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On the deviation of a parametric cubic spline interpolant from its data polygon

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Abstract

When fitting a parametric curve through a sequence of points, it is important in applications that the curve should not exhibit unwanted oscillations. In this paper we take the view that a good curve is one that does not deviate too far from the data polygon: the polygon formed by the data points. From this point of view, we study periodic cubic spline interpolation and derive bounds on the deviation with respect to three common choices of parameterization: uniform, chordal, and centripetal. If one wants small deviation, the centripetal spline is arguably the best choice among the three.
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1. Introduction

Cubic spline interpolation is a popular way to pass a smooth-looking curve through a sequence of points in arbitrary dimension. Given a sequence of points $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{n-1}$ in $\mathbb{R}^d$, where $d \geqslant 2$ and $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ are distinct, and defining $\mathbf{p}_n = \mathbf{p}_0$, we can choose any parameter values $t_0 < t_1 < \cdots < t_n$ in $\mathbb{R}$ and compute the unique periodic C^2 cubic spline curve $\mathbf{s} : [t_0, t_n] \to \mathbb{R}^d$ such that $\mathbf{s}(t_i) = \mathbf{p}_i$, $i = 0, 1, \dots, n$. Thus $\mathbf{s}$ is a parametric cubic polynomial on each interval $[t_i, t_{i+1}]$, $i = 0, 1, \dots, n-1$, and is C^2 at $t_1, \dots, t_{n-1}$ and $\mathbf{s}^{(k)}(t_n) = \mathbf{s}^{(k)}(t_0)$, $k = 1, 2$.

There remains the choice of parameter values $t_0, \dots, t_n$ and it is well known that they have a large influence on the shape of the resulting spline curve (Ahlberg et al., 1967; de Boor, 1978; Epstein, 1976; Farin, 1988; Floater 2005, 2006; Floater and Surazhsky, 2006; Foley and Nielson, 1989; Lee 1989, 1992; Marin, 1984; Penot, 1983). While some parameterization methods involve optimization and considerable computation, a simple approach is used frequently in practice: let $t_0 := 0$ and for some $\alpha \in [0, 1]$ let

$$t_{i+1} := t_i + |\mathbf{p}_{i+1} - \mathbf{p}_i|^\alpha, \quad i = 0, 1, \dots, n-1,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. The α-parameterizations with $\alpha = 0, 1/2, 1$ are known respectively as uniform, centripetal, and chordal (Ahlberg et al., 1967; Lee, 1989).

One advantage of the chordal spline is that it yields a fourth order approximation to a curve with a continuous fourth derivative with respect to arc length (Floater, 2006) due to the fact that the chord length $|\mathbf{p}_{i+1} - \mathbf{p}_i|$ is a sufficiently

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Table 1
Maximum μ_i and μ

Parameterization	μ_i	μ
Uniform	∞	$\leqslant 3/4$
Centripetal	$3/4$	$\leqslant 9/20$
Chordal	$3/4$	$3/4$

close approximation to the length of the curve between $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$. Thus chordal spline interpolation has the same order of accuracy as functional cubic spline interpolation (Beatson, 1986; Beatson and Chacko, 1992; Birkhoff and de Boor, 1964; de Boor, 1978; Hall and Meyer, 1976; Sharma and Meir, 1966). In comparison, numerical examples show that uniform and centripetal spline interpolation have second order accuracy at best. So approximation order puts the chordal spline in a good light, but this is only an asymptotic property, and the question remains whether the chordal spline is so effective when the points are sampled sparsely from a curve, or when the points are simply chosen by a designer and approximation is no longer relevant. Based on numerical examples, Lee (1989) argues that the centripetal spline often appears to stay closer to the data polygon (the polygon whose edges are $[\mathbf{p}_0, \mathbf{p}_1], [\mathbf{p}_1, \mathbf{p}_2], \ldots, [\mathbf{p}_{n-1}, \mathbf{p}_0]$) than the chordal spline, but there does not seem to be any mathematical evidence in the literature to support this. We note that the different but related issue of how far a Bezier or spline curve deviates from its *control* polygon has been studied and bounds involving second order differences in the control points have been derived by Nairn et al. (1999).

The purpose of this paper is to view 'deviation from the data polygon' as a measure of badness of the interpolating spline and to investigate how the three parameterizations: uniform, centripetal and chordal perform in this respect. A natural measure of local deviation is the Hausdorff distance between the ith cubic piece $\mathbf{s}|_{[t_i,t_{i+1}]}$ and the associated edge $[\mathbf{p}_i, \mathbf{p}_{i+1}]$. But since we expect this distance to be proportional to the length of this edge, we will consider the ratio

$$\mu_i(\mathbf{s}) = \frac{\operatorname{dist}(\mathbf{s}|_{[t_i,t_{i+1}]}, [\mathbf{p}_i, \mathbf{p}_{i+1}])}{|\mathbf{p}_{i+1} - \mathbf{p}_i|}.$$

Alternatively we can study the global deviation of the spline, and measure it relative to the maximum edge length,

$$\mu(\mathbf{s}) = \frac{\max_{0 \leqslant i \leqslant n-1} \operatorname{dist}(\mathbf{s}|_{[t_i,t_{i+1}]}, [\mathbf{p}_i, \mathbf{p}_{i+1}])}{\max_{0 \leqslant i \leqslant n-1} |\mathbf{p}_{i+1} - \mathbf{p}_i|}.$$

Note that $\mu(\mathbf{s}) \leqslant \max_i \mu_i(\mathbf{s})$. We take the view that a spline $\mathbf{s}$ is 'good' if both its μ_i and μ values are low.

We show that the maximum μ_i value of both the chordal and centripetal splines is $\mu_i = 3/4$. On the other hand, μ_i for a uniform spline can be arbitrarily high. Thus in terms of local deviation, the chordal and centripetal splines are equally good, and much better than the uniform one. We show that the maximum global deviation of the chordal spline is similarly $\mu = 3/4$ but the maximum value for the centripetal spline is at most $9/20$ (and may be lower). Thus if one wants a curve with both small local and global deviations, the centripetal spline is arguably the best choice among the three. These findings are summarized in Table 1.

2. Preliminaries

The definitions of μ_i and μ involve Hausdorff distance. As is well known, if $\mathbf{f}: [a, b] \to \mathbb{R}^d$ and $\mathbf{g}: [c, d] \to \mathbb{R}^d$ are two parametric curves, and $\phi: [a, b] \to [c, d]$ is some continuous monotonically increasing function then

$$\operatorname{dist}(\mathbf{f}, \mathbf{g}) \leqslant \sup_{a \leqslant u \leqslant b} \big|\mathbf{f}(u) - \mathbf{g}\big(\phi(u)\big)\big|.$$

Thus in studying the deviation between $\mathbf{s}$ and its data polygon, it helps to view the latter as a piecewise linear parametric curve, and we will use the representation $\mathbf{p}: [t_0, t_n] \to \mathbb{R}^d$ where

$$\mathbf{p}(t) = (1 - u)\mathbf{p}_i + u\mathbf{p}_{i+1}, \quad t_i \leqslant t \leqslant t_{i+1}, \tag{1}$$

and $u = (t - t_i)/(t_{i+1} - t_i)$, and the t_i are the same parameter values used to define $\mathbf{s}$. We will sometimes make use of the first and second order divided differences

$$[t_i, t_{i+1}]\mathbf{p} := \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{t_{i+1} - t_i}, \qquad [t_i, t_{i+1}, t_{i+2}]\mathbf{p} := \frac{[t_{i+1}, t_{i+2}]\mathbf{p} - [t_i, t_{i+1}]\mathbf{p}}{t_{i+2} - t_i}.$$

There are two well known ways of computing the spline interpolant and we will use both in our analysis. Here, it will help to use the Bernstein polynomials

$$B_{i,k}(x) = \binom{k}{i} x^i (1-x)^{k-i}, \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant i \leqslant k,$$

and the properties that if

$$f(x) = \sum_{i=0}^{k} c_i B_{i,k}(x),$$

then $f(0) = c_0$, $f(1) = c_k$, and

$$f'(x) = k \sum_{i=0}^{k-1} (c_{i+1} - c_i) B_{i,k-1}(x). \tag{2}$$

One method to find $\mathbf{s}$ is to represent it in each interval in terms of its values and *first* derivatives at the end points, and solve a linear system with $\mathbf{s}'(t_0), \dots, \mathbf{s}'(t_{n-1})$ as the unknowns. Specifically, defining $\mathbf{s}'_i = \mathbf{s}'(t_i)$, we can express $\mathbf{s}$ for $t \in [t_i, t_{i+1}]$, as

$$\mathbf{s}(t) = B_{0,3}(u)\mathbf{p}_i + B_{1,3}(u)(\mathbf{p}_i + \Delta t_i \mathbf{s}'_i/3) + B_{2,3}(u)(\mathbf{p}_{i+1} - \Delta t_i \mathbf{s}'_{i+1}/3) + B_{3,3}(u)\mathbf{p}_{i+1}, \tag{3}$$

with u as in (1) and $\Delta t_i := t_{i+1} - t_i$. This is easily confirmed by differentiating, using (2) and the fact that $\mathrm{d}/\mathrm{d}t = (1/\Delta t_i)(\mathrm{d}/\mathrm{d}u)$. Differentiation a second time and the assumption that $\mathbf{s}$ has a continuous second derivative at $t_0, \dots, t_{n-1}$ then leads to the familiar linear system of equations

$$\frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i}\mathbf{s}'_{i-1} + 2\mathbf{s}'_i + \frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i}\mathbf{s}'_{i+1} = \mathbf{b}_i, \quad i = 0, 1, \dots, n-1, \tag{4}$$

where

$$\mathbf{b}_i = 3\frac{\Delta t_i [t_{i-1}, t_i]\mathbf{p} + \Delta t_{i-1}[t_i, t_{i+1}]\mathbf{p}}{\Delta t_{i-1} + \Delta t_i}.$$

Here, $\mathbf{s}_{-1} = \mathbf{s}'_{n-1}$, $\mathbf{s}'_n = \mathbf{s}'_0$, $\mathbf{p}_{-1} = \mathbf{p}_{n-1}$, and $t_{-1} = t_0 + (t_n - t_{n-1})$.

Alternatively, we can express $\mathbf{s}$ in each interval in terms of its values and *second* derivatives at the end points and solve a linear system with $\mathbf{s}''(t_0), \dots, \mathbf{s}''(t_{n-1})$ as the unknowns. Letting $\mathbf{s}''_i := \mathbf{s}''(t_i)$, we have in $[t_i, t_{i+1}]$,

$$\mathbf{s}(t) = B_{0,3}(u)\mathbf{p}_i + B_{1,3}(u)\big((2\mathbf{p}_i + \mathbf{p}_{i+1})/3 - \Delta t_i^2(2\mathbf{s}''_i + \mathbf{s}''_{i+1})/18\big) \tag{5}$$

$$\qquad + B_{2,3}(u)\big((\mathbf{p}_i + 2\mathbf{p}_{i+1})/3 - \Delta t_i^2(\mathbf{s}''_i + 2\mathbf{s}''_{i+1})/18\big) + B_{3,3}(u)\mathbf{p}_{i+1}. \tag{6}$$

This can again be verified by differentiation and using (2). By the assumption that $\mathbf{s}$ has a continuous first derivative at $t_0, \dots, t_{n-1}$, one arrives at another familiar linear system,

$$\frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i}\mathbf{s}''_{i-1} + 2\mathbf{s}''_i + \frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i}\mathbf{s}''_{i+1} = \mathbf{c}_i, \quad i = 0, 1, \dots, n-1, \tag{7}$$

where $\mathbf{s}''_{-1} = \mathbf{s}''_{n-1}$, $\mathbf{s}''_n = \mathbf{s}''_0$, and

$$\mathbf{c}_i = 6[t_{i-1}, t_i, t_{i+1}]\mathbf{p}.$$

3. Local deviation

We start by bounding the local deviation of the chordal and centripetal splines. We show that both splines share the same property, that $\mu_i \leqslant 3/4$.

Theorem 1. *If* $\mathbf{s}$ *is a chordal spline then* $\mu_i(\mathbf{s}) \leqslant 3/4$.

Proof. We use (3) to represent $\mathbf{s}$ in $[t_i, t_{i+1}]$ in terms of its first derivatives. Due to the chordal parameterization,

$$\big|[t_j, t_{j+1}]\mathbf{p}\big| = \frac{|\mathbf{p}_{j+1} - \mathbf{p}_j|}{t_{j+1} - t_j} = 1,$$

and it follows that $|\mathbf{b}_j| \leqslant 3$ in (4). Thus, using a standard argument based on the diagonal dominance of the linear system (4) (see for example (de Boor, 1978, pp. 43–44)), it follows that

$$\max_{0\leqslant j\leqslant n-1} |\mathbf{s}'_j| \leqslant \max_{0\leqslant j\leqslant n-1} |\mathbf{b}_j| \leqslant 3. \tag{8}$$

Now to bound the Hausdorff distance between $\mathbf{s}$ and $\mathbf{p}$ in the interval $[t_i, t_{i+1}]$, it helps to represent $\mathbf{p}$ in the parametric form $\mathbf{q} : [t_0, t_n] \to \mathbb{R}^d$ where for $t \in [t_i, t_{i+1}]$,

$$\mathbf{q}(t) = B_{0,3}(u)\mathbf{p}_i + B_{1,3}(u)\mathbf{p}_i + B_{2,3}(u)\mathbf{p}_{i+1} + B_{3,3}(u)\mathbf{p}_{i+1}, \tag{9}$$

with u as in (1), because due to (3), the error

$$\mathbf{e}(t) := \mathbf{s}(t) - \mathbf{q}(t),$$

for t in $[t_i, t_{i+1}]$ takes on the simple form

$$\mathbf{e}(t) = \Delta t_i u(1-u)\big((1-u)\mathbf{s}'_i - u\mathbf{s}'_{i+1}\big).$$

Therefore,

$$\big|\mathbf{e}(t)\big| \leqslant \Delta t_i u(1-u) \max\big\{|\mathbf{s}'_i|, |\mathbf{s}'_{i+1}|\big\}, \tag{10}$$

and so

$$\big|\mathbf{e}(t)\big| \leqslant \frac{1}{4}|\mathbf{p}_{i+1} - \mathbf{p}_i| \max\big\{|\mathbf{s}'_i|, |\mathbf{s}'_{i+1}|\big\},$$

and the result follows from (8). □

Next we show that the constant 3/4 is the least possible. For some integer $m \geqslant 2$ and real $\omega > 0$, let $n = 2m$ and with $\mathbf{p}_i = (p_i, q_i)$, let

$$\begin{aligned} &p_i = 0, \quad q_i = \sum_{j=0}^{m-1-i} (-\omega)^j, \quad i = 0, \dots, m-1, \\ &p_i = 1, \quad q_i = q_{2m-1-i}, \quad i = m, \dots, 2m-1, \end{aligned} \tag{11}$$

and consider the deviation of $\mathbf{s}$ from the edge $[\mathbf{p}_{m-1}, \mathbf{p}_m] = [(0,1),(1,1)]$. The chordal parameter values satisfy $\Delta t_i = \omega^{|m-1-i|}$ for $0 \leqslant i \leqslant 2m-2$ and $\Delta t_{2m-1} = 1$. To obtain an extreme case of deviation suppose ω is small and m is large. The third column of Fig. 2 shows the chordal spline when $m = 4$ for $\omega = 1/10, 1/2, 1, 2, 10$ respectively. With $\mathbf{s}(t) = (x(t), y(t))$, the limit as $\omega \to 0$ of the system (4) with $i = 0, \dots, m-1$ in the y component is

$$2y'_0 + y'_1 = 3(-1)^m, \quad \text{and} \quad y'_{i-1} + 2y'_i = 3(-1)^{m-1-i}, \quad i = 1, \dots, m-1,$$

which has the unique solution

$$y'_i = 3(-1)^{m-1-i} + 4(-1)^m(-1/2)^i, \quad i = 0, \dots, m-1.$$

Thus $y'_{m-1} \to 3$ as $m \to \infty$, and similarly $y'_m \to -3$ as $m \to \infty$, and putting these values into (3) in $[t_{m-1}, t_m]$ with $t_* = (t_{m-1} + t_m)/2$ means that $\mathbf{s}(t_*) \to (1/2, 1 + 3/4)$ as $\omega \to 0$ and $m \to \infty$. We conclude that if ω is small enough and m large enough, the distance between the point $\mathbf{s}(t_*)$ and the nearest point of the polygon, $(1/2, 1)$, is arbitrarily close to 3/4. Next we consider centripetal splines.

Theorem 2. *If* $\mathbf{s}$ *is a centripetal spline then* $\mu_i(\mathbf{s}) \leqslant 3/4$.

Proof. We use the linear system in the second derivatives (7). Since

$$|\mathbf{c}_j| \leqslant 6\frac{|[t_j, t_{j+1}]\mathbf{p}| + |[t_{j-1}, t_j]\mathbf{p}|}{(t_{j+1} - t_j) + (t_j - t_{j-1})},$$

and

$$\big|[t_j, t_{j+1}]\mathbf{p}\big| = \frac{|\mathbf{p}_{j+1} - \mathbf{p}_j|}{t_{j+1} - t_j} = |\mathbf{p}_{j+1} - \mathbf{p}_j|^{1/2},$$

it follows that $|\mathbf{c}_j| \leqslant 6$ and therefore, by the same diagonal dominance argument used in the proof of the previous theorem,

$$\max_{0\leqslant j\leqslant n-1} |\mathbf{s}_j''| \leqslant \max_{0\leqslant j\leqslant n-1} |\mathbf{c}_j| \leqslant 6. \tag{12}$$

Now we can express the error as $\mathbf{e}(t) = \mathbf{s}(t) - \mathbf{p}(t)$, which implies that

$$\mathbf{e}(t) = -\Delta t_i^2 u(1-u)\big((1-u)(2\mathbf{s}_i'' + \mathbf{s}_{i+1}'') + u(\mathbf{s}_i'' + 2\mathbf{s}_{i+1}'')\big)/6. \tag{13}$$

Since

$$\big|\mathbf{e}(t)\big| \leqslant \Delta t_i^2 u(1-u)\frac{1}{2}\max\big\{|\mathbf{s}_i''|, |\mathbf{s}_{i+1}''|\big\},$$

it follows that

$$\big|\mathbf{e}(t)\big| \leqslant \frac{1}{8}|\mathbf{p}_{i+1} - \mathbf{p}_i| \max\big\{|\mathbf{s}_i''|, |\mathbf{s}_{i+1}''|\big\},$$

and the result follows from (12). □

For testing the sharpness of the constant 3/4 in Theorem 2 we again use the data set (11) but consider large ω as well as large m; see the second column of Fig. 2. The centripetal parameter values satisfy $\Delta t_i = \omega^{|m-1-i|/2}$ for $0 \leqslant i \leqslant 2m-2$ and $\Delta t_{2m-1} = 1$ and the limit as $\omega \to \infty$ of the second component of (7) with $i = 0, \ldots, m-1$ is

$$2y_0'' + y_1'' = 6(-1)^m, \quad \text{and} \quad y_{i-1}'' + 2y_i'' = 6(-1)^{m-i}, \quad i = 1, \ldots, m-1,$$

which has the solution

$$y_i'' = 6(-1)^{m-i}, \quad i = 0, \ldots, m-1.$$

We then have $y_{m-1}'' \to -6$ as $m \to \infty$, and similarly $y_{m-1}'' \to -6$ as $m \to \infty$, and putting these values into (5) implies $\mathbf{s}(t_*) \to (1/2, 1+3/4)$ as $\omega \to \infty$ and $m \to \infty$. Thus if ω and m are large enough, the distance between the point $\mathbf{s}(t_*)$ and the nearest point of the polygon, $(1/2, 1)$, is arbitrarily close to 3/4.

What about the local deviation of the uniform spline? In fact μ_i is unbounded and so from the point of view of local deviation one can firmly regard the uniform spline as inferior to the chordal and centripetal ones. To see this, let $\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}$ be any data set with $n \geqslant 2$, let $\mathbf{s}$ be the uniform spline interpolant, and consider the deviation of $\mathbf{s}$ from the edge $[\mathbf{p}_0, \mathbf{p}_1]$. If we replace the data point $\mathbf{p}_2$ by $\tilde{\mathbf{p}}_2$ and call the resulting spline $\tilde{\mathbf{s}}$, the difference $\mathbf{d}(t) = \tilde{\mathbf{s}}(t) - \mathbf{s}(t)$ satisfies

$$\mathbf{d}(t) = \phi(t)(\tilde{\mathbf{p}}_2 - \mathbf{p}_2),$$

where $\phi : [0, n] \to \mathbb{R}$ is the periodic cardinal spline satisfying $\phi(i) = \delta_{i,2}$, $0 \leqslant i \leqslant n$. A standard fact about cubic cardinal splines is that they have no zeros between the knots and so the value $\phi(1/2)$ is non-zero, and so we can make the difference vector $\mathbf{d}(1/2)$ arbitrarily large by simply changing $\tilde{\mathbf{p}}_2$. Hence the distance between the point $\tilde{\mathbf{s}}(1/2)$ and $\mathbf{s}(1/2)$ can be made arbitrarily large, and so too the distance between $\tilde{\mathbf{s}}(1/2)$ and $\mathbf{p}(1/2)$.

4. Global deviation

We now consider the global deviation. Using the fact that $\mu(\mathbf{s}) \leqslant \mu_i(\mathbf{s})$ for any interpolant $\mathbf{s}$, it follows from Theorem 1 that if $\mathbf{s}$ is a chordal spline then $\mu(\mathbf{s}) \leqslant 3/4$. Moreover, if $\omega < 1$ in Example (11) then the edge $[\mathbf{p}_{m-1}, \mathbf{p}_m]$ is the longest in the polygon (together with $[\mathbf{p}_{n-1}, \mathbf{p}_0]$) and so the example with small ω and large m shows that the bound 3/4 is again optimal.

Similarly, Theorem 2 shows that if $\mathbf{s}$ is a centripetal spline then $\mu(\mathbf{s}) \leqslant 3/4$. However, we can reduce this constant in the centripetal case. This might be expected because when Example (11) was used to show that 3/4 was optimal for the relative local deviation, we took ω to be large, and in particular larger than 1 which means that the edge $[\mathbf{p}_{m-1}, \mathbf{p}_m]$ is shorter than neighbouring edges.

Theorem 3. *If* $\mathbf{s}$ *is a centripetal spline then* $\mu(\mathbf{s}) \leqslant 9/20$.

Proof. Consider any interval $[t_i, t_{i+1}]$. Defining

$$\lambda = \frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i}, \qquad \mu = \frac{\Delta t_{i+1}}{\Delta t_i + \Delta t_{i+1}},$$

we write the two equations of the linear system (7) corresponding to i and $i+1$ in the form

$$2\mathbf{s}_i'' + (1-\lambda)\mathbf{s}_{i+1}'' = \boldsymbol{\alpha}, \tag{14}$$

$$(1-\mu)\mathbf{s}_i'' + 2\mathbf{s}_{i+1}'' = \boldsymbol{\beta}, \tag{15}$$

where $\boldsymbol{\alpha} = \mathbf{c}_i - \lambda\mathbf{s}_{i-1}''$ and $\boldsymbol{\beta} = \mathbf{c}_{i+1} - \mu\mathbf{s}_{i+2}''$. Inequality (12) implies

$$|\boldsymbol{\alpha}| \leqslant 6(1+\lambda), \qquad |\boldsymbol{\beta}| \leqslant 6(1+\mu).$$

Now, solving the 2×2 linear system (14)–(15) gives

$$\mathbf{s}_i'' = \big(2\boldsymbol{\alpha} - (1-\lambda)\boldsymbol{\beta}\big)/D,$$
$$\mathbf{s}_{i+1}'' = \big(2\boldsymbol{\beta} - (1-\mu)\boldsymbol{\alpha}\big)/D,$$

where $D = 4 - (1-\lambda)(1-\mu)$. Therefore, with (13) in mind,

$$2\mathbf{s}_i'' + \mathbf{s}_{i+1}'' = \big((3+\mu)\boldsymbol{\alpha} + 2\lambda\boldsymbol{\beta}\big)/D,$$
$$\mathbf{s}_i'' + 2\mathbf{s}_{i+1}'' = \big(2\mu\boldsymbol{\alpha} + (3+\lambda)\boldsymbol{\beta}\big)/D,$$

and so

$$|2\mathbf{s}_i'' + \mathbf{s}_{i+1}''| \leqslant 6(3+5\lambda+\mu+3\lambda\mu)/D, \qquad |\mathbf{s}_i'' + 2\mathbf{s}_{i+1}''| \leqslant 6(3+\lambda+5\mu+3\lambda\mu)/D,$$

and putting these bounds into (13) implies that for $t \in [t_i, t_{i+1}]$,

$$\big|\mathbf{e}(t)\big| \leqslant (\Delta t_i)^2 \frac{1}{4D} \max\{3+5\lambda+\mu+3\lambda\mu, 3+\lambda+5\mu+3\lambda\mu\}.$$

Let us now suppose that $\lambda \geqslant \mu$. Then

$$\big|\mathbf{e}(t)\big| \leqslant (\Delta t_i)^2 \frac{1}{4D}(3+5\lambda+\mu+3\lambda\mu),$$

which we can rewrite as

$$\big|\mathbf{e}(t)\big| \leqslant (\Delta t_i)^2 \left(\frac{1}{4} + \frac{\lambda(1+\mu)}{3+\lambda+\mu-\lambda\mu}\right).$$

It is easy to check that the expression in the brackets, being a rational linear function in μ is increasing in μ and so takes its maximum value when $\mu = \lambda$, hence

$$\big|\mathbf{e}(t)\big| \leqslant (\Delta t_i)^2 \left(\frac{1}{4} + \frac{\lambda}{3-\lambda}\right). \tag{16}$$

Now notice that since $\lambda \leqslant 1$, setting $\lambda = 1$ in the above inequality gives again the $3/4$ constant we had in Theorem 1. But now we can do better by exploiting the fact that the μ depends on *all* the polygon edge lengths, and in particular both $|\mathbf{p}_{i+1} - \mathbf{p}_i|$ and $|\mathbf{p}_i - \mathbf{p}_{i-1}|$. Thus, we consider the two cases $\lambda \leqslant 1/2$ and $\lambda \geqslant 1/2$. If $\lambda \leqslant 1/2$ we use (16) directly, giving

$$\big|\mathbf{e}(t)\big| \leqslant (\Delta t_i)^2 \left(\frac{1}{4} + \frac{1}{5}\right) \leqslant \frac{9}{20}|\mathbf{p}_{i+1} - \mathbf{p}_i|.$$

Conversely, if $\lambda \geqslant 1/2$, (16) gives

$$\big|\mathbf{e}(t)\big| \leqslant (\Delta t_i)^2 \frac{1}{4} + \Delta t_i \Delta t_{i-1} \frac{1-\lambda}{3-\lambda} \leqslant (\Delta t_i)^2 \frac{1}{4} + \Delta t_i \Delta t_{i-1} \frac{1}{5} \leqslant \frac{9}{20} \max_{j=i-1,i} |\mathbf{p}_{j+1} - \mathbf{p}_j|.$$

We have thus established the bound

$$\big|\mathbf{e}(t)\big| \leqslant \frac{9}{20} \max_{0 \leqslant j \leqslant n-1} |\mathbf{p}_{j+1} - \mathbf{p}_j| \tag{17}$$

for $t \in [t_i, t_{i+1}]$ for all $\lambda \in [0, 1]$ and all $\mu \in [0, \lambda]$. The remaining case that $\mu \geqslant \lambda$ can be treated in the same way and so the bound holds for all λ and μ in $[0, 1]$. $\square$

What can be said about the global deviation of the uniform spline? We next show that the global $3/4$ bound extends in fact to any α-parameterization in which $0 \leqslant \alpha \leqslant 1$, which includes the uniform case $\alpha = 0$. However, as we have already seen in the $\alpha = 1/2$ case, the $3/4$ bound is not optimal for general α.

Theorem 4. *If* $\mathbf{s}$ *is a spline with any* α*-parameterization,* $0 \leqslant \alpha \leqslant 1$*, then* $\mu(\mathbf{s}) \leqslant 3/4$.

Proof. With $M := \max_{0 \leqslant j \leqslant n-1} |\mathbf{p}_{j+1} - \mathbf{p}_j|$, we have $|[t_j, t_{j+1}]\mathbf{p}| \leqslant M^{1-\alpha}$ and therefore

$$\max_{0 \leqslant j \leqslant n-1} |\mathbf{s}'_j| \leqslant \max_{0 \leqslant j \leqslant n-1} |\mathbf{b}_j| \leqslant 3M^{1-\alpha}.$$

Then with $\mathbf{q}$ as in (9), Eq. (10) for $t \in [t_i, t_{i+1}]$ gives

$$\left|\mathbf{e}(t)\right| \leqslant \frac{1}{4} M^{\alpha} 3M^{1-\alpha}. \qquad \square$$

Thus $\mu \leqslant 3/4$ for all three parameterizations: chordal, centripetal, and uniform, but the only parameterization for which we know that $3/4$ is optimal is the chordal one.

5. Numerical examples

The data set

x	0	0	5	10	13	14	15	30	40	40
y	0	10	20	10	10	14	14	8	5	0

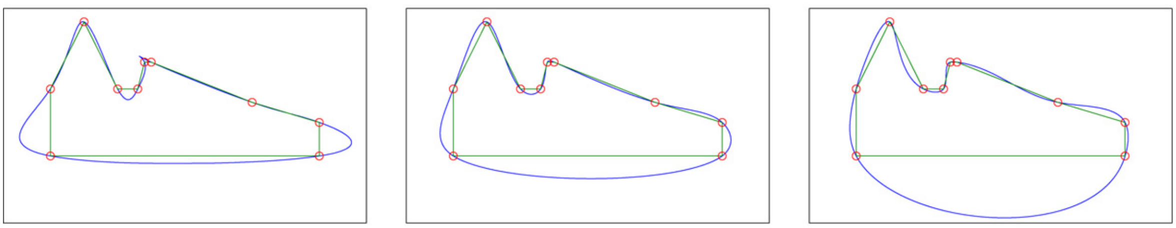
where $n = 10$, was used to generate the three splines: uniform, centripetal, and chordal. The maximum local and global deviations are:

Parameterization	$\max_i \mu_i$	μ
Uniform	1.166	0.120
Centripetal	0.265	0.084
Chordal	0.231	0.231

The three spline curves are shown in Fig. 1. Fig. 2 shows the three splines for the data set (11) with $m = 4$ and $n = 2m = 8$, and $\omega = 1/10, 1/2, 1, 2, 10$ respectively.

6. Final remarks

The bound $\mu \leqslant 9/20$ for the centripetal spline may not be the least possible and a topic for future research is to try to determine the least constant. Another question is whether there is some value of α, perhaps between $1/2$ and 1, for which the maximum μ_i is lower than $3/4$.



(a) (b) (c)

Fig. 1. Example data: (a) uniform, (b) centripetal, and (c) chordal.

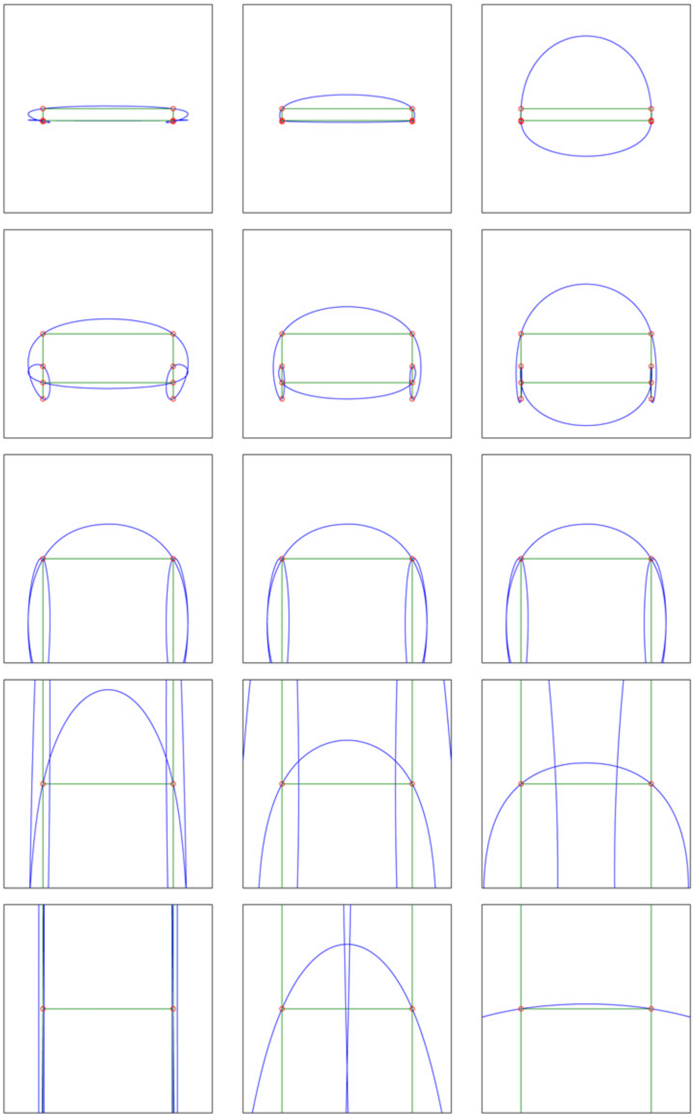


Fig. 2. Data set (11) with, top to bottom: $\omega = 1/10, 1/2, 1, 2, 10$ and left to right: uniform, centripetal, chordal.

One could also consider using the μ_i and μ values as measures of quality of other parametric interpolation methods. One could argue that a good method should have a bounded μ_i. On the other hand one should be careful about trying to *minimize* μ_i or μ. For example, if one fits a C^1 cubic spline, one can achieve $\mu_i = 0$ by forcing the spline to be equal to the polygon by setting the first derivative at each knot to be zero. This cannot happen though for a C^2 cubic spline. It would be interesting to know whether spline interpolants of higher degree, for example C^4 quintics, have a bounded μ_i for various α-parameterizations.

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