Parcijalne diferencijalne jednadžbe

2011/2012
60 sati

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Cilj: upoznati studente s osnovnim temama moderne teorije parcijalnih diferencijalnih jednadžbi. U prvom dijelu kolegija bavit ćemo se varijacijskom teorijom linearnih eliptičkih, paraboličkih i hiperboličkih parcijalnih diferencijalnih jednadžbi, dok je drugi dio posvećen metodama za nelinearne jednadžbame.

Teme:

- Primjeri PDJ, te pripadnih inicijalno-rubnih zadaća
- Fundamentalna rješenja. Klasična i slaba rješenja. Prostori Soboljeva.
- Varijacijska teorija linearnih eliptičkih PDJ. Egzistencija, jedinstvenost, regularnost rješenja, te princip maksimuma
- Varijacijska teorija linearnih paraboličkih PDJ. Evolucijski prostori, Galerkinova metoda. Egzistencija, jedinstvenost, regularnost rješenja, te princip maksimuma
- Varijacijska teorija linearnih hiperboličkih PDJ. Galerkinova metoda, konačna brzina širenja. Egzistencija, jedinstvenost, regularnost rješenja. Simetrični hiperbolički sustavi prvog reda.
- Teoremi fiksne točke i primjene na polulinearne i kvazilinearne jednadžbe
- Primjena metode monotonosti na nelinearne jednadžbe.
- Hiperbolički zakoni sačuvanja, šok valovi i entropijska rješenja.

Osnovna literatura

[1] Lawrence C. Evans: Partial differential equations, AMS, 1998.

Dodatna literatura

[1] H. Brezis: Analyse fonctionnelle, Masson, 1983.

[2] R. Dautray, J.-L. Lions: Mathematical analysis and numerical methods for science and technology, Springer, 1989–1991.

[3] J.-L. Lions: Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, 1969.

[4] J. Rauch, Partial differential equations, Springer, 1991.

[5] M. Renardy, R.C. Rogers: An Introduction to Partial Differential Equations, Texts in Applied Mathematics 13, Springer, 2003.

[6] R.E. Showalter: Monotone Operators in Banach Spaces and Nonlinead Partial Differential Equations, Mathematical Surveys and Monographs, Vol 49, AMS 1996.

1. Motivacija i primjeri parcijalnih diferencijalnih jednadžbi.
2. Klasifikacija parcijalnih diferencijalnih jednadžbi.
3. Klasična i slaba rješenja.
4. Slabe derivacije. Soboljevljevi prostori.
5. Linearne jednadžbe.
   - Linearne eliptičke jednadžbe 2. reda. Slaba rješenja. Lax Milgramov teorem. Principi maksimuma.
   - Linearne paraboličke jednadžbe 2. reda. Slaba rješenja. Princip maksimuma.
   - Linearne hiperboličke jednadžbe 2. reda. Slaba rješenja.
6. Nelinearne jednadžbe.
   - Metoda fiksne točke za polulinearne jednadžbe.
   - Skalarni hiperbolički zakon sačuvanja. Karakteristike. Šok valovi. Entropijska rješenja.

Osnovna literatura

L.C.Evans, Partial differential equations, AMS Graduate studies in mathematics, Vol 19, AMS, 1998.

Dodatna literatura

- M.Renardy, R.Rogers, An introduction to partial differential equations, Texsts in applied mathematics, Vol 13, Springer, 1993.
- J.Rauch, Partial differential equations, Springer, 1991.
- H.Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer, 2011.

PARCIJALNE DIFERENCIJALNE JEDNADŽBE

1. UVOD

<u>DEF:</u> IZRAZ

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0 \quad , \quad x \in U$$

NAZIVAMO PDJ k-TOG REDA, PRI ČEMU JE

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

ZADANA FUNKCIJA.

$u : U \to \mathbb{R}$ NEPOZNANICA

<u>NOTACIJA:</u>

- $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ – MULTI INDEKS
- $|\alpha| = \alpha_1 + \dots + \alpha_n$ – RED MULTINDEKSA
- $D^\alpha u(x) := \dfrac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x)$
- $D^k u(x) := \{ D^\alpha u(x) \mid |\alpha| = k \} \quad , \quad k \in \mathbb{N}_0$

  IMA IH n^k

  IDENTIFIKACIJA: $D^k u(x) \in \mathbb{R}^{n^k}$

- $|D^k u| = \left(\sum_{|\alpha| = k} |D^\alpha u|^2 \right)^{1/2}$ – NORMA NA $\mathbb{R}^{n^k}$
- $k = 1$ $\quad Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ – JACOBIJEVA M.
- $k = 2$

$$D^2 u = \begin{bmatrix} \partial_{x_1} \partial_{x_1} u & \dots & \partial_{x_1} \partial_{x_n} u \\ \vdots & & \vdots \\ \partial_{x_n} \partial_{x_1} u & \dots & \partial_{x_n} \partial_{x_n} u \end{bmatrix}$$ – HESSEOVA M.

- $\Delta u = \mathrm{tr}(D^2 u) = \sum^{n} \partial^2 u$ – LAPLACE[illegible]

RIJEŠITI PDJ:

- NAĆI SVA RJEŠENJA (KOJA ZADOVOLJAVAJU DODATNE (RUBNE/POČETNE) UVJETE) EKSPLICITNO
- POKAZATI EGZISTENCIJU I DRUGA SVOJSTVA RJEŠENJA

KLASIFIKACIJA:

(i) PDJ JE LINEARNA AKO JE OBLIKA

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u(x) = f(x)$$

f, (a_α, $|\alpha|\le k$) ZADANE FUNKCIJE.

$f=0$ PDJ HOMOGENA

NELINEARNE

(ii) SEMILINEARNA PDJ:

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0\left(D^{k-1}u(x), \dots, Du(x), u(x), x\right) = 0$$

(iii) KVAZILINEARNA PDJ:

$$\sum_{|\alpha|=k} a_\alpha\left(D^{k-1}u(x), \dots, Du(x), u(x), x\right) D^\alpha u(x) + a_0\left(D^{k-1}u(x), \dots, Du(x), u(x), x\right) = 0$$

(iv) "FULLY" NELINEARNA PDJ:

OVISI NELINEARNO O NAJVIŠIM DERIVACIJAMA

DEF: SUSTAV PDJ

$$\mathbb{F}(D^k u(x), \dots, Du(x), u(x), x) = 0, \quad x \in U$$

K-TOG REDA, PRI ČEMU JE

$$\mathbb{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \dots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$$

ZADANA FUNKCIJA

$u : U \to \mathbb{R}^m$, $u = (u^1, \dots, u^m)$ NEPOZNANICA/E

(BROJ NEPOZNANICA = BROJ JEDNADŽBI)

NAP:

- NEMA OPĆE TEORIJE
- FOKUS NA JDBE IZ PRIMJENA (IZ/IZVAN MATEMATIKE)
- ŠTO JE VIŠE NELINEARNA, TEŽA
- JDBE VIŠEG REDA, TEŽE
- SUSTAVI TEŽI OD JEDNADŽBI
- VIŠE NEPOZNANICA u, TEŽE
- RIJETKO IMAMO FORMULU ZA RJEŠENJE

Remark. We use "PDE" as an abbreviation for both "partial differential equation" and "partial differential equation*s*". □

1.2. EXAMPLES

There is no general theory known concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations that are important for applications within and outside of mathematics, with the hope that insight from the origins of these PDE can give clues as to their solutions.

Following is a list of many specific partial differential equations of interest in current research. This listing is intended merely to familiarize the reader with the names and forms of various famous PDE. To display most clearly the mathematical structure of these equations, we have mostly set relevant physical constants to unity. We will later discuss the origin and interpretation of many of these PDE.

Throughout $x \in U$, where U is an open subset of $\mathbb{R}^n$, and $t \geq 0$. Also $Du = D_x u = (u_{x_1}, \dots, u_{x_n})$ denotes the gradient of u with respect to the spatial variable $x = (x_1, \dots, x_n)$.

1.2.1. Single partial differential equations.

a. Linear equations.

1. *Laplace's equation*

RAVNOTEŽA TOPLINE
MEMBRANE

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0.$$

2. *Helmholtz's (or eigenvalue) equation*

$$-\Delta u = \lambda u.$$

3. *Linear transport equation*

MASS TRANSFER
HEAT TRANSFER
MOMENTUM TRANSFER

$$u_t + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

4. *Liouville's equation*

$$u_t - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

5. *Heat (or diffusion) equation*

$$u_t - \Delta u = 0.$$

6. *Schrödinger's equation*

$$iu_t + \Delta u = 0.$$

EVOLUCIJA KVANTNOG STANJA SISTEMA
U VALNA FUNKCIJA (F. STANJA)
GUSTOĆA VJEROJATNOSTI
ZA TRENUTAK t
I POLOŽAJ x

7. *Kolmogorov's equation*

$$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

KOMBINACIJA TRANSPORTA I DIFUZIJE

8. *Fokker–Planck equation*

$$u_t - \sum_{i,j=1}^{n} (a^{ij} u)_{x_i x_j} - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

9. *Wave equation*

$$u_{tt} - \Delta u = 0.$$

10. *Telegraph equation*

$$u_{tt} + du_t - u_{xx} = 0.$$

11. *General wave equation*

$$u_{tt} - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

12. *Airy's equation*

LIN KdV

$$u_t + u_{xxx} = 0.$$

13. *Beam equation*

$$u_t + u_{xxxx} = 0.$$

b. Nonlinear equations.

1. *Eikonal equation*

$$|Du| = 1.$$

OPTIKA

2. *Nonlinear Poisson equation*

$$-\Delta u = f(u).$$

3. *p-Laplacian equation*

$$\operatorname{div}(|Du|^{p-2}Du) = 0.$$

GENERALIZACIJA
MINIMIZACIJA $\int_\Omega |\nabla u|^p$

4. *Minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) = 0.$$

ZADAN U I ZADANI
R.U. TRAŽI SE
PLOHA MINIMALNE POVRŠINE

5. *Monge–Ampère equation*

$$\det(D^2u) = f.$$

GEOMETRIJA

6. *Hamilton–Jacobi equation*

$$u_t + H(Du, x) = 0.$$

NUŽAN UVJET ZA
EKSTREMALNOST, U VARIJACIJSKOM
RAČUNU

7. *Scalar conservation law*

$$u_t + \operatorname{div}\mathbf{F}(u) = 0.$$

JDBA GIBANJA
OČUVANJE MASE, MOMENTA, ...

8. *Inviscid Burgers' equation*

$$u_t + uu_x = 0.$$

MEHANIKA FLUIDA
(PLINOVI)
TRANSPORT

9. *Scalar reaction-diffusion equation*

$$u_t - \Delta u = f(u).$$

10. *Porous medium equation*

$$u_t - \Delta(u^\gamma) = 0.$$

11. *Nonlinear wave equations*

$$u_{tt} - \Delta u = f(u),$$
$$u_{tt} - \operatorname{div}\mathbf{a}(Du) = 0.$$

12. *Korteweg–deVries (KdV) equation*

$$u_t + uu_x + u_{xxx} = 0.$$

2D PLITKA VODA

1.2.2. Systems of partial differential equations.

a. Linear systems.

1. *Equilibrium equations of linear elasticity*

$$\mu\Delta\mathbf{u} + (\lambda + \mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

2. *Evolution equations of linear elasticity*

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda + \mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

3. *Maxwell's equations*

$$\begin{cases} \mathbf{E}_t = \operatorname{curl}\mathbf{B} \\ \mathbf{B}_t = -\operatorname{curl}\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0. \end{cases}$$

KLASIČNA ELEKTRO DINAMIKA
E - ELEKTRIČNO POLJE
B - MAGNETSKO

b. Nonlinear systems.

1. *System of conservation laws*

$$\mathbf{u}_t + \operatorname{div}\mathbf{F}(\mathbf{u}) = \mathbf{0}.$$

2. *Reaction-diffusion system*

$$\mathbf{u}_t - \Delta\mathbf{u} = \mathbf{f}(\mathbf{u}).$$

3. *Euler's equations for incompressible, inviscid flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

4. *Navier–Stokes equations for incompressible, viscous flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} - \Delta\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

See Zwillinger [**ZW**] for a much more extensive listing of interesting PDE.

DOBRO POSTAVLJENA ZADAĆA

(a) RJEŠENJE POSTOJI

(b) JEDINSTVENO JE

(c) NEPREKIDNO OVISI O ZADANIM PODACIMA

Ad(a) SMISAO RJEŠENJA:

- ANALITIČKA FUNKCIJA
- C^∞
- C^k, k - RED JEDNADŽBE

KLASIČNA RJEŠENJA (JAKA)

$\Delta u = 0$ u $\mathbb{R}^n$

- SLABA RJEŠENJA

$$u_t + F(u)_x = 0$$

MODELIRA PROBLEME MEHANIKE FLUIDA

KLASIČNA RJEŠENJA S VREMENOM POSTAJU PREKIDNA (ŠOKOVI)

ŽELIMO I PROMATRATI

KLASIČNA RJEŠENJA : TEŠKO POKAZATI EGZISTENCIJU

OBIČNO PUT : EGZISTENCIJA SLABIH (GENERALIZIRANIH) RJ.
PODATNA
REGULARNOST (GLATKOĆA) RJ.

Ad(b) ČESTO RJEŠENJ~~E~~ NIJ~~E~~ JEDINSTVENA

KLASIFIKACIJA I SVOJSTVA SU NAM CILJ

Ad(c) VAŽNO ZA PRIMJENE

2. ČETIRI PRIMJERA LINEARNIH PDJ

$u_t + b \cdot Du = 0$ TRANSPORTNA J.

$\Delta u = 0$ LAPLACEOVA J.

$u_t - \Delta u = 0$ J. PROVOĐENJA

$u_{tt} - \Delta u = 0$ VALNA J.

2.1. TRANSPORTNA JEDNADŽBA

$b \in \mathbb{R}^n$ ZADAN

$$u_t + b \cdot Du = 0 \quad u \ \mathbb{R}^n \times \langle 0, +\infty \rangle$$

NEPOZNANICA: $u : \underset{x}{\mathbb{R}^n} \times \underset{t}{[0, +\infty\rangle} \to \mathbb{R}$

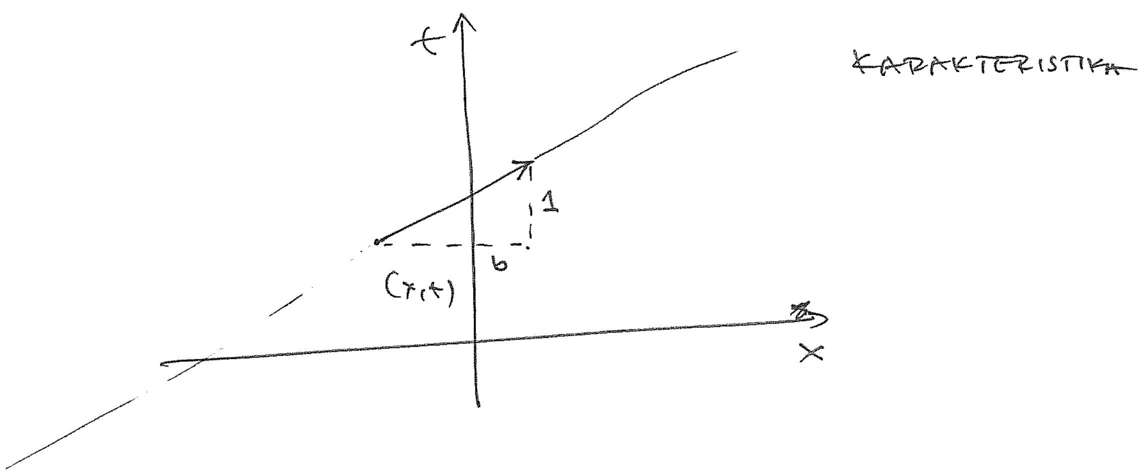
$$u(x,t)$$

$$Du(x,t) = D_x u(x,t) = (\partial_{x_1} u(x,t), \ldots, \partial_{x_n} u(x,t))$$

FIKSIRAM (x,t) I NEKA u ZADOVOLJAVA J. (KLASIČNO) u DOVOLJNO GLATKO

DEF: $z(s) = u(x+sb, t+s)$, $s \in \mathbb{R}$
$= u((x,t) + s(b,1))$

za $\mathbb{R} \times [0, +\infty\rangle$:



$$\dot{z}(s) = D_x u(x+sb, t+s) \cdot b + u_t(x+sb, t+s) = 0$$

JER JE u RJEŠENJE

$\Rightarrow$ z JE KONSTANTA

$\Rightarrow$ ZA DANI (x,t), $s \mapsto u((x,t)+s(b,1))$ JE KONSTANTA

$\Rightarrow$ RJEŠENJE JE KONSTANTNO DUŽ PRAVACA NOSENIH S $(b,1)$

$\Rightarrow$ AKO ZNAMO RJ U NEKOJ TOČKI PRAVCA $\Rightarrow$ ZNAMO RJ NA CIJELOM PRAVCU

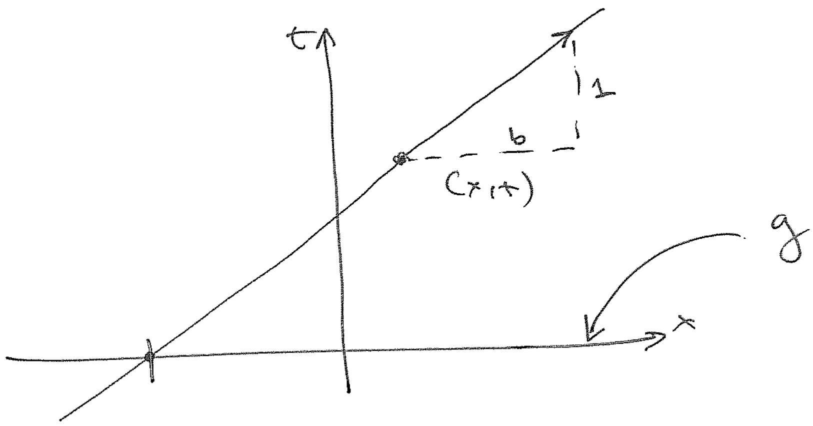
2.1.1. POČETNA ~~ZADAĆA~~ (INICIJALNA ZADAĆA)

CAUCHYJEVA ZADAĆA:

$b \in \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$ ZADANO

NAĆI $u : \mathbb{R}^n \times [0, +\infty\rangle \to \mathbb{R}$ T.D.

$$\text{C.Z.} \begin{cases} u_t + b \cdot Du = 0 & \text{u } \mathbb{R}^n \times \langle 0, +\infty\rangle \\ u = g & \text{na } \mathbb{R}^n \times \{0\} \end{cases}$$ (POČETNI UVJET)



PARAM. PRAVAC $f(s) = (x,t) + s(b,1)$

$$f(s) = (x_0, 0) = (x,t) + s(b,1) = (x+sb, t+s)$$

$t + s = 0 \Rightarrow s = -t$

$x_0 = x + sb \Rightarrow \boxed{x_0 = x - tb}$

$$u(x,t) = u(x-tb, 0) = g(x-tb)$$

$\uparrow$ IZ PRIJAŠNJEG SVOJSTVA $\qquad \nwarrow$ IZ P.U.

$$\boxed{u(x,t) = g(x-tb)} \qquad (*)$$

DAKLE: AKO JE u DOVOLJNO GLATKO $\Rightarrow u(x,t) = g(x-tb)$

OBRATNO, AKO JE g KLASE $C^1 \Rightarrow u(x,t) = g(x-tb)$ JE RJEŠENJE

NAP: AKO g NIJE $C^1 \Rightarrow$ NEMA C^1 RJEŠENJA C.Z.

I DALJE $(*)$ JE JEDINI RAZUMNI KANDIDAT ZA RJ.

OSTAJE PITANJE U KOJEM SMISLU ZADOVOLJAVA J.

2.1.2. NEHOMOGENA INICIJALNA ZADAĆA

ZADANI: $f: \mathbb{R}^n \times \langle 0, +\infty\rangle$, $g: \mathbb{R}^n \to \mathbb{R}$

$$\begin{cases} u_t + b \cdot Du = f & u \ \mathbb{R}^n \times \langle 0, +\infty\rangle \\ u = g & \text{na } \mathbb{R}^n \times \{0\} \end{cases}$$

FIKSIRAM (x,t), $u(x,t)$ RJ.

DEF: $z(s) = u(x+sb, t+s) = u((x,t) + s(b,1))$

$$\dot{z}(s) = Du(x+sb, t+s)\cdot b + u_t(x+sb, t+s) = f(x+sb, t+s) \quad \Big/ \int_{-t}^{0} ds$$

$$\int_{-t}^{0} \dot{z}(s)\, ds = \int_{-t}^{0} f(x+sb, t+s)\, ds = \left| \begin{matrix} t+s = p \\ ds = dp \end{matrix} \right| = \int_0^t f(x+(p-t)b, p)\, dp$$

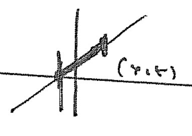
$\shortparallel$

$$z(0) - z(-t) = u(x,t) - u(x-tb, 0) = u(x,t) - g(x-tb)$$

$\Rightarrow$ RJ $\quad \boxed{u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\, ds}$ $\qquad$ PDJ $\rightsquigarrow$ ODJ

METODA KARAKTERISTIKA

ISTI ZAKLJUČCI KAO I PRIJE



2.2. LAPLACEOVA JEDNADŽBA

$$\Delta u = 0 \quad u \quad U \subseteq \mathbb{R}^n$$

$$\sum_{i=1}^{n} \partial_{x_i}^2 u = 0$$

NEHOMOGENA (POISSONOVA) J. : ZA $f: U \to \mathbb{R}$ ZADANU

$$-\Delta u = f \quad u \quad U$$

DEF: $u \in C^2(U)$ T.D. $\Delta u = 0$ ZOVE SE HARMONIJSKA FUNKCIJA

PORIJEKLO: u – GUSTOĆA SUPSTANCE (KONCENTRACIJA NEKOG KEMIJSKOG SPOJA)

F – FLUKS OD u: NPR.

$$F = -a Du \quad , \quad a > 0$$

SUSTAV U RAVNOTEŽI:

$$\int_{\partial V} F \cdot \nu \, dS = 0 \quad \text{ZA SVAKI } V \subset U$$

GAUSSOV TEOREM (TM O DIVERGENCIJI)

$$\int_V \operatorname{div} F \, dx = 0$$

PROIZVOLJNOST OD V

$$\operatorname{div} F = 0$$

$$-a \operatorname{div} Du = 0$$

$$\boxed{\Delta u = 0}$$

u

KEMIJSKA KONCENTRACIJA — FICKOV ZAKON DIFUZIJE

TEMPERATURA — FOURIEROV ZAKON PROVOĐENJA TOPLINE

ELEKTRIČNI POTENCIJAL — OHMOV ZAKON PROVOĐENJA

2.2.1 FUNDAMENTALNA RJEŠENJA

- TRAŽIMO NEKA SPECIJALNA RJEŠENJA
- SIMETRIJE

PR 1: $u \ldots \quad \Delta u = 0$

$$v(x) := u(x+c) \quad , \quad c \in \mathbb{R}^n$$

$$\Delta v(x) = \Delta u(x+c) = 0$$

PR 2: $u \ldots \quad \Delta u = 0$

$$Q \in O(n)$$

$$v(x) := u(Qx)$$

$$\partial_{x_i} v(x) = Du(Qx)\, Qe_i$$

$$\partial^2_{x_i} v(x) = D^2 u(Qx)\, Qe_i \cdot Qe_i = Q^T D^2 u(Qx) Q\, e_i \cdot e_i$$

$$\Delta v(x) = \operatorname{tr}\left(Q^T D^2 u(Qx) Q\right) = \operatorname{tr}\left(D^2 u(Qx)\right) =$$

$$= \Delta u(Qx) = 0$$

LAPLACEOVA J. INVARIJANTNA NA ROTACIJE

TRAŽIMO RJ. ZA $U = \mathbb{R}^n$ OBLIKA

$$u(x) = v(|x|) \quad \left(v(r) \ , \ r(x) = |x| = \sqrt{\sum_{i=1}^n x_i^2} \right)$$

$$\frac{\partial r}{\partial x_i}(x) = \frac{1}{2}\left(\sum_{i=1}^n x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{r(x)} \quad , \quad x \neq 0$$

$$\partial_{x_i} u(x) = v'(r(x)) \frac{\partial r}{\partial x_i}(x) = v'(r(x)) \frac{x_i}{r(x)}$$

$$\partial^2_{x_i} u(x) = v''(r(x)) \frac{x_i^2}{r(x)^2} + v'(r(x)) \left(\frac{1}{r(x)} - \frac{x_i^2}{r(x)^3} \right)$$

$$\Delta u(x) = v''(r(x)) + v'(r(x)) \frac{n-1}{r(x)}$$

$$\Delta u(x) = 0 \iff v''(r(x)) + v'(r(x)) \frac{n-1}{r(x)} = 0$$

ZAMJENA VAR

$$\iff \quad v''(r) + v'(r) \frac{n-1}{r} = 0$$

$$\frac{v''}{v'} = \frac{1-n}{r}$$

$$\| $$

$$(\log v')' \quad \Rightarrow \log v'(r) = (1-n)\log r + \log a$$

$$v'(r) = a r^{1-n} = \frac{a}{r^{n-1}}$$

DAKLE:

$$v(r) = \begin{cases} b \log r + c, & n = 2 \\ \dfrac{b}{r^{n-2}} + c, & n \geq 3 \end{cases}$$

DEF: FUNKCIJA

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log|x|, & n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

NAZIVA SE FUNDAMENTALNO RJEŠENJE L.J.

- $\alpha(n)$ - VOLUMEN $K(0,1) \subseteq \mathbb{R}^n$.
- KONSTANTE SMO ODABRALI PRIGODNO

VRIJEDI:

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$$

$$|D^2\Phi(x)| \leq \frac{C}{|x|^n}$$

NEHOMOGENA J.

$x \neq 0 \qquad x \longmapsto \Phi(x)$ HARMONIJSKA

$x \neq y \qquad x \longmapsto \Phi(x-y)$ HARMONIJSKA

$f \in C_c^2(\mathbb{R}^n)$, $x \neq y \qquad x \longmapsto \Phi(x-y) f(y)$ HARMONIJSKA

DEF:

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \quad , \quad x \in \mathbb{R}^n$$

NIJE HARMONIJSKA !

NAIME NE VRIJEDI

$$\Delta u(x) = \int_{\mathbb{R}^n} \underset{\uparrow}{\Delta_x \Phi(x-y)} f(y)\, dy$$

INTEGRAL NIJE DEFINIRAN

TM 1 u ZADOVOLJAVA

$$u \in C^2(\mathbb{R}^n)$$
$$-\Delta u = f \quad u \ \mathbb{R}^n$$

DOK: 1) $u \in C^2(\mathbb{R}^n)$

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-z) f(z) dz = \begin{vmatrix} y = x-z \\ dy = -dz \end{vmatrix} = \int_{\mathbb{R}^n} \Phi(y) f(x-y) dy$$

$$\Rightarrow D^2 u(x) = \int_{\mathbb{R}^n} \Phi(y) D_x^2 f(x-y) dy$$

2) $\varepsilon > 0$

$$\Delta u(x) = \int_{B(0,\varepsilon)} \Phi(y) \underbrace{\Delta_x f(x-y)}_{\leq C} dy + \int_{\partial B(0,\varepsilon)} \Phi(y) \underbrace{\frac{\partial f}{\partial \nu}(x-y)}_{\leq C} dS(y) \qquad \to 0$$

$2\times$P.I.

$$- \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y) dS(y) + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta \Phi(y) f(x-y) dy}_{= 0}$$

$$\frac{\partial \Phi}{\partial \nu}(y) = D\Phi(y)\cdot \nu(y)$$

VANJSKA JEDINIČNA NA $\partial B(0,\varepsilon)$

$$\nu(y) = -\frac{y}{|y|} = -\frac{y}{\varepsilon}$$

$$D\Phi(y) = \frac{1}{n(n-2)\alpha(n)}(-n+2)\frac{1}{|y|^{n-1}}\frac{y}{|y|} = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n} = -\frac{1}{n\alpha(n)}\frac{y}{\varepsilon^n}$$

$$\Rightarrow \frac{\partial \Phi}{\partial \nu}(y) = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$$

$$\Rightarrow -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,dS(y) = -\int \frac{1}{n\alpha(n)\varepsilon^{n-1}} f(x-y)\,dS(y)$$

$$= -\frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} f(y)\,dS(y)$$

$n\alpha(n) \leftarrow |S^{n-1}|$, $\subseteq \mathbb{R}^n$

$$= -⨍_{\partial B(x,\varepsilon)} f(y)\,dS(y) \longrightarrow -f(x)$$

$$\Rightarrow \quad -\Delta u = f$$

FORMALNO: $\quad -\Delta\Phi = \delta_0 \quad u\ \mathbb{R}^n$

$$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta_x \Phi(x-y) f(y)\,dy$$

$$= \int_{\mathbb{R}^n} \delta_x f(y)\,dy = f(x)$$

TM 2 (TEOREM SREDNJE VRIJEDNOSTI)

AKO JE $u \in C^2(U)$ HARMONIJSKA, TADA ZA SVAKU $K(x,r) \subset U$

$$u(x) = \fint_{\partial B(x,r)} u \, dS = \fint_{B(x,r)} u \, dy .$$

POK: 1. JEDNAKOST

$$\phi(r) := \fint_{\partial B(x,r)} u(y) \, dS(y) = \fint_{\partial B(0,1)} u(x+rz) \, dS(z)$$

JEDINIČNA VANJSKA NORMALA ν

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz) \cdot z \, dS(z) = \fint_{\partial B(x,r)} Du(y) \left(\frac{y-x}{r}\right) dS(y)$$

$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y) \, dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y) \, dS(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y) \, dy = 0$$

$\Rightarrow$ ϕ JE KONSTANTA

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} \fint_{\partial B(x,t)} u(y) \, dS(y) = u(x)$$

$\Rightarrow$ 1. JEDNAKOST

2. JEDNAKOST

~~POLARNE KOORDINATE~~

$$\int_{B(x,r)} u(y) \, dy = \int_0^r \left(\int_{\partial B(x,s)} u \, dS \right) ds = \int_0^r u(x) \, n\alpha(n) s^{n-1} \, ds$$

$$= u(x) \, n\alpha(n) \frac{r^n}{n} = \underbrace{\alpha(n) r^n}_{V_n(r)} u(x)$$

$\square$

TM3 (OBRAT) AKO $u \in C^2(U)$ ZADOVOLJAVA

$$u(x) = \fint_{\partial B(x,r)} u \, dS , \quad B(x,r) \subset U$$

$\Rightarrow$ u JE HARMONIJSKA

POK: IZ $\Rightarrow$ $\phi(r)$ JE KONSTANTA.

PRETP. u NIJE HARMONIJSKA: $\Delta u \neq 0 \Rightarrow \exists B(x,r) \subset U$ T.D. $\Delta u > 0$

$$0 = \phi'(r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y) \, dy > 0 \Rightarrow \Leftarrow$$

$\square$

TM 4 (PRINCIP MAKSIMUMA)

NEKA JE $U \subseteq \mathbb{R}^n$ OTVOREN I OGRANIČEN

NEKA JE $u \in C^2(U) \cap C(\bar{U})$ HARMONIJSKA U U.

TADA:

(i) $$\max_{\bar{U}} u = \max_{\partial U} u$$ PRINCIP MAKSIMUMA

(ii) AKO JE U POVEZAN I $\exists\, x_0 \in U$ T.D.

$$u(x_0) = \max_{\bar{U}} u$$

TADA JE u KONSTANTA NA U — JAKI PRINCIP MAKSIMUMA

NAP: TM VRIJEDI I ZA "MIN" $(-u)$

DOK: PRETP: $\exists\, x_0 \in U$ TD $u(x_0) = \max_{\bar{U}} u =: M$

PROMATRAMO: $$S = \{x \in U : u(x) = M\} \ni x_0$$

1) S RELATIVNO ZATVOREN U U
2) $x_0 \in S \Rightarrow S \neq \emptyset$
3) NEKA JE $z \in S$ T.J. $u(z) = M$

$$M = u(z) = \fint_{B(z,r)} u \, dy \le M$$

(ZA r DOVOLJNO MALI DA JE $B(z,r) \subset U$)

KAKO VRIJEDI $=$ $\Rightarrow u \equiv M$ NA $B(z,r)$

$\Rightarrow B(z,r) \subset S \Rightarrow S$ JE OTVOREN

$\Rightarrow S = U$

NAP: U POVEZAN, $u \in C^2(U) \cap C(\bar{U})$: $\begin{cases} \Delta u = 0 & \text{u } U \\ u = g & \text{na } \partial U \end{cases}$

AKO JE $g \geq 0 \Rightarrow \min_{\partial U} g \geq 0$

$\overset{(i)}{\Rightarrow} \min_{\bar{U}} u = \min_{\partial U} u = \min_{\partial U} g \geq 0 \Rightarrow u \geq 0$ NA U

$\overset{(ii)}{\Rightarrow} \exists\, x_0 \in U$ $u(x_0) = 0 = \min_{\bar{U}} u$

$\Rightarrow u = 0$ NA $\bar{U}$

AKO JE g NEGDJE $> 0 \Rightarrow\!\!\!\!\!\!/$

$\Rightarrow u > 0$ NA U

TM5 (JEDNSTVENOST RUBNE ZADAĆE)

NEKA JE $g \in C(\partial U)$, $f \in C(U)$.

TADA $\exists$ NAJVIŠE JEDNO RJEŠENJE $u \in C^2(U) \cap C(\bar{U})$ OD

$$\begin{cases} -\Delta u = f & \text{u } U \\ u = g & \text{na } \partial U \end{cases}$$

DOK: $u, \tilde{u}$ ZADOVOLJAVAJU R.Z.

DEF: $w_{\pm} = \pm(u - \tilde{u})$

$$\Rightarrow \quad -\Delta w_{\pm} = 0 \quad \text{u } U$$
$$w_{\pm} = 0 \quad \text{na } \partial U$$

PRINCIP MAKSIMUMA:

ZA w_+: $\max\limits_{\bar{U}} w_+ = \max\limits_{\partial U} w_+ = 0$

ZA w_-: $\max\limits_{\bar{U}} w_- = \max\limits_{\partial U} w_- = 0$

DAKLE:

$$\max_{\bar{U}} (u - \tilde{u}) = 0$$

$$\max_{\bar{U}} -(u - \tilde{u}) = 0$$
$$\parallel$$
$$-\min (u - \tilde{u})$$

$$0 = \min(u - \tilde{u}) \leq \max(u - \tilde{u}) = 0$$

$$\Rightarrow = \qquad \Rightarrow u = \tilde{u}$$