# 7. LINEARNE EVOLUCIJSKE JEDNADŽBE



EVOLUCIJSKE JEDNADŽBE ... $t$ - VARIJABLA VRIJEME

- ENERGETSKA METODA
- FOURIEROVA TRANSFORMACIJA
- POLUGRUPE

## 7.1. PARABOLIČKE JEDNADŽBE 2. REDA

- POOPĆENJE JEDNADŽBE PROVOĐENJA

$$u_t = \Delta u$$

### 7.1.1. DEFINICIJE

- $U \subseteq \mathbb{R}^n$ OTVOREN, OGRANIČEN
- $U_T = U \times \langle 0, T]$ , $T > 0$

ZADAĆA

$$\begin{cases} u_t + Lu = f & u \quad U_T \\ u = 0 & na \quad \partial U \times [0,T] \\ u = g & na \quad U \times \{0\} \end{cases}$$

ZADAĆA (*)

ZADANO

$f : U_T \to \mathbb{R}$

$g : U \to \mathbb{R}$

$Lu = -\operatorname{div}(A\nabla u) + b \cdot \nabla u + cu$ (DIV-OBLIK)

ILI

$Lu = -A \cdot H_u + b \cdot \nabla u + cu$ (NEDIV-OBLIK)

$A : U_T \to M_n(\mathbb{R})$

$b : U_T \to \mathbb{R}^n$ (OVISE I O VREMENU)

$c : U_T \to \mathbb{R}$

TRAŽI SE:

$u : U_T \to \mathbb{R}$, $u(x,t)$

DEF: OPERATOR $\frac{\partial}{\partial t} + L$ JE (UNIFORMNO) PARABOLIČKI AKO $\exists\, \theta > 0$ T.D.

$$A(x,t)\xi \cdot \xi \geq \theta |\xi|^2, \quad (x,t) \in U_T, \ \xi \in \mathbb{R}^n$$

NAP: $\forall t \in [0,T]$ $A(\cdot, t)$ JE UNIFORMNO ELIPTIČKI.

PR:

$$A = I, \quad b = 0, \quad c = 0, \quad f = 0$$

$$L = -\Delta$$

$$u_t = \Delta u$$

NAP: PARABOLIČKE J. NPR. OPISUJU POJAVE VREMENSKE EVOLUCIJE GUSTOĆE $u$ (NPR. KEMIJSKE KONCENTRACIJE) U $U$.

$\operatorname{div}(A \nabla u)$ — DIFUZIJA

$b \cdot \nabla u$ — TRANSPORT

$c u$ — NASTANAK / NESTANAK TVARI

JAVLJAJU SE I U VJEROJATNOSTI

## DEFINICIJA SLABOG RJEŠENJA

PRETP:

$$A \in L^\infty(U_T; M_n(\mathbb{R})), \quad A = A^T$$

$$b \in L^\infty(U_T; \mathbb{R}^n)$$

$$c \in L^\infty(U_T; \mathbb{R})$$

$$f \in L^2(U_T; \mathbb{R})$$

$$g \in L^2(U)$$

OZNAKA:

$$B[u, v; t] := \int_U A(\cdot, t) \nabla u \cdot \nabla v + b(\cdot, t) \cdot \nabla u \, v + c(\cdot, t) u v$$

— BILINEARNA FORMA, $u, v \in H_0^1(U)$

MOTIVACIJA

PRETP: $u$ ... DOVOLJNO GLATKO RJ. $(*)$

$(x,t) \mapsto u(x,t)$ SHVATIMO ~~KAO~~ $t \mapsto u(t) \in H_0^1(U)$

TJ. POISTOVJETIMO $u(t)(x) = u(x,t)$

ISTO IZA $f(t)(x) = f(x,t)$

$$u_t + Lu = f \quad \Big| \; v \in H_0^1(U) \quad \int_U$$

$$\int_U u_t v + \underbrace{Lu\, v}_{\text{P.I.}} = \int_U f v, \quad v \in H_0^1(U)$$

$$(u', v) + B[u, v; t] = (f, v), \quad v \in H_0^1(U)$$

$$u'(t) = \frac{\partial u}{\partial t}(\cdot, t)$$

$$(u'(t), v) + B[u(t), v; t] = (f(t), v), \quad v \in H_0^1(U)$$

$(\cdot, \cdot)$ $L^2$ SKALARNI PRODUKT!

$$\Rightarrow (u'(t), v) = -B[u(t), v; t] + (f(t), v), \quad v \in H_0^1(U)$$

$$= + \int_U \underbrace{-A(\cdot,t)\nabla u(t)}_{= h_x(t)} \cdot \nabla v + \underbrace{\big(f(t) - b(\cdot,t)\nabla u(t) - c(\cdot,t)u(t)\big)}_{= h^0} v$$

$h_x^{(t)} \in L^2(U)$ s.s.t $h^0(t) \in L^2(U)$

$$\Rightarrow |(u'(t), v)| \le \|h_x^{(t)}\|_{L^2(U)} \|\nabla v\|_{L^2(U)} + \|h^0(t)\|_{L^2(U)} \|v\|_{L^2(U)}$$

$$\Rightarrow \quad |(u'(t),v)| \le C\left(\|h_x(t)\|_{L^2(U)} + \|h^0(t)\|_{L^2(U)}\right)\|v\|_{H_0^1(U)}$$

$$\le C\left(\|u\|_{H_0^1(U)} + \|f\|_{L^2(U)}\right)\|v\|_{H_0^1(U)}, \quad v \in H_0^1(U)$$

$\Rightarrow$ $u'(t)$ JE NEPREKIDAN L.F. NA $H_0^1(U)$ ZA S.S. $t$

$\Rightarrow$ $u'(t) \in H^{-1}(U)$ S.S. $t$

$$\Rightarrow \quad (u'(t),v) = {}_{H^{-1}(U)}\langle u'(t), v\rangle_{H_0^1(U)}$$

NAP: PRETPOSTAVILI SMO DA JE $u$ DOVOLJNO GLATKA.

DEF: FUNKCIJU

$$u \in L^2(0,T; H_0^1(U)) \text{ T.D. } u' \in L^2(0,T; H^{-1}(U))$$

NAZIVAMO SLABIM RJEŠENJEM ZADAĆE (*) AKO

(i) $\langle u', v\rangle + B[u,v;t] = (f,v)$, $v \in H_0^1(U)$, S.S. $t \in [0,T]$

(ii) $u(0) = g$

NAP: $\left.\begin{array}{l} u \in L^2(0,T; H_0^1(U)) \\ u' \in L^2(0,T; H^{-1}(U)) \end{array}\right\}$ TH 3 U §5.9.2 $\Rightarrow u \in C([0,T]; L^2(U))$

$\Rightarrow$ (ii) IMA SMISLA!

## 7.1.2. EGZISTENCIJA SLABOG RJEŠENJA

GALERKINOVA METODA!

- KONSTRUIRAMO KONAČNO-DIMENZIONALNE APROKSIMACIJE
- KONVERGIRAJU K S. RJEŠENJU $\Rightarrow$ POSTOJI!

NEKA JE $(w_k)_k$ OB OD $H_0^1(U)$

OHB OD $L^2(U)$

TAKVA POSTOJI!

SVOJSTVENI VEKTORI OD $-\Delta$

~~NEN.~~

DEF:

$$u_m : [0,T] \to H_0^1(U)$$

$$u_m(t) = \sum_{k=1}^{m} d_m^k(t)\, w_k$$

PRI ČEMU $(d_m^k)_{k=1,\dots,m}$ ZADOVOLJAVAJU:

$$\text{(ODJ)} \quad \begin{cases} (u_m', w_k) + B[u_m, w_k; t] = (f, w_k), & k=1,\dots,m \\ & t\in[0,T] \\ d_m^k(0) = (g, w_k), & k=1,\dots,m \end{cases}$$

$(\cdot,\cdot)$ SKALARNI PRODUKT U $L^2(U)$.

OVAJ PROBLEM JE KONAČNO-DIMENZIONALNA APROKSIMACIJA ORIGINALNOG PROBLEMA NA PROSTORU $L\{w_1,\dots,w_k\}$

TEOREM 1 $\forall m \in \mathbb{N}$ $\exists!$ $u_m$ ZADANOG OBLIKA KOJI ZADOVOLJAVA (ODJ)

DOK:

$$(u_m'(t), w_k) = \left(\sum_{j=1}^{m} (d_m^j)'(t)\, w_j, w_k\right)$$

$$= \sum_{j=1}^{m} (d_m^j)'(t) \underbrace{(w_j, w_k)}_{\substack{\| \\ \delta_{jk}}} \quad \text{ONB u } L^2(U)$$

$$= (d_m^k)'(t)$$

$$B[u_m(t), w_k; t] = \int_U A(\cdot,t) \nabla\left(\sum_{j=1}^{m} (d_m^j)(t)\, w_j\right) \cdot \nabla w_k$$

$$= \sum_{j=1}^{m} d_m^j(t) \int_U A(\cdot,t) \nabla w_j \cdot \nabla w_k$$

$$L^\infty(0,T) \ni M_m(t) := \left(\int_U A(\cdot,t) \nabla w_j \cdot \nabla w_k\right)_{k,j=1,\dots,m}$$

$$= \left(M_m(t)\, D_m(t)\right)_k$$

$$D_m(t) := \begin{bmatrix} d_m^1(t) \\ \vdots \\ d_m^m(t) \end{bmatrix}$$

$$F_m(t) := \begin{bmatrix} \int_U f(t)\, w_1 \\ \vdots \\ \int_U f(t)\, w_k \end{bmatrix} \Rightarrow F \in L^2(0,T)$$

(ODJ) $\Rightarrow$

$$D_m'(t) + M_m(t) D_m(t) = F_m(t)$$

SUSTAV ODJ 1. REDA, LINEARAN!

DEF: $G := \begin{bmatrix} (g, w_1) \\ \vdots \\ (g, w_k) \end{bmatrix}$

C. ZADAĆA ZA SUSTAV ODJ 1. REDA

$\Rightarrow \quad D_m(0) = G$

$\Rightarrow$ $\exists!$ RJEŠENJE, APSOLUTNO NEPREKIDNO, ~~SUSTAVA~~ C. ZADAĆE ZA SUSTAV.

PRI TOME JE SUSTAV ZADOVOLJEN S.S. $t$.

# ENERGETSKE OCJENE

- ŽELIMO PUSTITI $m \to \infty$ U $u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k$

  I IZLIMESIRATI JEDNADŽBU KOJU ZADOVOLJAVA $u_m$

  DA BI ZA LIMES $u$ UTVRDILI DA ZADOVOLJA (i), (ii)

  TJ. DA JE SLABO RJEŠENJE

- ZA TO NAM TREBA DA JE $u_m$ KONVERGENTAN U ODGOVARAJUĆIM TOPOLOGIJAMA

- TOME SLUŽE ENERGETSKE OCJENE $\Rightarrow$ SLABE KONVERGENCIJE

<u>TEOREM 2</u> $\exists\, C > 0$ $(C(U, T, L))$ T.D.

$$\max_{t \in [0,T]} \|u_m(t)\|_{L^2(U)} + \|u_m\|_{L^2(0,T;H_0^1(U))} + \|u_m'\|_{L^2(0,T;H^{-1}(U))}$$

$$\leq C\left(\|f\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)}\right)$$

$\forall\, m \in \mathbb{N}$.

DOK: $u_m$ ZADOVOLJAVA:

$$(u_m', w_k) + B[u_m, w_k; t] = (f, w_k) \quad , \text{s.s. } t \in [0,T], \ k=1,\dots,m$$

MNOŽIMO SOBU S $d_m^k$ (k-TU) I SUMIRAMO PO $k=1,\dots,m$

$$(u_m', u_m) + B[u_m, u_m; t] = (f, u_m) \qquad \text{s.s. } t \in [0,T]$$

- U DIJELU O ELIPTIČKIM JEDNADŽBAMA §6.2.2 POKAZALI SMO DA $\exists\ \beta > 0, \gamma \geq 0$ T.D.

$$\beta \|u(t)\|^2_{H_0^1(U)} \leq B[u(t), u(t); t] + \gamma \|u(t)\|^2_{L^2(U)} \quad , \ t \in [0,T], \ m \in \mathbb{N}.$$

- $$|(f, u(t))| \leq \frac{1}{2}\|f\|^2_{L^2(U)} + \frac{1}{2}\|u(t)\|^2_{L^2(U)} \quad , \ \text{s.s. } t \in [0,T]$$

- $$(u_m'(t), u_m(t)) = \frac{1}{2}\frac{d}{dt}\left(\|u_m(t)\|^2_{L^2(U)}\right) \qquad \text{s.s. } t \in [0,T]$$

$\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\left(\|u_m(t)\|^2_{L^2(U)}\right) + B[u_m(t), u_m(t); t] + \gamma\|u_m(t)\|^2_{L^2(U)}$$

$$\leq \frac{1}{2}\|f(t)\|^2_{L^2(U)} + \frac{1}{2}\|u_m(t)\|^2_{L^2(U)} + \gamma\|u_m(t)\|^2_{L^2(U)}$$

$\Rightarrow$ $$\boxed{\frac{d}{dt}\|u_m(t)\|^2_{L^2(U)} + 2\beta\|u_m(t)\|^2_{H_0^1(U)} \leq \|f(t)\|^2_{L^2(U)} + (1+\gamma)\|u_m(t)\|^2_{L^2(U)}}$$

$$\Rightarrow \quad \frac{d}{dt}\|u_m(t)\|^2_{L^2(U)} \leq \|f(t)\|^2_{L^2(U)} + (1+\gamma)\|u_m(t)\|^2_{L^2(U)}$$

OZNAČIMO: $\eta(t) := \|u_m(t)\|^2_{L^2(U)}$

$$\xi(t) := \|f(t)\|^2_{L^2(U)}$$

$$\Rightarrow \quad \eta'(t) \leq (1+\gamma)\,\eta(t) + \xi(t) \qquad \text{s.s. } t\in[0,T]$$

GRONWALLOVA NEJEDNAKOST (DIFERENCIJALNI OBLIK)

$$\eta'(t) \leq \phi(t)\,\eta(t) + \psi(t) \qquad \text{s.s } t$$

$\eta \geq 0$, $\eta$ APSOLUTNO NEPREKIDNA NA $[0,T]$

$\phi, \psi \geq 0$ INTEGRABILNE NA $[0,T]$

$$\Rightarrow \quad \eta(t) \leq e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)\,ds\right] \quad t\in[0,T]$$

$$\Rightarrow \quad \eta(t) \leq e^{\int_0^t (1+\gamma)ds}\left[\eta(0) + \int_0^t \xi(s)ds\right]$$

$$= e^{(1+\gamma)t}\Big[\underset{\#}{\|u_m(0)\|^2_{L^2(U)}} + \underbrace{\int_0^t \|f(s)\|^2_{L^2(U)}ds}_{\|f\|^2_{L^2(0,T;\,L^2(U))}}\Big]$$

$$\|u_m(0)\|^2_{L^2(U)} = \sum_{k=1}^{m} d_m^k(0)^2 = \sum_{k=1}^{m}(g, w_k)^2$$

S DRUGE STRANE

$$\|g\|^2_{L^2(U)} = \sum_{k=1}^{\infty}(g, w_k)^2 \qquad \text{(PARSEVALOVA JEDNAKOST)}$$

$$\Rightarrow \quad \|u_m(0)\|^2_{L^2(U)} \leq \|g\|^2_{L^2(U)}$$

$$\Rightarrow \quad \|u_m(t)\|^2_{L^2(U)} \le \underbrace{e^{(\gamma+1)t}}_{\le C} \left[ \|g\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T;L^2(U))} \right] \quad \text{ZA } t \in [0,T]$$

$$\Rightarrow \quad \boxed{\max_{t\in[0,T]} \|u_m(t)\|^2_{L^2(U)} \le C\left(\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T;L^2(U))}\right)}$$

VRATIM SE U $\square$ NA 221 DOBIVAM I (NAKON INTEGRACIJE $\int_0^T$) NAIME $\frac{d}{dt}\|u_m(t)\|^2_{L^2(U)}$ MOŽE BITI $<0$

$$\|u_m(T)\|^2_{L^2(U)} + 2\beta \int_0^T \|u_m(t)\|^2_{H^1_0(U)}\,dt$$

$$\le \|f\|^2_{L^2(0,T;L^2(U))} + (1+\gamma)\underbrace{\|u_m\|^2_{L^2(0,T;L^2(U))}}_{\left(\le C(\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T;L^2(U))}) \text{ ODOZGO}\right)}$$

$$+ \|u_m(0)\|^2_{L^2(U)}$$

$$\Rightarrow \quad \boxed{\|u_m\|^2_{L^2(0,T;H^1_0(U))} \le C\left(\|f\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\right)}$$

OSTAJE DOKAZATI OCJENU ZA $\|u_m'\|_{L^2(0,T;H^{-1}(U))}$

$$(u_m'(t), w_k) + B[u_m(t), w_k; t] = (f(t), w_k) \quad , \text{ s.s. } t\in[0,T],\ k=1,\dots,m$$

PO LINEARNOSTI PROŠIRIMO NA $L\{w_1,\dots,w_m\}$.

NEKA JE $v \in H_0^1(U)$ , $\|v\|_{H_0^1(U)} \leq 1$

ZA $u_m'$ TREBAMO OCJENU NA $H^{-1}(U) = H_0^1(U)'$ (FUNKCIONALI)

DJELOVANJE NA $v$

$$v = v^1 + v^2 \ , \qquad v^1 \in L\{w_1,\dots,w_m\}$$

$$v^2 \perp L\{w_1,\dots,w_m\} \quad \text{U } L^2(U) \quad (\text{ALI I } H_0^1(U))$$

$$\Rightarrow \quad \cancel{\phantom{xx}} \quad v = \sum_{k=1}^{\infty} (v, w_k) w_k \qquad \text{KVG U } H_0^1(U)$$

$$= \underbrace{\sum_{k=1}^{m} (v, w_k) w_k}_{v^1} + \underbrace{\sum_{k=m+1}^{\infty} (v, w_k) w_k}_{v^2}$$

$$\|v\|^2_{H_0^1(U)} = \|v^1\|^2_{H_0^1(U)} + \|v^2\|^2_{H_0^1(U)}$$

$$\Rightarrow \quad \|v^1\|_{H_0^1(U)} \leq \|v\|_{H_0^1(U)} \leq 1$$

$$(u_m'(t), v^1) + B[u_m(t), v^1; t] = (f(t), v^1) \qquad \text{s.s. } t\in[0,T]$$

S DRUGE STRANE

$$\langle u_m'(t), v\rangle = (u_m'(t), v) = (u_m', v^1) = (f(t), v^1) - B[u_m(t), v^1; t]$$

$$\Rightarrow |\langle u_m'(t), v\rangle| \leq \|f(t)\|_{L^2(U)} \|v^1\|_{L^2(U)} + C\|u_m(t)\|_{H_0^1(U)} \|v^1\|_{H_0^1(U)}$$

$$\leq C\left(\|f(t)\|_{L^2(U)} + \|u_m(t)\|_{H_0^1(U)}\right)\|v\|_{H_0^1(U)}$$

$$\Rightarrow \quad \|u_m'(t)\|_{H^{-1}(U)} \leq C\left(\|f(t)\|_{L^2(U)} + \|u_m(t)\|_{H_0^1(U)}\right) \qquad |^2 \ \Big| \int_0^T dt$$

$$\Rightarrow \quad \|u_m'\|^2_{L^2(0,T;H^{-1}(U))} \leq C\left(\|f\|^2_{L^2(0,T;L^2(U))} + \|u_m\|^2_{L^2(0,T;H_0^1(U))}\right)$$

$$\leq C\left(\|f\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\right)$$

EGZISTENCIJA I JEDINSTVENOST

$m \longrightarrow +\infty$

TEOREM 3 $\exists !$ SLABO RJEŠENJE ZADAĆE

$$u_t + Lu = f \quad \text{u } U_T$$
$$u = 0 \quad \text{NA } \partial U \times [0,T]$$
$$u = g \quad \text{NA } U \times \{0\}$$

DOK: ZA NIZ $u_m$ OGRANIČENO JE (NEOVISNO O $m$)

$$\|u_m\|_{L^\infty(0,T;L^2(U))}, \quad \|u_m\|_{L^2(0,T;H_0^1(U))}, \quad \|u_m'\|_{L^2(0,T;H^{-1}(U))}$$

$\Rightarrow \exists$ PODNIZ $(u_{m_\ell})_\ell$ I FUNKCIJA $u$ T.D.

$$u \in L^2(0,T;H_0^1(U))$$
$$u' \in L^2(0,T;H^{-1}(U))$$

TAKVI DA VRIJEDI

$$u_{m_\ell} \longrightarrow u \quad \text{SLABO u } L^2(0,T;H_0^1(U))$$
$$u_{m_\ell}' \longrightarrow u' \quad \text{SLABO u } L^2(0,T;H^{-1}(U))$$

NEKA JE $N \in \mathbb{N}$ I $d^k \in C^1([0,T])$ $k=1,\dots,N$.

DEF:

$$v(t) = \sum_{k=1}^{N} d^k(t)\, w_k.$$

ZA $m \geq N$ PROMATRAMO:

$$(u_m'(t), w_k) + B[u_m(t), w_k; t] = (f(t), w_k), \quad \text{s.s. } t \in [0,T], \; k=1,\dots,N$$

MNOŽIM SA $d^k(t)$, $\sum_{k=1}^{N}$, $\int_0^T dt$

$$\int_0^T (u_m'(t), v(t))\,dt + \int_0^T B[u_m(t), v(t); t]\,dt = \int_0^T (f(t), v(t))\,dt$$

UZMEM SAMO PODNIZ

$$\int_0^T \langle u_{m_\ell}'(t), v(t)\rangle dt + \int_0^T B[u_{m_\ell}(t), v(t); t]\, dt = \int_0^T (f(t), v(t))\, dt$$

$\downarrow$ 2. KVG $\qquad\qquad \downarrow$ 1. KVG $\qquad\qquad \|$

$$\int_0^T \langle u'(t), v(t)\rangle dt + \int_0^T B[u(t), v(t); t]\, dt = \int_0^T (f(t), v(t))\, dt$$

OVO VRIJEDI $\forall \quad v(t) = \sum_{k=1}^{N} d^k(t) w_k$

OVE FUNKCIJE ČINE GUST SKUP U $L^2(0,T; H_0^1(U))$

$\Rightarrow$ JEDNADŽBA VRIJEDI ZA SVAKI $L^2(0,T; H_0^1(U))$

SAD ODABEREMO $v(t) = w\, \varphi(t)$, $\quad w \in H_0^1(U)$, $\varphi \in C^\infty([0,T])$

$$\int_0^T \langle u'(t), w\rangle \varphi(t)\, dt + \int_0^T B[u(t), w; t]\varphi(t)\, dt = \int_0^T (f(t), w)\varphi(t)\, dt$$

$$\Rightarrow \quad \langle u'(t), w\rangle + B[u(t), w; t] = (f(t), w), \quad w \in H_0^1(U) \text{ s.s. } t \in [0,T]$$

DOBILI SMO JEDNADŽBU!

OSTAJE POČETNI UVJET.

$u(0)$ MORA IMATI SMISAO.

ZNAMO: $\left.\begin{array}{l} u \in L^2(0,T; H_0^1(U)) \\ u' \in L^2(0,T; H^{-1}(U)) \end{array}\right\} \xRightarrow{\text{TM3 §5.9.2}} u \in C([0,T]; L^2(U))$

ODOZGO

$$\int_0^T \langle u'(t), v(t)\rangle dt + \int_0^T B[u(t), v(t); t]\, dt = \int_0^T (f(t), v(t))\, dt$$

$$\forall \in L^2(0,T; H_0^1(U))$$

ODABEREM $v \in C^1([0,T]; H_0^1(U))$, $v(T)=0$ &

PARCIJALNO INTEGRIRAM PO VREMENU

$$\boxed{\begin{aligned} -\int_0^T \langle v'(t), u(t)\rangle\,dt + \cancel{(u(T), v(T))}^{=0} - (u(0), v(0)) \\ + \int_0^T B[u(t), v(t); t]\,dt = \int_0^T (f(t), v(t))\,dt \end{aligned}}$$

ZA $u_{m_\ell}$ IMAMO

$$\int_0^T \langle u_{m_\ell}'(t), v(t)\rangle\,dt + \int_0^T B[u_{m_\ell}(t), v(t); t]\,dt = \int_0^T (f(t), v(t))\,dt$$

$$v = \sum_{k=1}^{N} d^k(t)\, w_k, \quad d^k(T)=0$$

~~PUSTIM~~ NAPRAVIM P.I KAO GORE

$$-\int_0^T \langle v'(t), u_{m_\ell}(t)\rangle\,dt + \int_0^T B[u_{m_\ell}(t), v(t); t]\,dt = \int_0^T (f(t), v(t))\,dt$$

$$+ \cancel{(u_{m_\ell}(T), v(T))} - (u_{m_\ell}(0), v(0))$$

PUSTIM LIMES $\ell \to +\infty$

$$\boxed{\begin{aligned} -\int_0^T \langle v'(t), u(t)\rangle\,dt + \int_0^T B[u(t), v(t); t]\,dt = \int_0^T (f(t), v(t))\,dt \\ - (g, v(0)) \end{aligned}}$$

NAIME $u_m(0) = \sum_{k=1}^{m} d_m^k(0)\, w_k = \sum_{k=1}^{m} (g, w_k)\, w_k \longrightarrow g \quad L^2(U)$

ODUZIMANJEM ZA $v = \sum_{k=1}^{N} d^k(t)\, w_k$, $d^k(T)=0$ SLIJEDI,

$$(u(0), v(0)) = (g, v(0))$$

$$\Rightarrow (u(0) - g, w_k) = 0 \qquad k \in \mathbb{N}$$

$$\Rightarrow u(0) = g \quad u \; L^2(U)\,!$$

TEOREM 4 SLABO RJEŠENJE ZADAĆE

$$u_t + Lu = f \quad \text{u } U_T$$
$$u = 0 \quad \text{NA } \partial U \times [0,T]$$
$$u = g \quad \text{NA } U \times \{0\}$$

JE JEDINSTVENO.

DOK: ZBOG LINEARNOSTI ZADAĆE DOVOLJNO JE PROMATRATI $f \equiv 0$, $g \equiv 0$

U DOKAZU TM 3 DOBILI SMO

$$\int_0^T \langle u'(t), v(t) \rangle \, dt + \int_0^T B[u(t), v(t); t] \, dt = \int_0^T (f(t), v(t)) \, dt$$
$$v \in L^2(0,T; H_0^1(U))$$

UVRSTIMO U (SLABO RJEŠENJE) ZA $f \equiv 0$

$$\int_0^T \langle u'(t), u(t) \rangle \, dt + \int_0^T B[u(t), u(t); t] \, dt = 0$$
$$\int_0^T \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{L^2(U)} dt + \int_0^T B[u(t), u(t); t] \, dt = 0$$

$$\frac{1}{2} \|u(T)\|^2_{L^2(U)} + \int_0^T B[u(t), u(t); t] \, dt = 0$$

$$\|u(T)\|^2_{L^2(U)} = -2 \int_0^T B[u(t), u(t); t] \, dt$$

$$B[u(t), u(t); t] \geq \beta \|u(t)\|^2_{H_0^1} - \gamma \|u(t)\|^2_{L^2(U)} \geq -\gamma \|u(t)\|^2_{L^2(U)}$$

$$\Rightarrow \quad -B[u(t), u(t); t] \leq \gamma \|u(t)\|^2_{L^2(U)}$$

$$\Rightarrow \quad \|u(T)\|^2_{L^2(U)} \leq 2\gamma \int_0^T \|u(t)\|^2_{L^2(U)} dt$$

$$\eta(t) = \|u(t)\|^2_{L^2(U)}$$

$$\eta(T) \leq 2\gamma \int_0^T \eta(t)\, dt + 0$$

GRONWALLOVA NEJEDNAKOST (INTEGRÁLNÍ)

$$\Rightarrow \quad \eta(T) \leq 0\left(1 + 2\gamma t e^{2\gamma t}\right) = 0$$

$$\Rightarrow \quad \eta(T) = 0$$

$$\|u(T)\|^2_{L^2(U)} = 0$$

$$\Rightarrow \quad u \equiv 0 \qquad (T \text{ LIBOVOLNÉ})$$