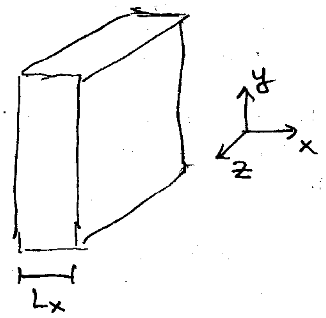
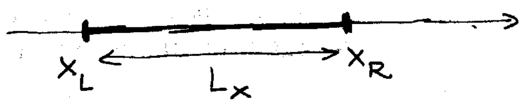
PLOČA – 1. DIM. MODEL

Zadana je tanka ploča debljine L_x, uz pretpostavku da su preostale dvije dimenzije ploče mnogo veće od L_x (tj. beskonačne



Projekt na x osi je



1. dim. jednadžba provođenja ima oblik:

$$\rho c \cdot \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\lambda \cdot \frac{\partial T}{\partial x}\right] \quad \text{za } x \in [x_L, x_R]$$

uz zadani početni uvjet za $t = 0$

$$T(x, 0) = T_0(x) \quad , x \in [x_L, x_R]$$

i promatramo temperaturu $T(x,t)$ za $t > 0$, u nekom intervalu $t \in [\emptyset, t_{final}]$.

Pretpostavljamo uniformne mreže u x i t smjeru

$$h_x = \frac{x_R - x_L}{M_x} = \frac{L_x}{M_x} \quad , \text{ čvorovi } x_i = x_L + i \cdot h_x \; , \; i = 0, -, M_x$$

$$\tau = \frac{t_{final}}{N_t} \quad , \text{ čvorovi } t_n = n \cdot \tau \quad , \; n = 0, -, N_t$$

Sve funkcije $f(T, \rho, c, \lambda)$ u točki (x_i, t_n) označavamo indeksima

$$f(x_i, t_n) = f_i^n$$

– Pravo rješenje T aproksimiramo mrežnom funkcijom U

$$T(x_i, t_n) \approx U_i^n$$

gdje će U biti rješenje aproksimativne - diferencijske jednadžbe.

Implicitna metoda - jednadžba

Diskretiziramo jednadžbu u točki (x_i, t_{n+1})
- u t - diferencijom unatrag
- u x - simetričnom (drugom) diferencijom

$$\frac{\partial U}{\partial t}(x_i, t_{n+1}) \approx \frac{U_i^{n+1} - U_i}{\tau} \quad \text{s lok. pogreškom } O(\tau)$$

$$\frac{\partial}{\partial x}\left[\lambda \cdot \frac{\partial U}{\partial x}\right](x_i, t_{n+1}) \approx \frac{\left[\lambda \frac{\partial U}{\partial x}\right](x_{i+\frac{1}{2}}, t_{n+1}) - \left[\lambda \frac{\partial U}{\partial x}\right](x_{i-\frac{1}{2}}, t_{n+1})}{h_x}$$

$$\approx \frac{\lambda_{i+\frac{1}{2}}^{n+1}\left[U_{i+1}^{n+1} - U_i^{n+1}\right] - \lambda_{i-\frac{1}{2}}^{n+1}\left[U_i^{n+1} - U_{i-1}^{n+1}\right]}{h_x^2} \quad \text{s lok. pogr. } O(h_x^2)$$

Diskretizirana jednadžba ima oblik:

$$(\rho c)_i^{n+1} \cdot \frac{U_i^{n+1} - U_i^n}{\tau} = \frac{\lambda_{i+\frac{1}{2}}^{n+1}\left[U_{i+1}^{n+1} - U_i^{n+1}\right] - \lambda_{i-\frac{1}{2}}^{n+1}\left[U_i^{n+1} - U_{i-1}^{n+1}\right]}{h_x^2}$$

Sve vrijednosti s indeksom $n+1$ pišemo s lijeve, a sve vrijednosti s indeksom n s desne strane; i množimo s h_x^2:

$$\boxed{\begin{aligned} -\lambda_{i-\frac{1}{2}}^{n+1} U_{i-1}^{n+1} + \left[(\rho c)_i^{n+1} \cdot \frac{h_x^2}{\tau} + \lambda_{i-\frac{1}{2}}^{n+1} + \lambda_{i+\frac{1}{2}}^{n+1}\right] \cdot U_i^{n+1} - \lambda_{i+\frac{1}{2}}^{n+1} \cdot U_{i+1}^{n+1} &= \\ = (\rho c)_i^{n+1} \cdot \frac{h_x^2}{\tau} \cdot U_i^n & \\ \text{za } i = 1, \ldots, M_x - 1 & \end{aligned}}$$

Lokalna pogreška diskretizacije je $O(\tau + h_x^2)$ u aproksimaciji jednadžbe, odnosno, integracijom po t, imamo lokalnu pogrešku diskretizacije $O(\tau^2 + \tau h_x^2)$ u aproksimaciji funkcije.

Napomena: kod iterativnog rješenja nelinearnog problema, svi koeficijenti s indeksom $n+1$ ~~na lijevoj strani~~ računaju se iz prethodne aproksimacije rješenja u zadanoj točki (s indeksom $n+1$).

Implicitna metoda – rubni uvjeti

Lijevi rub $x_L = x_\emptyset$:

Opći oblik rubnog uvjeta je:

$$a_L \cdot T(x_L, t) + b_L \cdot \left[\lambda \frac{\partial T}{\partial x}\right](x_L, t) = f_L$$

gdje su a_L, b_L, f_L općenito funkcije temperature i/ili vremena. Promatramo ih kao funkcije vremena.

Diskretizacija RU u točki (x_L, t_{n+1}), uz $x_L = x_\emptyset$:

$$\left[\lambda \frac{\partial U}{\partial x}\right](x_0, t_{n+1}) \approx \lambda_\emptyset^{n+1} \cdot \frac{U_1^{n+1} - U_{-1}^{n+1}}{2h_x}$$

što daje jednadžbu:

$$a_L^{n+1} \cdot U_\emptyset^{n+1} + b_L^{n+1} \cdot \lambda_\emptyset^{n+1} \cdot \frac{U_1^{n+1} - U_{-1}^{n+1}}{2h_x} = f_L^{n+1}$$

s lokalnom pogreškom diskretizacije $O(h_x^2)$.

- Ako je $b_L^{n+1} = \emptyset$ – Dirichletov RU, onda je

$$\boxed{U_\emptyset^{n+1} = \frac{f_L^{n+1}}{a_L^{n+1}}} .$$

- Ako je $b_L^{n+1} \neq \emptyset$ – Neumannov ili mješoviti RU, onda dobivamo relaciju za U_{-1}^{n+1}

$$U_{-1}^{n+1} = U_1^{n+1} + \frac{2h_x}{b_L^{n+1} \cdot \lambda_\emptyset^{n+1}} \cdot \left[a_L^{n+1} \cdot U_\emptyset^{n+1} - f_L^{n+1}\right],$$

koju treba uvrstiti u diskretiziranu jednadžbu u točki (x_L, t_{n+1}), koja glasi:

$$-\lambda_{-\frac{1}{2}}^{n+1} \cdot U_{-1}^{n+1} + \left[(\rho c)_\emptyset^{n+1} \cdot \frac{h_x^2}{\tau} + \lambda_{-\frac{1}{2}}^{n+1} + \lambda_{\frac{1}{2}}^{n+1}\right] \cdot U_0^{n+1} - \lambda_{\frac{1}{2}}^{n+1} \cdot U_1^{n+1} =$$

$$= (\rho c)_0^{n+1} \cdot \frac{h_x^2}{\tau} \cdot U_0^n .$$

Dobivamo jednadžbu:

$$\boxed{\left[(\rho c)_{\emptyset}^{n+1}\cdot\frac{h_x^2}{\tau}+\lambda_{-\frac{1}{2}}^{n+1}\cdot\left(1-\frac{2h_x\, a_L^{n+1}}{b_L^{n+1}\cdot\lambda_{\emptyset}^{n+1}}\right)+\lambda_{\frac{1}{2}}^{n+1}\right]\cdot U_{\emptyset}^{n+1}-\left(\lambda_{-\frac{1}{2}}^{n+1}+\lambda_{\frac{1}{2}}^{n+1}\right)U_1^{n+1}= =(\rho c)_{\emptyset}^{n+1}\cdot\frac{h_x^2}{\tau}\cdot U_0^n-\lambda_{-\frac{1}{2}}^{n+1}\cdot\frac{2h_x\cdot f_L^{n+1}}{b_L^{n+1}\cdot\lambda_0^{n+1}}}$$

- Newtonov zakon hlađenja u $x_L = x_\emptyset$: $\left(u\ x_L,\ \frac{\partial T}{\partial n}=-\frac{\partial T}{\partial x}\right)$

$$-\left[\lambda\frac{\partial T}{\partial x}\right](x_L,t)=-\alpha_L\cdot\left[T(x_L,t)-T_{exL}(t)\right]$$

što odgovara standardnom zapisu

$$\alpha_L\cdot T(x_L,t)+\left[\lambda\cdot\frac{\partial T}{\partial x}\right](x_L,t)=\alpha_L\cdot T_{exL}(t)$$

$\Rightarrow$
$a_L=\alpha_L$
$b_L=-1$
$f_L=\alpha_L\cdot T_{exL}(t)$.

Alternativa:
$a_L=\emptyset$,
$b_L=-1$,
$f_L=-\alpha_L\left[T_{old}(x_L,t)-T_{exL}(t)\right]$

Odgovarajuća jednadžba je:

$$\boxed{\left[(\rho c)_{\emptyset}^{n+1}\cdot\frac{h_x^2}{\tau}+\lambda_{-\frac{1}{2}}^{n+1}\cdot\left(1+\frac{2h_x\,\alpha_L^{n+1}}{\lambda_{\emptyset}^{n+1}}\right)+\lambda_{\frac{1}{2}}^{n+1}\right]\cdot U_0^{n+1}-\left(\lambda_{-\frac{1}{2}}^{n+1}+\lambda_{\frac{1}{2}}^{n+1}\right)\cdot U_1^{n+1}= =(\rho c)_{\emptyset}^{n+1}\cdot\frac{h_x^2}{\tau}\cdot U_0^n+\lambda_{-\frac{1}{2}}^{n+1}\cdot\frac{2h_x\cdot\alpha_L^{n+1}\cdot T_{exL}^{n+1}}{\lambda_{\emptyset}^{n+1}}}$$

- Uokvirene jednadžbe treba dodati linearnom sustavu kao jednadžbe za $i=\emptyset$, ovisno o vrsti RU.

Desni rub $x_R = x_{M_x}$:

Opći oblik rubnog uvjeta je:

$$a_R \cdot T(x_R, t) + b_R \cdot \left[\lambda \frac{\partial T}{\partial x}\right](x_R, t) = f_R .$$

gdje su a_R, b_R, f_R općenito funkcije temperature i/ili vremena. Promatramo ih kao funkcije vremena.

Diskretizacija RU u točki (x_R, t_{n+1}), uz $x_R = x_{M_x}$:

$$\left[\lambda \frac{\partial U}{\partial x}\right](x_{M_x}, t_{n+1}) \approx \lambda_{M_x}^{n+1} \cdot \frac{U_{M_x+1}^{n+1} - U_{M_x-1}^{n+1}}{2h_x}$$

što daje jednadžbu:

$$a_R^{n+1} \cdot U_{M_x}^{n+1} + b_R^{n+1} \cdot \lambda_{M_x}^{n+1} \cdot \frac{U_{M_x+1}^{n+1} - U_{M_x-1}^{n+1}}{2h_x} = f_R^{n+1}$$

s lokalnom pogreškom diskretizacije $\mathcal{O}(h_x^2)$.

- Ako je $b_R^{n+1} = \emptyset$ – Dirichletov RU, onda je:

$$\boxed{U_{M_x}^{n+1} = \frac{f_R^{n+1}}{a_R^{n+1}} .}$$

- Ako je $b_R^{n+1} \neq \emptyset$ – Neumannov ili mješoviti RU, onda dobivamo relaciju za $U_{M_x+1}^{n+1}$:

$$U_{M_x+1}^{n+1} = U_{M_x-1}^{n+1} + \frac{2h_x}{b_R^{n+1} \cdot \lambda_{M_x}^{n+1}} \cdot \left[f_R^{n+1} - a_R^{n+1} \cdot U_{M_x}^{n+1} \right] ,$$

koju treba uvrstiti u diskretiziranu jednadžbu u točki (x_R, t_{n+1}), koja glasi:

$$-\lambda_{M_x-\frac{1}{2}}^{n+1} \cdot U_{M_x-1}^{n+1} + \left[(\varrho c)_{M_x}^{n+1} \cdot \frac{h_x^2}{\tau} + \lambda_{M_x-\frac{1}{2}}^{n+1} + \lambda_{M_x+\frac{1}{2}}^{n+1} \right] \cdot U_{M_x}^{n+1} - \lambda_{M_x+\frac{1}{2}}^{n+1} \cdot U_{M_x+1}^{n+1} =$$

$$= (\varrho c)_{M_x}^{n+1} \cdot \frac{h_x^2}{\tau} \cdot U_{M_x}^{n}$$

Dobivamo jednadžbu:

$$\boxed{\begin{aligned}&-\left(\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\right)\cdot U^{n+1}_{M_x-1}+\left[(\varrho c)^{n+1}_{M_x}\cdot\frac{h_x^2}{\tau}+\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\left(1+\frac{2h_x a_R^{n+1}}{b_R^{n+1}\cdot\lambda^{n+1}_{M_x}}\right)\right]\cdot U^{n+1}_{M_x}=\\&=(\varrho c)^{n+1}_{M_x}\cdot\frac{h_x^2}{\tau}\cdot U^n_{M_x}+\lambda^{n+1}_{M_x+\frac{1}{2}}\cdot\frac{2h_x f_R^{n+1}}{b_R^{n+1}\cdot\lambda^{n+1}_{M_x}}\end{aligned}}$$

- Newtonov zakon hlađenja u $x_R = x_{M_x}$:

$$\left[\lambda\frac{\partial T}{\partial x}\right](x_R,t) = -\alpha_R\left[T(x_R,t)-T_{exR}(t)\right]$$

što odgovara standardnom zapisu:

$$\alpha_R\cdot T(x_R,t)+\left[\lambda\frac{\partial T}{\partial x}\right](x_R,t) = \alpha_R\cdot T_{exR}(t)$$

s

$$a_R = \alpha_R$$
$$b_R = 1$$
$$f_R = \alpha_R\cdot T_{exR}(t)$$

alternativa:

$$a_R = \emptyset$$
$$b_R = 1$$
$$f_R = -\alpha_R\left[T_{old}(x_R,t)-T_{exR}(t)\right]$$

Odgovarajuća jednadžba je:

$$\boxed{\begin{aligned}&-\left(\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\right)\cdot U^{n+1}_{M_x-1}+\left[(\varrho c)^{n+1}_{M_x}\cdot\frac{h_x^2}{\tau}+\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\left(1+\frac{2h_x\alpha_R^{n+1}}{\lambda^{n+1}_{M_x}}\right)\right]\cdot U^{n+1}_{M_x}=\\&=(\varrho c)^{n+1}_{M_x}\cdot\frac{h_x^2}{\tau}\cdot U^n_{M_x}+\lambda^{n+1}_{M_x+\frac{1}{2}}\cdot\frac{2h_x\cdot\alpha_R^{n+1}\cdot T_{exR}^{n+1}}{\lambda^{n+1}_{M_x}}\end{aligned}}$$

- Uokvirene jednadžbe treba dodati linearnom sustavu kao jednadžbe za $i = M_x$, ovisno o vrsti RU.

Crank-Nicolson metoda - jednadžba

Diskretiziramo jednadžbu u točki $(x_i, t_{n+\frac{1}{2}})$:
- u t - simetričnom (centralnom) diferencijom
- u x - simetričnom (dugom) diferencijom

$$\frac{\partial U}{\partial t}(x_i, t_{n+\frac{1}{2}}) \approx \frac{U_i^{n+1} - U_i^n}{\tau} \quad \text{s lok. pogreškom } \mathcal{O}(\tau^2)$$

$$\frac{\partial}{\partial x}\left[\lambda \cdot \frac{\partial U}{\partial x}\right](x_i, t_{n+\frac{1}{2}}) \approx \frac{1}{2}\left(\frac{\partial}{\partial x}\left[\lambda \frac{\partial U}{\partial x}\right](x_i, t_{n+1}) + \frac{\partial}{\partial x}\left[\lambda \frac{\partial U}{\partial x}\right](x_i, t_n)\right)$$

što je linearna interpolacija u t, s lokalnom pogreškom aproksimacije također $\mathcal{O}(\tau^2)$ i diskretizacijom

$$\frac{\partial}{\partial x}\left[\lambda \cdot \frac{\partial U}{\partial x}\right](x_i, t_{n+\ell}) \approx \frac{\left[\lambda \frac{\partial U}{\partial x}\right](x_{i+\frac{1}{2}}, t_{n+\ell}) - \left[\lambda \frac{\partial U}{\partial x}\right](x_{i-\frac{1}{2}}, t_{n+\ell})}{h_x}$$

$$\approx \frac{\lambda_{i+\frac{1}{2}}^{n+\ell}\left[U_{i+1}^{n+\ell} - U_i^{n+\ell}\right] - \lambda_{i-\frac{1}{2}}^{n+\ell}\left[U_i^{n+\ell} - U_{i-1}^{n+\ell}\right]}{h_x^2}$$

za $\ell = 0, 1$, s lokalnom pogreškom $\mathcal{O}(h_x^2)$.

Diskretizirana jednadžba ima oblik:

$$(\varrho c)_i^{n+\frac{1}{2}} \cdot \frac{U_i^{n+1} - U_i^n}{\tau} = \frac{1}{2h_x^2} \cdot \Big[\lambda_{i+\frac{1}{2}}^{n+1}\left[U_{i+1}^{n+1} - U_i^{n+1}\right] - \lambda_{i-\frac{1}{2}}^{n+1}\left[U_i^{n+1} - U_{i-1}^{n+1}\right] + $$
$$+ \lambda_{i+\frac{1}{2}}^{n}\left[U_{i+1}^{n} - U_i^{n}\right] - \lambda_{i-\frac{1}{2}}^{n}\left[U_i^{n} - U_{i-1}^{n}\right] \Big]$$

Sve vrijednosti s indeksom $n+1$ pišemo s lijeve, a sve vrijednosti s indeksom n s desne strane i množimo s $2h_x^2$:

$$\boxed{\begin{aligned} &-\lambda_{i-\frac{1}{2}}^{n+1} U_{i-1}^{n+1} + \left[(\varrho c)_i^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} + \lambda_{i-\frac{1}{2}}^{n+1} + \lambda_{i+\frac{1}{2}}^{n+1}\right] U_i^{n+1} - \lambda_{i+\frac{1}{2}}^{n+1} \cdot U_{i+1}^{n+1} = \\ &= \lambda_{i-\frac{1}{2}}^{n} \cdot U_{i-1}^{n} + \left[(\varrho c)_i^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} - \lambda_{i-\frac{1}{2}}^{n} - \lambda_{i+\frac{1}{2}}^{n}\right] \cdot U_i^{n} + \lambda_{i+\frac{1}{2}}^{n} \cdot U_{i+1}^{n} \\ &\qquad \text{za } i = 1, \ldots, M_x - 1 \end{aligned}}$$

Lokalna pogreška diskretizacije je $\mathcal{O}(\tau^2 + h_x^2)$ u aproksimaciji jednadžbe.
Integracijom po t, imamo lokalnu pogrešku diskretizacije $\mathcal{O}(\tau^3 + \tau h_x^2)$ u aproksimaciji funkcije.

Napomena: kod iterativnog rješavanja nelinearnog problema, svi koeficijenti s indeksom n+1 na lijevoj strani, računaju se iz prethodne aproksimacije rješenja u zadanoj ~~točki~~ (s indeksom n+1).

Vrijednosti $(\varrho c)_i^{n+\frac{1}{2}}$ mogu se računati linearnom interpolacijom

$$(\varrho c)_i^{n+\frac{1}{2}} = \frac{1}{2}\left[(\varrho c)_i^n + (\varrho c)_i^{n+1}\right]$$

gdje se $(\varrho c)_i^{n+1}$ računa iz prethodne aproksimacije rješenja.

Moguće je $(\varrho c)_i^{n+\frac{1}{2}}$ računati i iz $U_i^{n+\frac{1}{2}} = \frac{1}{2}\left[U_i^n + \underset{\text{prošli}}{\underset{\downarrow}{U_i^{n+1}}}\right]$.

Crank-Nicolson metoda - rubni uvjeti

Diskretizirana jednadžba u rubnim točkama sadrži vrijednosti U^{n+1} i U^n u točkama koje mogu ići preko ruba (na pr. U_{-1}^{n+1} i U_{-1}^n).

Zbog toga treba koristiti zadane rubne uvjete na oba vremenska nivoa t_n i t_{n+1}, za eliminaciju takvih točaka.

Lijevi rub $x_L = x_\emptyset$:

Opći oblik rubnog uvjeta je

$$a_L \cdot T(x_L, t) + b_L \cdot \left[\lambda \cdot \frac{\partial T}{\partial x}\right](x_L, t) = f_L .$$

Diskretizacija RU u točkama $(x_L, t_{n+\ell})$, uz $x_L = x_0$:

$$\left[\lambda \frac{\partial U}{\partial x}\right](x_0, t_{n+\ell}) \approx \lambda_0^{n+\ell} \cdot \frac{U_1^{n+\ell} - U_{-1}^{n+\ell}}{2h_x}, \quad \ell = \emptyset, 1$$

što daje jednadžbe

$$a_L^{n+\ell} \cdot U_0^{n+\ell} + b_L^{n+\ell} \cdot \lambda_0^{n+\ell} \cdot \frac{U_1^{n+\ell} - U_{-1}^{n+\ell}}{2h_x} = f_L^{n+\ell}, \quad \ell = \emptyset, 1$$

s lokalnom pogreškom diskretizacije $\sigma(h_x^2)$.

— Ako je $b_L^{n+1} = \emptyset$ - Dirichletov RU, onda je

$$\boxed{U_0^{n+1} = \frac{f_L^{n+1}}{a_L^{n+1}}} .$$

(tada se u jednadžbi za $i=1$, koristi ranije izračunati U_0^n)

— Ako je $b_L^{n+1} \neq \emptyset$ - Neumannov ili mješoviti RU, onda dobivamo relaciju za U_{-1}^{n+1}:

$$U_{-1}^{n+1} = U_1^{n+1} + \frac{2h_x}{b_L^{n+1} \cdot \lambda_0^{n+1}} \cdot \left[a_L^{n+1} \cdot U_\emptyset^{n+1} - f_L^{n+1}\right]$$

koju treba uvrstiti u diskretiziranu jednadžbu u točki $(x_L, t_{n+\frac{1}{2}})$, koja glasi:

$$-\lambda_{-\frac{1}{2}}^{n+1} U_{-1}^{n+1} + \left[(\rho c)_{\emptyset}^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} + \lambda_{-\frac{1}{2}}^{n+1} + \lambda_{\frac{1}{2}}^{n+1}\right] U_0^{n+1} - \lambda_{\frac{1}{2}}^{n+1} \cdot U_1^{n+1} =$$

$$= \lambda_{-\frac{1}{2}}^{n} \cdot U_{-1}^{n} + \left[(\rho c)_{\emptyset}^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} - \lambda_{-\frac{1}{2}}^{n} - \lambda_{\frac{1}{2}}^{n}\right] \cdot U_0^{n} + \lambda_{\frac{1}{2}}^{n} U_1^{n} .$$

Ova jednadžba sadrži i U_{-1}^n. Tu vrijednost treba eliminirati iz rubnog uvjeta u (x_L, t_n), što je moguće samo ako je $b_L^n \neq 0$. U protivnom, za $b_L^n = 0$, ta vrijednost nije korektno definirana. Tada U_{-1}^n treba izračunati na pr. ekstrapolacijom iz točaka $U_0^n, U_1^n, \ldots$ (koristeći interpolacioni polinom ili metodu najmanjih kvadrata).
Ovdje pretpostavljamo da je U_{-1}^n već izračunat iz RU ili ekstrapolacije.

Dobivamo jednadžbu:

$$\boxed{\begin{aligned}&\left[(\rho c)_{\emptyset}^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} + \lambda_{-\frac{1}{2}}^{n+1}\left(1 - \frac{2h_x \cdot a_L^{n+1}}{b_L^{n+1} \cdot \lambda_0^{n+1}}\right) + \lambda_{\frac{1}{2}}^{n+1}\right] U_0^{n+1} - \left(\lambda_{-\frac{1}{2}}^{n+1} + \lambda_{\frac{1}{2}}^{n+1}\right) U_1^{n+1} = \\ &= \lambda_{-\frac{1}{2}}^{n} U_{-1}^{n} + \left[(\rho c)_{\emptyset}^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} - \lambda_{-\frac{1}{2}}^{n} - \lambda_{\frac{1}{2}}^{n}\right] U_0^{n} + \lambda_{\frac{1}{2}}^{n} U_1^{n} - \lambda_{-\frac{1}{2}}^{n+1} \cdot \frac{2h_x f_L^{n+1}}{b_L^{n+1} \cdot \lambda_0^{n+1}}\end{aligned}}$$

- Newtonov zakon hlađenja u $x_L = x_\emptyset$:

$$-\left[\lambda \frac{\partial T}{\partial x}\right](x_L, t) = -\alpha_L \left[T(x_L, t) - T_{exL}(t)\right]$$

ili

$$\alpha_L \cdot T(x_L, t) + \left[\lambda \frac{\partial T}{\partial x}\right](x_L, t) = \alpha_L \cdot T_{exL}(t)$$

$\Rightarrow$

$$a_L = \alpha_L$$
$$b_L = -1$$
$$f_L = \alpha_L \cdot T_{exL}(t).$$

Odgovarajuća jednadžba je:

$$\boxed{\begin{aligned}&\left[(\rho c)_0^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} + \lambda_{-\frac{1}{2}}^{n+1}\left(1 + \frac{2h_x \alpha_L^{n+1}}{\lambda_0^{n+1}}\right) + \lambda_{\frac{1}{2}}^{n+1}\right] \cdot U_0^{n+1} - \left(\lambda_{-\frac{1}{2}}^{n+1} + \lambda_{\frac{1}{2}}^{n+1}\right) \cdot U_1^{n+1} = \\ &= \lambda_{-\frac{1}{2}}^{n} \cdot U_{-1}^{n} + \left[(\rho c)_{\emptyset}^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} - \lambda_{-\frac{1}{2}}^{n} - \lambda_{\frac{1}{2}}^{n}\right] \cdot U_0^{n} + \lambda_{\frac{1}{2}}^{n} U_1^{n} + \lambda_{-\frac{1}{2}}^{n+1} \cdot \frac{2h_x \alpha_L^{n+1} \cdot T_{exL}^{n+1}}{\lambda_{\emptyset}^{n+1}}\end{aligned}}$$

pri čemu U_{-1}^n treba korektno izračunati.

- Dobivene jednadžbe treba dodati linearnom sustavu, kao jednadžbe za $i = \emptyset$, ovisno o vrsti RU.

Desni rub $x_R = x_{M_x}$:

Opći oblik rubnog uvjeta je:

$$a_R \cdot T(x_R, t) + b_R \cdot \left[\lambda \cdot \frac{\partial T}{\partial x}\right](x_R, t) = f_R$$

Diskretizacija RU u točki (x_R, t_{n+1}), uz $x_R = x_{M_x}$:

$$\left[\lambda \frac{\partial U}{\partial x}\right](x_{M_x}, t_{n+1}) \approx \lambda_{M_x}^{n+1} \cdot \frac{U_{M_x+1}^{n+1} - U_{M_x-1}^{n+1}}{2h_x}$$

što daje jednadžbu

$$a_R^{n+1} \cdot U_{M_x}^{n+1} + b_R^{n+1} \cdot \lambda_{M_x}^{n+1} \cdot \frac{U_{M_x+1}^{n+1} - U_{M_x-1}^{n+1}}{2h_x} = f_R^{n+1}$$

s lokalnom pogreškom diskretizacije $\mathcal{O}(h_x^2)$.

– Ako je $b_R^{n+1} = 0$ – Dirichletov RU, onda je

$$\boxed{U_{M_x}^{n+1} = \frac{f_R^{n+1}}{a_R^{n+1}}}.$$

– Ako je $b_R^{n+1} \neq 0$ – Neumannov ili mješoviti RU, onda dobivamo relaciju za $U_{M_x+1}^{n+1}$:

$$U_{M_x+1}^{n+1} = U_{M_x-1}^{n+1} + \frac{2h_x}{b_R^{n+1} \cdot \lambda_{M_x}^{n+1}} \cdot \left[f_R^{n+1} - a_R^{n+1} \cdot U_{M_x}^{n+1}\right],$$

koju treba uvrstiti u diskretiziranu jednadžbu u točki $(x_R, t_{n+\frac{1}{2}})$, koja glasi:

$$-\lambda_{M_x-\frac{1}{2}}^{n+1} U_{M_x-1}^{n+1} + \left[(\rho c)_{M_x}^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} + \lambda_{M_x-\frac{1}{2}}^{n+1} + \lambda_{M_x+\frac{1}{2}}^{n+1}\right] \cdot U_{M_x}^{n+1} - \lambda_{M_x+\frac{1}{2}}^{n+1} U_{M_x+1}^{n+1} =$$

$$= \lambda_{M_x-\frac{1}{2}}^{n} U_{M_x-1}^{n} + \left[(\rho c)_{M_x}^{n+\frac{1}{2}} \cdot \frac{2h_x^2}{\tau} - \lambda_{M_x-\frac{1}{2}}^{n} - \lambda_{M_x+\frac{1}{2}}^{n}\right] \cdot U_{M_x}^{n} + \lambda_{M_x+\frac{1}{2}}^{n} \cdot U_{M_x+1}^{n}.$$

Opet pretpostavljamo da je $U_{M_x+1}^{n}$ korektno izračunat iz RU ili ekstrapolacijom.

Dobivamo jednadžbu:

$$\boxed{\begin{aligned}&-\left(\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\right)\cdot U^{n+1}_{M_x-1}+\left[(\rho c)^{n+\frac{1}{2}}_{M_x}\cdot\frac{2h_x^2}{\tau}+\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\cdot\left(1+\frac{2h_x a_R^{n+1}}{b_R^{n+1}\cdot\lambda^{n+1}_{M_x}}\right)\right]\cdot U^{n+1}_{M_x}=\\&=\lambda^{n}_{M_x-\frac{1}{2}}\cdot U^{n}_{M_x-1}+\left[(\rho c)^{n+\frac{1}{2}}_{M_x}\cdot\frac{2h_x^2}{\tau}-\lambda^{n}_{M_x-\frac{1}{2}}-\lambda^{n}_{M_x+\frac{1}{2}}\right]\cdot U^{n}_{M_x}+\lambda^{n}_{M_x+\frac{1}{2}}\cdot U^{n}_{M_x+1}\\&\quad+\lambda^{n+1}_{M_x+\frac{1}{2}}\cdot\frac{2h_x f_R^{n+1}}{b_R^{n+1}\cdot\lambda^{n+1}_{M_x}}\end{aligned}}$$

- Newtonov zakon hlađenja u $x_R = x_{M_x}$:

$$\left[\lambda\frac{\partial T}{\partial x}\right](x_R,t) = -\alpha_R\left[T(x_R,t)-T_{exR}(t)\right]$$

ili

$$\alpha_R\cdot T(x_R,t)+\left[\lambda\frac{\partial T}{\partial x}\right](x_R,t) = \alpha_R\cdot T_{exR}(t)$$

s

$$a_R=\alpha_R$$
$$b_R = 1$$
$$f_R = \alpha_R\cdot T_{exR}(t).$$

Odgovarajuća jednadžba je:

$$\boxed{\begin{aligned}&-\left(\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\right)\cdot U^{n+1}_{M_x-1}+\left[(\rho c)^{n+\frac{1}{2}}_{M_x}\cdot\frac{2h_x^2}{\tau}+\lambda^{n+1}_{M_x-\frac{1}{2}}+\lambda^{n+1}_{M_x+\frac{1}{2}}\cdot\left(1+\frac{2h_x\alpha_R^{n+1}}{\lambda^{n+1}_{M_x}}\right)\right]\cdot U^{n+1}_{M_x}=\\&=\lambda^{n}_{M_x-\frac{1}{2}}\cdot U^{n}_{M_x-1}+\left[(\rho c)^{n+\frac{1}{2}}_{M_x}\cdot\frac{2h_x^2}{\tau}-\lambda^{n}_{M_x-\frac{1}{2}}-\lambda^{n}_{M_x+\frac{1}{2}}\right]\cdot U^{n}_{M_x}+\lambda^{n}_{M_x+\frac{1}{2}}\cdot U^{n}_{M_x+1}\\&\quad+\lambda^{n+1}_{M_x+\frac{1}{2}}\cdot\frac{2h_x\cdot\alpha_R^{n+1}\cdot T_{exR}^{n+1}}{\lambda^{n+1}_{M_x}}\end{aligned}}$$

- Uokvirene jednadžbe treba dodati linearnom sustavu, kao jednadžbe za $i = M_x$, ovisno o vrsti RU.