Quadratic Lagrange spectrum

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Abstract

We study the quadratic Lagrange spectrum defined by Parkkonen and Paulin by considering the approximation by quadratic numbers whose regular continued fraction expansion is ultimately periodic with the same period as a fixed quadratic number or its Galois conjugate. We improve the upper bound on the approximation constants involved thereby proving a conjecture stated by Bugeaud.

1 Introduction

In this paper the superscript $^\sigma$ denotes the Galois conjugate of a quadratic number. Let α_0 be a fixed real quadratic irrational number and $\mathcal{E}_{\alpha_0}$ the set of quadratic numbers whose regular continued fraction expansion is ultimately periodic with the same period as α_0 or α_0^σ.

Following Parkkonen and Paulin [3] and Bugeaud [2], we observe the quantity

$$c_{\alpha_0}(\xi) := \liminf_{\alpha \in \mathcal{E}_{\alpha_0} \,:\, |\alpha - \alpha^\sigma| \to 0} 2\frac{|\xi - \alpha|}{|\alpha - \alpha^\sigma|}$$

for a real number ξ not in $\mathbb{Q} \cup \mathcal{E}_{\alpha_0}$. This approximation constant $c_{\alpha_0}(\xi)$ of ξ by elements of $\mathcal{E}_{\alpha_0}$ is always finite, as proved by Parkkonen and Paulin [3]. It follows immediately from the definition that

$$c_{\alpha_0+k}(\xi + k') = c_{\alpha_0}(\xi)$$

for any integers k and k'. Define by

$$Sp_{\alpha_0} := \{c_{\alpha_0}(\xi) : \xi \in \mathbb{R} \setminus (\mathbb{Q} \cup \mathcal{E}_{\alpha_0})\}$$

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the so called *quadratic Lagrange spectrum* of α_0.

Parkkonen and Paulin [3, 4] showed that Sp_{α_0} is a closed subset of $[0, (1+\sqrt{2})\sqrt{3}]$. Bugeaud [2] proved that Sp_{α_0} is contained in $[0, 1/2]$. Denote by K_{α_0} the maximum of Sp_{α_0}. Bugeaud also proved that for α_0 ending in an infinite string of identical digits in its continued fraction expansion, $Sp_{\alpha_0} \subset [0, \frac{3}{\sqrt{5}} - 1]$ with $K_{(1+\sqrt{5})/2} = \frac{3}{\sqrt{5}} - 1$ and

$$\lim_{m \to +\infty} K_{(m+\sqrt{m^2+4})/2} = \frac{3}{\sqrt{5}} - 1.$$

Recall that the continued fraction expansion of $(m+\sqrt{m^2+4})/2$ is $[m, m, \ldots] = [\overline{m}]$.

In view of these facts, Bugeaud proposed as an open problem to show that $-1 + 3/\sqrt{5}$ is a common upper bound for all values of K_{α_0}, where α_0 is a quadratic irrationality. We show that this is in fact the case.

Theorem 1. *For any real quadratic irrational number α_0, the maximum K_{α_0} of Sp_{α_0} satisfies*

$$0 < K_{\alpha_0} \leq \frac{3}{\sqrt{5}} - 1.$$

The strict inequality $0 < K_{\alpha_0}$ in the previous theorem follows from Corollary 2.3 in [2].

Moreover, we generalize the other aforementioned result by Bugeaud [2].

Theorem 2. *Let u, v be positive integers such that $u > 1/0.0008$ and $v \geq u$. Then for $\beta_{u,v} := [\overline{u, v}]$, we have*

$$\lim_{v \to +\infty} K_{\beta_{u,v}} = \frac{3}{\sqrt{5}} - 1 - \frac{2/\sqrt{5}}{u}.$$

The previous theorem shows that $-1+3/\sqrt{5}$ is in the second derived set of

$$\{K_{\alpha_0} : \alpha_0 \text{ real quadratic irrationality}\}.$$

We introduce the following notation

$$
\begin{aligned}
a_i &\in \mathbb{Z}, \quad a_i \geq 1 \text{ for } i \geq 1,\\
\xi &= [a_0, a_1, a_2, \ldots] \notin \mathcal{E}_{\alpha_0},\\
p_r/q_r &= [a_0, a_1, \ldots, a_r], \text{ for } r \geq 1 \text{ (convergents to } \xi),\\
d_r &= [a_r, a_{r-1}, \ldots, a_1],\\
D_r &= [a_{r+1}, a_{r+2}, \ldots],\\
b_i &\in \mathbb{Z}_{\geq 1},\ i \in \{1, \ldots, s\},\\
\alpha_0 &= [0, \overline{b_1, \ldots, b_s}],\\
\tau_j &= [\overline{b_j, b_{j+1}, \ldots, b_{j-1}}],\ \tau_j' = [\overline{b_{j-1}, b_{j-2}, \ldots, b_j}],\ j \in \{1, \ldots, s\}\\
&\text{(we permute } (b_1, b_2, \ldots, b_s) \text{ and } (b_s, b_{s-1}, \ldots, b_1) \text{ cyclically)},\\
\varphi &= \frac{1+\sqrt{5}}{2}, \quad \kappa = -1 + \frac{3}{\sqrt{5}} = \frac{2(2-\varphi)}{\varphi + \varphi^{-1}} = 0.3416\ldots,\\
M &= \limsup_{i \geq 1} a_i, \quad N = \max_{1 \leq j \leq s} b_j.
\end{aligned}
$$

We tacitly assume that s is minimal. A theorem by Galois (see [5, Satz 3.6] or [2]) says that $\tau_j^\sigma = -1/\tau_j'$. This shows that τ_j^σ and τ_j' are equivalent numbers and thus have the same period in the continued fraction expansion. Therefore, two quadratic numbers which are Galois conjugates to each other always have inverse periods (see [5, Satz 3.8]). It is clear now that $\mathcal{E}_{\alpha_0} = \mathcal{E}_{\tau_1} = \cdots = \mathcal{E}_{\tau_s} = \mathcal{E}_{\tau_1'} = \cdots = \mathcal{E}_{\tau_s'}$. This is also the reason why we could take any pre-period for α_0 in the notation above without loss of generality.

Let k be an integer and $\tau \in T := \{\tau_1, \ldots, \tau_s, \tau_1', \ldots, \tau_s'\}$ such that $\tau + k \geq 1$. We presume that k is chosen so that such a τ exists. We have $\mathcal{E}_{\tau+k} = \mathcal{E}_\tau = \mathcal{E}_{\alpha_0}$ since, for example, $\tau_1 + k = [b_1 + k, \tau_2]$.

For $r \geq 1$, the quadratic number

$$\alpha_r := [a_0, a_1, \ldots, a_r, \tau + k]$$

is a quite good approximation to ξ in $\mathcal{E}_{\alpha_0}$ and

$$\ell_\tau(\xi) := \liminf_{r \to +\infty} 2 \frac{|\xi - \alpha_r|}{|\alpha_r - \alpha_r^\sigma|}$$

is greater than or equal to $c_{\alpha_0}(\xi)$. Note that

$$\alpha_r^\sigma = [a_0, a_1, \ldots, a_r, \tau^\sigma + k].$$

Taking into account all possible choices for τ, we see that

$$c_{\alpha_0}(\xi) \leq \min_{\substack{\tau\in T \\ \tau+k\geq 1}} \ell_\tau(\xi). \tag{1}$$

It is not at all clear whether an equality holds in (1) for some k since to validate that claim the quantities $2|\xi-\alpha|/|\alpha-\alpha^\sigma|$ would have to be checked for α of the form

$$[a_0, a_1, \ldots, a_r, a_1', \ldots, a_t', \tau],$$

where $\tau \in T$, $t \geq 1$ and $a_1', \ldots, a_t'$ are positive integers. This is not easy to do.

By a well known theorem in the theory of continued fractions (see e.g. Theorem 1.7 in [1] or p. 32 in [5]), we have

$$\alpha_r = \frac{p_r(\tau+k)+p_{r-1}}{q_r(\tau+k)+q_{r-1}} \quad \text{and} \quad \alpha_r^\sigma = \frac{p_r(\tau^\sigma+k)+p_{r-1}}{q_r(\tau^\sigma+k)+q_{r-1}},$$

which implies

$$\begin{aligned} |\alpha_r-\alpha_r^\sigma| &= \left|\frac{(p_rq_{r-1}-p_{r-1}q_r)\big((\tau+k)-(\tau^\sigma+k)\big)}{\big(q_r(\tau+k)+q_{r-1}\big)\big(q_r(\tau^\sigma+k)+q_{r-1}\big)}\right| \\ &= \frac{\tau-\tau^\sigma}{\left|\big(q_r(\tau+k)+q_{r-1}\big)\big(q_r(\tau^\sigma+k)+q_{r-1}\big)\right|}. \end{aligned} \tag{2}$$

Observe that

$$\begin{aligned} |\xi-\alpha_r| &= \left|\frac{p_rD_r+p_{r-1}}{q_rD_r+q_{r-1}} - \frac{p_r(\tau+k)+p_{r-1}}{q_r(\tau+k)+q_{r-1}}\right| \\ &= \frac{|\tau+k-D_r|}{(q_rD_r+q_{r-1})\big(q_r(\tau+k)+q_{r-1}\big)}. \end{aligned} \tag{3}$$

Combining (2) with (3) and using $q_r/q_{r-1} = d_r$, we get

$$2\frac{|\xi-\alpha_r|}{|\alpha_r-\alpha_r^\sigma|} = \frac{2|\tau+k-D_r|\cdot|\tau^\sigma+k+\frac{1}{d_r}|}{(\tau-\tau^\sigma)(D_r+\frac{1}{d_r})}. \tag{4}$$

This is what Bugeaud obtained (see [2], formula (3.4) and its modifications on pp. 994-995). Theorem 1 will now follow by (1) if the inequality

$$\frac{2|\tau+k-D_r|\cdot|\tau^\sigma+k+\frac{1}{d_r}|}{(\tau-\tau^\sigma)(D_r+\frac{1}{d_r})} \leq \kappa \tag{5}$$

holds for some integer k, for $\tau \in \{\tau_1, \dots, \tau_s, \tau'_1, \dots, \tau'_s\}$ such that $\tau + k \geq 1$ and for infinitely many integers $r \geq 1$.

If $\tau = \tau_j$, we put $\tau' = \tau'_j$ and if $\tau = \tau'_j$, we set $\tau' = \tau_j$.

By the already mentioned theorem of Galois which states that $\tau^\sigma = -1/\tau'$, proving (5) becomes equivalent to showing that

$$\frac{2|k + \tau - D_r| \cdot |k - \frac{1}{\tau'} + \frac{1}{d_r}|}{(\tau + \frac{1}{\tau'})(D_r + \frac{1}{d_r})} \tag{6}$$

is less than κ. In the next section we show that this is true thereby proving Theorem 1. In the last section, we prove Theorem 2.

2 Proof of Theorem 1

In order to prove Theorem 1, we distinguish between three different cases according to the size of the partial quotients in the continued fraction expansion of ξ and α_0. These are the possibilities: $\min\{M, N\} \geq 3$, $\min\{M, N\} = 2$ and $\min\{M, N\} = 1$. The case $\min\{M, N\} = 1$ has already been solved by Bugeaud [2], but we include it for completeness. Observe that by interchanging (τ, τ') with (D_r, d_r) and k with $-k$ in (6), we obtain exactly the same quantity. This remark will be useful for making our proofs shorter, since we can assume without loss of generality that $M \geq N$ and treat the case $N \geq M$ in an analogous manner. Thus, we only note the necessary modifications at the end of each subsection.

2.1 Case $N \geq 3$

Suppose that $M \geq N \geq 3$. Take $j \in \{1, \dots, s\}$ such that $b_j = N$ and put $\tau = \tau_j$. Choose r large enough such that $a_{r+1} = M$ and $a_l \leq M$ for $l \geq r$. By taking for k the integer closest to $1/\tau' - 1/d_r$, we ensure that $k \in \{-1, 0, 1\}$ and

$$2\left|k - \frac{1}{\tau'} + \frac{1}{d_r}\right| \leq 1.$$

Since $\tau \in [N, N+1]$, $D_r \in [M, M+1]$, we have $|k + \tau - D_r| \leq M - N + 2 \leq M - 1$. We also have $\tau' < N + 1$, $d_r < M + 1$. Hence, the value of (6) is smaller than

$$\frac{M-1}{(N + \frac{1}{N+1})(M + \frac{1}{M+1})} < \frac{1}{N + \frac{1}{N+1}} \leq \frac{1}{3 + \frac{1}{3+1}} = 0.307\ldots < \kappa.$$

The case when $N > M \geq 3$ is dealt with in the same way.

2.2 Case $N = 2$

Suppose that $M \geq N = 2$. Take $k \in \{-1, 0, 1\}$ such that $|k - 1/\tau' + 1/d_r| \leq 1/2$. We distinguish between two possible cases:

(A) $s = 1$, $\tau = \tau' = [\overline{2}] = 1 + \sqrt{2}$.

(B) $1 \in \{b_1, \ldots, b_s\}$.

In case (B), we can choose $j \in \{1, \ldots, s\}$ such that $b_j = 2$, $b_{j+1} = 1$ and then put $\tau = \tau_j = [2, 1, \ldots]$, so that $\tau \geq [2, 1, 1, 3] = 18/7 > 1 + \sqrt{2}$.

In either case $\max\{\tau, \tau'\} \leq [2, 1, 2, 1] = 11/4$.

If $M = 2$, then by the same argument, we also have $D_r \in [1 + \sqrt{2}, 11/4]$ for infinitely many r, thus (6) is less than

$$\frac{1 + |\tau - D_r|}{\tau D_r} \leq \frac{1 + (\frac{11}{4} - 1 - \sqrt{2})}{(1 + \sqrt{2})^2} = 0.229\ldots < \kappa.$$

Therefore, assume $M \geq 3$. For r such that $a_{r+1} \geq 3$, value of (6) is less than

$$\frac{1 + D_r - \tau}{(\tau + \frac{4}{11})D_r}.$$

This is smaller than κ if and only if $D_r(1 - \kappa(\tau + 4/11)) \leq \tau - 1$ and since $\tau \in [1 + \sqrt{2}, 11/4]$, the last inequality certainly holds if

$$D_r < \frac{(1 + \sqrt{2}) - 1}{1 - \kappa(1 + \sqrt{2} + \frac{4}{11})} = 27.744\ldots.$$

For this reason, we can assume that there are only finitely many $r \geq 1$ such that $3 \leq a_{r+1} \leq 26$ and this implies $M \geq 27$.

In the case (B), (6) is less than

$$\frac{1}{\tau + \frac{1}{\tau'}} \leq \frac{1}{\frac{18}{7} + \frac{4}{11}} = 0.3407\ldots < \kappa.$$

In the case (A) when $\tau = \tau' = 1 + \sqrt{2}$, since $D_r \geq 27$, we have $|k + \tau - D_r| < D_r + 1/d_r$ which implies (6) is less than $\frac{1}{\sqrt{2}}|k + 1 - \sqrt{2} + 1/d_r|$ and this is smaller than κ if and only if

$$-k + \sqrt{2} - 1 - \kappa\sqrt{2} \leq \frac{1}{d_r} \leq -k + \sqrt{2} - 1 + \kappa\sqrt{2}. \tag{7}$$

This holds for some $k \in \{-1, 0, 1\}$ if and only if (recall that $d_r > 1$)

$$d_r \leq \frac{1}{\sqrt{2}(1 - \kappa)} = 1.07404\ldots = [1, 13, \ldots]$$

$$\text{or } d_r \geq \frac{1}{\sqrt{2}(1 + \kappa) - 1} = 1.11437\ldots = [1, 8, \ldots].$$

But from our assumption

$$d_r \notin \Big[\frac{1}{\sqrt{2}(1-\kappa)}, \frac{1}{\sqrt{2}(1+\kappa)-1}\Big]$$

for r large enough since $a_r, a_{r-1} \in \{1,2\} \cup \{27, 28, \dots\}$.

We have finished the case when $M \geq N = 2$. The case $N > M = 2$ is done analogously.

2.3 Case $N = 1$

Let $N = 1$. Then $M = 1$ is not possible since $\xi \notin \mathcal{E}_{\alpha_0}$. Thus, $M \geq 2$. Here we have $\tau = \tau' = \varphi = (1+\sqrt{5})/2$ and we want to prove

$$\frac{2|k+\varphi-D_r| \cdot |k - \frac{1}{\varphi} + \frac{1}{d_r}|}{(\varphi + \frac{1}{\varphi})(D_r + \frac{1}{d_r})} \leq \kappa = \frac{2}{\varphi + \frac{1}{\varphi}}(2-\varphi)$$

or

$$\frac{|k+\varphi-D_r| \cdot |k - \frac{1}{\varphi} + \frac{1}{d_r}|}{D_r + \frac{1}{d_r}} \leq 2 - \varphi. \tag{8}$$

If $D_r \in [1, \frac{\varphi}{2-\varphi}]$, then $|\varphi - D_r| \leq (\varphi-1)D_r$ and since $d_r \geq 1$, $|-\frac{1}{\varphi} + \frac{1}{d_r}| \leq \frac{1}{\varphi}$ holds. Thus, for $k=0$, the left hand side of (8) is less than (recall that $\varphi^2 - \varphi - 1 = 0$)

$$\frac{(\varphi-1)D_r \cdot \frac{1}{\varphi}}{D_r} = 2 - \varphi,$$

which is what we wanted.

If $D_r > \varphi/(2-\varphi) = 4.236\dots$ and $k=1$, the inequality (8) becomes

$$(D_r - \varphi - 1)\Big(\frac{1}{d_r} + 1 - \frac{1}{\varphi}\Big) \leq (2-\varphi)\Big(D_r + \frac{1}{d_r}\Big),$$

or, after simplification, $D_r \leq d_r + 3$.

Now, if $D_r \leq d_r + 3$ for infinitely many r, then we are finished. Otherwise, for some r_0 and all $r \geq r_0$, the inequality $D_r > d_r + 3$ is valid. This implies that for $r \geq r_0$,

$$a_{r+1} \geq D_r - 1 > d_r + 2 \geq a_r + 2,$$

so for $r \to +\infty$, necessarily $d_r, D_r \to +\infty$ and now $k = 1$ with this limit case gives equality in (8).

The case when $M = 1$ is done analogously while observing that the final situation where the partial quotients of τ would be unbounded cannot happen since τ has a periodic continued fraction.

3 Proof of Theorem 2

Using the previously established notation, let $s = 2$, $b_1 = u$, $b_2 = v$, where integers u and v satisfy $v > u > 1/0.0008$. Here $\tau_1 = \beta_{u,v} = [\overline{u,v}]$ and $\tau_2 = [\overline{v,u}] = \tau_1'$. For ξ such that $M = \limsup_{i\geq 1} a_i \geq 2$, the first and second part of the proof of Theorem 1 in the previous section, show that (6) is smaller than $\kappa - 0.0008$ and therefore smaller than

$$\frac{3}{\sqrt{5}} - 1 - \frac{2/\sqrt{5}}{u}$$

for some $k \in \{-1, 0, 1\}$, $\tau \in \{\tau_1, \tau_2\}$ and infinitely many positive integers r. Note that in these two parts (Subsections 2.1 and 2.2) with easy modification, the worst upper bound which we obtain is less than $0.3408 < \kappa - 0.0008$. Therefore, the only case worth looking at is $M = 1$, or, without loss of generality, by removing a finite prefix of the continued fraction expansion of ξ, we can take $\xi = \varphi$.

Putting $k = -1$, which we can since u is large, (6) becomes

$$\frac{2(\tau - \varphi - 1)(1 + \frac{1}{\tau'} - \frac{1}{\varphi})}{(\varphi + \frac{1}{\varphi})(\tau + \frac{1}{\tau'})}. \tag{9}$$

for $r \to +\infty$. Note that for $k = 0$, the value of (6) is certainly larger than (9). The function (9) in (τ, τ') is increasing in τ and decreasing in τ', so the smallest value is obtained for $(\tau, \tau') = (\tau_1, \tau_2) = (\tau_1, \tau_1')$. If we let $v \to +\infty$, we get for (9) the value

$$\kappa\Big(1 - \frac{\varphi + 1}{u}\Big) = \kappa - \frac{2/\sqrt{5}}{u}. \tag{10}$$

What is left to prove is that this is the best upper bound that we can achieve with the chosen τ. In other words, if we approximate $\xi = \varphi$ by

$$\alpha = [\underbrace{1, \ldots, 1}_{r}, a_1', \ldots, a_t', \tau],$$

where $a_1' \neq 1$, the value of $2|\xi - \alpha|/|\alpha - \alpha^\sigma|$ will not be smaller than (10) (for $v \to +\infty$). Denote by $p/q = [a_1', \ldots, a_t']$ and $p'/q' = [a_1', \ldots, a_{t-1}']$, or, if $t = 1$, $p' = 1$, $q' = 0$ as is usual in the theory of regular continued fractions. Reasoning in the same way that led us to (5), what we have to show is the following inequality

$$\frac{2\left|\frac{p\tau+p'}{q\tau+q'} - \varphi\right| \cdot \left|\frac{p\tau^\sigma+p'}{q\tau^\sigma+q'} + \frac{1}{\varphi}\right|}{\left|\frac{p\tau+p'}{q\tau+q'} - \frac{p\tau^\sigma+p'}{q\tau^\sigma+q'}\right|(\varphi + \frac{1}{\varphi})} \geq \kappa - \frac{2/\sqrt{5}}{u}$$

when $v \to +\infty$. Since $|pq' - p'q| = 1$ and $\tau^\sigma = -1/\tau'$, this simplifies to

$$\frac{|p\tau + p' - \varphi(q\tau + q')| \cdot |p\tau^\sigma + p' + \varphi^{-1}(q\tau^\sigma + q')|}{|\tau - \tau^\sigma|} \geq 2 - \varphi - \frac{1}{u}$$

or

$$\frac{|(p - \varphi q)\tau + (p' - \varphi q')| \cdot | - (p + \varphi^{-1}q) + (p' + \varphi^{-1}q')\tau'|}{\tau\tau' + 1} \geq 2 - \varphi - \frac{1}{u} \quad (11)$$

If $t = 1$, then $p' = 1$, $q' = 0$, $q = 1$ and we should prove

$$L := \frac{|(p - \varphi)\tau + 1| \cdot | - (p + \varphi^{-1}) + \tau'|}{\tau\tau' + 1} \geq 2 - \varphi - \frac{1}{u}. \quad (12)$$

Looking at the numerator of L as an absolute value of a quadratic function in $p \in \mathbb{Z}$, it becomes clear that the minimum of L can only be attained for integers closest to the zeroes of this function, i.e. for

$$p \in \{\lfloor \varphi - \tau^{-1} \rfloor, \lceil \varphi - \tau^{-1} \rceil, \lfloor \tau' - \varphi^{-1} \rfloor, \lceil \tau' - \varphi^{-1} \rceil\},$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are, respectively, the largest integer not greater than the real number x and the smallest integer not less than x.

Since we have $\{\tau, \tau'\} = \{[\overline{u, v}], [\overline{v, u}]\}$ and u, v are large, for $\tau = [\overline{u, v}]$, we are left with checking $p \in \{1, 2, v - 1, v\}$, while for $\tau = [\overline{v, u}]$, we should check $p \in \{1, 2, u - 1, u\}$.

For $\tau = [\overline{u, v}]$, when $v \to +\infty$, we have

$$L \to \begin{cases} \varphi - 1 - \frac{1}{u} & \text{if } p \in \{1, v\} \\ 2 - \varphi + \frac{1}{u} & \text{if } p \in \{2, v - 1\} \end{cases}.$$

For $\tau = [\overline{v, u}]$, when $v \to +\infty$, we have

$$L \to \begin{cases} \varphi - 1 - \frac{1}{u} & \text{if } p \in \{1, u\} \\ 2 - \varphi - \frac{1}{u} & \text{if } p \in \{2, u - 1\} \end{cases}.$$

Thus we are left with the case when $t \geq 2$. Substituting

$$p' - \varphi q' = \frac{q'}{q}(p - \varphi q) \pm \frac{1}{q}, \quad p + \varphi^{-1}q = \frac{q}{q'}(p' + \varphi^{-1}q') \mp \frac{1}{q'}$$

into (11), we obtain

$$\frac{\left|(p - \varphi q)\left(\tau + \frac{q'}{q}\right) \pm \frac{1}{q}\right| \cdot \left|(p' + \varphi^{-1}q')\left(\tau' - \frac{q}{q'}\right) \pm \frac{1}{q'}\right|}{\tau\tau' + 1} \geq 2 - \varphi - \frac{1}{u} \quad (13)$$

Now

$$\frac{p}{q} = [a'_1, \dots, a'_t], \quad \frac{p'}{q'} = [a'_1, \dots, a'_{t-1}], \quad \frac{q}{q'} = [a'_t, \dots, a'_2].$$

Since $a'_1 \neq a_{r+1} = 1$, it follows $a'_1 \geq 2$. Also, $a'_t \neq \lfloor \tau' \rfloor$ because otherwise the periodic part of the continued fraction of α would start with a'_t. Thus $p - \varphi q \geq q(2-\varphi)$ and so

$$\frac{\left|(p-\varphi q)\left(\tau + \frac{q'}{q}\right) \pm \frac{1}{q}\right|}{\tau + \frac{1}{\tau'}} \geq q\Big(2 - \varphi - \frac{1}{u}\Big).$$

Therefore, there remains to be proved that

$$\frac{\left|(p' + \varphi^{-1}q')\left(\tau' - \frac{q}{q'}\right) \pm \frac{1}{q'}\right|}{\tau'} \geq \frac{1}{q},$$

or, after using $p' \geq 2q'$,

$$(2+\varphi^{-1})qq'\Big|\tau' - \frac{q}{q'}\Big| \geq \tau' + \frac{q}{q'},$$

which follows if the inequality

$$\big((2+\varphi^{-1})qq' - 1\big)\Big|\tau' - \frac{q}{q'}\Big| \geq \frac{2q}{q'} \tag{14}$$

is valid.

If $|\tau' - \frac{q}{q'}| \geq \frac{1}{q'^2}$, then the left hand side of (14) is greater than

$$(2+\varphi^{-1})\frac{q}{q'} - \frac{1}{q'^2} \geq \frac{2q}{q'}.$$

Otherwise, $|\tau' - \frac{q}{q'}| < \frac{1}{q'^2}$ implies, according to a result by Fatou (see [1, p. 11]), that $\frac{q}{q'}$ is a convergent $\frac{P_n}{Q_n}$ of τ' or a secondary convergent of τ' of the following type

$$\frac{P_{n+1}+P_n}{Q_{n+1}+Q_n} \quad \text{or} \quad \frac{P_{n+2}-P_{n+1}}{Q_{n+2}-Q_{n+1}}.$$

All of these fractions belong to the closed interval with endpoints $\frac{P_n}{Q_n}$ and $\frac{P_{n+1}}{Q_{n+1}}$ which would require $a'_t = \lfloor \frac{q}{q'} \rfloor = \lfloor \tau' \rfloor$ for $n \geq 0$ and this is not possible. The only option left to check corresponds to the case when $n = -1$, that is when

$$\frac{q}{q'} = \frac{\lfloor \tau' \rfloor + 1}{1}.$$

Since u and v are large integers, τ' is very close to $\lfloor \tau' \rfloor$ which allows us to easily show that (14) holds in this case as well. Hence, we are finished with the proof of Theorem 2.

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