(O)BSD i Riemannova hipoteza

originalna

Definirajmo:

$$\mathrm{Prod}(E,x) = \prod_{\substack{p \mid N \\ p \le x}} \frac{1}{1 - a_p/p} \cdot \underbrace{\prod_{\substack{p \nmid N \\ p \le x}} \frac{1}{1 - a_p/p + 1/p}}_{\prod_{\substack{p \le x \\ p \nmid N}} \frac{1}{\# E(\mathbb{F}_p)/p}}$$

e.k. / $\mathbb{Q}$

konduktor N

OBSD: $\mathrm{Prod}(E,x) \sim \dfrac{C}{(\log x)^r}$

gdje je $r = \mathrm{rank}(E)$.

Q: Je li OBSD $\iff$ BSD ?

weak OBSD: $\mathrm{Prod}(E,x) \sim \dfrac{C}{(\log x)^r}$

za neki $r \in \mathbb{R}$ (ne nužno $r = \mathrm{rank}(E)$)

Teorem (Goldfeld; Sur les...)

konsti se modularnost

Neka je $E/\mathbb{Q}$ e.k. Ako je

(inicijalno je to bila pretp.)

$$\mathrm{Prod}(E,x) \sim C/(\log x)^r \text{ kad } x \to \infty$$

za neki $C>0$ i $r \in \mathbb{R}$ onda

!

i) $L(E,s)$ zadovoljava Riemannovu hipotezu
($tj.\ L(E,s) \neq 0$ za $\mathrm{Re}\, s > 1$)

ii) $r = \operatorname{ord}_{s=1} L(E,s)$

Eulerova konstanta $\gamma = 0.5772\ldots$

iii) $$C = \frac{L^{(r)}(E,1)}{r!} \cdot \frac{1}{\sqrt{2} \cdot e^{r\gamma}}$$

Napomena: Ako je $r=0$, teorem implicira:

ako formalan Eulerov produkt "za" $L(E,1)$ konvergira

onda ne konvergira u $L(E,1)$ nego u

$$L(E,1)/\sqrt{2} \quad !$$

nigdje ovdje se ne spominje alg. rang!

Što je s obratom?

Za $p \nmid N$ neka su α_p i β_p svojstvene vrijednosti "Frobeniusa": $\alpha_p + \beta_p = a_p$ i $\alpha_p \beta_p = p$

Definirajmo:

$$\psi_E(x) = \sum_{\substack{p^k \leq x \\ p \nmid N}} (\alpha_p^k + \beta_p^k) \log p \quad \Longleftrightarrow \quad \sum_{p^k \leq x}{}' \frac{\alpha_p^k + \beta_p^k}{k} = o(x)$$

Kuo, Murty

Teorem 1 (K. Conrad: Partial Euler Products...)

weak OBSD $\Longleftrightarrow \psi_E(x) = o(x \log x)$

$\Leftarrow$ alg. rang se nigdje ne spominje

Napomena: RH za $L(E,s)$ je ekvivalentna s $\psi_E(x) = O(x(\log x)^2)$ tako da je weak OBSD "jača" od RH.

$$\text{BSD} \overset{?}{\cancel{\Longrightarrow}} \text{OBSD}$$

$\hookrightarrow$ i bez tvrdnji o alg. rangu

Analogija s RH: Neka je $\psi(x) = \sum_{n \leq x} \Lambda(n)$.

Tada je RH $\iff$ $\psi(x) - x = O(\sqrt{x}(\log x)^2)$

dok je Montgomery "predložio" da vrijedi

bolja ocjena $\psi(x) - x = O(\sqrt{x}(\log\log\log x)^2)$.

Možda onda za $\psi_E(x)$ vrijedi $O(x(\log\log\log x)^2)$

što bi implicirало weak OBSD (bez dijela o

alg. rangu).

Od kuda dolazi $\psi_E(x)$ i slične sume?

Teorem?

$N_p = p+1-a_p = \frac{1}{p}(p-\alpha_p)(p-\beta_p)$

$p \nmid N$ $\Downarrow$

$$\Rightarrow \prod_{p \leq x}{}' = \prod_{p \leq x}{}' \left(1-\frac{\alpha_p}{p}\right)\left(1-\frac{\beta_p}{p}\right) \quad / -\log$$

$$\Rightarrow -\sum_{p \leq x} \log\left[\left(1-\frac{\alpha_p}{p}\right)\left(1-\frac{\beta_p}{p}\right)\right] = \sum_{p \leq x}{}' \sum_{k=1}^{\infty} \frac{\alpha_p^k + \beta_p^k}{k p^k}$$

Ili slično:

$$-\frac{L_E'(s)}{L_E(s)} = \sum_{n=1}^{\infty} \frac{c_n \Lambda(n)}{n^s}$$

gdje je
$$c_n = \begin{cases} \alpha_p^k + \beta_p^k & \text{ako } n=p^k \text{ i } p \nmid N \\ \alpha_p^k & \text{ako } n=p^k \text{ i } p \mid N \\ 0 & \text{inače.} \end{cases}$$

Standardna situacija iz analitičke teorije brojeva - koristimo Perronovu formulu jer nas zanima suma $\sum_{n=1}^{\infty} c_n \Lambda(n)$, odnosno $\sum (\alpha_p^k + \beta_p^k) \log p$

suma po nultočkama ρ od $L_E(s)$

$$\frac{1}{2\pi i} \int_C -\frac{L_E'(s)}{L_E(s)} \cdot \frac{x^s}{s} \, ds = -\sum_{|\gamma|<R} n_\rho \frac{x^\rho}{\rho} - \sum_{m<U} \operatorname{Res}_{s=-m} \left[\frac{L_E'(s)}{L_E(s)} \cdot \frac{x^s}{s} \right]$$

n_ρ = kratnost nultočke.

$\rho = \beta + i\gamma$

C = pravokutnik $c-iR, c+iR, -U+iR, -U-iR$

$\Longrightarrow$ Suma ovisi o nultočkama $L_E(s)$!

$$\sum_{n\le x} c_n \Lambda(n) = \ldots = -\sum_{|\gamma|<R} n_\rho \frac{x^\rho}{\rho} + \sum_{m<u} \frac{x^{-m}}{m}$$

error term

$$+ \frac{1}{2\pi i}\int_{C\setminus J_1} \frac{L_E'(s)}{L_E(s)}\cdot\frac{x^s}{s}\,ds + E(x,u,R)$$

$= \ldots$ $J_1 : c-iR \to c+iR$

Kim, Murty: From The Birch and Swinnerton-Dyer conjecture to Nagao's conjecture

Teorem: Ako limes

$$\lim_{x\to+\infty}\frac{1}{\log x}\sum_{p<x}\frac{a_p\log p}{p} \quad \text{postoji}$$

tada je RH za $L_E(s)$ točna i limes je jednak $-r + \frac{1}{2}$ gdje je $r = \mathrm{ord}_{s=1} L_E(s)$