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Congruences for sporadic sequences and modular forms for non-congruence subgroups

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Abstract In 1979, in the course of the proof of the irrationality of $\zeta(2)$ Apéry introduced numbers b_n that are, surprisingly, integral solutions of the recursive relation

$$(n+1)^2u_{n+1} - (11n^2+11n+3)u_n - n^2u_{n-1} = 0.$$

Indeed, b_n can be expressed as $b_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$.

Zagier performed a computer search of the first 100 million triples $(A,B,C) \in \mathbb{Z}^3$ and found that the recursive relation generalizing b_n

$$(n+1)^2u_{n+1} - (An^2+An+B)u_n + Cn^2u_{n-1} = 0,$$

with the initial conditions $u_{-1} = 0$ and $u_0 = 1$ has (non-degenerate, i.e. $C(A^2-4C) \neq 0$) integral solution for only six more triples (whose solutions are so called sporadic sequences).

Stienstra and Beukers showed that for the prime $p \geq 5$

$$b_{(p-1)/2} \equiv \begin{cases} 4a^2 - 2p \pmod p \text{ if } p = a^2+b^2, \text{ a odd} \\ 0 \pmod p \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

Recently, Osburn and Straub proved similar congruences for all but one of the six Zagier's sporadic sequences (three cases were already known to be true by the work of Stienstra and Beukers) and conjectured the congruence for the sixth sequence (which is a solution of the recursion determined by triple $(17, 6, 72)$).

In this paper we prove that remaining congruence by studying Atkin and Swinnerton-Dyer congruences between Fourier coefficients of a certain cusp form for non-congurence subgroup.

Keywords Apéry numbers · modular forms · Atkin and Swinnerton-Dyer congruences · non-congruence subgroups

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1 Introduction

In 1979, in the course of his famous proof of the irrationality of $\zeta(3)$ and $\zeta(2)$ Apéry [1] introduced numbers $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ and $b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$. These numbers, which were important for the proof, are integral solutions of the recursive relations

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0, \quad \text{and}$$

$$(n+1)^2 u_{n+1} - (11n^2 + 11n + 3)u_n - n^2 u_{n-1} = 0$$

respectively. The integrality came as a big surprise since to calculate a_n (or b_n) in each step one has to divide by n^3 (or n^2) so a priori one would expect that these numbers have denominators of the size $n!^3$ (or $n!^2$). Inspired by Beukers [3], Zagier [19] performed a computer search of the first 100 million triples $(A, B, C) \in \mathbb{Z}^3$ and found that the recursive relation generalizing b_n

$$(n+1)u_{n+1} - (An^2 + An + B)u_n + Cn^2 u_{n-1} = 0,$$

with the initial conditions $u_{-1} = 0$ and $u_0 = 1$ has (non-degenerate i.e. $C(A^2 - 4C) \neq 0$) an integral solution for only six more triples (whose solutions are so called sporadic sequences)

$$(0, 0, -16), (7, 2, -8), (9, 3, 27), (10, 3, 9), (12, 4, 32) \text{ and } (17, 6, 72).$$

Interestingly, Stienstra and Beukers [15] showed that the generating function of Apéry's numbers b_n is a holomorphic solution of the Picard-Fuchs differential equation of the elliptic K3-surface $\mathcal{S} : X(Y-Z)(Z-X) - t(X-Y)YZ = 0$ (other sporadic sequences are related in this way to K3 surfaces as well, see [19]). Using this connection they also proved that for primes $p \geq 5$

$$b_{(p-1)/2} \equiv \begin{cases} 4a^2 - 2p \pmod{p} \text{ if } p = a^2 + b^2, \text{ a odd} \\ 0 \pmod{p} \text{ if } p \equiv 3 \pmod{4}. \end{cases}$$

Here one can interpret the right-hand side of the congruences as a p-th Fourier coefficient of a certain CM modular form of weight 3 whose L-function is a factor of the zeta function of $\mathcal{S}$. (Later Beukers [3] proved a similar result for the numbers a_n - this time relating them to the coefficients of a Hecke eigenform of weight 4.) For a beautiful survey of these results see [20].

Recently, Osburn and Straub [11] proved similar congruences for all but one of the six Zagier's sporadic sequences (three cases were already known to be true by the work of Stienstra and Beukers) and conjectured the congruence for the sixth sequence $F(n)$ (which is a solution of the recursion determined by the triple $(17, 6, 72)$. In this paper we prove that remaining congruence.

Denote by

$$F(n) = \sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^3,$$

the sporadic sequence corresponding to triple $(17, 6, 72)$. For $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$, consider the newform

$$f(\tau) = \sum_{n=0}^{\infty} \gamma(n) q^n = q - 2q^2 + 3q^3 + \cdots \in S_3\left(\Gamma_0(24), \left(\frac{-6}{\cdot}\right)\right).$$

Our main result is the following theorem.

Theorem 1 *For all primes $p > 2$ we have*

$$F\left(\frac{p-1}{2}\right) \equiv \gamma(p) \pmod{p}.$$

Remark 1 One can check that $f(\tau)$ is CM form such that for primes $p \not\equiv 5, 11 \pmod{24}$

$$\gamma(p) = \begin{cases} 2(a^2 - 6b^2) \text{ if } p = a^2 + 6b^2 \\ 0 \text{ if } \left(\frac{-6}{p}\right) = -1. \end{cases}$$

In Section 2 using the method of Beukers [3, Proposition 3.] and Verrill [18, Theorem 1.1] we reduce Theorem 1 to showing that the weight three cusp form (for a non-congruence subgroup Γ_2 of $\Gamma_1(6)$)

$$g(\tau) = q^{1/2} + \frac{3}{2}q^{3/2} - \frac{9}{8}q^{5/2} - \frac{85}{16}q^{7/2} - \frac{981}{128}q^{9/2} + \cdots \in S_3(\Gamma_2),$$

satisfies a three-term Atkin and Swinnerton-Dyer congruence relation with respect to $f(\tau)$ for all primes $p > 3$ (see Proposition 2). A similar idea was used previously by the author [7] in proving three term congruence relations for some multinomial sums by employing Atkin and Swinnerton-Dyer congruence relations satisfied by the Fourier coefficients of certain weakly holomorphic modular forms (but for congruence subgroups).

In Section 3 we explain how using Scholl's theory [12] we can reduce Proposition 2 to the equivalence of two strictly compatible families of ℓ-adic Galois representations: $\tilde{\rho}_\ell$ isomorphic to ℓ-adic realization of the motive associated to the space of cusp forms $S_3(\Gamma_2)$, and ρ'_ℓ attached to the newform $f(\tau) \otimes \left(\frac{-1}{\cdot}\right)$ by Deligne's work.

In Section 4 and Section 5 we prove that these two ℓ-adic families are isomorphic by showing that they are isomorphic to a third ℓ-adic family ρ_ℓ which is constructed from the explicit model of the universal family of elliptic curves over a modular curve of Γ_2.

2 Elliptic surfaces, modular forms and the proof of Theorem 1

Consider the modular rational elliptic surface attached to $\Gamma_1(6)$ (see third example in [18, Section 4.2.2.])

$$\mathcal{W} : (x + y)(x + z)(y + z) - 8xyz = \frac{1}{t}xyz,$$

with fibration $\phi : \mathcal{W} \to P^1$, $(x, y, z, t) \mapsto t$. For $t \notin \{\infty, 0, -\frac{1}{9}, -\frac{1}{8}\}$ the preimage $\phi^{-1}(t)$ is an elliptic curve with a distinguished point of order 6. Denote by $f' = t\frac{d}{dt}f$. The Picard-Fuchs differential equation associated to this elliptic surface

$$(8t+1)(9t+1)P(t)'' + t(144t+17)P(t)' + 6t(12t+1)P(t) = 0,$$

has a holomorphic solution around $t = 0$

$$P(t) = \sum_{n=0}^{\infty}(-1)^n F(n)t^n.$$

(Our notation is slightly different from [18, Section 4.2.2.] since $F(n) = (-1)^n c_n$, with c_n defined in [18].) We can identify t with $t(\tau/6)$ where

$$t(\tau) = \frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)}, \quad \tau \in \mathbb{H}$$

is a modular function for $\Gamma_0(6)$. It follows that $P(\tau) := \sum_{n=0}^{\infty}(-1)^n F(n)t(\tau)^n$ is a weight one modular form for $\Gamma_1(6)$.

Now consider a 2-cover $\mathcal{S}$ of $\mathcal{W}$, a K3-surface given by the equation

$$\mathcal{S} : (x+y)(x+z)(y+z) - 8xyz = \frac{1}{s^2}xyz \tag{1}$$

where $t = s^2$. Then $s(\tau) = \sqrt{\frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)}}$ is a corresponding modular function for an index two genus zero subgroup $\Gamma_2 \subset \Gamma_1(6)$.

By identifying the s-line with the modular curve $X(\Gamma_2)$, we can identify singular fibers of the K3-surface $\mathcal{S}$ with cusps of the modular curve $X(\Gamma_2)$. More precisely, using Tate's algorithm one finds that Kodaira types of singular fibers at $s = \infty, 0, \pm\frac{i}{2\sqrt{2}}$ and $\pm\frac{i}{3}$ are $I_2, I_{12}, I_3, I_3, I_2$ and I_2 respectively. Hence all the cusps of $X(\Gamma_2)$ are regular.

In general, for a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of genus g such that $-I \notin \Gamma$ and k odd, [14, Theorem 2.25] gives the formula for the dimension of $S_k(\Gamma)$

$$\dim S_k(\Gamma) = (k-1)(g-1) + \frac{1}{2}(k-2)r_1 + \frac{1}{2}(k-1)r_2 + \sum_{i=1}^{j}\frac{e_i - 1}{2e_i},$$

where r_1 is the number of regular cusps, r_2 is the number of irregular cusps, and $e_i's$ are the orders of elliptic points. Since Γ_2 has no elliptic points ($\Gamma_1(6)$ is a free group), we have that $\dim S_3(\Gamma_2) = 1$.

Our starting point for studying congruences involving $F(n)$ is the following proposition of Beukers [3].

Proposition 1 (Beukers) *Let p be a prime and*

$$\omega(t) = \sum_{n=1}^{\infty} b_n t^{n-1} dt$$

a differential form with $b_n \in \mathbb{Z}_p$. Let $t(q) = \sum_{n=1}^{\infty} A_n q^n$, $A_n \in \mathbb{Z}_p$, and suppose

$$\omega(t(q)) = \sum_{n=1}^{\infty} c_n q^{n-1} dq.$$

Suppose there exist $\alpha_p, \beta_p \in \mathbb{Z}_p$ with $p|\beta_p$ such that

$$b_{mp^r} - \alpha_p b_{mp^{r-1}} + \beta_p b_{mp^{r-2}} \equiv 0 \pmod{p^r}, \quad \forall m, r \in \mathbb{N}.$$

Then

$$c_{mp^r} - \alpha_p c_{mp^{r-1}} + \beta_p c_{mp^{r-2}} \equiv 0 \pmod{p^r}, \quad \forall m, r \in \mathbb{N}.$$

On the other hand, if A_1 is p-adic unit then the second congruence implies the first, and we have that $b_p \equiv \alpha_p b_1 \pmod{p}$.

If we apply the previous proposition to the differential form

$$\omega(s) = \sum_{n=0}^{\infty} (-1)^n F(n) s^{2n} ds,$$

and $s(q)$ - the q-expansion of modular function $s(\tau)$, we obtain that $\omega(s(q)) = \sum_{n=1}^{\infty} c_n q^{(n-1)/2} dq$, where c_n are Fourier coefficients of weight 3 cusp form $g(\tau) \in S_3(\Gamma_2)$

$$g(q) = 2P(q) q \frac{d}{dq} s(q) = q^{1/2} + \frac{3}{2} q^{3/2} - \frac{9}{8} q^{5/2} - \frac{85}{16} q^{7/2} - \frac{981}{128} q^{9/2} + \cdots = \sum_{n=1}^{\infty} c_n q^{n/2}.$$

Remark 2 a) For $p = 2$ the Fourier coefficients of $s(q)$ are not p-integral so we cannot use Proposition 1.

b) It is well known that the differential operator $q\frac{d}{dq}$ maps modular functions to meromorphic modular forms of weight 2. Holomorphicity and cuspidality of $g(\tau)$ then follow since the zeros of $P(\tau)$ cancel out the poles of $s(\tau)$.

c) Since Fourier coefficients of $g(\tau)$ have unbounded denominators, it follows that Γ_2 is non-congruence subgroup of $\Gamma_1(6)$. For congruence subgroups the Hecke eigenforms (which form the basis for the space of cuspforms) have Fourier coefficients that are algebraic integers.

We will show that, for all primes $p > 3$, the cusp form $g(\tau)$ satisfies a three term Atkin and Swinnerton-Dyer congruence relation with respect to the quadratic twist of the newform $f(\tau) = \sum_{n=1}^{\infty} \gamma(n) e^{2\pi i \tau}$ by $\left(\frac{-1}{\cdot}\right)$. Hence, since the first coefficient in the q-expansion of $s(q)$ is $A_1 = 1$, Theorem 1 follows from the second part of Proposition 1 and the following proposition.

Proposition 2 *Let $p > 3$ be a prime. Then for all $m, r \in \mathbb{N}$, we have that*

$$c_{mp^r} - \left(\frac{-1}{p}\right) \gamma(p) c_{mp^{r-1}} + \left(\frac{-6}{p}\right) p^2 c_{mp^{r-2}} \equiv 0 \pmod{p^{2r}}.$$

3 Atkin and Swinnerton-Dyer congruences for $S_3(\Gamma_2)$

For a finite index non-congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and a prime p, we say that weight k cusp form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in S_k(\Gamma, \overline{\mathbb{Z}_p})$ satisfies an Atkin and Swinnerton-Dyer (ASD) congruence at p if there exist an algebraic integer A_p and a root of unity μ_p such that for all non-negative integers m and r we have

$$a_{mp^r} - A_p a_{mp^{r-1}} + \mu_p p^{k-1} a_{mp^{r-2}} \equiv 0 \pmod{p^{(k-1)r}}. \tag{2}$$

(In our example $a_n{}'s$ and $A_p{}'s$ are rational integers, and $\mu_p = \pm 1$.)

In the absence of the useful theory of Hecke operators for non-congruence subgroups, such $f(\tau)$ can be regarded as Hecke eigenfunction at prime p. A discovery of these congruences by Atkin and Swinnerton-Dyer [2] initiated a systematic study of modular forms for non-congruence subgroups. For more information see the survey article by Li, Long and Yang [9].

In the case when the space of cusp forms is one dimensional and generated by $f(\tau)$ (which is the case for $S_3(\Gamma_2)$ and $g(\tau)$), Scholl [12] proved that the ASD congruence holds for all but finitely many p. The congruences were obtained by embedding the module of cusp forms into a certain de Rham cohomology group $DR(\Gamma, k)$ which is the de Rham realization of the motive associated to the relevant space of modular forms. At a good prime p, crystalline theory endows $DR(\Gamma, k) \otimes \mathbb{Z}_p$ with a Frobenius endomorphism whose action on q-expansion gives rise to Atkin and Swinnerton-Dyer congruences, i.e. if $T^2 - A_p T + \mu_p p^2$ is a characteristic polynomial of Frobenius acting on $DR(\Gamma, k) \otimes \mathbb{Z}_p$ then congruence (2) holds (A_p is the trace of Frobenius). See [8, Section 2] for the summary of these results.

To calculate the trace of Frobenius A_p, following Scholl [12, Sections 4 and 5], we associate to the subgroup Γ_2 a strictly compatible family of ℓ-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\tilde{\rho}_\ell$, that is isomorphic to ℓ-adic realization of the motive associated to the space of cusp forms $S_3(\Gamma_2)$. From [13, 2.7. Proposition] and the algebraic relation between $s(\tau/6)$ and modular j-invariant $j(\tau)$

$$(1+6s^2)^3(1+18s^2+84s^4+24s^6)^3 - s^{12}(1+8s^2)^3(1+9s^2)^2 j = 0,$$

it follows that $\tilde{\rho}_\ell$ is unramified outside $2, 3$ and ℓ.

In particular, for $\ell = 2$ and prime $p > 3$ we have that [12, Theorem 5.4.]

$$A_p = trace(\tilde{\rho}_2(\mathrm{Frob}_p)) \text{ and } \mu_p p^2 = \det(\tilde{\rho}_2(\mathrm{Frob}_p)). \tag{3}$$

4 Compatible families of ℓ-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Denote by ρ'_ℓ a strictly compatible family of two dimensional ℓ-adic Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to the newform $f(\tau) \otimes \left(\frac{-1}{\cdot}\right)$ by the work of Deligne [5]. Hence,

$$\mathrm{trace}(\rho'_\ell(\mathrm{Frob}_p)) = \left(\frac{-1}{p}\right)\gamma(p) \text{ and } \det(\rho'_\ell(\mathrm{Frob}_p)) = \left(\frac{-6}{p}\right)p^2, \tag{4}$$

for primes $p \neq 2, 3$ and ℓ.

We will prove that representations ρ'_ℓ and $\tilde{\rho}_\ell$ are isomorphic by showing that both of them are isomorphic to the representation ρ_ℓ which we define now. Proposition 2 then follows from (3) and (4).

Let $X(\Gamma_2)^0$ be the complement in $X(\Gamma_2)$ of the cusps. Denote by i the inclusion of $X(\Gamma_2)^0$ into $X(\Gamma_2)$, and by $h' : \mathcal{S} \to X(\Gamma_2)^0$ the restriction of the elliptic surface $h : \mathcal{S} \to X(\Gamma_2)$ to $X(\Gamma_2)^0$. For a prime ℓ we obtain a sheaf

$$\mathcal{F}_\ell = R^1 h'_* \mathbb{Q}_\ell$$

on $X(\Gamma_2)^0$, and also a sheaf $i_* \mathcal{F}_\ell$ on $X(\Gamma_2)$ (here $\mathbb{Q}_\ell$ is the constant sheaf on the elliptic surface $\mathcal{S}$, and R^1 is derived functor). The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $\mathbb{Q}_\ell$-vector space

$$W = H^1_{et}(X(\Gamma_2) \otimes \overline{\mathbb{Q}}, i_* \mathcal{F}_\ell)$$

defines ℓ-adic representation ρ_ℓ. This representation is unramified outside $2, 3$ and ℓ. By an argument similar to [10, Proposition 5.1.], ρ_ℓ is isomorphic to $\tilde{\rho}_\ell$ up to a twist by quadratic character.

Using (1), we can calculate $trace(\rho_l(\mathrm{Frob}_p))$ and

$$\begin{aligned}\det(\rho_l(\mathrm{Frob}_p)) &= \frac{1}{2}((trace(\rho_l(\mathrm{Frob}_p))^2 - trace(\rho_l(\mathrm{Frob}_p^2))) \\ &= \frac{1}{2}((trace(\rho_l(\mathrm{Frob}_p))^2 - trace(\rho_l(\mathrm{Frob}_{p^2})))\end{aligned}$$

for $p \neq 2, 3, \ell$ using the following well known result (for the proof see Theorem 10 in [6]).

Theorem 2 *Let $q = p^s$ be a power of prime $p \neq 2, 3, \ell$. The following are true:*

(1) We have that

$$trace(Frob_q | W) = - \sum_{t \in X(\Gamma_2)(\mathbb{F}_q)} trace(Frob_q | (i_* \mathcal{F}_\ell)_t).$$

(2) If the fiber $E_t := h^{-1}(t)$ is smooth, then

$$trace(Frob_q | (i_* \mathcal{F}_\ell)_t) = trace(Frob_q | H^1(E_t, \mathbb{Q}_\ell)) = q + 1 - \#E_t(\mathbb{F}_q).$$

(3) If the fiber E_t^j is singular, then

$$trace(Frob_q | (i_* \mathcal{F}_\ell)_t) = \begin{cases} 1 & \text{if the fiber is split multiplicative,} \\ -1 & \text{if the fiber is nonsplit multiplicative,} \\ 0 & \text{if the fiber is additive.} \end{cases}$$

5 Serre-Faltings method and proof of Proposition 2

We will prove the following proposition.

Proposition 3 *For every prime ℓ the representations ρ_ℓ and ρ'_ℓ are isomorphic.*

Since the families are strictly compatible, by Chebotarev density theorem it is enough to prove that ρ_2 and ρ'_2 are isomorphic. We apply the method of Serre and Faltings as formulated in [13, Section 5].

Theorem 3 *For a finite set of primes S of $\mathbb{Q}$, let $\chi_1, \dots, \chi_r$ be a maximal independent set of quadratic characters of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside S, and G a subset of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the map $(\chi_1, \dots, \chi_r) : G \to (\mathbb{Z}/2\mathbb{Z})^r$ is surjective.*

Let $\sigma, \sigma' : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{GL}_2(\mathbb{Q}_2)$ be continuous semisimple representations unramified away from S, whose images are pro-2-groups. If for every $g \in G$

$$trace(\sigma(g)) = trace(\sigma'(g)) \text{ and } \det(\sigma(g)) = \det(\sigma'(g)),$$

then σ and σ' are isomorphic.

Proposition 4 *Images of representations ρ_2 and ρ'_2 are pro-2-groups.*

Proof We can assume that the images of both representations are contained in $\mathbb{GL}_2(\mathbb{Z}_2)$. It is enough to prove that the images of their mod 2 reductions have order 2 (since the kernel of the natural homomorphism $\mathbb{GL}_2(\mathbb{Z}/2^{k+1}\mathbb{Z}) \to \mathbb{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ is a 2-group). For primes $p \in \{7, 11, 13\}$ using Theorem 2 and an explicit model for the surface $\mathcal{S}$, we compute that

$$\text{trace}(\rho_2(\text{Frob}_p)) = \left(\frac{-1}{p}\right)\gamma(p) \text{ and } \det(\rho_2(\text{Frob}_p)) = \left(\frac{-6}{p}\right)p^2.$$

Moreover, if $\left(\frac{-6}{p}\right) = -1$, we find that $\gamma(p) = 0$ and the eigenvalues of $\rho_2(\text{Frob}_p)$ are $\pm p\sqrt{-1}$ from which it follows that mod 2 reduction of $\rho_2(\text{Frob}_p)$ has order 2. If $\left(\frac{-6}{p}\right) = 1$, then the eigenvalues mod 2 are equal, and mod 2 reduction of $\rho_2(\text{Frob}_p)$ is trivial.

Since the group $\mathbb{GL}_2(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to the symmetric group S_3, if we assume that the mod 2 image is not of order two, then it must be the whole group. In that case, denote by L a S_3 Galois extension of $\mathbb{Q}$ cut out by mod 2 reduction of ρ_2 (i.e. L is the fixed field of the kernel of the mod 2 reduction of ρ_2). Then L contains a unique quadratic field K which is unramified outside 2 and 3 in which 7 and 11 split and 13 is inert. It follows that $K = \mathbb{Q}(\sqrt{-6})$. We know by the Hermite-Minkowski theorem that there are finitely many S_3 extensions of $\mathbb{Q}$ unramified outside 2 and 3, and using LMFDB [16] we find that there is only one such field $\mathbb{Q}(x)$, where $x^6 - 3x^2 + 6 = 0$, whose Galois group contains $\mathbb{Q}(\sqrt{-6})$. This field contains a cubic field $F = \mathbb{Q}(s)$, where $s^3 + 3s - 2 = 0$. One finds that 7 is inert in F, hence $\rho_2(\text{Frob}_7)$ has order 3. This is impossible since $\text{trace}(\rho_2(\text{Frob}_7)) = 10$ is an even number which implies that mod 2 reduction of $\rho_2(\text{Frob}_7)$ has order 1 or 2. □

To apply Theorem 3 for $S = \{2,3\}$ we choose characters

$$\chi_1 = \left(\frac{-1}{p}\right), \chi_2 = \left(\frac{2}{p}\right), \chi_3 = \left(\frac{3}{p}\right),$$

and $G = \{\mathrm{Frob}_p : 31 \le p \le 73, \text{ for } p \text{ prime}\}$. Using Theorem 2 and (4) we can check that

$$\mathrm{trace}(\rho_2(g)) = \mathrm{trace}(\rho_2'(g)) \text{ and } \det(\rho_2(g)) = \det(\rho_2'(g)),$$

for all $g \in G$, hence Proposition 3 follows.

To prove Proposition 2 (and consequently Theorem 1), we need to show that representations ρ_ℓ and $\tilde{\rho}_\ell$ are isomorphic. In particular, it is enough to prove this claim for $\ell = 2$. By an argument similar to [10, Proposition 5.1.], it follows that ρ_2 is isomorphic to $\tilde{\rho}_2$ up to a twist by a quadratic character. Since both representations are unramified outside 2 and 3, this character is an element of the group generated by characters χ_1, χ_2 and χ_3. For every nontrivial χ from that group, we can find a prime $p > 3$ such that $\chi(p) = -1$, and numerically check that ASD congruence relation for the Fourier coefficients of $g(\tau)$

$$c_{mp^r} - \chi(p)\left(\frac{-1}{p}\right)\gamma(p)c_{mp^{r-1}} + \left(\frac{-6}{p}\right)p^2 c_{mp^{r-2}} \equiv 0 \pmod{p^{2r}},$$

does not hold for some choice of m and r. The claim follows.

All the computations in this paper were done in SageMath [17] and Magma [4].

6 Future work

It is natural to ask do similar congruences exist for the numbers $F(\frac{p-1}{n})$, when $n > 2$ and $p \equiv 1 \pmod{n}$ is a prime?

For example when $n = 3$, one can show that for $p \equiv 1 \pmod 3$ we have

$$F\left(\frac{p-1}{3}\right) \equiv \mathrm{trace}(\rho_\ell(\mathrm{Frob}_p)) \pmod{p},$$

where ρ_ℓ is a four-dimensional Galois representation associated to the 3-cover of the elliptic surface $\mathcal{W}$ defined by $t = s^3$ (see Section 4). More generally, the numbers $F\left(\frac{mp^r-1}{3}\right)$ satisfy the expected three term congruence relations.

We will investigate these congruences in a paper under preparation.

Here is another direction for future work. Almkvist, Zudilin and Cooper searched for parameters (A,B,C,D) such that

$$(n+1)^3 u_{n+1} - (2n+1)(An^2 + An + B)u_n + n(Cn^2 + D)u_{n-1} = 0, \quad (5)$$

with initial conditions $u_{-1} = 0$ and $u_0 = 1$, produces only integer solutions. For example, $(A,B,C,D) = (17,5,1,0)$ yields the Apery numbers a_n. In total, there are nine sporadic cases for (5). For each of these nine cases, is there a congruence similar to Theorem 1 except the modular form is of weight 4?

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