$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{c \cdot q^2}$$

$$\uparrow$$

$c > 1$

Teorem 6.8. Neka su $\frac{p_{n-1}}{q_{n-1}}$ i $\frac{p_n}{q_n}$ dvije uzastopne konv. od α.

Tada barem jedna od njih zadovoljava nejednakost $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.

Dokaz: Budući da se α nalazi između dvije uzastp. konvergente.

$$\underbrace{\left|\alpha - \frac{p_n}{q_n}\right|} + \underbrace{\left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right|} = \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n \cdot q_{n-1}} < \underbrace{\frac{1}{2q_n^2}} + \underbrace{\frac{1}{2q_{n-1}^2}}$$

Prema tome, vrijedi

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2} \text{ ili } \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{2q_{n-1}^2} \quad \blacksquare$$

Teorem 6.9. (Borel)

Neka su $\frac{p_{n-2}}{q_{n-2}}$, $\frac{p_{n-1}}{q_{n-1}}$ i $\frac{p_n}{q_n}$ tri uzastopne konverg. od α.

Tada barem jedna od njih zadovoljava nejednakost

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\, q^2}.$$

$(\alpha_i = [a_i, \alpha_{i+1}])$

<u>Dokaz:</u> Neka je $\alpha = [a_0, a_1, a_2, \ldots]$, $\alpha_i = [a_i, a_{i+1}, \ldots]$ i $\beta_i = \frac{q_{i-2}}{q_{i-1}}$.

Dokazat ćemo

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2(\underbrace{\alpha_{n+1} + \beta_{n+1}})}.$$

Imamo $\alpha = [a_0, a_1, \ldots, a_n, \alpha_{n+1}]$ pa je

$$q_n\alpha - p_n = q_n \frac{\alpha_{n+1} \cdot p_n + p_{n-1}}{\alpha_{n+1} \cdot q_n + q_{n-1}} - p_n \overset{\text{(d.z.)}}{=} \frac{(-1)^n}{\alpha_{n+1} q_n + q_{n-1}} \qquad (**)$$

Stoga je

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2(\alpha_{n+1} + \beta_{n+1})}.$$ Pokazat ćemo da

ne postoji $n \in \mathbb{N}$ t.d. za $i = n-1, n, n+1$ vrijedi

$$\alpha_i + \beta_i \le \sqrt{5} \qquad (*).$$

Pretp. da $(*)$ vrijedi za $i = n-1, n$. Tada je

$$\underset{[a_{n-1}, \alpha_n]}{\alpha_{n-1}} = a_{n-1} + \frac{1}{\alpha_n} \quad \text{i} \quad \frac{1}{\beta_n} = \frac{q_{n-1}}{q_{n-2}} = \frac{a_{n-1} \cdot q_{n-2} + q_{n-3}}{q_{n-2}} = a_{n-1} + \frac{q_{n-3}}{q_{n-2}} = a_{n-1} + \beta_{n-1}$$

slijedi $\frac{1}{\alpha_n} + \frac{1}{\beta_n} = \alpha_{n-1} + \beta_{n-1} \le \sqrt{5}$. $\qquad \beta_n \ge \frac{\sqrt{5}-1}{2} \Rightarrow \beta_n > \frac{\sqrt{5}-1}{2}$

$$\Rightarrow 1 = \alpha_n \cdot \frac{1}{\alpha_n} \le \left(\sqrt{5} - \beta_n\right)\left(\sqrt{5} - \frac{1}{\beta_n}\right) \Leftrightarrow \beta_n^2 - \sqrt{5}\beta_n + 1 \le 0$$

$\alpha_n + \beta_n \le \sqrt{5}$

Kad bi (*) vrijedilo za $i = n, n+1 \Rightarrow \beta_{n+1} > \frac{\sqrt{5}-1}{2}$

$$\Rightarrow \underset{n}{1} \leq a_n = \frac{q_n}{q_{n-1}} - \frac{q_{n-2}}{q_{n-1}} = \frac{1}{\beta_{n+1}} - \beta_n < \frac{2}{\sqrt{5}-1} - \frac{\sqrt{5}-1}{2} = 1$$

d.z. $\Rightarrow \Leftarrow$

jer: $\frac{a_n \cdot q_{n-1} + q_{n-2}}{q_{n-1}} = \frac{q_n}{q_{n-1}}$

$\blacksquare$.

Teorem 6.10. Pretp. da α ima razvoj u verižni razlomak oblika $\alpha = [a_0, a_1, \ldots, a_k, 1, 1, 1, \ldots, 1, \ldots]$

Tada je $\underline{\lim_{n \to \infty} \left| \alpha - \frac{p_n}{q_n} \right| \cdot q_n^2 = \frac{1}{\sqrt{5}}}$

$\forall C > \sqrt{5}$ i

To znači da za takav α postoji konačno mnogo $\frac{p_n}{q_n}$ t.d.

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{\cancel{\sqrt{5}} \cdot q_n^2 \cdot C}$$

Dokaz: skripta.

Teorem 6.11. (Legendre) Neka su p i $q \in \mathbb{Z}$ t.d. $q \geq 1$ i $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$. Tada je $\frac{p}{q}$ neka konvergenta od α.

Dokaz: Možemo pretp. da je $\alpha \neq \frac{p}{q}$. Tada je

$$\alpha - \frac{p}{q} = \frac{\varepsilon \cdot \mu}{q^2}, \text{ gdje je } 0 < \mu < \frac{1}{2} \text{ i } \varepsilon = \pm 1.$$

Neka je $\frac{p}{q} = [b_0, \ldots, b_{n-1}]$ razvoj od $\frac{p}{q}$ u verižni razlomak gdje je n odabran tako da vrijedi $(-1)^{n-1} = \varepsilon$. To uvijek možemo postići jer je

$$[a_0, a_1, \ldots, a_m] = [a_0, a_1, \ldots, a_m - 1, 1]$$

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_m - 1 + \frac{1}{1}}}}$$

Def.: ω s

$$\alpha = \frac{\omega p_{n-1} + p_{n-2}}{\omega q_{n-1} + q_{n-2}}.$$ Tada je $\alpha = [b_0, b_1, \ldots, b_{n-1}, \omega]$.

Dovoljno je pokazati da je $\omega > 1$. Po formuli (**)

$$\frac{\varepsilon \vartheta}{q^2} = \alpha - \frac{p}{q} = \alpha - \frac{p_{n-1}}{q_{n-1}} = \frac{1}{q_{n-1}}\left(\alpha q_{n-1} - p_{n-1}\right) = \frac{1}{q_{n-1}} \cdot \frac{(-1)^{n-1}}{\omega q_{n-1} + q_{n-2}}$$

$$\Rightarrow \vartheta = \frac{q_{n-1}}{\omega q_{n-1} + q_{n-2}} \Rightarrow \omega = \frac{1}{\vartheta} - \underbrace{\frac{q_{n-2}}{q_{n-1}}}_{\text{d.š.} \leq 1} \Rightarrow \omega > 2 - 1 = 1 \checkmark$$

Neka je $\omega = [b_n, b_{n+1}, \ldots]$. Budući da je $\omega > 1$, svi b_j-ovi su prirodni brojevi.

$\Rightarrow \alpha = [b_0, \ldots, b_{n-1}, \omega] = [b_0, b_1, \ldots, b_n, \ldots]$ je razvoj u jednostavni verižni razlomak od α, $\frac{p}{q} = \frac{p_{n-1}}{q_{n-1}} = [b_0, \ldots, b_{n-1}]$ je konvergent od α ∎

Teorem 6.12. (Hurwitz).

(i) Za svaki iracionalni α postoji ∞ $\frac{p}{q}$ t.d.

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5} \cdot q^2}$$

(ii) Tvrdnja (i) ne vrijedi ukoliko se $\sqrt{5}$ zamijeni

s bilo kojom konstantom $A > \sqrt{5}$ $\rightsquigarrow$ $\alpha = [1, 1, 1, 1, \dots, 1]$

(d.z.)

(Tm. 6.10. $\left|\alpha - \frac{p_n}{q_n}\right| \cdot q_n^2 \to \sqrt{5}$ $\rightsquigarrow$)