$$\sum_{n\le x}\varphi(n) \sim f(x) \quad \text{ako i samo ako} \quad \lim_{x\to+\infty}\frac{\sum_{n\le x}\varphi(n)}{f(x)} = 1.$$

Lema 5.5. $$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Dokaz: Ideja: $]_2$ za $x\in[0,\pi]$ $0\le \sin x < x < \operatorname{tg} x$ slijedi

$$\Rightarrow \quad 0\le \operatorname{ctg} x < \frac{1}{x} < \frac{1}{\sin x} \quad /^2$$

$$\frac{1}{\sin^2 x}$$

$$\Rightarrow \quad 0\le \operatorname{ctg}^2 x < \frac{1}{x^2} < 1+\operatorname{ctg}^2 x$$

za $x=n$ $\Rightarrow \operatorname{ctg}^2 n < \frac{1}{n^2} < 1+\operatorname{ctg}^2 n \Rightarrow \sum_{n=1}^{N}\operatorname{ctg}^2 n < \sum_{n=1}^{N}\frac{1}{n^2} < \sum_{n=1}^{N}(1+\operatorname{ctg}^2 n)$

$x=\frac{n\pi}{2N+1}$ .....

Pokažimo da $\forall N\in\mathbb{N}$ vrijedi

$$\sum_{n=1}^{N}\operatorname{ctg}^2\frac{n\pi}{2N+1}=\frac{N(2N-1)}{3}$$

Iz de Moivre formule $\left((e^{i x})^m = e^{i x m} ,\ (\cos x + i \sin x)^m = \cos mx + i \sin mx \right)$

$$\cos mx + i \sin mx = \sin^m x \, (i + \operatorname{ctg} x)^m$$

$m = 2N+1$ imag. dio pa je

$$\sin (2N+1) x = \sin^{2N+1} x \, F(\operatorname{ctg}^2 x) \quad \text{gdje je}$$

$$F(x) = \binom{2N+1}{1} x^N - \binom{2N+1}{3} x^{N-1} + \dots + (-1)^N$$

Ako je $x = \frac{n\pi}{2N+1}$ onda $0 = F(\operatorname{ctg}^2 x)$ pa vidimo

da su $\operatorname{ctg}^2 \frac{n\pi}{2N+1}$; $n = 1, \dots, N$ (sve) nultočke polinoma F.

Po Vietovim formulama je sada

$$\sum_{n=1}^{N} \operatorname{ctg}^2 \frac{n\pi}{2N+1} = \frac{\binom{2N+1}{3}}{\binom{2N+1}{1}} = \frac{N(2N-1)}{3}.$$

Uvrstimo li $x = \frac{n\pi}{2N+1}$ u $\operatorname{ctg}^2 x < \frac{1}{x^2} < 1 + \operatorname{ctg}^2 x$

i sumiramo dobivamo

$$\frac{N(2N-1)}{3} < \sum_{n=1}^{N} \frac{(2N+1)^2}{n^2\pi^2} < N + \frac{N(2N-1)}{3} \quad \Big/ \cdot \frac{\pi^2}{(2N+1)^2}$$

$$\begin{array}{ccccc}
\frac{\pi^2}{3}\cdot\frac{2N^2-N}{4N^2+4N+1} & < & \sum\limits_{n=1}^{N}\frac{1}{n^2} & < & \frac{\pi^2}{3}\cdot\frac{2N^2+2N}{4N^2+4N+1} \\
\Big\downarrow {\scriptstyle N\to+\infty} & & \Big| & & \Big\downarrow {\scriptstyle N\to+\infty} \\
\frac{\pi^2}{6} & \le & \sum\limits_{n=1}^{\infty}\frac{1}{n^2} & = & \frac{\pi^2}{6}
\end{array}$$

$\blacksquare$.

Propozicija:

1) $\displaystyle\sum_{n\le x} \tau(n) = x\ln x + O(x)$

2) $\displaystyle\sum_{n\le x} \sigma(n) = \frac{1}{12}\pi^2 x^2 + O(x\ln x)$

3) $\displaystyle\sum_{n\le x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x\ln x)$

$f(x) = O(g(x))$ ako $\exists\, C>0$ t.d. $|f(x)| \le C\cdot g(x)\ \forall x$

ogr.

npr. $\lfloor x\rfloor = x - \{x\} = x + O(1)$

također $\displaystyle\sum_{n\le x} \frac{1}{n} = \ln x + O(1)$

Dokaz:

$O(x) + O(x) = O(x)$

1) $$\sum_{n\le x} \tau(n) = \sum_{n\le x}\sum_{d\mid n} 1 = \sum_{d\le x} 1\cdot\left\lfloor \frac{x}{d}\right\rfloor = \sum_{d\le x}\left(\frac{x}{d} + O(1)\right)$$

$$= x\cdot \sum_{d\le x}\frac{1}{d} + O(x) = x\cdot(\ln x + O(1)) + O(x)$$

$$= x\cdot\ln x + O(x)$$

2) $$\sum_{n\le x}\sigma(n) = \sum_{n\le x}\sum_{d\mid n}\frac{n}{d} = \sum_{d\le x}\frac{1}{\cancel{d}}\sum_{m\le \frac{x}{d}} m\cdot\cancel{d} = \sum_{d\le x}\sum_{m\le\frac{x}{d}} m$$

Nadalje je $$\sum_{m\le\frac{x}{d}} m = \frac{1}{2}\left\lfloor\frac{x}{d}\right\rfloor\cdot\left(\left\lfloor\frac{x}{d}\right\rfloor+1\right) = \frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right) \quad \text{(d. z.)}$$

Sada je $$\sum_{d\le x}\frac{1}{d^2} - \underbrace{\sum_{d=1}^{\infty}\frac{1}{d^2}}_{=\pi^2/6} \overset{\text{dz.}}{=} O\left(\int_x^{\infty}\frac{1}{t^2}\,dt\right) = O\left(\frac{1}{x}\right).$$

$$\sum_{n\le x}\sigma(n) = \sum_{d\le x}\left[\frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)\right] = \frac{\pi^2}{12}\cdot x^2 + O(x) + O(x\ln x)$$

$$= \frac{\pi^2}{12}x^2 + O(x\ln x)$$

3) Vrijedi

$$\sum_{n\le x}\varphi(n)=\sum_{n\le x}\sum_{d|n}\mu(d)\frac{n}{d}=\sum_{d\le x}\frac{\mu(d)}{\cancel{d}}\sum_{m\le \frac{x}{d}} m\cdot \cancel{d} \quad \frac{1}{2}\left(\frac{x}{d}\right)^2+O\left(\frac{x}{d}\right)$$

$$\sum_{d\le x}\frac{\mu(d)}{d^2}=?$$

Nadalje, $\sum_{d\le x}\frac{\mu(d)}{d^2}=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O\left(\frac{1}{x}\right)$

d.z. (Cauch. ab. fn.)

$$\sum_{d_1=1}^{\infty}\frac{\mu(d_1)}{d_1^2}\cdot\underbrace{\sum_{d_2=1}^{\infty}\frac{1}{d_2^2}}_{=\frac{\pi^2}{6}}=[m=d_1d_2]=\sum_{m=1}^{\infty}\frac{1}{m^2}\cdot\sum_{d|m}\mu(d)=\sum_{m=1}^{\infty}\frac{1}{m^2}\nu(m)=1$$

d.z.

$$\Rightarrow\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}=\frac{6}{\pi^2}, \quad \Longrightarrow \sum_{n\le x}\varphi(n)=\sum_{d\le x}\left[\frac{\mu(d)}{2}\cdot\left(\frac{x}{d}\right)^2+O\left(\frac{x}{d}\right)\right]$$

$$=\frac{3}{\pi^2}\cdot x^2+O(x\ln x)$$

$(+O(x))$

■

Napomena: Budući da je

$$\sum_{n\le x}\varphi(n)\sim\frac{3}{\pi^2}x^2 \quad \text{i} \quad \sum_{n\le x} n \sim \frac{1}{2} x^2$$

prethodni rezultat se može interpretirati i tako da kažemo da je vjerojatnost da su dva "slučajno" odabrana prirodna broja rel. prosta jednaka

$$\lim_{N\to\infty}\frac{\sum_{n\le N}\varphi(n)}{\sum_{n\le N} n} \sim \frac{\frac{3}{\pi^2}\cdot \cancel{N^2}}{\frac{1}{2}\cdot\cancel{N^2}} = \frac{6}{\pi^2}\approx 0.6079\ldots$$

$\blacksquare$