0,T))} + \|f_t\|_{L^2(\mathbb{R}^n\times(0,T))}\right)$$

$\Rightarrow$ DESNA STRANA OCJENJENA SA:

$\|f\|_{L^2}$, $\|f_t\|_{L^2}$, $\|D^2 g\|_{L^2}$

$\Rightarrow$ IMAMO $\boxed{u_t \in L^\infty(0,T;L^2(\mathbb{R}^n)), \quad Du_t \in L^2}$

3. KORAK 3

DODATNO

$$-\Delta u = f - u_t$$

$$\int_{\mathbb{R}^n} (\Delta u)^2 = \int_{\mathbb{R}^n} (f - u_t)^2 \leq C \int_{\mathbb{R}^n} (f^2 + u_t^2)\, dx$$

u P.I. $\qquad\qquad \Uparrow$

$$\int_{\mathbb{R}^n} |D^2 u|^2 \qquad\qquad L^\infty(0,T; L^2(\mathbb{R}^n))$$

ograničen

$$\Rightarrow \quad \boxed{D^2 u \in L^\infty(0,T; L^2(\mathbb{R}^n))}$$

NAP: OVO NIJE DOKAZ! ZAKLJUČAK SMO PRETPOSTAVILI

DOKAZ: GALERKIN

PRETPOSTAVKE: $(w_k)_{k\in\mathbb{N}}$ POTPUN SKUP SVOJSTVENIH FUNKCIJA ZA $-\Delta$ NA $H_0^1(U)$

$U \subseteq \mathbb{R}^n$ OGRANIČEN, OTVOREN, ∂U GLADAK

A, b, c GLATKE NA $\overline{U}$, NEOVISNE O t!

TEOREM 5

(i) ~~NEKA~~ JE $g \in H^1_0(U)$, $f \in L^2(0,T; L^2(U))$.

NEKA JE $u \in L^2(0,T; H^1_0(U))$, $u' \in L^2(0,T; H^{-1}(U))$

SLABO RJEŠENJE ~~ZADAĆE~~

$$\begin{aligned} u_t + Lu &= f && \text{u } U_T \\ u &= 0 && \text{na } \partial U \times [0,T] \\ u &= g && \text{na } U \times \{0\} \end{aligned}$$

TADA:

$$u \in L^2(0,T; H^2(U)) \cap L^\infty(0,T; H^1_0(U))$$

$$u' \in L^2(0,T; L^2(U)) \qquad \text{(1. KORAK)}$$

I VRIJEDI: $\exists C(U,T,L) > 0$ T.D.

$$\operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|_{H^1_0(U)} + \|u\|_{L^2(0,T; H^2(U))} + \|u'\|_{L^2(0,T; L^2(U))}$$

$$\leq C \left(\|f\|_{L^2(0,T; L^2(U))} + \|g\|_{H^1_0(U)} \right)$$

(ii) AKO JOŠ VRIJEDI $g \in H^2(U)$, $f' \in L^2(0,T; L^2(U))$

TADA:

$$u \in L^\infty(0,T; H^2(U)) \qquad \text{(3. KORAK)}$$

$$u' \in L^\infty(0,T; L^2(U)) \cap L^2(0,T; H^1_0(U)) \qquad \text{(2. KORAK)}$$

$$u'' \in L^2(0,T; H^{-1}(U))$$

I VRIJEDI:

$$\operatorname*{ess\,sup}_{t \in [0,T]} \left(\|u(t)\|_{H^2(U)} + \|u'(t)\|_{L^2(U)} \right) + \|u'\|_{L^2(0,T; H^1_0(U))} + \|u''\|_{L^2(0,T; H^{-1}(U))}$$

$$\leq C \left(\|f\|_{H^1(0,T; L^2(U))} + \|g\|_{H^2(U)} \right)$$

DOK: 1. KORAK GALERKINOVE APROKSIMACIJE

$$u_m(t) := \sum_{k=1}^{m} d_m^k(t)\, w_k$$

ZADOVOLJAVAJU

NEMA OVISNOSTI O t!

C. ZADAĆA LINEARNA SJ. ... $H^1(0,T;\mathbb{R}^m)$

$$\left|\begin{aligned} &(u_m'(t), w_k) + B[u_m(t), w_k; t] = (f(t), w_k), \quad k=1,\dots,m \\ &d_m^k(0) = (g, w_k), \quad k=1,\dots,m \end{aligned}\right.$$

MNOŽIM JDBU SA $d_m^{k\,\prime}(t)$ I $\sum_{k=1}^{m}$:

$$(u_m'(t), u_m'(t)) + B[u_m(t), u_m'(t)] = (f(t), u_m'(t)) \quad \text{s.s. } t\in[0,T]$$

$$B[u_m(t), u_m'(t)] = \int_U A\nabla u_m(t)\cdot \nabla u_m'(t)\,dx$$

$$+ \int_U b\cdot \nabla u_m(t)\, u_m'(t) + c\, u_m(t)\, u_m'(t)\,dx$$

$$= \tilde{A} + \tilde{B}$$

SIMETRIČNOST OD A

$$\tilde{A} = \int_U A\nabla u_m(t)\cdot \nabla u_m'(t)\,dx = \frac{d}{dt}\left(\frac{1}{2}\int_U A\nabla u_m(t)\cdot\nabla u_m(t)\,dx\right)$$

$$= \frac{1}{2}\frac{d}{dt} A[u_m(t), u_m(t)]$$

$$A[u,v] = \int_U A\nabla u\cdot\nabla v\,dx$$

$$|\tilde{B}| = \left| \int_U b \cdot \nabla u_m(t)\, u_m'(t) + c\, u_m(t)\, u_m'(t) \right|$$

$$\leq \int_U |b|\, |\nabla u_m(t)|\, |u_m'(t)| + |c|\, |u_m(t)|\, |u_m'(t)|$$

$$\leq C\, \|u_m(t)\|_{H_0^1(U)} \|u_m'(t)\|_{L^2(U)}$$

$$\leq \frac{C}{\varepsilon} \|u_m(t)\|^2_{H_0^1(U)} + \varepsilon \|u_m'(t)\|^2_{L^2(U)} \qquad \forall \varepsilon > 0 \quad \text{s.s. } t$$

$$|(f(t), u_m'(t))| \leq \|f(t)\|_{L^2(U)} \|u_m'(t)\|_{L^2(U)} \leq \frac{C}{\varepsilon} \|f\|^2_{L^2(U)} + \varepsilon \|u_m'(t)\|^2_{L^2(U)} \qquad \text{s.s. } t \quad \forall \varepsilon > 0$$

ENERGETSKA JEDNAKOST ZA u_m PIŠEMO:

$$\underline{\|u_m'(t)\|^2_{L^2(U)}} + \frac{d}{dt}\left(\frac{1}{2} A[u_m(t), u_m(t)]\right)$$

$$= (f(t), u_m'(t)) - \tilde{B}$$

$$\leq \frac{C}{\varepsilon}\left(\|u_m(t)\|^2_{H_0^1(U)} + \|f\|^2_{L^2(U)}\right) + \underline{2\varepsilon \|u_m'(t)\|^2_{L^2(U)}}$$

INTEGRIRAM $\int_0^t dt$ $\Big|$ $\longrightarrow$ $\varepsilon = \frac{1}{4}$

$$\int_0^t \|u_m'(t)\|^2_{L^2(U)}\, dt + A[u_m(t), u_m(t)]$$

$$\leq A[u_m(0), u_m(0)] + C\left(\int_0^t \|u_m(\tau)\|^2_{H_0^1(U)}\, d\tau + \int_0^t \|f(\tau)\|^2_{L^2(U)}\, d\tau\right)$$

OCIJENIM $t=T$ NA DESNU STRANU

$$\int_0^t \|u_m'(\tau)\|^2_{L^2(U)}\,d\tau + A[u_m(t),u_m(t)]$$

$$\le \int_U A\nabla g\cdot\nabla g\,dx + C\Big(\underbrace{\int_0^T \|u_m(\tau)\|^2_{H_0^1(U)}\,d\tau}_{} + \int_0^T \|f(\tau)\|^2_{L^2(U)}\,d\tau\Big)$$

$$\le C\Big(\|g\|^2_{H_0^1(U)} + \overbrace{\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T;L^2(U))}}^{\text{TM2}\ \wedge}\Big)$$

$$\le C\left(\|g\|^2_{H_0^1(U)} + \|f\|^2_{L^2(0,T;L^2(U))}\right)$$

$\forall$ s.s. t

$$\Rightarrow \int_0^T \|u_m'(\tau)\|^2_{L^2(U)}\,d\tau \le C\left(\|g\|^2_{H_0^1(U)} + \|f\|^2_{L^2(0,T;L^2(U))}\right)$$

$$A[u_m(t),u_m(t)] \le C\left(\quad -\ \text{"}\ - \quad\right)$$

$\vee\!/$ s.s. t

$$\Rightarrow \theta \int_\Omega \|u_m(t)\|^2_{H_0^1(U)} \qquad \Big|\ \operatorname*{ess\,sup}_{t\in[0,T]}$$

$$\Rightarrow \|u_m'\|^2_{L^2(0,T;L^2(U))} \le C\left(\|g\|^2_{H_0^1(U)} + \|f\|^2_{L^2(0,T;L^2(U))}\right)$$

$$\|u_m\|^2_{L^\infty(0,T;H_0^1(U))} \le C\left(\|g\|^2_{H_0^1(U)} + \|f\|^2_{L^2(0,T;L^2(U))}\right)$$

NA PODNIZU

$$u_{m_\ell}' \longrightarrow u' \in L^2(0,T;L^2(U))$$

$$u_{m_\ell} \xrightarrow{*} u \in L^\infty(0,T;H_0^1(U))$$

DODATNO u, u' ZADOVOLJAVAJU:

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_{H_0^1(U)} + \|u'\|_{L^2(0,T;L^2(U))} \leq C\left(\|g\|_{H_0^1(U)} + \|f\|_{L^2(0,T;L^2(U))}\right)$$

3. KORAK ZA (i)

OSTAJE: $u \in L^2(0,T;H^2(U))$ & OCJENA

DOK: ZNAMO DA OVAJ LIMES ZADOVOLJAVA (VIDI DOKAZ TM 3)

$$(u'(t), v) + B[u(t), v] = (f(t), v), \qquad v \in H_0^1(U)$$

NAP: SAD ZNAMO DA JE $u' \in L^2(0,T;L^2(U))$ OK

$$\Rightarrow \quad B[u(t), v] = (\underbrace{f(t) - u'(t)}_{h(t)}, v), \quad v \in H_0^1(U)$$

$h(t) \in L^2(U)$ s.s $t \in [0,T]$

ELIPTIČKA REGULARNOST $\Rightarrow$ $u(t) \in H^2(U)$ s.s $t \in [0,T]$ &

TM 4 §6.3.2.

$$\|u(t)\|^2_{H^2(U)} \leq C\left(\|h(t)\|^2_{L^2(U)} + \|u(t)\|^2_{L^2(U)}\right)$$
$$\leq C\left(\|f(t)\|^2_{L^2(U)} + \|u(t)\|^2_{L^2(U)} + \|u'(t)\|^2_{L^2(U)}\right)$$

INTEGRIRAM PO $t \in [0,T)$

$$\|u\|^2_{L^2(0,T;H^2(U))} \leq C\left(\|f\|^2_{L^2(0,T;L^2(U))} + \underbrace{\|u\|^2_{L^2(0,T;L^2(U))} + \|u'\|^2_{L^2(0,T;L^2(U))}}_{\leq \dots}\right)$$

$\Rightarrow$ OK

$\Rightarrow$ (i) DOKAZAN!

AD (ii)

4. KORAK : TREBA $g \in H^2(U) \cap H_0^1(U)$, $f \in H^1(0,T; L^2(U))$

$$(u_m'(t), w_k) + B[u_m(t), w_k] = \underbrace{(f(t), w_k)}_{\in H^1(0,T)} \qquad k = 1, \dots, m$$

LINEARNA ZADAĆA

$$\Rightarrow \quad d_m^k \in H^2(0,T) \quad \Rightarrow \quad u_m \in H^2(0,T; H_0^1(U))$$

$$u_m' \in H^1(0,T; H_0^1(U)) \Rightarrow \in C(0,T; H_0^1(U))$$

MOGU DERIVIRATI PO t!

$$(u_m''(t), w_k) + B[u_m'(t), w_k] = (f'(t), w_k) \qquad k = 1, \dots, m$$

MNOŽIM s $d_m^{k'}(t)$ i $\sum_{k=1}^{m}$

$$(u_m''(t), u_m'(t)) + B[u_m'(t), u_m'(t)] = (f'(t), u_m'(t))$$

ISTO KAO PRIJE (ZADATAK ZA ZADAĆU)

$$\sup_{t \in [0,T]} \|u_m'(t)\|^2_{L^2(U)} + \|u_m'\|^2_{L^2(0,T; H_0^1(U))}$$

$$\le C \left(\|f'\|^2_{L^2(0,T; L^2(U))} + \|u_m'(0)\|^2_{L^2(U)} \right) \qquad (*)$$

VRIJEDI

$$(u_m'(0), w_k) = (f(0), w_k) - B[u_m(0), w_k]$$

$$\le \|f(0)\|_{L^2(U)} \|w_k\|_{L^2(U)} + C \|u_m(0)\|_{H^2(U)} \|w_k\|_{L^2(U)}$$

$$\Rightarrow \quad \|u_m'(0)\|_{L^2(U)} \le C \Big(\underbrace{\|f(0)\|_{L^2(U)}}_{\le \|f\|_{H^1(0,T; L^2(U))}} + \|u_m(0)\|_{H^2(U)} \Big)$$

STOGA (*) POSTAJE:

$$\sup_{t\in[0,T]} \|u_m'(t)\|_{L^2(U)} + \|u_m'\|_{L^2(0,T;H_0^1(U))}$$

$$\leq C\left(\|f\|^2_{H^1(0,T;L^2(U))} + \|u_m(0)\|^2_{H^2(U)}\right) \qquad (**)$$

TREBA OCIJENITI

$$u_m(0) = \sum_{k=1}^m d_m^k(0)\, w_k \quad , \quad w_k \text{ su } C^\infty \ : \ w_k\big|_{\partial U} = 0$$

$$\Rightarrow -\Delta w_k\big|_{\partial U} = \lambda_k w_k\big|_{\partial U} = 0$$

$$\Rightarrow \Delta u_m(0)\big|_{\partial U} = 0 \ , \quad u_m(0)\big|_{\partial U} = 0$$

$$\|u_m(0)\|^2_{H^2(U)} \underset{\text{P.I.}}{\leq} C\|\Delta u_m(0)\|^2_{L^2(U)} = C\left(\Delta u_m(0), \Delta u_m(0)\right)$$

$$\overset{\text{P.I.}}{=} -C\left(\nabla u_m(0), \nabla\Delta u_m(0)\right) + C\int_{\partial U} \nabla u_m(0)\cdot n\, \underbrace{\Delta u_m(0)}_{=0}$$

SKALARNI u $L^2(U;\mathbb{R}^n)$

$$\overset{\text{P.I.}}{=} C\left(u_m(0), \Delta^2 u_m(0)\right) - C\int_{\partial U} \underbrace{u_m(0)}_{=0} \nabla\Delta u_m(0)\cdot n$$

$$\Rightarrow \|u_m(0)\|^2_{H^2(U)} \leq C\left(u_m(0), \Delta^2 u_m(0)\right)$$

$$= C\left(u_m(0), \sum_{k=1}^m d_m^k(0)\, \underbrace{\Delta^2 w_k}_{\lambda_k^2 w_k}\right)$$

$$= C\sum_{k=1}^m d_m^k(0)\,\lambda_k^2\, \underbrace{(u_m(0), w_k)}_{=(g, w_k)}$$

POČETNI UVJET ZA d_m^k

$$= C\left(g, \sum_{k=1}^m d_m^k(0)\lambda_k^2 w_k\right) = C\left(g, \Delta^2 u_m(0)\right)$$

$$\overset{2\times\text{P.I.}}{=} C\left(\Delta g, \Delta u_m(0)\right)$$

$$\Rightarrow \quad \|u_m(0)\|^2_{H^2(U)} \le C\,(\Delta g, \Delta u_m(0))$$

$$\le \tfrac{1}{2}\|u_m(0)\|^2_{H^2(U)} + C\|\Delta g\|^2_{L^2(U)} \qquad \text{(GEN. CAUCHY)}$$

$$\le \tfrac{1}{2}\|u_m(0)\|^2_{H^2(U)} + C\|g\|^2_{H^2(U)}$$

$$\Rightarrow \quad \|u_m(0)\|_{H^2(U)} \le C\|g\|_{H^2(U)}$$

IZ (**) DOBIVAMO

$$\boxed{\sup_{t\in[0,T]} \|u_m'(t)\|_{L^2(U)} + \|u_m'\|_{L^2(0,T;H^1_0(U))} \le C\left(\|f\|^2_{H^1(0,T;L^2(U))} + \|g\|^2_{H^2(U)}\right)}$$

6. KORAK

$$B[u_m(t), w_k] = (f(t) - u_m'(t), w_k) \quad k = 1,\dots,m$$

MNOŽIMO S $\lambda_k d_m^k(t)$, $\sum_{k=1}^{m}$

$$B[u_m(t), -\Delta u_m(t)] = (f(t) - u_m'(t), -\Delta u_m(t)) \quad t\in[0,T]$$

NEPR. PO t

$\exists \beta > 0, \gamma \ge 0$ T.D.

TV! $\beta\|u\|^2_{H^2(U)} \le (Lu, -\Delta u) + \gamma\|u\|^2_{L^2(U)}, \quad u\in H^2(U)\cap H^1_0(U)$

JER JE $\quad B[u_m(t), -\Delta u_m(t)] \overset{\text{P.I. \& } \Delta u_m|_{\partial U}=0}{=} (Lu_m(t), -\Delta u_m(t))$

$$\Rightarrow \quad \beta\|u_m(t)\|^2_{H^2(U)} \le \left(\|f(t)\|_{L^2(U)} + \|u_m'(t)\|_{L^2(U)}\right)\underbrace{\|\Delta u_m(t)\|_{L^2(U)}}_{\le \|u_m(t)\|_{H^2(U)}} + \underbrace{\|u_m(t)\|^2_{L^2(U)}}_{\le \|u_m(t)\|_{L^2(U)}\cdot\|u_m(t)\|_{H^2(U)}}$$

$$\Rightarrow \|u_m(t)\|_{H^2(U)} \le C\left(\|f(t)\|_{L^2(U)} + \|u_m'(t)\|_{L^2(U)} + \|u_m(t)\|_{L^2(U)}\right)$$

$\square \Rightarrow \|u_m'(t)\|_{L^2}$ JE KONTROLIRAN S $\|f\|_{H^1(0,T;L^2(U))}, \|g\|_{H^2(U)}$

IZ (i) $\|u_m(t)\|_{L^2(U)}$ ISTO

$\sup \|f(t)\|_{L^2(U)}$ JE KONTROLIRAN S $\|f\|_{H^1(0,T;L^2(U))}$

$$\Rightarrow \boxed{\sup_{t\in[0,T]} \|u_m(t)\|_{H^2(U)} \leq C\left(\|f\|_{H^1(0,T;L^2(U))} + \|g\|_{H^2(U)}\right)}$$

IZ OVE DVIJE $\square$ FORMULE $\Rightarrow$ NA LIMESU VRIJEDE OCJENE & $u \in L^\infty(0,T;H^2(U))$

$u' \in L^\infty(0,T;L^2(U))$

$u' \in L^2(0,T;H_0^1(U))$

7. KORAK u'' & OCJENA

$v \in H_0^1(U)$, $\|v\|_{H_0^1(U)} \leq 1$, $v = v^1 + v^2$, $v^1 \in L\{w_1,\dots,w_m\}$

$v^2 \perp L\{w_1,\dots,w_m\}$ u $L^2(U)$

$$\langle u_m''(t), v\rangle = (u_m''(t), v) = (u_m''(t), v^1) = (f'(t), v^1) - B[u_m'(t), v^1]$$

$$\Rightarrow |\langle u_m''(t), v\rangle| \leq \|f'(t)\|_{L^2(U)} \underbrace{\|v^1\|_{L^2(U)}}_{\leq 1} + C\|u_m'(t)\|_{H_0^1(U)} \underbrace{\|v\|_{H_0^1(U)}}_{\leq 1}$$

$$\Rightarrow \|u_m''(t)\|_{H^{-1}(U)} \leq C\left(\|f'(t)\|_{L^2(U)} + \|u_m'(t)\|_{H_0^1(U)}\right)$$

$$\Rightarrow \int_0^T \|u_m''(t)\|^2_{H^{-1}(U)} \leq C\left(\int_0^T \|f'(t)\|^2_{L^2(U)} + \int_0^T \|u_m'(t)\|^2_{H_0^1(U)}\right)$$

$$\Rightarrow \|u_m''\|^2_{L^2(0,T;H^{-1}(U))} \leq C\left(\|f\|^2_{H^1(0,T;L^2(U))} + \underbrace{\|u_m'\|^2_{L^2(0,T;H_0^1(U))}}_{\leq \|f\|^2_{H^1(0,T;L^2(U))} + \|g\|^2_{H^2(U)}}\right)$$

$\Rightarrow u'' \in L^2(0,T;H^{-1}(U))$ & OCJENA