$$\pi(x) = \#\{p \text{ prost} \mid p < x\}$$

Teorem o prostim brojevima:

$$\pi(x) \sim \frac{x}{\ln x} \quad \left(\Leftrightarrow \lim_{x \to +\infty} \frac{\pi(x) \cdot \ln x}{x} = 1\right)$$

Gauss (Hadamard, Poussin 1896) ⇝ Riem. $\zeta(s)$

(1949. Selberg.)

Mi ćemo dokazati Čebiševljev teorem:

<u>Teorem:</u> Postoje konstante $A, B > 0$ t.d.

za sve x-eve veće od neke konst.

$$A \frac{x}{\ln x} \leq \pi(x) \leq B \cdot \frac{x}{\ln x} \quad \forall x >> 0$$

Def. Von Mangeldtova fj. $\Lambda(n)$, za $n \in \mathbb{N}$ je def. s

$$\Lambda(n) = \begin{cases} \ln p & \text{ako je } n = p^m \\ 0 & \text{inače.} \end{cases}$$

główny związek: $\dfrac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \dfrac{\Lambda(n)}{n^s}$ $\rightsquigarrow$ $\psi(x) := \sum_{n \le x} \Lambda(n)$

$\updownarrow$

$\pi(x) = \sum_{p \le n} 1 \quad \longleftrightarrow \quad \Theta(x) = \sum_{p < x} \ln p$

$f(x) = O(g(x)) \iff f(x) \le \underset{\sim}{K} \cdot g(x) \quad \forall x >> 0.$

Lema 1: $\Theta(x) = O(x)$ ...

Dokaz:

$$\binom{2n}{n} \le (1+1)^{2n} = 2^{2n}$$

$$\binom{2n}{n} = \frac{2n(2n-1)\cdots(2n-(n-1))}{n!}$$

$\Downarrow$

$n < p < 2n \Rightarrow p \mid \binom{2n}{n}$

$$\Rightarrow \prod_{n<p<2n} p \,\Big|\, \binom{2n}{n} \le 2^{2n} \quad / \ln$$

$$\sum_{n<p<2n} \ln p \le 2n \ln 2$$

$$\boxed{\Theta(2n) - \Theta(n) \le 2n \ln 2}$$

$$
+\begin{cases}
\theta(2n)-\cancel{\theta(n)} \le 2n\ln 2 \\
\cancel{\theta(n)}-\cancel{\theta(\tfrac{n}{2})} \le n\ln 2 \\
\cancel{\theta(\tfrac{n}{2})}-\theta(\tfrac{n}{4}) \le \tfrac{n}{2}\ln 2 \\
\vdots
\end{cases}
$$

teleskopiranj

$$\Longrightarrow \theta(2n) \le 4n\ln 2$$

$$\Longrightarrow \theta(x) = O(x) \quad \blacksquare$$

Lema 3 $\Rightarrow \exists K>0$ t.d.

$$\underbrace{\sum_{p<\sqrt{x}} \ln p}_{} + \sum_{\sqrt{x}<p<x} \ln \cancel{m} \le Kx \quad \forall x>>0$$

$$\sum_{p<\sqrt{x}} \ln p = O(\sqrt{x}\cdot \ln\sqrt{x}) = O(\sqrt{x}\ln x)$$

$$\Rightarrow \sum_{\sqrt{x}<p<x} \ln p \le Kx + O(\sqrt{x}\ln x)$$

$$\sum_{\sqrt{x}<p<x} \ln p \ge (\pi(x)-\pi(\sqrt{x}))\ln\sqrt{x}$$

$$\Rightarrow \quad Kx + O(\sqrt{x}\ln x) \geq \big(\pi(x) - \pi(\sqrt{x})\big)\ln\sqrt{x} \geq \big(\pi(x) - \sqrt{x}\big)\ln\sqrt{x}$$

$$\Rightarrow \quad \pi(x)\cdot\frac{1}{2}\ln x \leq \frac{1}{2}\sqrt{x}\ln x + Kx + O(\sqrt{x}\ln x) \quad \Big/ : \ln x\cdot\frac{1}{2}$$

$$\Rightarrow \quad \pi(x) \leq \ln x + 2K\frac{x}{\ln x} + O(\sqrt{x}) = 2K\frac{x}{\ln x} + O(\sqrt{x})$$

$\Rightarrow \exists B, M' > 0$

s.t. $\quad \pi(x) \leq B\cdot\dfrac{x}{\ln x} \quad \forall x > M'$.

Lema 2: $\ln n = \sum_{d|n} \Lambda(d)$

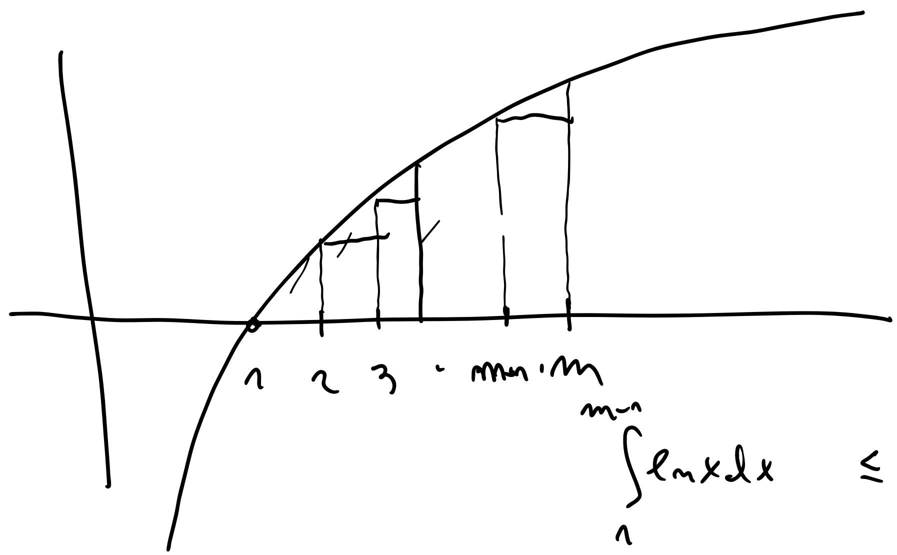
Dokaz: $n = \prod p_i^{\alpha_i} \Rightarrow \ln n = \sum \alpha_i \ln p_i = \sum_{d|n} \Lambda(d)$

$(d = p_i^k)$

Lema 3: $\sum_{n \le x} \ln n = x \ln x - x + O(\ln x)$

Dokaz: Cauchyjev integralni kriterij

$$\sum_{n=1}^{m-1} \ln n \le m \ln m - m \le \sum_{n=1}^{m} \ln n$$



$$\int_1^{m-1} \ln x \, dx \le \ln 1 + \ln 2 + \dots + \ln(m-1) \le \int_1^{m} \ln x \, dx = m \ln m - m$$

$$\Rightarrow \quad m\ln m - m - \sum_{n=1}^{m-1} \ln n \le \ln m \quad \Rightarrow \quad \sum_{n=1}^{m-1} \ln n = m \ln m - m + O(\ln m) \quad / + \ln m$$

$$\Rightarrow \sum_{n=1}^{m} \ln n = m\ln m - m + O(\ln m) \;\ldots \;\Rightarrow\; \ldots \sum_{n\le x} \ln x = x \ln x - x + O(\ln x)$$

d.z.

$\square$

Lema 2 i Lema 3 $\Rightarrow$

$$\sum_{n\le x} \ln n = \sum_{n\le x} \sum_{d\mid n} \Lambda(d) = \sum_{d\le x} \Lambda(d) \sum_{\substack{n\le x \\ d\mid n}} 1 = \sum_{d\le x} \Lambda(d)\cdot \left\lfloor \frac{x}{d} \right\rfloor$$

$$= \sum_{d\le x} \Lambda(d)\left(\frac{x}{d} + O(1)\right) = x \sum_{d\le x} \frac{\Lambda(d)}{d} + O(1)\cdot \underbrace{\sum_{d\le x} \Lambda(d)}_{\psi(x)}$$

Lema 4: $\psi(x) = O(x)$

<u>Dokaz:</u> $\psi(x) - \vartheta(x) = \sum_{\substack{p^k < x \\ k>1}} \ln p < \sum_{\substack{p^k < x \\ k>1}} \ln x = \ln x \sum_{\substack{p^k < x \\ k>1}} 1$

Q: Koliko ima potencija prostih brojeva $p^k \le x$ gdje je $k > 1$?

$$p^k < x \overset{k>1}{\Rightarrow} p < \sqrt{x}$$

$$\Rightarrow \sum_{\substack{p^k < x \\ k>1}} 1 < \sqrt{x} \cdot \ln x \Rightarrow \psi(x) = \theta(x) + O(\sqrt{x} \ln x)$$

$$= O(x).$$

$$\boxed{\psi(x) = \Psi(x) = \sum_{n \le x} \Lambda(n)}$$

$$x \sum_{d \le x} \frac{\Lambda(d)}{d} + O(1) \cdot \sum_{d \le x} \Lambda(d) = x \sum_{d \le x} \frac{\Lambda(d)}{d} + O(x) \quad / : x$$

$$\|$$

$$\sum_{n \le x} \ln n$$

$$\|$$

$$x \ln x - x + O(\ln x)$$

$$\Rightarrow \sum_{d \le x} \frac{\Lambda(d)}{d} + O(1) = \ln x - 1 + O\left(\frac{\ln x}{x}\right)$$

$$\Rightarrow \sum_{d \le x} \frac{\Lambda(d)}{d} = \ln x + O(1)$$

$$\Rightarrow \boxed{\sum_{p\le x} \frac{\ln p}{p} = \ln x + O(1)}$$

zakaj? (D.Z.)

(red $\sum_{d\le x} \frac{\Lambda(d)}{d}$ ni konvergent.)

d ni prost $\rightsquigarrow \sum \frac{\ln p}{p^2} \ldots$

tj. $\exists\, C>0$ t.d. $\forall x >> 0$

$$\sum_{p\le x} \frac{\ln p}{p} - \ln x \le C \qquad \forall x > c'$$

Posebno za $A>0$ .. $x = \frac{x}{A}$

$$\Rightarrow \sum_{p\le \frac{x}{A}} \frac{\ln p}{p} - \ln\frac{x}{A} \le C \quad ; \quad \forall x > c' \cdot A$$

Za A dovolj velik obstaja $C''>0$ t.d.

$$\sum_{\frac{x}{A}<p<x} \frac{\ln p}{p} > C'' \quad \forall x >> 0 \quad \text{(d.z.)}$$

$$\Rightarrow c'' < \sum_{\frac{x}{A} < p < x} \frac{\ln p}{p} < \left(\pi(x) - \pi(x/A)\right) \frac{\ln \frac{x}{A}}{\frac{x}{A}} < \pi(x) \cdot \frac{\ln}{x/A}$$

$$\Rightarrow \pi(x) > \frac{c''}{A} \cdot \frac{x}{\ln x} \quad \forall x >> 0 .$$

$\blacksquare$.