6.4. PRINCIP MAKSIMUMA

- BAZIRANO NA: u KLASE $C^2(U)$ u $x_0 \in U \Rightarrow Du(x_0)=0$ & $D^2(x_0) \leq 0$
- $D^2u(x_0) \leq 0$ NEGATIVNO SEMIDEFINITNA
- TOČKOVNI ZAKLJUČCI (RAZLIČITO OD PRETHODNIH ODJELJAKA)
- $u \in C^2(U)$ NAM TREBA (TEORIJA REGULARNOSTI POKRIĆE)
- L U NEDIVERGENTNOJ FORMI

$$Lu = -A \cdot D^2u + b \cdot \nabla u + cu$$

A - UNIFORMNO ELIPTIČNA FUNKCIJA

A - SIMETRIČNA (BSOMP)

6.4.1 SLABI PRINCIP MAKSIMUMA

<u>TM 1</u> (SPM) $U \subseteq \mathbb{R}^n$ OTVOREN I OGRANIČEN.

NEKA JE: $u \in C^2(U) \cap C(\overline{U})$ & $c \equiv 0$ u U

(i) AKO JE $Lu \leq 0$ u $U \Rightarrow \max\limits_{\overline{U}} u = \max\limits_{\partial U} u$

(ii) AKO JE $Lu \geq 0$ u $U \Rightarrow \min\limits_{\overline{U}} u = \min\limits_{\partial U} u$

<u>NAP.</u>: (i) PODRJEŠENJE (SUBSOLUTION) (POSTIŽE MAX NA ∂U)

(ii) NADRJEŠENJE (SUPERSOLUTION) (POSTIŽE MIN NA ∂U)

DOK: (i)

1. KORAK: PRETPOSTAVIMO $Lu < 0$ u U

PRETP DA SE MAX NE POSTIŽE NA RUBU!

$$\exists x_0 \in U \quad \text{T.D.} \quad u(x_0) = \max_{\bar{U}} u$$

u - KLASE C^2 & NUŽNI UVJET ZA LOKALNI MAX:

$$Du(x_0) = 0 \quad \& \quad D^2u(x_0) \leq 0.$$

2. KORAK:

$A(x_0)$ - SIMETRIČNA, POZITIVNO DEFINITNA

$\Rightarrow \exists\, O \quad O^T O = I \quad$ T.D. $\quad O A(x_0) O^T = D$ - DIJAGONALNA

DEF: $\tilde{u}(y) = u(x_0 + O^T(y - x_0))$

$$\nabla \tilde{u}(y) = \nabla u(x_0 + O^T(y - x_0))\, O^T$$

$$\nabla \tilde{u}(y)^T = O\, \nabla u(x_0 + O^T(y - x_0))^T$$

$$H_{\tilde{u}}(y) = O\, H_u(x_0 + O^T(y - x_0))\, O^T$$

$$D^2 \tilde{u}(y) = O\, D^2 u(x_0 + O^T(y - x_0))\, O^T$$

$$D^2 \tilde{u}(x_0) = O\, D^2 u(x_0)\, O^T \leq 0$$

RAČUNAMO:

$$A(x_0) \cdot D^2 u(x_0) = A(x_0)\, O^T D^2 \tilde{u}(x_0) O = O A(x_0) O^T \cdot D^2 \tilde{u}(x_0)$$

$$= D \cdot D^2 \tilde{u}(x_0) \leq 0$$

3. KORAK:

$\Rightarrow$
$$Lu(x_0) = -A(x_0) \cdot D^2 u(x_0) + \underbrace{b(x_0) \cdot Du(x_0)}_{0} \geq 0$$

$$\Rightarrow\Leftarrow \quad Lu < 0 \text{ u } U$$

4. KORAK: $Lu \le 0$ u U

~~DEF~~: $u^\varepsilon(x) := u(x) + \varepsilon e^{\lambda x_1}$, $x \in U$

$\varepsilon > 0$

$\lambda > 0$ - ODABRAT ĆEMO

$$Lu^\varepsilon(x) = Lu(x) + \varepsilon L(e^{\lambda x_1})$$

$$\le \varepsilon L(e^{\lambda x_1}) = \varepsilon\left(-A(x)\cdot D^2(e^{\lambda x_1}) + b\cdot D(e^{\lambda x_1})\right)$$

$$\le \varepsilon\left(-A(x)\cdot e^{\lambda x_1}\lambda^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix} + b\cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e^{\lambda x_1}\lambda\right)$$

$$= \varepsilon\left(-a_{11}(x)\lambda^2 e^{\lambda x_1} + b_1\lambda e^{\lambda x_1}\right)$$

$$= \varepsilon e^{\lambda x_1}\left(-\underset{\theta}{\underset{\vee\!\!\!\vee}{a_{11}(x)}}\lambda^2 + b_1\lambda\right)$$

$$\le \varepsilon e^{\lambda x_1}\left(-\theta\lambda^2 + b_1\lambda\right)$$

$$\le \varepsilon e^{\lambda x_1}\left(-\theta\lambda^2 + \|b\|_{L^\infty}\lambda\right) < 0 \text{ u } U$$

ZA λ DOVOLJNO VELIK

SAD SMO U UVJETIMA 1.-3. KORAKA $\Rightarrow$

$$\max_{\overline{U}} u^\varepsilon = \max_{\partial U} u^\varepsilon$$

PUSTIMO $\varepsilon \to 0$ $\Rightarrow$ $\max_{\overline{U}} u = \max_{\partial U} u$.

(ii) NEKA JE u NADRJEŠENJE $\Rightarrow$ $-u$ JE PODRJEŠENJE

$$\Rightarrow \max_{\overline{U}} -u = \max_{\partial U} -u$$

$$\Rightarrow \min_{\overline{U}} u = \min_{\partial U} u$$

SAD IDEMO NA SLUČAJ $c \geq 0$

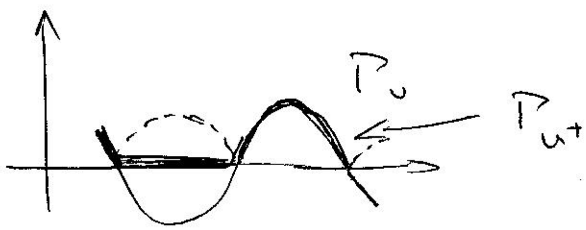
TM 2 (SPM, $c \geq 0$)

NEKA JE $u \in C^2(U) \cap C(\bar{U})$ & $c \geq 0$ u U

(i) AKO JE $Lu \leq 0$ u U $\Rightarrow$ $\max\limits_{\bar{U}} u \leq \max\limits_{\partial U} u^+$

(ii) AKO JE $Lu \geq 0$ u U $\Rightarrow$ $\min\limits_{\bar{U}} u \geq -\max\limits_{\partial U} u^-$

$u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$



$u = u^+ - u^-$

NAP: ZA u T.D. $Lu = 0$ u U

$$\max_{\bar{U}} |u| = \max_{\partial U} |u|$$

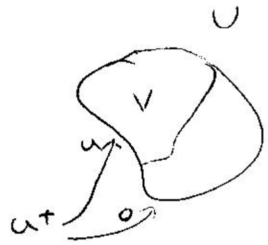
$\leq$: (i) $\max\limits_{\bar{U}} u \leq \max\limits_{\partial U} u^+ \leq \max\limits_{\partial U} |u|$

(ii) $\max\limits_{\bar{U}} -u \leq \max\limits_{\partial U} u^- \leq \max\limits_{\partial U} |u|$

DOK:

(i) NEKA JE $Lu \leq 0$ u U

DEF: $V = \{x \in U : u(x) > 0\}$ OTVOREN



$K u := Lu - cu \leq -cu \leq 0$ NA V

TM1 $\Rightarrow$ $\max\limits_{\bar{V}} u = \max\limits_{\partial V} u = \max\limits_{\partial U} u^+$

$\Rightarrow$ TV ZA $V \neq \emptyset$

AKO JE $V = \emptyset$ $\Rightarrow$ $u \leq 0$ $\Rightarrow$ $\max\limits_{\bar{U}} u \leq 0 = \max\limits_{\partial U} u^+$

(ii) $Lu \geq 0$ u U $\Rightarrow$ $L(-u) \leq 0$ u U

$$\Rightarrow \underbrace{\max_{\bar{U}}(-u)}_{= -\min\limits_{\bar{U}} u} \leq \max_{\partial U} \underbrace{(-u)^+}_{= u^-}$$

6.4.2 JAKI PRINCIP MAKSIMUMA

LEMA (HOPFOVA LEMA)

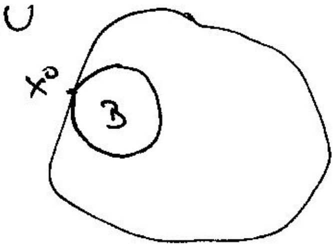
NEKA JE $u \in C^2(U) \cap C^1(\bar{U})$ A $c \equiv 0$ U U.

NEKA JE $Lu \le 0$ U U

$\exists\, x^0 \in \partial U$ T.D. $u(x^0) > u(x)$, $x \in U$

NEKA U ZADOVOLJAVA UVJET KUGLE U x^0:

$\exists$ KUGLA $B \subset U$ T.D. $x^0 \in \partial B$



TADA

$$\frac{\partial u}{\partial \nu}(x^0) > 0$$

(ν JEDINIČNA NORMALA NA B, VANJSKA U TOČKI x^0)

AKO JE $c \ge 0$ U U ZAKLJUČAK VRIJEDI UZ $u(x^0) \ge 0$.

NAP:

1) POANTA JE U STROGOJ NEJEDNAKOSTI.

$\frac{\partial u}{\partial \nu}(x^0) \ge 0$ VRIJEDI I NAČE

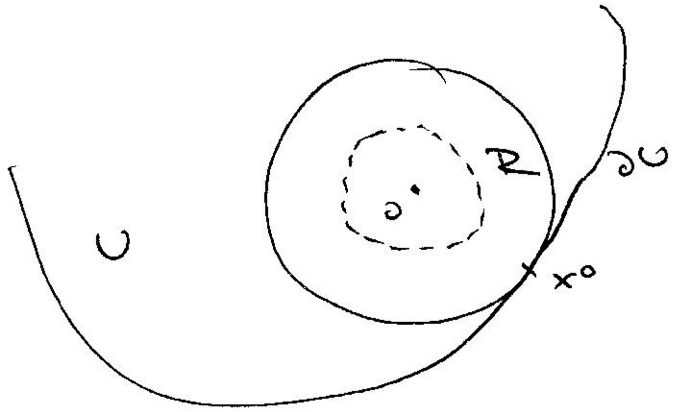
2) AKO JE RUB KLASE C^2 UVJET KUGLE JE ZADOVOLJEN ZA SVE TOČKE

DOK: 1. KORAK

NEKA JE $c \geq 0$ I $u(x) \geq 0$

B = KUGLA IZ UVJETA. (RADIJUSA r)

TRANSLATIRAM KOORDINATNI SUSTAV U SREDIŠTE B.



$$R = B^\circ(0,r) \setminus B(0,\tfrac{r}{2})$$

DEF: $v(x) = e^{-\lambda|x|^2} - e^{-\lambda r^2}$, $x \in B = B(0,r)$

$\lambda > 0$ IZABRAT ĆEMO GA

$$v\big|_{\partial B(0,r)} = 0$$

RAČUNAM:

$$(Lv)(x) = -A(x) \cdot D^2 v(x) + b(x) \cdot Dv(x) + c(x)\, v(x)$$

$$Dv(x) = -2\lambda e^{-\lambda|x|^2} x$$

$$D^2 v(x) = +4\lambda^2 e^{-\lambda|x|^2}\, x \otimes x - 2\lambda e^{-\lambda|x|^2} I$$

$$\Rightarrow (Lv)(x) = -A(x) \cdot \cancel{x} \otimes x \; 4\lambda^2 e^{-\lambda|x|^2} + 2\lambda e^{-\lambda|x|^2} A \cdot I$$

$$+ b(x) \cdot x \left(-2\lambda e^{-\lambda|x|^2}\right) + c(x)\left(e^{-\lambda|x|^2} - e^{-\lambda r^2}\right)$$

$$= e^{-\lambda|x|^2}\left(-A(x)x \cdot x \; 4\lambda^2 + 2\lambda \operatorname{tr} A - 2\lambda\, b(x) \cdot x + c(x)\right)$$

$$- \underbrace{c(x)\, e^{-\lambda r^2}}_{\geq 0}$$

KVADRATNI POLINOM U λ

$$\leq e^{-\lambda|x|^2}\left(-\theta |x|^2 4\lambda^2 + 2\lambda \operatorname{tr} A + 2\lambda |b(x)||x| + c(x)\right)$$

ZA DOVOLJNO VELIKI λ : $x \in R$ (UDALJIM GA OD ISHODIŠTA)

$$(Lv)(x) \leq 0, \qquad x \in R$$

2. KORAK

$$u(x^0) > u(x), \quad x \in U$$

$$\Rightarrow u(x^0) \geq u(x) + \varepsilon v(x), \quad x \in \partial B(0, r/2)$$

$$\& \quad u(x^0) \geq u(x) + \varepsilon v(x), \quad x \in \partial B(0, r)$$

TU JE $v = 0$

ZAJEDNO: $u(x^0) \geq u(x) + \varepsilon v(x), \quad x \in \partial R$

3. KORAK

$$L(u + \varepsilon v - u(x^0)) = \underbrace{Lu}_{\leq 0} + \varepsilon \underbrace{Lv}_{\leq 0} - \underbrace{L(u(x^0))}_{= -cu(x^0)} \leq -cu(x^0) \leq 0 \quad u \; R$$

$$\text{TM1} \Rightarrow \max_{\overline{R}} (u + \varepsilon v - u(x^0)) = \max_{\partial R} (u + \varepsilon v - u(x^0)) \leq 0$$

$\uparrow$ 2. KORAK

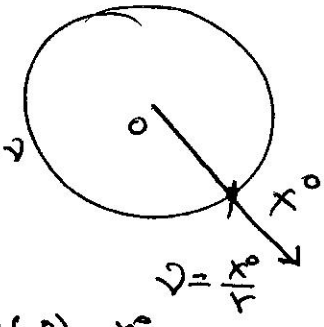
$$\Rightarrow u(x) + \varepsilon v(x) - u(x^0) \leq 0 \quad x \in \overline{R}$$

$$u(x^0) + \varepsilon v(x^0) - u(x^0) = 0$$

PROMATRAM

$$f(\rho) = u\left(\rho \frac{x^0}{r}\right) + \varepsilon v\left(\rho \frac{x^0}{r}\right) - u(x^0)$$

$\Rightarrow f$ JE RASTUĆA U $\rho = r$



$$0 \leq f'(r) = Du\left(\rho \frac{x^0}{r}\right) \cdot \nu + \varepsilon Dv\left(r \frac{x^0}{r}\right) \cdot \nu$$

$$\Rightarrow \frac{\partial u}{\partial \nu}(x^0) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x^0) = -\varepsilon Dv(x^0) \cdot \frac{x^0}{r}$$

$$= -\varepsilon \left(-2\lambda e^{-\lambda |x^0|^2} x^0\right) \cdot \frac{x^0}{r}$$

$$= 2\varepsilon \lambda e^{-\lambda r^2} r > 0$$

TM3 (SPM)

Neka je $u \in C^2(U) \cap C(\bar{U})$ & $c \equiv 0$ u U

Neka je U otvoren, ograničen i povezan.

(i) Ako je $Lu \leq 0$ u U & u postiže max na $\bar{U}$ u U

$\Rightarrow$ $u \equiv \text{const}$ u U

(ii) Ako je $Lu \geq 0$ u U & u postiže min na $\bar{U}$ u U

$\Rightarrow$ $u \equiv \text{const}$ u U

Dok: (i) $M := \max\limits_{\bar{U}} u$

$C := \{x \in U : u(x) = M\}$ $\qquad U = C \cup V$

$V := \{x \in U : u(x) < M\}$ $\qquad C \cap V = \emptyset$

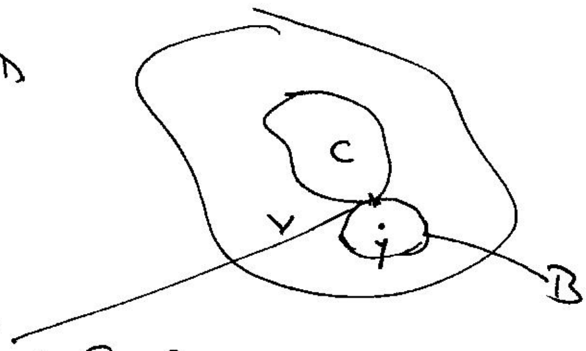
Pretp $V \neq \emptyset$ (V je otvoren)

$\Rightarrow$ $\exists y \in V$ t.d. $d(y, C) < d(y, \partial U)$

B najveća otv. kugla t.d.

y - centar

$B \subset V$



$\Rightarrow$ $\exists$ $x^0 \in C$ & $x^0 \in \partial B$

V zadovoljava uvjet kugle u x^0

$\Rightarrow$ Lema (Hopf) $\Rightarrow$ $\frac{\partial u}{\partial \nu}(x^0) > 0$

$\Rightarrow\Leftarrow$ x^0 je točka max!

$\Rightarrow V = \emptyset$

TM 4 (JPM & $c \geq 0$)

NEKA JE $u \in C^2(U) \cap C(\bar{U})$ & $c \geq 0$ u U.

NEKA JE U POVEZAN.

(i) AKO JE $Lu \leq 0$ u U & u POSTIŽE MAX u $\bar{U}$ ≥ 0 u U

$\Rightarrow$ $u = \text{const}$ u U

(ii) AKO JE $Lu \geq 0$ u U & u POSTIŽE MIN u $\bar{U}$ ≤ 0 u U

$\Rightarrow$ $u = \text{const}$ u U.

DOKAZ: SLIČNO, HOPFOVA LEMA (ii) ZA $c \geq 0$.

6.4.3. HARNACKOVA NEJEDNAKOST

TM 5 (HARNACKOVA NEJEDNAKOST)

NEKA JE $u \geq 0$ I u JE KLASE C^2 I $Lu = 0$ u U

$V \subset\subset U$ POVEZAN.

TADA $\exists$ $C > 0$ T.D. $(C(V, L))$

$$\sup_V u \leq C \inf_V u$$

6.5. SVOJSTVENA ZADAĆA

NEKA JE U - OTVOREN, OGRANIČEN

$$\boxed{\begin{array}{l} \text{NAĆI SVE } \lambda \in \mathbb{R} \quad 0 \neq u \text{ T.D.} \\ \qquad Lu = \lambda u \quad \text{u } U \\ \qquad u = 0 \quad \text{na } \partial U \end{array}}$$

λ - SVOJSTVENA VRIJEDNOST

u - PRIPADNI, SVOJSTVENI VEKTOR

POKAZALI SMO : NAJVIŠE PREBROJIVO SVOJSTVENIH VRIJEDNOSTI,
KONAČNE KRATNOSTI
BEZ GOMILIŠTA

NAREDNI REZULTATI POOPĆUJU TEOREME ZA MATRICE

NPR: SIMETRIČNA REALNA MATRICA => REALNE SVOJSTVENE
&
ONB SV. VEKTORA

POZITIVNA M. => POZITIVNE SV. VR.

MATRICA POZITIVNIH ELEMENATA => IMA POZITIVNU SV. VR.
I VEKTOR S POZITIVNIM
KOMPONENTAMA

6.5.1. SVOJSTVENE VRIJEDNOSTI SIMETRIČNOG ELIPTIČKOG OPERATORA

$$Lu = -\operatorname{div}(A\nabla u)$$

$A \in C^\infty(\overline{U})$, A SIMETRIČNA, A UNIFORMNO ELIPTIČNA

$$A(x)\xi\cdot\xi \geq \theta\,\xi^2,\ \xi\in\mathbb{R}$$

PRIPADNA BILINEARNA FORMA

$$B[u,v] = \int_\Omega A\nabla u\cdot\nabla v = B[v,u]$$

SIMETRIČNA!

<u>TM1</u> (SVOJSTVENA ~~Z~~ZADAĆA SEO)

(i) SVE SVOJSTVENE VRIJEDNOSTI OD L SU REALNE

(ii) BROJIMO LI KRATNOSTI (KONAČNE):

$$\Sigma = \{\lambda_k : k\in\mathbb{N}\}$$

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

$$\lambda_k \longrightarrow +\infty \quad \text{KAD } k\longrightarrow+\infty$$

(iii) $\exists$ ONB $(w_k)_{k\in\mathbb{N}}$ u $L^2(U)$, $w_k \in H^1_0(U)$ JE SV. VEKTOR λ_k

$$\begin{aligned} Lw_k &= \lambda_k w_k && \text{u } U \\ w_k &= 0 && \text{na } \partial U \end{aligned} \qquad k\in\mathbb{N}$$

<u>NAP.</u> TEORIJA REGULARNOSTI & $A\in C^\infty(\overline{U}; M_n(\mathbb{R}))$

$$Lw_k = \lambda_k w_k \in H^1_0 \Rightarrow w_k \in H^3$$

$$\Rightarrow w_k \in H^5$$

$$\vdots$$

DOK: KORAK 1: ZA $f \in L^2(U)$ PROMATRAMO

$$Lu = f \quad u\ U$$
$$u = 0 \quad \text{na } \partial U$$

DEF: $\quad Sf := u \qquad S : L^2(U) \to H^1_0(U) \hookrightarrow L^2(U)$

S OGRANIČEN, LINEARAN, KOMPAKTAN OPERATOR

$$S : L^2(U) \to L^2(U)$$

KORAK 2: S JE SIMETRIČAN

DOK: $f, g \in L^2(U)$. TREBA POKAZATI $(Sf, g)_{L^2} = (f, Sg)_{L^2}$

SVOJSTVO:

$$B[Sf, v] = (f, v) \quad , v \in H^1_0(U)$$
$$B[Sg, v] = (g, v) \quad , v \in H^1_0(U)$$

STAVIM: $\quad v = Sg \quad$ u 1.

$\quad v = Sf \quad$ u 2.

SIMETRIČNOST B

$$(f, Sg) \underset{1.}{=} B[Sf, Sg] = B[Sg, Sf] \underset{2.}{=} (g, Sf) \;//$$

KORAK 3:

$$(Sf, f) = B[Sf, Sf] \geq \|Sf\|_{H^1_0(U)} \geq 0$$

$\Rightarrow S \geq 0$

SPEKTRALNI TEOREM ZA S POVLAČI TVRDNJU

$\mu_k \to 0$ BEZ KONAČNIH GOMILIŠTA

ONB

$$\lambda_k = \frac{1}{\mu_k} : \qquad S w_k = \mu_k w_k$$

$$L S w_k = \mu_k L w_k$$

$$\overset{\|}{w_k} \quad \Rightarrow \quad L w_k = \frac{1}{\mu_k} w_k$$

λ_k

SV. VEKTORI ISTI!

DEF: $\lambda_1 > 0$ GLAVNA SV. VRIJEDNOST OD L
(PRINCIPAL EIGENVALUE)

TM 2 (VARIJACIJSKI PRINCIP ZA SV. ZADAĆU)

(i) $\lambda_1 = \min \{ B[u,u] : u \in H^1_0(U), \|u\|_{L^2(U)} = 1 \}$

(ii) MIN SE POSTIŽE ZA w_1 KOJA JE POZITIVNA I

$$L w_1 = \lambda_1 w_1 \quad \text{u } U$$
$$w_1 = 0 \quad \text{na } \partial U$$

(iii) AKO JE $u \in H^1_0(U)$ NEKO SLABO RJ.

$$Lu = \lambda_1 u \quad \text{u } U$$
$$u = 0 \quad \text{na } \partial U$$

$\Rightarrow \quad u = \text{const}\, w_1$

NAP.: – (iii) KRATNOST GLAVNE SV. VR. JE 1

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

– $$\lambda_1 = \min_{\substack{u \in H^1_0(U) \\ u \neq 0}} \frac{B[u,u]}{\|u\|^2_{L^2(U)}}$$ – RAYLEIGHJEV KVOCIJENT

– $B[w_k, w_l] = \lambda_k (w_k, w_l) = \delta_{kl} \lambda_k$

$\Rightarrow (w_k)$ ORTOGONALNA FAMILIJA I U SKALARNOM PRODUKTU $B[u,v]$

ŠTOVIŠE

$$B\left[\frac{w_k}{\sqrt{\lambda_k}}, \frac{w_k}{\sqrt{\lambda_k}} \right] = 1$$

$\left(\frac{w_k}{\sqrt{\lambda_k}} \right)$ ONB u $B(u,v]$
FAMILIJA

DOK:

TV: $\left(\frac{v_k}{\sqrt{\lambda_k}}\right)_{k\in\mathbb{N}}$ JE BAZA U $B[u,v]$ ZA $H^1_0(U)$

DOK: TREBA PROVJERITI POTPUNOST

$$B\left[\frac{v_k}{\sqrt{\lambda_k}}, u\right] = 0 \quad k\in\mathbb{N} \overset{?}{\Rightarrow} u = 0$$

$$\parallel$$

$$\lambda_k\left(\frac{v_k}{\sqrt{\lambda_k}}, u\right)_{L^2}$$

$$\Rightarrow (v_k, u)_{L^2} = 0 \Rightarrow \underline{\underline{u = 0}}$$

$\underline{(v_k) \text{ OHB U } L^2(U)}$

$$\Rightarrow u\in L^2(U) \Rightarrow u = \sum_{k=1}^{\infty} (u, v_k)_{L^2} v_k \quad \leftarrow \text{RED KVG U } L^2$$

$$\|u\|^2_{L^2(U)} = \sum_{k=1}^{\infty} (u, v_k)^2 \quad \text{PARSEVALOVA JEDNAKOST}$$

$\underline{\left(\frac{v_k}{\sqrt{\lambda_k}}\right)_k \text{ OHB U } B[u,v] \text{ ZA } H^1_0(U)}$

$$\Rightarrow u\in H^1_0(U) \Rightarrow u = \sum_{k=1}^{\infty} B\left[u, \frac{v_k}{\sqrt{\lambda_k}}\right]\frac{v_k}{\sqrt{\lambda_k}}$$

RED KVG U $H^1_0(U)$

ODNOSNO ZA $u\in H^1_0(U)$ $\quad u = \sum_{k=1}^{\infty} (u, v_k) v_k$

RED KVG U $L^2(U)$

OHB JEDINSTVENOST RASTAVA U BAZI

$$\frac{1}{\lambda_k} B[u, v_k] = (u, v_k)_{L^2} \quad (\text{OČITO})$$

PA I RED KVG ZAPRAVO U $H^1_0(U)$

KORAK 4:

$$B[u,u] = B\left[u, \sum_{k=1}^{\infty} (u, w_k) w_k\right] \underset{\substack{\uparrow \\ \text{RED KVG U } H_0^1(U)}}{=} \sum_{k=1}^{\infty} (u, w_k) \underbrace{B[u, w_k]}_{\lambda_k (u, w_k)}$$

$$= \sum_{k=1}^{\infty} \underset{\substack{\vee| \\ \lambda_1}}{\lambda_k} (u, w_k)_{L^2}^2 \geq \lambda_1 \sum_{k=1}^{\infty} (u, w_k)_{L^2}^2 = \lambda_1 \|u\|_{L^2}^2$$

ZA u T.D. $\|u\|_{L^2} = 1$

$\Rightarrow \quad B[u,u] \geq \lambda_1$

$\Rightarrow \quad \lambda_1 \leq \inf \{ B[u,u] : u \in H_0^1(U), \|u\|_{L^2} = 1 \}$

ZA $u = w_1$ DOBIVAMO

$$B[w_1, w_1] = \lambda_1 (w_1, w_1)_{L^2} = \lambda_1$$

PA SE INF $\quad 1 \quad = \quad$ POSTIŽU $\quad \Rightarrow$ (i)

KORAK 5: $\quad u \in H_0^1(U)$, $\|u\|_{L^2} = 1$.

u JE SLABO RJ. $\begin{cases} Lu = \lambda_1 u & \text{u } U \\ u = 0 & \text{na } \partial U \end{cases} \iff B[u,u] = \lambda_1$

DOK: $\boxed{\Rightarrow}$ $\quad B[u,u] = \lambda_1 (u,u) = \lambda_1$

$\boxed{\Leftarrow}$ $\quad B[u,u] = \lambda_1 = \lambda_1 (u,u) = \lambda_1 \sum_{k=1}^{\infty} (u, w_k)^2$

$$B\left[\sum_{k=1}^{\infty} (u, w_k) w_k, \sum_{\ell=1}^{\infty} (u, w_\ell) w_\ell\right] = \sum_{k,\ell=1}^{\infty} (u, w_k) (u, w_\ell) B[w_k, w_\ell]$$

$$= \sum_{k,\ell=1}^{\infty} (u, w_k)(u, w_\ell) \delta_{k\ell} \lambda_k = \sum_{k=1}^{\infty} (u, w_k)^2 \lambda_k$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \underset{\substack{\vee| \\ 0}}{(u, w_k)^2} \underset{\substack{\vee| \\ 0}}{(\lambda_k - \lambda_1)} = 0$$

$\Rightarrow \quad (u, w_k)(\lambda_k - \lambda_1) = 0 \quad \forall k$

$\Rightarrow (u, w_k) = 0 \quad \lambda_k \gneq \lambda_1$ (PRVIH m JE λ_1!)

$$\Rightarrow \quad u = \sum_{k=1}^{m} (u, w_k)\, w_k$$

$$Lu = \sum_{k=1}^{m} (u, w_k)\, \underbrace{L w_k}_{\lambda_1 w_k} = \lambda_1 u$$

KORAK 6: NEKA JE $u \in H_0^1(U)$ SLABO RJ $\begin{cases} Lu = \lambda_1 u \\ u = 0 \end{cases}$

$\Rightarrow \quad u > 0$ u U ILI $u < 0$ u U

DOK. NEKA JE $\|u\|_{L^2} = 1$

$$\alpha := \int_U (u^+)^2 dx$$

$$\beta := \int_U (u^-)^2 dx$$

$$\Rightarrow \quad \alpha + \beta = \int_U (u^+)^2 + (u^-)^2 dx = \int_U |u|^2 dx = 1$$

VRIJEDI: $u^{\pm} \in H_0^1(U)$

$$Du^+ = \begin{cases} Du & \text{s.s. NA } \{u \geq 0\} \\ 0 & \text{s.s NA } \{u \leq 0\} \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{s.s. NA } \{u \geq 0\} \\ -Du & \text{s.s. NA } \{u \leq 0\} \end{cases}$$

$B[u^+, u^-] = 0$ NOSAČI IMAJU PRESJEK U SKUPU MJERE 0

$u = u^+ - u^-$

SL.P. $Lu = \lambda_1 u$

$$B[u,u] = \lambda_1 (u,u) = \lambda_1$$

$$\lambda_1 = B[u^+ - u^-, u^+ - u^-] = B[u^+, u^+] + B[u^-, u^-]$$

$$\geq \lambda_1 \|u^+\|^2_{L^2(U)} + \lambda_1 \|u^-\|^2_{L^2(U)} = \lambda_1(\alpha + \beta) = \lambda_1$$

$\Rightarrow$ SVE SU JEDNAKOSTI

$$B[u^+, u^+] = \lambda_1 \|u^+\|^2_{L^2(U)}$$

$$B[u^-, u^-] = \lambda_1 \|u^-\|^2_{L^2(U)}$$

KOROLAR $\Rightarrow$ $\begin{cases} Lu^+ = \lambda_1 u^+ \\ u^+ = 0 \end{cases}$

$\begin{cases} Lu^- = \lambda_1 u^- \\ u^- = 0 \end{cases}$

~~STROGE NEJEDNAKOSTI~~

$$A \in C^\infty(\bar{U}; M_n(\mathbb{R})) \Rightarrow u^+ \in C^\infty(U)$$

$$Lu^+ = \lambda_1 u^+ \geq 0 \quad u \ U$$

$\Rightarrow u^+$ NADRJEŠENJE

JPM $\Rightarrow$ $u^+ > 0$ u U ILI $u^+ \equiv \text{const} = 0$

TO NIJE

SLIČNO ZA u^-

$\Rightarrow$ $u = u^+ - u^-$

& JEDAN JE $\equiv 0$!

KORAK 8 : u i $\tilde{u}$ NETRIVIJALNA SLABA RJEŠENJA

$$\begin{cases} Lu = \lambda_1 u \\ u = 0 \end{cases} \qquad \begin{cases} L\tilde{u} = \lambda_1 \tilde{u} \\ \tilde{u} = 0 \end{cases}$$

u i $\tilde{u}$ $\neq 0$ U INTERIORU, NEPR

$$\int_U u \neq 0 \quad \& \quad \int_U \tilde{u} \neq 0$$

$\Rightarrow \exists \chi \in \mathbb{R}$ TD

$$\int_U u = \chi \int_U \tilde{u}$$

$$\Rightarrow \int_U (u - \chi \tilde{u}) = 0$$

ALI $u - \chi \tilde{u}$ JE SV. VEKTOR ZA λ_1 (LINEARNOST)

$\Rightarrow$ $u - \chi \tilde{u} \equiv 0$

$\Rightarrow u = \chi \tilde{u}$ $\Rightarrow$ KRATNOST 1 !

6.5.2. SVOJSTVENE VRIJEDNOSTI NESIMETRIČNIH ELIPTIČKIH OPERATORA

$$Lu = -A \cdot D^2 u + b \cdot Du + cu$$

$A \in C^\infty(\bar{U})$, $b \in C^\infty(\bar{U})$, $c \in C^\infty(\bar{U})$

∂U GLADAK

A - SIMETRIČNA

$c \geq 0$

<u>TM 3</u> (GLAVNA SV. VR)

(i) $\exists\ \lambda_1 \in \mathbb{R}$ SV. VR OD L NA $H_0^1(U)$.

AKO JE $\lambda \in \mathbb{C}$ SV. VR $\quad Re(\lambda) \geq \lambda_1$

(ii) $\exists\ w_1$ SV. VEKTOR ZA λ_1 KOJI JE POZITIVAN NA U

(iii) λ_1 JE KRATNOSTI 1.