5.9. DRUGI PROSTORI FUNKCIJA

5.9.1. PROSTOR H^{-1}

DEF: $H^{-1}(U) = $ DUAL OD $H^1_0(U)$

$=$ PROSTOR OGRANIČENIH LINEARNIH FUNKCIONALA NA $H^1_0(U)$

DJELOVANJE OZNAČAVAMO: $\langle \, , \, \rangle$ ILI $\langle \, , \, \rangle_{H^{-1}, H^1_0}$

$f \in H^{-1}(U)$, $u \in H^1_0(U)$ $\quad \langle f, u \rangle$

DEF: ZA $f \in H^{-1}(U)$ DEF:

$$\|f\|_{H^{-1}(U)} := \sup_{\substack{u \in H^1_0(U) \\ \|u\|_{H^1_0} \leq 1}} \langle f, u \rangle$$

- TO JE NORMA
- OPERATORSKA NORMA

TM 1 (KARAKTERIZACIJA H^{-1})

(i) NEKA JE $f \in H^{-1}(U)$. TADA $\exists$ $f^0, f^1, \dots, f^n \in L^2(U)$ T.D.

$$\langle f, v \rangle = \int_U \left(f^0 v + \sum_{i=1}^n f^i v_{x_i} \right) dx, \quad v \in H^1_0(U)$$

(ii)

$$\|f\|_{H^{-1}(U)} = \inf_{\substack{g^0, \dots, g^n \in L^2(U) \\ 12 \ (i)}} \left(\int_U \sum_{i=0}^n |g^i|^2 dx \right)^{1/2}$$

NAP: PISAT ĆEMO

$$f = f^0 - \sum_{i=1}^n f^i_{x_i}$$

KAD VRIJEDI (i)

NAP: $f^0, \dots, f^n$ NISU JEDINSTVENI!

DOK: (i) ZA $u, v \in H^1_0(U)$ DEF:

$$(u, v) = \int_U Du \cdot Dv + uv \, dx$$

TO JE SKALARNI PRODUKT NA $H^1_0(U)$ (HILBERTOV).

RIESZOV TM REPREZENTACIJE:

$$\forall f \in H^{-1}(U) \quad \exists u \in H^1_0(U) \quad \langle f, v \rangle = (u, v)$$

TJ.

$$\langle f, v \rangle = \int_U Du \cdot Dv + uv$$

$$\Rightarrow \quad f^0 = u \qquad f^i = u_{x_i}, \quad i = 1, \dots, n.$$

(ii)

$$|\langle f, v \rangle| \underset{\text{S-C-B}}{\leq} \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} + \|u\|_{L^2(U)} \|v\|_{L^2(U)} \quad \text{NET.}$$

$$\leq \|u\|_{H^1(U)} \|v\|_{H^1(U)}$$

$$= \left(\int_U |Du|^2 + |u|^2 \right)^{1/2} \|v\|_{H^1(U)}$$

$$= \left(\int_U \sum_{i=0}^n |f^i|^2 \right)^{1/2} \|v\|_{H^1(U)}$$

$$\Rightarrow \quad \|f\|_{H^{-1}(U)} \leq \left(\int_U \sum_{i=0}^n |f^i|^2 \right)^{1/2}$$

NEKA SU $g^0, \dots, g^n \in L^2(U)$ NEKI DRUGI PA VRIJEDI

$$\langle f, v \rangle = \int_U \left(g^0 v + \sum_{i=1}^n g^i v_{x_i} \right) dx$$

$$\int_U Du \cdot Du + uu = \langle f, u \rangle \leq \int_U \left(g^0 u + \sum_{i=1}^n g^i u_{x_i} \right) dx$$

$$\|u\|^2_{H^1(U)} \qquad \leq \left(\int_U \sum_{i=0}^n |g^i|^2 \right)^{1/2} \|u\|_{H^1(U)}$$

$$\Longrightarrow \quad \|u\|_{H^1(U)} \underline{\underline{\leq}} \underline{\underline{\left(\sum_{i=0}^{n} \int_U |g^i|^2\right)^{1/2}}}$$

$$\| \quad \left(\sum_{i=0}^{n} \int_U |f^i|^2\right)^{1/2}$$

~~ZA~~ OVAKO ~~DEF~~ f^i NAJMANJA JE "L^2" NORMA

$$\Longrightarrow \quad \underline{\|f\|_{H^{-1}(U)}} \leq \inf_{\substack{g^0,\dots,g^n \in L^2 \\ \text{iz (i)}}} \left(\sum_{i=0}^{n} \int_U |g^i|^2\right)^{1/2} = \left(\sum_{i=0}^{n} \int_U |f^i|^2\right)^{1/2} \quad (*)$$

STAVIMO LI $v = \dfrac{u}{\|u\|_{H_0^1(U)}}$ ~~DO~~ $\langle f, v\rangle$ DOBIVAMO

$$\langle f, v\rangle = \left(\int_U Du \cdot Du + u \cdot u\right) \cdot \frac{1}{\|u\|_{H_0^1(U)}} = \|u\|_{H_0^1(U)}$$

$$= \left(\sum_{i=0}^{n} \int_U |f^i|^2\right)^{1/2}$$

$$\Longrightarrow \quad \|f\|_{H^{-1}(U)} = \sup_{\substack{u \in H_0^1(U) \\ \|u\|_{H^1} \leq 1}} \langle f, u\rangle \geq \langle f, v\rangle = \left(\sum_{i=0}^{n} \int_U |f^i|^2\right)^{1/2} \quad (**)$$

$(*)$ i $(**)$ $\Longrightarrow$ JEDNAKOST

5.9.2. PROSTORI KOJI UKLJUČUJU VRIJEME

VAŽNI ZA PARABOLIČKE I HIPERBOLIČKE ZADAĆE

NEKA JE $(X, \| \|)$ BANACHOV.

<u>~~NAP~~:</u> POGLEDAJTE DODATAK E.5 ZA POJMOVE IZMJERIVOSTI I INTEGRABILNOSTI ZA FUNKCIJE

$$f : [0,T] \to X$$

<u>DEF:</u> ZA $p \in [1, +\infty]$

$$L^p(0,T;X) = \left\{ u : [0,T] \to X : \ u \text{ IZMJERIVA I } \|u\|_{L^p(0,T;X)} < \infty \right\}$$

GDJE JE

$$\|u\|_{L^p(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|^p \, dt \right)^{1/p}, & p \in [1, +\infty) \\ \operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|, & p = +\infty \end{cases}$$

<u>DEF:</u>

$$C([0,T];X) = \left\{ u : [0,T] \to X : u \text{ NEPREKIDNA}, \ \|u\|_{C(0,T;X)} < \infty \right\}$$

GDJE JE

$$\|u\|_{C([0,T];X)} := \max_{t \in [0,T]} \|u(t)\|.$$

<u>NAP:</u> $L^p(0,T;X)$, $C([0,T];X)$ SU BANACHOVI PROSTORI

NEKAD SE ZOVU BOCHNEROVI PROSTORI

DEF: NEKA JE $u \in L^1(0,T;X)$. NEKA JE $v \in L^1(0,T;X)$ T.D.

$$\int_0^T \phi'(t)\, u(t)\, dt = -\int_0^T \phi(t)\, v(t)\, dt, \quad \phi \in C_c^\infty(0,T)$$

TADA ~~je~~ v ZOVEMO SLABA DERIVACIJA OD u I PIŠEMO

$$u' = v.$$

SAD MOŽEMO DEF. PROSTORE SOBOLJEVA

DEF:

$$W^{1,p}(0,T;X) := \{ u \in L^p(0,T;X) : u' \in L^p(0,T;X) \}$$

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|^p + \|u'(t)\|^p \right)^{1/p}, & p \in [1,+\infty) \\ \operatorname{ess\,sup}_{t \in [0,T]} \left(\|u(t)\| + \|u'(t)\| \right), & p = +\infty. \end{cases}$$

NAP:

- UZ OVU NORMU $W^{1,p}(0,T;X)$ JE BANACHOV PROSTOR
- $H^1(0,T;X) = W^{1,2}(0,T;X)$ UOBIČAJENA OZNAKA.

TM2 NEKA JE $p \in [1,+\infty]$ I $u \in W^{1,p}(0,T;X)$. TADA

(i) $u \in C([0,T];X)$ (UZ IZBOR PREDSTAVNIKA KLASE)

(ii) $$u(t) = u(s) + \int_s^t u'(\tau)\,d\tau, \quad \forall\, 0 \le s \le t \le T \qquad (N-L)$$

(iii) $\exists C>0$, OVISAN O T S.T.
$$\max_{t\in[0,T]} \|u(t)\| \le C \|u\|_{W^{1,p}(0,T;X)}$$
(NEPREKIDNOST ULAGANJA)

NAP: $n=1$, PA JE ZAPRAVO PRIRODNO

DOK: PROŠIRIMO u S 0 NA $\langle -\infty, 0\rangle \cup \langle T, +\infty\rangle$

DEF: $u^\varepsilon = \eta_\varepsilon * u$, η_ε - IZGLAĐIVAČ NA $\mathbb{R}$

PREVIJ (KAO U 1. TM. O APROKSIMACIJI)

$\Rightarrow$ $u^{\varepsilon\prime} = \eta_\varepsilon * u'$ NA $\langle \varepsilon, T-\varepsilon\rangle$

SVOJSTVA ISTA KAO PRIJE (NPR $u^\varepsilon, (u^\varepsilon)' \in C([0,T];X)$)

$$\Rightarrow \quad u^\varepsilon \longrightarrow u \quad L^p(0,T;X)$$
$$(u^\varepsilon)' \longrightarrow u' \quad L^p(0,T;X)$$

N-L VRIJEDI ZA NEPREKIDNE: $u^\varepsilon, u^{\varepsilon\prime}$

$$u^\varepsilon(t) = u^\varepsilon(s) + \int_s^t u^{\varepsilon\prime}(\tau)\,d\tau \quad , 0 < s < t < T$$

$\varepsilon \to 0$ (NA PODNIZU!)

$$\Rightarrow \quad u(t) = u(s) + \int_s^t u'(\tau)\,d\tau \quad \text{s.s.} \Rightarrow \text{(ii)}$$

ZA $u' \in L^p(0,T;X) \Rightarrow t \longmapsto \int_0^t u'(\tau)\,d\tau$ NEPREKIDNA $\Rightarrow$ (i)

$$\|u(t)\| = \frac{1}{T}\int_0^T \|u(t)\|\,ds \le \frac{1}{T}\int_0^T \|u(s)\|\,ds + \frac{1}{T}\int_0^T \Big\|\int_s^t u'(\tau)\,d\tau\Big\|\,ds$$
$$\le C(T)\|u\|_{L^p} + \frac{1}{T}\int_0^T (t-s)^{1/2}\|u'\|_{L^p}\,ds \le C(T)\|u\|_{W^{1,p}} \quad \text{(iii)}$$

GELJFANDOVA TROJKA

V, H HILBERTOVI PROSTORI

$V \subset H$ GUST I NEPREKIDNO ULOŽEN $\quad i : V \to H \,,\; i(v) = v$

$\qquad \| i(v) \|_H \leq C \|v\|_V \,,\; v \in V$

$f \in H^*$ = OGRANIČEN L.F.

$v \in V \Rightarrow v \in H$

$\Rightarrow \quad {}_{H^*}\langle f, v \rangle_H$ DOBRO DEFINIRANO

$$\left| {}_{H^*}\langle f, v \rangle_H \right| \leq \|f\|_{H^*} \|v\|_H \leq C \|f\|_{H^*} \|v\|_V \,, \; v \in V$$

$v \longmapsto {}_{H^*}\langle f, v \rangle_H$

$\Rightarrow$ TO DEFINIRA OGRANIČEN L.F. NA V! & $\|f\|_{V^*} \leq C \|f\|_{H^*}$

$\Rightarrow \quad f \;\; (tj. \; i^*(f)) \in V^*$

ZNAČI

$$i^* : H^* \longrightarrow V^* \quad \text{NEPREKIDNO ULOŽENO}$$

$$V \hookrightarrow H \quad \Longrightarrow \quad H^* \hookrightarrow V^*$$

IDENTIFICIRAMO $\quad H \cong H^*$

$f \in H^*, v \in H \qquad {}_{H^*}\langle f, v \rangle_H = (f, v)_H \qquad$ SKALARNI PRODUKT

AKO JE $v \in V$ & $f \in H^* \Rightarrow f \in V^*$

$${}_{V^*}\langle f, v \rangle_V = {}_{H^*}\langle f, v \rangle_H = (f, v)_H$$

NPR:

TIPIČNO: $\quad V = H^1_0(U)$

$\qquad H = L^2(U)$

$\qquad V^* = H^{-1}(U)$

$$H^1_0(U) \hookrightarrow L^2(U) \hookrightarrow H^{-1}(U)$$

$f \in L^2(U), v \in H^1_0(U) \qquad {}_{H^{-1}}\langle f, v \rangle_{H^1_0} = (f, v)_{L^2} = \int_U f v \, dx$

TM3 NEKA JE $u \in L^2(0,T; H^1_0(U))$ & $u' \in L^2(0,T; H^{-1}(U))$.

(i) TADA JE $u \in C([0,T]; L^2(U))$ (~~ZA~~ PREDSTAVNIKA KLASE)

(ii) PRESLIKAVANJE $t \longmapsto \|u(t)\|^2_{L^2(U)}$

JE APSOLUTNO NEPREKIDNA I

$$\frac{d}{dt}\|u(t)\|^2_{L^2(U)} = 2\langle u'(t), u(t)\rangle \quad \text{s.s. } t \in [0,T]$$

(iii) VRIJEDI: $\exists C(T) > 0$

$$\max_{t\in[0,T]} \|u(t)\|_{L^2(U)} \le C(T)\left(\|u\|_{L^2(0,T; H^1_0(U))} + \|u'\|_{L^2(0,T;H^{-1}(U))}\right)$$

DOK: KORAK 1: PROŠIRIMO u u $\mathbb{R}$ NULOM (KAO U TM2)

DEF $u^\varepsilon = \eta_\varepsilon * u$

$$\frac{d}{dt}\|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(U)} = \frac{d}{dt}\left(\left(u^\varepsilon(t)-u^\delta(t), u^\varepsilon(t)-u^\delta(t)\right)_{L^2(U)}\right)$$

$$= 2\,{}_{H^{-1}(U)}\langle u^{\varepsilon\prime}(t) - u^{\delta\prime}(t), u^\varepsilon(t) - u^\delta(t)\rangle_{H^1_0(U)}$$

INTEGRIRAM $+\int_s d\tau$. N-L $\Rightarrow$

$$\|u^\varepsilon(t)-u^\delta(t)\|^2_{L^2(U)} = \|u^\varepsilon(s)-u^\delta(s)\|^2_{L^2(U)} + 2\int_s^t {}_{H^{-1}}\langle u^{\varepsilon\prime}(\tau) - u^{\delta\prime}(\tau), u^\varepsilon(\tau)-u^\delta(\tau)\rangle_{H^1_0}\, d\tau$$

$\forall\, s,t \in [0,T]$

NEKA JE $s \in (0,T)$ BILO KOJA ZA KOJU VRIJEDI:

$$u^\varepsilon(s) \longrightarrow u(s) \quad u \quad L^2(U)$$

~~SLJEDI~~

DODATNO OCJENIM (YOUNGOVA NEJEDNAKOST)

$$\|u^\varepsilon(t)-u^\delta(t)\|^2_{L^2(U)} \le \|u^\varepsilon(s)-u^\delta(s)\|^2_{L^2(U)} + \int_s^t\left(\|u^{\varepsilon\prime}(\tau)-u^{\delta\prime}(\tau)\|^2_{H^{-1}(U)} + \|u^\varepsilon(\tau)-u^\delta(\tau)\|^2_{H^1_0(U)}\right)d\tau$$

$$\le \|u^\varepsilon(s)-u^\delta(s)\|^2_{L^2(U)} + \int_0^T\left(\| \quad \|^2 + \| \quad \|^2\right)d\tau$$

$$\Rightarrow \sup_{t\in[0,T]} \| u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(U)} \le \| u^\varepsilon(s) - u^\delta(s)\|^2_{L^2(U)} + \int_0^T \left(\|\ \|^2_{H^{-1}} + \|\ \|^2_{H^1_0}\right) d\tau$$

UZMEM $\limsup_{\varepsilon,\delta\to 0}$

$$\limsup_{\varepsilon,\delta\to 0} \sup_{t\in[0,T]} \| u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(U)} \le 0$$

NAIME
$$u^\varepsilon \longrightarrow u \quad L^2(0,T; H^1_0(U))$$
$$u^{\varepsilon\prime} \longrightarrow u' \quad L^2(0,T; H^{-1}(U))$$

$$\Rightarrow \limsup_{\varepsilon,\delta\to 0} \| u^\varepsilon - u^\delta\|_{C([0,T]; L^2(U))} = 0$$

$\Rightarrow$ CAUCHYJEVOST FAMILIJE u^ε U $C([0,T]; L^2(U))$

$\Rightarrow$ $u^\varepsilon \longrightarrow v$ u $C([0,T]; L^2(U))$

$\Rightarrow$ $v = u$ s.s. $\Rightarrow$ (i)

KORAK 2: STAVIM $u^\delta = 0$ U GORNJEM RAČUNU

$$\Rightarrow \| u^\varepsilon(t)\|^2_{L^2(U)} = \| u^\varepsilon(s)\|^2_{L^2(U)} + 2\int_s^t \langle u^{\varepsilon\prime}(\tau), u^\varepsilon(\tau)\rangle_{H^{-1}, H^1_0} d\tau$$

PUSTIM $\varepsilon \to 0$. ISTO ZA LIMES!

$\Rightarrow$ APSOLUTNA NEPREKIDNOST

KORAK 3: OCJENA KAO NA VRHU $\Rightarrow$ (ii)!

TM 4 NEKA JE $U \subseteq \mathbb{R}^n$ OTVOREN, OGRANIČEN I ∂U KLASE C^1.
NEKA JE $m \in \mathbb{N}$, $u \in L^2(0,T; H^{m+2}(U))$, $u' \in L^2(0,T; H^m(U))$

(i) TADA JE $u \in C([0,T]; H^{m+1}(U))$ (PREDSTAVNIK KLASE)

(ii)
$$\max_{t\in[0,T]} \|u(t)\|_{H^{m+1}(U)} \leq C\left(\|u\|_{L^2(0,T;H^{m+2}(U))} + \|u'\|_{L^2(0,T;H^m(U))}\right)$$

$C(T, U, m)$.

DOK: KORAK 1 $m=0$:

$$u \in L^2(0,T; H^2(U)), \quad u' \in L^2(0,T; L^2(U))$$

UZMEM $V \supset\supset U$ I PROŠIRIMO u

$$\bar{u} = Eu$$

PO NEKOJ NAPOMENI ZA $\phi \in H^2(U)$ $E\phi \in H^2(V)$ S KOMPAKTNIM NOSAČEM U V

$$\Rightarrow \bar{u} \in L^2(0,T; H^2(V))$$

KORAK 2 KAD BI $\bar{u}$ BIO GLADAK

$$\left|\frac{d}{dt}\int_V |D\bar{u}|^2 dx\right| = 2\left|\int_V D\bar{u}\cdot D\bar{u}' dx\right| \overset{P.I.}{=} 2\left|\int_V \Delta\bar{u}\, \bar{u}' dx\right|$$

$$\leq C\left(\|\bar{u}\|^2_{H^2(V)} + \|\bar{u}'\|^2_{L^2(V)}\right)$$

$$\left|\int_V |D\bar{u}(t)|^2 dx - \int_V |D\bar{u}(s)|^2 dx\right| \leq C\left(\|\bar{u}\|^2_{L^2(0,T;H^2(V))} + \|\bar{u}'\|_{L^2(0,T;L^2(V))}\right)$$

SVOJSTVA E

$$\leq C\left(\|u\|^2 + \|u'\|_{L^2}\right)$$

$\|\;\|^2_{H^1}$

TO MOGU KONTROLIRATI KAO PRIJE

$\Rightarrow$ OCJENA

KORAK 3 ~~KAD~~ $\bar{u}$ NIJE GLADAK ... IZGLAĐIVANJE

KORAK 4 $m \geq 1$ $v := D^\alpha u$, $|\alpha| \leq m$ I ZA v PRIMIJENIM REZULTAT