SKICA DOKAZA:

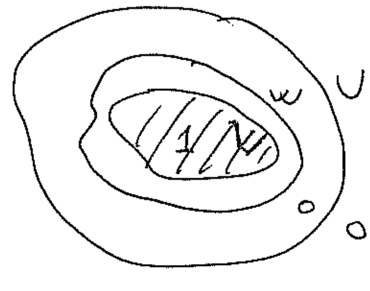
1) FIKSIRAMO $V \subset\subset U$ i $W : V \subset\subset W \subset\subset U$

$\zeta \equiv 1$ na V

$\zeta \equiv 0$ na $\mathbb{R}^n \setminus W$

$0 \le \zeta \le 1$

CUT-OFF FUNKCIJA



2) $H^1_0(U)$? u JE SLABO RJ. $Lu = f$:

$$B[u,v] = (f,v), \quad v \in H^1_0(U)$$

$$\Rightarrow \int_U A \nabla u \cdot \nabla v = \int_U \tilde{f} v \qquad (*)$$

$$\tilde{f} = f - b \cdot \nabla u - cu \in L^2(U)$$

3) NEKA JE $|h| > 0$ DOVOLJNO MALI (MANJI OD $d(\partial W, \partial U)$)

$$v := -D_k^{-h}(\zeta^2 D_k^h u), \quad D_k^h u(x) = \frac{u(x+he_k) - u(x)}{h}$$

NOSAČ U W PODIJELJENA RAZLIKA

$v \in H^1_0(U) \Rightarrow$ UBACIM U SLABU FORMULACIJU $(*)$

L.S. OCIJENIM ODOZDO
D.S. -"- ODOZGO $\Rightarrow$

$$\int_V |D_k^h Du|^2 dx \le \int_U \zeta^2 |D_k^h Du|^2 dx \le C\left(\|f\|^2_{L^2} + \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\right)$$

$k = 1, \dots, n$, $|h| > 0$ DOVOLJNO MALI

⇐) TH 3, (ii) u §5.8.2 $\Rightarrow$ $Du \in H^1_{loc}(U)$ $\Rightarrow$ $u \in H^2_{loc}(U)$ i

i

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right)$$

⇒) ISTI DOKAZ ZA $V \subset\subset W$ (DAKLE W IGRA ULOGU U)

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right)$$

NOVA CUT-OFF FUNKCIJA

$$\begin{cases} \zeta \equiv 1 & \text{NA } W, \\ \zeta \equiv 0 & \text{NA } \mathbb{R}^n \setminus U \\ 0 \leq \zeta \leq 1 & \end{cases}$$

$v = \zeta^2 u$ u $(*)$ $\Rightarrow$ $\displaystyle\int_U \zeta^2 |Du|^2 dx \leq C \int_U f^2 + u^2$

$$\geq \int_W |Du|^2 dx$$

$$\Rightarrow \quad \|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(W)} + \|u\|_{L^2(W)}\right)$$

KORISTIMO PRETHODNI REZULTAT VIŠE PUTA DA DOBIJEMO REGULARNOST VIŠEG REDA

TM 2 (UNUTARNJA REGULARNOST VIŠEG REDA)

NEKA JE $m \in \mathbb{N} \cup \{0\}$ I

$$A \in C^{m+1}(U; M_n(\mathbb{R})),\ b \in C^{m+1}(U; \mathbb{R}^n),\ c \in C^{m+1}(U;\mathbb{R})$$

$$f \in H^m(U)$$

NEKA JE $u \in H^1(U)$ SLABO RJ. $Lu = f$ U U.

TADA:

$$u \in H^{m+2}_{loc}(U)$$

$$\forall V \subset\subset U \quad \exists C > 0 \quad \text{T.D.} \quad \|u\|_{H^{m+2}(V)} \le C\left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}\right)$$

$$C(U, V, m, L)$$

DOK: $m=0$ TM1

$m=1$ A, b, c KLASE C^2, $f \in H^1$

$$Lu = f \Rightarrow \partial_{x_k}(Lu) = \partial_{x_k} f$$

$$\partial_{x_k}\left(-\operatorname{div}(A\nabla u) + b\cdot\nabla u + cu\right) = \partial_{x_k} f$$

$$-\operatorname{div}\left(\partial_{x_k}A\,\nabla u + A\nabla\partial_{x_k}u\right) + \partial_{x_k}b\cdot\nabla u + b\cdot\nabla\partial_{x_k}u + \partial_{x_k}c\,u + c\,\partial_{x_k}u = \partial_{x_k} f$$

DEF. $\tilde{u} = \partial_{x_k} u$

$$-\operatorname{div}(A\nabla\tilde{u}) + b\cdot\nabla\tilde{u} + c\tilde{u} = \underbrace{\partial_{x_k} f}_{\in L^2(U)} + \underbrace{\operatorname{div}\big(\underbrace{\partial_{x_k}A}_{C^1}\,\underbrace{\nabla u}_{H^1_{loc}}\big)}_{\in L^2_{loc}} - \underbrace{\partial_{x_k}b}_{C^1}\cdot\underbrace{\nabla u}_{L^2} - \underbrace{\partial_{x_k}c}_{C^1}\,\underbrace{u}_{L^2}$$

$$\in L^2_{loc}$$

TM1

$$\Rightarrow \tilde{u} \in H^2_{loc}(U) \quad \forall k = 1,\dots,n \Rightarrow u \in H^3_{loc}(U)$$

$$\|\tilde{u}\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|Du\|_{L^2(U)}\right) \overset{TM1}{\le} C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

$$\|u\|_{H^3(V)} \le \|u\|_{L^2(V)} + \underline{\|Du\|_{L^2(V)} + \|D^2u\|_{L^2(V)} + \|D^3u\|_{L^2(V)}} \le \|u\|_{L^2(V)} + \|Du\|_{H^2(V)}$$

$$\le C(\|f\|_{L^2} + \|u\|_{L^2})$$

DOKAZ : INDUKCIJOM

TM 3 (C^∞ U UNUTRAŠNJOSTI)

NEKA JE

$$A \in C^\infty(U; M_n(\mathbb{R})),\ b \in C^\infty(U; \mathbb{R}^n),\ c \in C^\infty(U; \mathbb{R})$$

$$f \in C^\infty(U)$$

$$u \in H^1(U) \text{ SLABO RJ. } Lu = f \text{ u } U.$$

TADA

$$u \in C^\infty(U).$$

NAP: NEMA NIKAKVOG UVJETA ∂U ILI PONAŠANJA RJ. NA NJEMU.

$\Rightarrow$ SINGULARITETI S RUBA NE PRENOSE SE U INTERIOR.

DOK: TM2 $\Rightarrow$ $u \in H^m_{loc}(U)$ $\forall m \in \mathbb{N}$.

TM ULAGANJA $\Rightarrow$ $u \in C^k$, $k \in \mathbb{N}$.

6.3.2 REGULARNOST NA U

TM 4 (H^2) NEKA JE

$$A \in C^1(\bar{U}; M_n(\mathbb{R})), \quad b \in L^\infty(U;\mathbb{R}^n), \quad c \in L^\infty(U;\mathbb{R})$$

$$f \in L^2(U)$$

$u \in H_0^1(U)$ SLABO RJ. $\quad \begin{cases} Lu = f & u\ U \\ u = 0 & na\ \partial U \end{cases}$

∂U KLASE C^2.

TADA JE

$$u \in H^2(U)$$

$\exists C > 0$ T.D. $\quad \|u\|_{H^2(U)} \le C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right)$

$C(U,L)$

NAP: AKO JE $u \in H_0^1(U)$ JEDINSTVENO RJ $\Rightarrow \lambda = 0 \notin \Sigma$

TM 6 $\Rightarrow \|u\|_{L^2(U)} \le C \|f\|_{L^2(U)}$

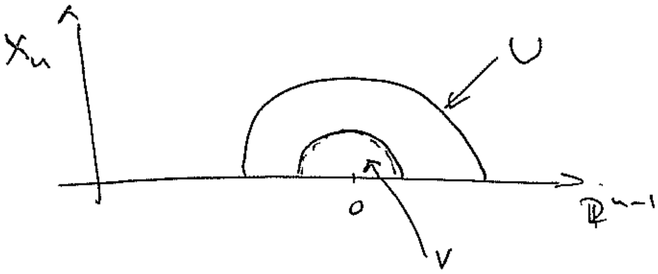
$$\Rightarrow \quad \|u\|_{H^2(U)} \le C \|f\|_{L^2(U)}$$

(ii) TREBA GLATKOĆA RUBA, TJ. PARAMETRIZACIJE

SKICA DOKAZA:

*) $\boxed{U = B^\circ(0,1) \cap \mathbb{R}^n_+}$

$V = B^\circ(0,\tfrac{1}{2}) \cap \mathbb{R}^n_+$

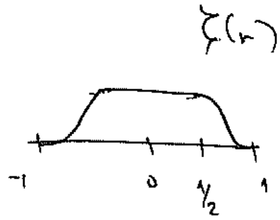


CUT-OFF:

$$\zeta \equiv 1 \quad B(0,\tfrac{1}{2})$$

$$\zeta \equiv 0 \quad \mathbb{R}^n \setminus B(0,1)$$

$$0 \le \zeta \le 1$$



$u \in H^1_0(U)$ SLABO RJ $\begin{cases} Lu = f & u\ U \\ u = 0 & na\ \partial U \end{cases}$

$$B[u,v] = (f,v), \quad v \in H^1_0(U)$$

ZAPIŠEMO KAO:

$$\int_U A \nabla u \cdot \nabla v = \int_U \tilde{f} v, \quad v \in H^1_0(U) \qquad (*)$$

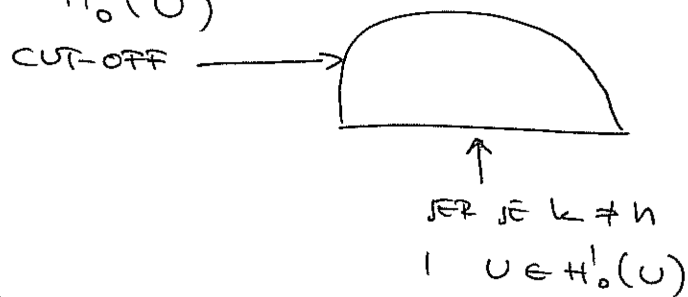
$$\tilde{f} = f - b \cdot \nabla u - cu$$

DEF:

$$v := -D_k^{-h}(\zeta^2 D_k^h u) \qquad k = 1, \dots, n-1 \ !$$

DOBRO DEF NA U ZA h DOVOLJNO MALI

SVOJSTVO: $v \in H^1_0(U)$



$\Rightarrow$ v UVRSTIM U VARIJACIJSKU JEDNADŽBU (*)

ISTI RAČUN KAO PRIJE

$$\int_U A Du \cdot Dv = \int_U \tilde{f} \cdot v \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 + C \int_U f^2 + u^2 + |Du|^2$$

$$\vee\!/$$

$$\frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 - C \int_U |Du|^2$$

$$\Rightarrow \quad \int_V |D_k^h Du|^2 \leq C \int_U \left(f^2 + u^2 + |Du|^2\right)$$

TH3 u §5.8.2 (NAPOMENA IZA)

$\Rightarrow$

$$(Du)_{x_k} = D(u_{x_k}) \in L^2(V)$$

$$\Rightarrow \quad u_{x_k} \in H^1(V) \ \& \ \sum_{\substack{k,l=1 \\ k+l<2n}}^{n} \|u_{x_k x_l}\|_{L^2(V)} \leq C\left(\|f\|_{L^2} + \|u\|_{H^1}\right)$$

FALI: $\quad u_{x_n} \in H^1(V) \quad \& \quad \|u_{x_n x_n}\|_{L^2(V)} \leq \dots$

OD RANIJE ZNAMO $\quad u \in H^2_{loc}(U)$

$\Rightarrow \quad Lu = f \quad$ VRIJEDI S.S.

$$-\operatorname{div}(A \nabla u) + b \cdot \nabla u + cu = f \qquad \text{s.s. } x \in U$$

$$\operatorname{div}(A\nabla u) = \sum_{ij} \partial_i (a_{ij} \partial_j u) = \sum_{ij} \partial_i a_{ij} \partial_j u + a_{ij} \partial_i \partial_j u$$

$$= (\operatorname{div} A^T) \cdot \nabla u + A \cdot H_u$$

$$\Rightarrow \quad -A \cdot H_u = f - \left(b - (\operatorname{div} A^T)\right) \cdot \nabla u - cu$$

$$\Rightarrow \quad A \cdot H_u = -f + (b - \operatorname{div} A^T) \cdot \nabla u + cu$$

$$a_{nn} u_{x_n x_n} = f - \sum_{\substack{i,j=1 \\ i+j<2n}}^{n} a_{ij} u_{x_i x_j} + (b - \operatorname{div} A^i) \cdot \nabla u + cu$$

$L^2 \qquad L^\infty \qquad L^2 \qquad L^2 \qquad L^2$

$a_{nn}(x) \geq \theta > 0$ ZBOG UNIFORMNE POZITIVNE DEFINITNOSTI

I JOŠ a_{nn} NEPREKIDNA

$\Downarrow$

$\frac{1}{a_{nn}} \in L^\infty$

$\Rightarrow u_{x_n x_n} \in L^2(V)$

$\Rightarrow u \in H^2(V)$

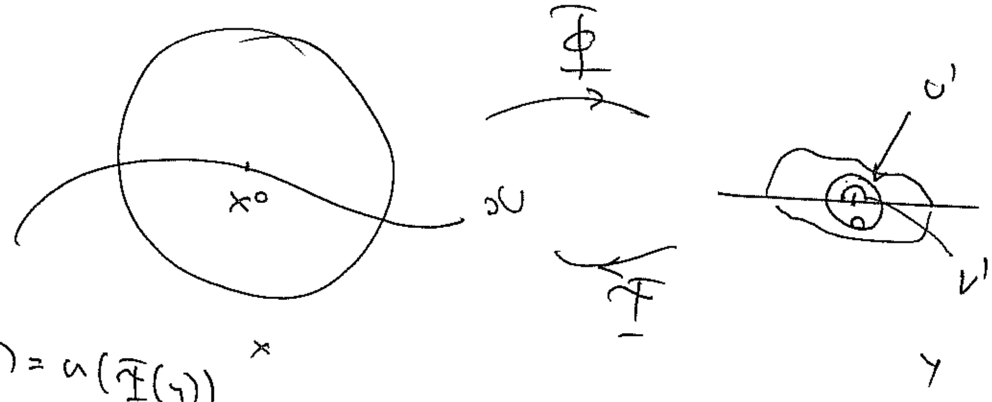
TAKOĐER OCJENA:

$$|u_{x_n x_n}| \leq C \left(\sum_{\substack{i,j=1 \\ i+j<2n}}^{n} |u_{x_i x_j}| + |Du| + |u| + |f| \right)$$

$\Rightarrow$

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right)$$ ZAMJENA S L^2 KAO U TM 1

b) OPĆENITI U



DEF: $u'(y) = u(\Psi(y))$

$\Rightarrow u' \in H^1(U')$, $u' = 0$ na $\partial U' \cap \{y_n = 0\}$

$\Rightarrow L' u' = f'$ u U'

$f'(y) := f(\Psi(y))$

$L' u' := -\operatorname{div}(A' \nabla u') + b' \cdot \nabla u' + c' u'$

A', b', c' SE IZRAŽAVAJU POMOĆU A, b, c i Φ

ZAMJENA VARIJABLI U JDBI

$\Rightarrow L'$ JE UNIFORMNO ELIPTIČAN OPERATOR (TJ. A' JE UNIF. >0)

$\Rightarrow$ MOŽEMO PRIMIJENITI a)

$$\Rightarrow u' \in H^2(V')$$

$$\|u'\|_{H^2(V')} \leq C\left(\|f'\|_{L^2(U')} + \|u'\|_{L^2(U')}\right)$$

$$\Rightarrow u(x) = u'(\Phi(x)), \quad u \in H^2(V) \qquad V = \Phi(V')$$

&

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

c) ∂U KOMPAKTAN

POKRIVAMO GA S KONAČNO KUGLI (IZ b))

SUMIRAMO REZULTATE + LOC OCJENA $\Rightarrow$ TV

TU TREBAM Φ KLASE C^2 (TJ. RUB)

TM5 (REGULARNOST VIŠEG REDA)

NEKA JE $m \in \mathbb{N}$ I NEKA JE

$$A \in C^{m+1}(\bar{U}, M_n(\mathbb{R})),\ b \in C^{m+1}(\bar{U};\mathbb{R}^n),\ c \in C^{m+1}(\bar{U})$$

$$f \in H^m(U)$$

$u \in H^1_0(U)$ SLABO RJ. $\begin{cases} Lu = f & u\ U \\ u = 0 & \text{na } \partial U \end{cases}$

∂U KLASE C^{m+2}

TADA

$$u \in H^{m+2}(U)$$

$$\exists C(U,m,L) > 0 \text{ T.D. } \|u\|_{H^{m+2}(U)} \leq C\left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}\right)$$

NAP: (i) AKO $\lambda = 0 \notin \Sigma$ TJ. u JE JEDINSTVENO RJ.

$$\Rightarrow \quad \|u\|_{H^{m+2}(U)} \leq C\,\|f\|_{H^m(U)}$$

(ii) NE MOŽEMO KAO PRIJE JER $\partial_{x_k} u$ NE ZADOVOLJAVA RUBNI UVJET $k = 1, \dots, n$

ALI MOŽEMO ZA $\partial_{x_k} u$ $k = 1, \dots, n-1$!

(iii) POGLEDAJTE DOKAZ SAMI

TM 6 (C^∞) NEKA JE

$$A \in C^\infty(\bar{U}; M_n(\mathbb{R})),\ b \in C^\infty(\bar{U}, \mathbb{R}^n),\ c \in C^\infty(\bar{U})$$

$$f \in C^\infty(\bar{U})$$

$u \in H^1_0(U)$ SLABO RJ. $\begin{cases} Lu = f & u\ U \\ u = 0 & \text{na } \partial U \end{cases}$

∂U JE KLASE C^∞

TADA JE

$$u \in C^\infty(\bar{U}).$$

DOK: KAO PRIJE. TM5 $\Rightarrow$ $u \in H^m(U)$, $m \in \mathbb{N}$

TM ULAGANJA $\Rightarrow$ $u \in C^k(\bar{U})$, $k \in \mathbb{N}$.